Emergent centrality in rank-based supplanting process

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We propose a stochastic process of interacting many agents, which is inspired by rank-based supplanting dynamics commonly observed in a group of Japanese macaques. In order to characterize the breaking of permutation symmetry with respect to agents' rank in the stochastic process, we introduce a rank-dependent quantity, *overlap centrality*, which quantifies how often a given agent overlaps with the other agents. We give a sufficient condition in a wide class of the models such that overlap centrality shows perfect correlation in terms of the agents' rank in the zero-supplanting limit. We also discuss a singularity of the correlation in the case of interaction induced by a Potts energy.

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I. INTRODUCTION

One of the promising candidates for going a step further in studying a many-body system is to construct a lattice model of interacting elements or agents which describes a many-body system. This strategy is applied not only to equilibrium systems [1] but also nonequilibrium systems [2]. Such attention has been given to nonequilibrium lattice models such as driven lattice gas [3], the asymmetric simple exclusion process (ASEP) [4], the ABC model [5], the zero range process [6,7], etc., where an emergent macroscopic property such as phase transition is one of the topics to be elucidated. Recent developments on active matter have focused on experimentally realizable systems such as colloidal or biological systems showing various phase transitions such as flocking transition, lane formation, or motility-induced phase separation [8–10]. In the framework of statistical physics, it is of interest to look for an analytically tractable and minimal model for such phenomena [11–13].

Apart from such model-based studies, in the context of network theory, the concept of centrality plays one of the important roles in studying a given network induced by a many-body system consisting of inhomogeneous agents. Centrality has been particularly used in the literature of social network analysis to characterize which element on a network is the most influential. Depending on the purpose of network analysis, various measures of centrality such as degree centrality, closeness centrality, PageRank, eigenvector centrality, etc., have been proposed and found to be useful to characterize network structures [14–17].

As an example of inhomogeneous agents, primate species often live through interacting with members in a group [18]. It has been reported in a primate species that the individuals, which we call agents, with high rank in social dominance tend to have high rank also in the eigenvector centrality of adjacency matrix for a graph composed from agents' positions [19,20]. In particular, Japanese macaques (*Macaca fuscata*) form a group living together, and each agent in a group has its rank along the linear social dominance in the group, leading to rank-dependent repulsion between two agents [21,22]. This is the so-called *supplanting* phenomenon which we mainly focus on.

In this paper, we focus on one of the intersections between lattice models and network theory from the viewpoint of inhomogeneous agents. We propose a type of nonequilibrium lattice model, which is inspired by a supplanting phenomenon occurring between two agents in a group of Japanese macaques. The main objective of this paper is to show that a certain type of macroscopic correlation appears when a supplanting process, which is a rank-based interaction with broken detailed balance, is added to an equilibrium system. It turns out that this problem can be mapped to computing a type of centrality, which we call *overlap centrality*, for a complete graph derived naturally from the correlations of agents' positions.

This paper consists of five sections. In Sec. II we introduce a class of models with broken permutation symmetry that we study in this paper. In Sec. III focusing on the case of the model where the interaction induced by the Potts energy is assumed, we provide a brief review of the equilibrium properties, introduce overlap centrality, and compute it by exact diagonalization of transition matrix. In Sec. IV we provide the proof for the main result that overlap centrality characterizing how often a given agent overlaps with the other agent shows perfect correlation with respect to the ranking of the given agent in the zero-supplanting limit. This result holds rather generally, which is not limited to the case where the Potts energy is assumed as the source of an equilibrium interaction. Further, a conjecture about the existence of a singularity of the correlation is discussed for the case of the Potts energy. In Sec. V as concluding remarks, we summarize the results and some subjects of future considerations.

II. MODEL

Let $N \geqslant 2$ be the number of agents, and $L \geqslant 3$ be the length of the one-dimensional lattice $X := \mathbb{Z}/L\mathbb{Z} =$

$\{0, 1, \ldots, L-1\}$. Let us denote by $i \in \{1, 2, \ldots, N\}$ an agent, and by the integer $x_i \in X$ the position of agent i. We also regard the number i identifying an agent as the *rank* of that agent. We say that rank i is *higher* (*lower*) than j if $i < j$ ($i > j$); for example, rank i is higher than rank $i+1$. Let us also write the collection of elements a_i labeled by $1 \leqslant i \leqslant n$ as $(a_i)_{i=1}^n$ or the bold symbol $\boldsymbol{a}$. In particular we write a set of positions of agents as $\boldsymbol{x} = (x_i)_{i=1}^N$. Hereafter, we call $\boldsymbol{x} = (x_i)_{i=1}^N$ a *configuration*. We consider a hopping map $f_i^\pm$ such that $f_i^\pm \boldsymbol{x} := (x_j \pm \delta(i,j))_{j=1}^N$, where $\delta(i,j)$ is the Kronecker delta. Note that the periodic boundary condition in terms of positions is automatically assumed by definition of X.

A. Equilibrium dynamics

We consider a general class of energy function $E(\boldsymbol{x}) = E(x_1, x_2, \ldots, x_N)$ which is permutation symmetric in the following sense:

$$E(x_1, x_2, \ldots, x_N) = E(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) \tag{1}$$

for any permutation $\sigma \in \mathfrak{S}_N$ of N elements, where $\mathfrak{S}_N$ is the symmetric group of order N. In addition, let β be a parameter determining the magnitude of the energy including the sign $\pm$.

As an example of the models belonging to the above class, one can consider the following L-state Potts energy $E(\boldsymbol{x})$ on the complete graph where each agent connects with all agents [23]:

$$E(\boldsymbol{x}) = -\frac{2(L-1)\log(L-1)}{L-2}\frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N \delta(x_i, x_j). \tag{2}$$

This case means that an agent interacts with the other agents only if they have overlaps. In this sense, this Potts model on the complete graph is equivalent to the agents with an on-site interaction in one dimension, which may be a simple model to describe interacting agents. The coefficient of (2) is adjusted so that the phase transition point β_c in the equilibrium state is equal to 1, which we will discuss in more detail in Sec. III A.

Let us consider a Markov process with discrete time t, where during one time step between t and $t+1$, only one of the following possible transitions may occur. The transition probability $T_0(\boldsymbol{x} \to f_i^\pm \boldsymbol{x})$ from each configuration $\boldsymbol{x}$ to the configuration $f_i^\pm \boldsymbol{x}$ for any agent i is

$$T_0(\boldsymbol{x} \to f_i^\pm \boldsymbol{x}) = \frac{1}{2N}\frac{1}{1+\exp[\beta \mathcal{D}_i^\pm E(\boldsymbol{x})]}, \tag{3}$$

where

$$\mathcal{D}_i^\pm E(\boldsymbol{x}) := E(f_i^\pm \boldsymbol{x}) - E(\boldsymbol{x}). \tag{4}$$

This leads to that the joint probability $P_t(\boldsymbol{x})$ of configuration $\boldsymbol{x}$ at time t satisfies the following master equation:

$$P_{t+1}(\boldsymbol{x}) = \sum_{\boldsymbol{x}' \neq \boldsymbol{x}} P_t(\boldsymbol{x}')T_0(\boldsymbol{x}' \to \boldsymbol{x}) + P_t(\boldsymbol{x})\left[1 - \sum_{\boldsymbol{x}' \neq \boldsymbol{x}} T_0(\boldsymbol{x} \to \boldsymbol{x}')\right], \tag{5}$$

where the summation over $\boldsymbol{x}'$ is done for all of the possible configurations such that $T_0(\boldsymbol{x} \to \boldsymbol{x}')$ and $T_0(\boldsymbol{x}' \to \boldsymbol{x})$ are defined above.

The Gibbs distribution

$$P_{\mathrm{can}}(\boldsymbol{x}) := \frac{1}{Z_N(\beta)}\exp[-\beta E(\boldsymbol{x})], \tag{6}$$

where $Z_N(\beta) := \sum_{\boldsymbol{x}} \exp[-\beta E(\boldsymbol{x})]$, is the stationary solution $P_{\mathrm{st}}(\boldsymbol{x})$ of the master equation (5), satisfying

$$\sum_{\boldsymbol{x}'} P_{\mathrm{st}}(\boldsymbol{x}')T_0(\boldsymbol{x}' \to \boldsymbol{x}) = P_{\mathrm{st}}(\boldsymbol{x})\sum_{\boldsymbol{x}' \neq \boldsymbol{x}} T_0(\boldsymbol{x} \to \boldsymbol{x}'), \tag{7}$$

because the Gibbs distribution satisfies the detailed balance condition:

$$P_{\mathrm{can}}(\boldsymbol{x})T_0(\boldsymbol{x} \to \boldsymbol{x}') = P_{\mathrm{can}}(\boldsymbol{x}')T_0(\boldsymbol{x}' \to \boldsymbol{x}), \tag{8}$$

for any pair $\boldsymbol{x}, \boldsymbol{x}'$ realized by the above dynamics.

B. Broken detailed balance by supplanting

Next, we consider to add a supplanting process to the equilibrium dynamics introduced above, which breaks the detailed balance condition.

Let us imagine a process in which an agent i hops to a position $y_i = x_i \pm 1$ from position x_i in accordance with the equilibrium transition probability T_0, and then an agent at y_i, which has a rank j such that $i < j$, is stochastically forced to hop to the position $y_i + 1$ for $d = +$, or $y_i - 1$ for $d = -$. In this process, an agent with a higher rank i supplants another agent with a lower rank j. This is why we call such a process the *supplanting process*. Note that the configuration $\boldsymbol{x}$ turns to be the configuration $f_j^d f_i^\pm \boldsymbol{x}$ through the whole supplanting process. See Fig. 1 for a graphical reference of the process. For convenience, let us introduce the following set:

$$S(\boldsymbol{x}, i, \pm) := \{i < j \leqslant N \mid x_j = x_i \pm 1\}, \tag{9}$$

which is a set of every agent whose rank is lower than i at position $x_i \pm 1$. That is, those agents could be supplanted by the agent i when the agent i hops to position $x_i \pm 1$.

Suppose that every agent in $S(\boldsymbol{x}, i, \pm)$ has the same chance to be chosen as the supplanted one, and that the direction d of hopping by supplanting is determined with equal probability $1/2$. Explicitly, we define the transition probability from $\boldsymbol{x}$ to $f_j^d f_i^\pm \boldsymbol{x}$ for $d \in \{+, -\}$ and $j \in S(\boldsymbol{x}, i, \pm)$:

$$T\left(\boldsymbol{x} \to f_j^d f_i^\pm \boldsymbol{x}\right) = \frac{p}{2}\frac{1}{1 + p\#S(\boldsymbol{x}, i, \pm)}T_0(\boldsymbol{x} \to f_i^\pm \boldsymbol{x}), \tag{10}$$

where $p \in \mathbb{R}_+ := [0, \infty)$ is the parameter of supplanting rate and $\#S$ is the number of all the elements of a set S. When $p \to 0$, supplanting rarely occurs, and when $p \to \infty$, supplanting almost always occurs. Note that at most one agent is supplanted in a single transition regardless of the value of p.

On the other hand, for the case of $j \notin S(\boldsymbol{x}, i, \pm)$, it holds that

$$T\left(\boldsymbol{x} \to f_j^d f_i^\pm \boldsymbol{x}\right) = 0. \tag{11}$$

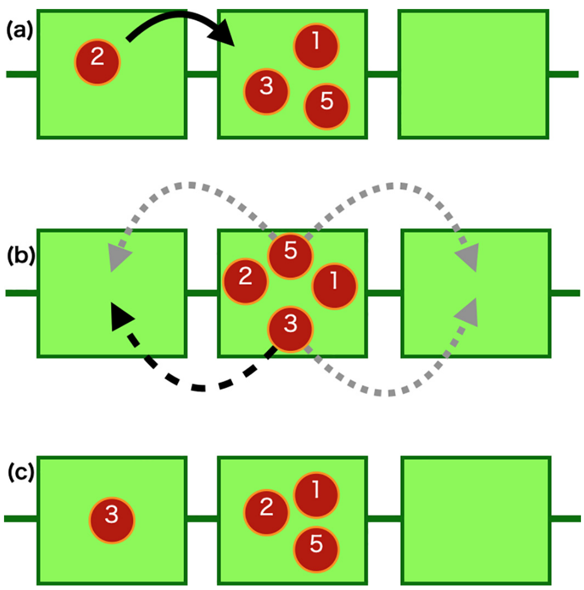

FIG. 1. Schematic illustration of a transition step from a configuration described by (a) to a configuration described by (c) in the model. From (a) to (b), agent 2 hops to the right site, and from (b) to (c), agent 3 among two possibly supplanted agents, which are 3 and 5, is supplanted by agent 2 and hops to the left site. Four arrows in (b) mean that, in the above transition, two agents 3 and 5 in $S(\boldsymbol{x}, 2, +)$ could be supplanted, and the direction of supplanting process could be either to the left or to the right.

Further, the probability of transition $T(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x})$ is modified from T_0 as

$$T(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x}) = \frac{1}{1 + p\#S(\boldsymbol{x}, i, \pm)} T_0(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x}). \tag{12}$$

Totally, the following holds:

$$T_0(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x}) = T(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x}) + \sum_{\substack{j\in S(\boldsymbol{x},i,\pm)\\ d=\pm}} T\left(\boldsymbol{x} \to f_j^{d} f_i^{\pm}\boldsymbol{x}\right). \tag{13}$$

Then the master equation for the joint probability $P_t(\boldsymbol{x})$ governing the above stochastic process is as follows:

$$P_{t+1}(\boldsymbol{x}) = \sum_{\boldsymbol{x}'} P_t(\boldsymbol{x}')T(\boldsymbol{x}' \to \boldsymbol{x}) + P_t(\boldsymbol{x})\left[1 - \sum_{\boldsymbol{x}'\neq \boldsymbol{x}} T(\boldsymbol{x} \to \boldsymbol{x}')\right], \tag{14}$$

where the summation over $\boldsymbol{x}'$ is done for all of the possible configurations such that $T(\boldsymbol{x}' \to \boldsymbol{x})$ and $T(\boldsymbol{x} \to \boldsymbol{x}')$ are defined. Since $T(\boldsymbol{x} \to f_i^{\pm}\boldsymbol{x})$ is positive for any $\boldsymbol{x}$ and i with finite β, p, N, L, and E, any state can reach any state in this stochastic process; that is, the stochastic process defined above is an irreducible Markov process.

In the presence of $p > 0$, the supplanting process does not hold the detailed balance condition because of the asymmetric property in terms of agents' rank; when a supplanting occurs in one step, i.e., an agent supplants another agent, the reverse process never occurs in any single step. Thus, obviously the stationary solution $P_{\rm st}(\boldsymbol{x})$ is no longer the Gibbs distribution of a given energy function. Note that in the limit of $p \to 0$, the detailed balance condition in terms of a given energy function is recovered. Thus, we can also regard p as the strength of violation of detailed balance condition.

III. THE CASE OF THE POTTS ENERGY

In this section we focus on the case of the Potts energy defined by (2). We briefly review the known equilibrium properties and compute nonequilibrium stationary distribution by exact diagonalization of the transition matrix corresponding to the master equation (14). Further, we introduce overlap centrality and its correlation with agents' rank, which are calculated using the computed stationary distribution.

A. Computation of partition function at equilibrium

As a preliminary, we consider the equilibrium case with $p = 0$. The equilibrium ferromagnetic Potts model on the complete graph has two phases: one is the ordered phase for stronger interaction, and another is the disordered phase for weaker interaction, which are separated by a first-order transition point if $L \geqslant 3$ [23]. In the context of this paper, the ordered state can be regarded as the condensate state of agents, that is, the state where all agents are located at the same position.

Let us look in more detail at the computation of the above results. When $p = 0$ and interaction strength β is positive ($\beta > 0$), corresponding to the case of attractive interaction, the partition function $Z_N(\beta) := \sum_{\boldsymbol{x}} \exp[-\beta E(\boldsymbol{x})]$ for $N \to \infty$ can be explicitly expressed, by which the equilibrium transition point is computed exactly. Concretely, by performing the Stratonovich-Hubbard transformation [24,25] with

$$\exp\left[\frac{K\beta}{2N}\sum_{x\in X}\left(\sum_{i=1}^{N}\delta(x_i, x)\right)^2\right] = \prod_{x\in X}\sqrt{\frac{NK\beta}{2\pi}}\int_{\mathbb{R}} dq \exp\left[-\frac{NK\beta}{2}q^2 + K\beta q\sum_{i=1}^{N}\delta(x_i, x)\right], \tag{15}$$

we obtain

$$Z_N(\beta) = \left(\frac{NK\beta}{2\pi}\right)^{L/2}\int_{\mathbb{R}^L} d^L q \exp[-N\beta f_\beta(\boldsymbol{q})], \tag{16}$$

$$f_\beta(\boldsymbol{q}) = \frac{K}{2}\sum_{x\in X} q_x^2 - \beta^{-1}\log\left(\sum_{x\in X}\exp\left(K\beta q_x\right)\right), \tag{17}$$

where

$$K = \frac{2(L-1)\log(L-1)}{L-2}. \tag{18}$$

For $N \gg 1$, the minimal value of $f_\beta(\boldsymbol{q})$ as a function of the order parameter $\boldsymbol{q}$ behaves effectively as the free energy density of the Potts model as follows:

$$\lim_{N\to\infty}\frac{-\beta^{-1}\log Z_N(\beta)}{N} = \min_{\boldsymbol{q}\in\mathbb{R}^L} f_\beta(\boldsymbol{q}). \tag{19}$$

Taking $\frac{\partial f_\beta}{\partial q_x}(\boldsymbol{q}) = 0$ to minimize $f_\beta(\boldsymbol{q})$, we obtain the stationary condition

$$q_y \exp(-K\beta q_y) = \left(\sum_{x\in X} \exp(K\beta q_x)\right)^{-1} \tag{20}$$

for each $y \in X$. From (20), we see that

$$q_x \exp(-K\beta q_x) = q_y \exp(-K\beta q_y), \tag{21}$$

for any $x, y \in X$. This indicates that there is a constant c such that $q_x \exp(-K\beta q_x) = c$, which has at most two real solutions a, b as an equation for q_x. Then we have $q_x \in \{a, b\}$ for each $x \in X$. From (20), we also have

$$\sum_{x\in X} q_x = 1. \tag{22}$$

Thus, a necessary condition for the order parameter $\boldsymbol{q}$ to minimize $f_\beta(\boldsymbol{q})$ is described below. Keeping with (21) and (22), one of the following conditions (i) and (ii) is satisfied:

(1) It holds that $\boldsymbol{q} = \tilde{\boldsymbol{q}}^{(0)} := \frac{1}{L}(1, 1, \dots, 1)$.

(2) There exist an integer $n \in \{1, \dots, L-1\}$ and two distinct real numbers $a_n = a_n(\beta), b_n = b_n(\beta)$ satisfying

$$a_n \exp(-K\beta a_n) = b_n \exp(-K\beta b_n), \tag{23}$$

$$n a_n + (L-n) b_n = 1. \tag{24}$$

Moreover, n components of $\boldsymbol{q}$ are a_n and remaining $(L-n)$ components of $\boldsymbol{q}$ are b_n.

For example, if $n = 1$, the solutions are described by $\boldsymbol{q} = \tilde{\boldsymbol{q}}^{(\alpha)}$ for $1 \leqslant \alpha \leqslant L$, where

$$\tilde{q}_x^{(\alpha)} := \begin{cases} a_1(\beta) & (\text{if } x \equiv \alpha \mod L) \\ b_1(\beta) & (\text{if } x \not\equiv \alpha \mod L) \end{cases}. \tag{25}$$

In Ref. [26] it has been shown that the set of the global minimum points $\boldsymbol{q}$ of $f_\beta(\boldsymbol{q})$ corresponds to the case of $\boldsymbol{q} = \tilde{\boldsymbol{q}}^{(0)}$ or $n = 1$, depending on β. Concretely, the set is described as

$$\begin{cases} \{\tilde{\boldsymbol{q}}^{(1)}(\beta), \tilde{\boldsymbol{q}}^{(2)}(\beta), \dots, \tilde{\boldsymbol{q}}^{(L)}(\beta)\} & (\text{if } 0 < \beta < 1) \\ \{\tilde{\boldsymbol{q}}^{(0)}, \tilde{\boldsymbol{q}}^{(1)}(1), \dots, \tilde{\boldsymbol{q}}^{(L)}(1)\} & (\text{if } \beta = 1) \\ \{\tilde{\boldsymbol{q}}^{(0)}\} & (\text{if } \beta > 1). \end{cases} \tag{26}$$

Note that Eqs. (23) and (24) with $n = 1$ determine the value $a_1 \neq b_1$ uniquely, and the resulting functions $a_1(\beta), b_1(\beta)$ are differentiable in the region $0 < \beta < 1$.

The expectation value of energy density is also expressed as

$$\lim_{N\to\infty} \frac{\langle E \rangle_{\text{can}}}{N} = \begin{cases} \frac{\partial}{\partial\beta} \beta f_\beta(\tilde{\boldsymbol{q}}^{(1)}(\beta)) & (\text{if } 0 < \beta < 1) \\ \frac{\partial}{\partial\beta} \beta f_\beta(\tilde{\boldsymbol{q}}^{(0)}) & (\text{if } \beta > 1). \end{cases} \tag{27}$$

Thus, one can show that the energy density (27) exhibits a discontinuous jump at $\beta = 1$, which is the phase transition point of the Potts model.

B. State vector description

Let us move onto the model with general $p \geqslant 0$. In this case, we need to explicitly consider the dynamics in order to compute the stationary distribution of the model. We would like to describe the stochastic process by transition matrices with some basic linear operators. For more detailed description and derivation, see Appendix A 3.

Let H_X be the one-agent state space, which is of dimension L. It is considered as a complex vector space with inner product $\langle\cdot|\cdot\rangle$ and has an orthonormal basis $\{|x\rangle \mid x \in X\}$ over the set of complex numbers $\mathbb{C}$. Then the N-times self-tensored space $H_X^{\otimes N}$ can be identified with the N-agent state spaces, which is of dimension L^N. For a configuration of agents $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in X^N$, the corresponding state vector is $|\boldsymbol{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle$. The space $H_X^{\otimes N}$ has a natural inner product induced by $\langle\cdot|\cdot\rangle$, and the set of the state vectors $\{|\boldsymbol{x}\rangle\}_{\boldsymbol{x}\in X^N}$ is an orthonormal basis. We use the same symbol $\langle\cdot|\cdot\rangle$ to write the inner product on $H_X^{\otimes N}$.

We associate the probability $P(\boldsymbol{x})$ for agents' configuration $\boldsymbol{x}$ with a state $|P\rangle \in H_X^{\otimes N}$ as follows:

$$\langle \boldsymbol{x}|P\rangle = P(\boldsymbol{x}) \tag{28}$$

or

$$|P\rangle = \sum_{\boldsymbol{x}\in X^N} P(\boldsymbol{x}) |\boldsymbol{x}\rangle. \tag{29}$$

For a given state $|P_t\rangle$ at time t, the time evolution of the state is described as follows:

$$|P_{t+1}\rangle = \widehat{T} |P_t\rangle, \tag{30}$$

where $\widehat{T}$ is a transition matrix on $H_X^{\otimes N}$ such that (30) is equivalent to the master equation (14) for the joint probability. Similarly, $\widehat{T}_0$ is the transition matrix $\widehat{T}$ when $p = 0$.

We introduce some basic operators. A hopping map $f_i^\pm$ to the right (resp. the left) corresponds to the operator defined by $\widehat{\Delta}_i^\pm$; for $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in X^N$,

$$\begin{aligned} \widehat{\Delta}_i^+ |\boldsymbol{x}\rangle &= |f_i^+ \boldsymbol{x}\rangle \\ &= |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_i + 1\rangle \otimes \cdots \otimes |x_N\rangle, \end{aligned} \tag{31}$$

$$\begin{aligned} \widehat{\Delta}_i^- |\boldsymbol{x}\rangle &= |f_i^- \boldsymbol{x}\rangle \\ &= |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_i - 1\rangle \otimes \cdots \otimes |x_N\rangle. \end{aligned} \tag{32}$$

Next we define projection operators. For a site $y \in X$ and an agent $1 \leqslant i \leqslant N$, we define $\widehat{\Xi}_i^y$ by

$$\widehat{\Xi}_i^y |\boldsymbol{x}\rangle = \begin{cases} |\boldsymbol{x}\rangle & (\text{if } x_i = y) \\ 0 & (\text{if } x_i \neq y), \end{cases} \tag{33}$$

for $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in X^N$. Then for a configuration $\boldsymbol{y} \in X^N$, we define the operator $\widehat{\Xi}^{\boldsymbol{y}}$ as $\prod_{1\leqslant i\leqslant N} \widehat{\Xi}_i^{y_i}$. Finally, we denote $\widehat{id}_H$ (resp. $\widehat{id}_H^{\otimes N}$) as the identity operator on H_X (resp. $H_X^{\otimes N}$).

Using these notions, we can describe the transition matrices $\widehat{T}_0$ and $\widehat{T}$. First, $\widehat{T}_0$ is

$$\begin{aligned} \widehat{T}_0 = \sum_{\substack{1\leqslant i\leqslant N \\ d=\pm}} \sum_{\boldsymbol{x}\in X^N} \Bigg[& T_0(\boldsymbol{x} \to f_i^d \boldsymbol{x}) \widehat{\Delta}_i^d \\ & + \left(\frac{1}{2N} - T_0(\boldsymbol{x} \to f_i^d \boldsymbol{x}) \right) \widehat{id}_H^{\otimes N} \Bigg] \widehat{\Xi}^{\boldsymbol{x}}. \end{aligned} \tag{34}$$

Then $\widehat{T}$ is written as

$$\widehat{T} = \sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} \sum_{\boldsymbol{x}\in X^N} \left[\sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} T\big(\boldsymbol{x}\to f_j^{d'} f_i^d \boldsymbol{x}\big)\widehat{\Delta}_j^{d'}\widehat{\Delta}_i^d + T\big(\boldsymbol{x}\to f_i^d\boldsymbol{x}\big)\widehat{\Delta}_i^d + \left(\frac{1}{2N} - T_0\big(\boldsymbol{x}\to f_i^d\boldsymbol{x}\big)\right)\widehat{id}_H^{\otimes N}\right]\widehat{\Xi}^{\boldsymbol{x}} \tag{35}$$

$$= \widehat{T}_0 + \sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} \sum_{\boldsymbol{x}\in X^N} \sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} T\big(\boldsymbol{x}\to f_j^{d'} f_i^d \boldsymbol{x}\big)\big(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\big)\widehat{\Delta}_i^d\widehat{\Xi}^{\boldsymbol{x}}, \tag{36}$$

where we used (13). For the definition of coefficients, see (3), (10), and (12).

On this setting, the transition matrix $\widehat{T} = \widehat{T}(\beta, p)$ is naturally regarded as a linear operator on $H_X^{\otimes N}$. Let $|P(\beta, p)\rangle$ be the unique stationary state of $\widehat{T}(\beta, p)$ satisfying

$$\widehat{T}(\beta, p)|P(\beta, p)\rangle = |P(\beta, p)\rangle. \tag{37}$$

One can show that, since $\widehat{T}$ is irreducible, $|P(\beta, p)\rangle$ exists and is uniquely determined by the Perron-Frobenius theorem.

For the latter discussion, let us consider the symmetry of the transition matrices $\widehat{T}_0$ and $\widehat{T}$. We introduce permutation operators $\widehat{\Pi}_\sigma$ on $H_X^{\otimes N}$. For a given element $\sigma \in \mathfrak{S}_N$ of the symmetry group $\mathfrak{S}_N$ of agents, we define

$$\widehat{\Pi}_\sigma|\boldsymbol{x}\rangle := |\sigma^{-1}(\boldsymbol{x})\rangle, \tag{38}$$

where $\sigma^{-1}(\boldsymbol{x}) := (x_{\sigma^{-1}(j)})_{j=1}^N$. Then we have

$$\widehat{\Pi}_\sigma^\dagger\widehat{T}_0\widehat{\Pi}_\sigma = \widehat{T}_0, \tag{39}$$

$$\widehat{\Pi}_\sigma^\dagger\widehat{T}\widehat{\Pi}_\sigma \neq \widehat{T}, \tag{40}$$

where $\widehat{\Pi}_\sigma^\dagger$ is the Hermitian conjugate of $\widehat{\Pi}_\sigma$ [see (A54) and (A74)]. Note that $\widehat{\Pi}_\sigma$ is unitary: $\widehat{\Pi}_\sigma^\dagger = \widehat{\Pi}_\sigma^{-1} = \widehat{\Pi}_{\sigma^{-1}}$. In the sense of relation (39), the equilibrium dynamics described by $\widehat{T}_0$ holds permutation symmetry. In contrast, the whole dynamics by $\widehat{T}$ breaks the permutation symmetry, as described by (40).

C. Exact diagonalization of transition matrices

We perform exact diagonalization for the transition matrix $\widehat{T}$ to obtain the eigenvalues and their corresponding eigenvectors. Thus, the stationary distribution corresponds to the eigenvector with the maximum real part, which is 1, of the eigenvalue. Note that the number of the states is L^N, which gets exponentially large as a function of N.

As shown in Fig. 2 with $p = 0$, at $\beta = 0$, the joint probability of each configuration shows the same value. As β is increased from 0, the joint probability of condensate configuration (0,0,0) is much higher than that of the other configurations. Conversely, as β is decreased from 0, the joint probability of the configuration (0,1,2), where all the agents are separated, is much higher than that of the other configurations. On the other hand, the joint probability of configuration with a pair of two agents overlapping at the same site and the other located at a different site such as (1, 0, 0), (0, 1, 0), (0, 0, 1) does not depend on the pair for $p = 0$.

When $p = 10$, the joint probabilities of configurations (1,0,0), (0,1,0), (0,0,1) are distinct. Concretely, those probabilities with $\beta = 2$ and $\beta = -1$ increase and decrease, respectively, in the order of (1,0,0), (0,1,0), and (0,0,1). This means that higher-ranked agents (resp. lower-ranked agents) tend to overlap more frequently for $\beta = 2$ (resp. for $\beta = -1$). This can be interpreted as a typical consequence of supplanting process.

In order to discuss how the configuration is condensed, let us introduce the normalized expectation value of the Potts energy in terms of a probability distribution $P(\boldsymbol{x})$ as follows:

$$M := \frac{1}{N^2}\sum_{\boldsymbol{x}}\sum_{i=1}^N\sum_{j=1}^N \delta(x_i, x_j)P(\boldsymbol{x}). \tag{41}$$

Note that by definition, M takes 1 as the maximum value in the case of $P(\boldsymbol{x}) = \prod_{k=1}^N \delta(x, x_k)$. In Fig. 3, using the computation of the stationary distribution by the exact diagonalization, M is shown as a function of p and β.

Relating to Sec. III A, in the case of the equilibrium distribution corresponding to the case with $p = 0$, M is rewritten

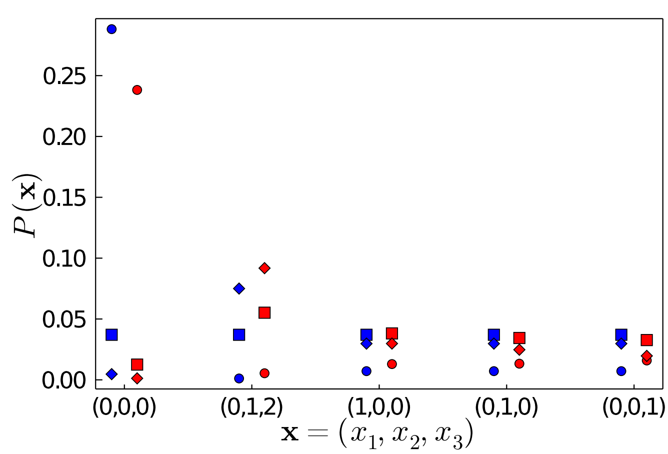

FIG. 2. Probabilities of the configurations determined by stationary distribution for $\beta = 2$ (circles), $\beta = 0$ (rectangles), and $\beta = -1$ (diamonds); $p = 0$ (blue or the left at each column) and $p = 10$ (red or the right); $N = L = 3$. The state of (0,0,0) means that three agents are located at the same site. The state of (0,1,2) means that each of three agents is located at the different position, respectively. The other three states mean that two agents are located at the same site, and another agent is located at one of the different sites.

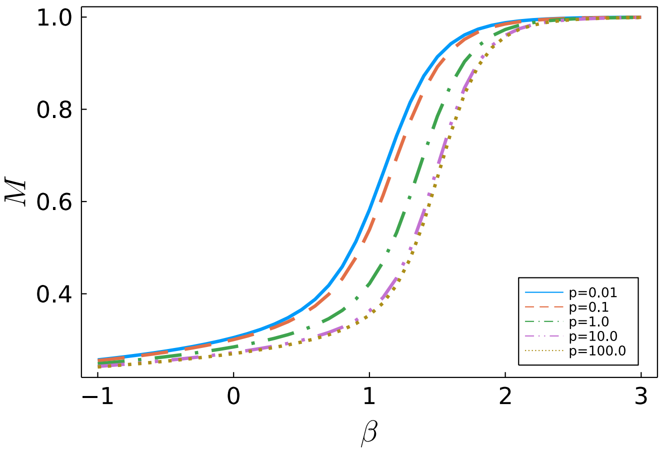

FIG. 3. The β dependence of the normalized expectation value of the energy $M = \frac{1}{N^2}\sum_{\boldsymbol{x}}\sum_{i,j}\delta(x_i, x_j)\langle \boldsymbol{x}|P(\beta, p)\rangle$ for various values of p with $L = 6, N = 6$.

by using $\langle E\rangle_{\mathrm{can}}$ as

$$M = -\frac{2}{NK}\sum_{\boldsymbol{x}} E(\boldsymbol{x})\frac{e^{-\beta E(\boldsymbol{x})}}{Z_N(\beta)} = -\frac{2}{K}\frac{\langle E\rangle_{\mathrm{can}}}{N}, \tag{42}$$

which means that M is also discontinuous at the equilibrium phase transition point $\beta = 1$ in the thermodynamic limit $N \to \infty$.

D. Overlap centrality and its correlation coefficient

In order to characterize the correlation among agents, we may consider the neighbor matrix $\mathcal{R} := (r_{ij})_{i,j}$ defined as

$$r_{ij} := \sum_{\boldsymbol{x}} \delta(x_i, x_j)P(\boldsymbol{x}). \tag{43}$$

For example, if $P(\boldsymbol{x})$ is the uniform distribution then $r_{ij} = 1/L$, and if $P(\boldsymbol{x}) = \prod_{k=1}^{N}\delta(x, x_k)$ then $r_{ij} = 1$. The latter gives the maximum value of r_{ij}. Note that $r_{jj} = 1$ for any j.

The entry r_{ij} means how often agent i and j are located at the same site under the distribution $P(\boldsymbol{x})$. In Fig. 4 we show heatmaps of neighbor matrices computed from the stationary distribution. It demonstrates that at $\beta = 1$, the agents with higher rank have more overlaps with the other agents, and conversely at $\beta = -1$, the agents with lower rank have more overlaps with the other agents.

In order to quantify how often a given agent overlaps with the other agents in total, we introduce the overlap centrality as a function of rank i using the entries of the neighbor matrix:

$$O_i := \sum_{\substack{1\leqslant j\leqslant N\\ j\neq i}} r_{ij}. \tag{44}$$

That is, we regard the agent i having larger value of the overlap centrality as a more influential agent compared to the other agents having lower values of the overlap centrality. When the probability distribution $P(\boldsymbol{x})$ is permutation symmetric, i.e., $P(\sigma(\boldsymbol{x})) = P(\boldsymbol{x})$ for any $\sigma \in \mathfrak{S}_N$, the overlap centrality does not depend on rank i. Note that

$$M = \frac{1}{N^2}\sum_{i=1}^{N} O_i + \frac{1}{N} \tag{45}$$

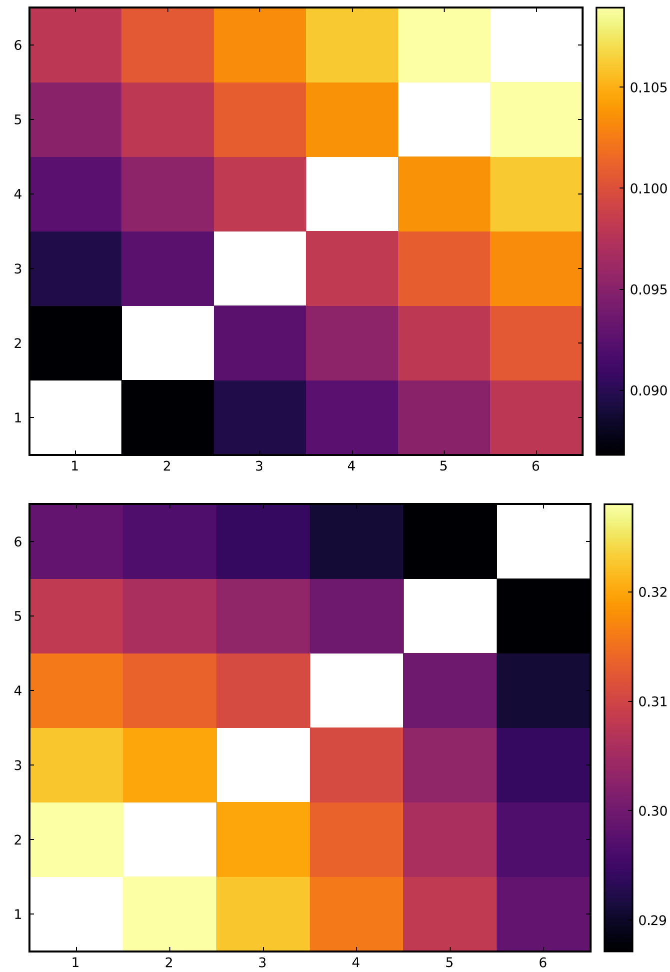

FIG. 4. Heatmap of neighbor matrix $\mathcal{R}$ determined by the stationary distribution with the diagonal components left out. The color corresponds to r_{ij} for pair of two agents (i, j). Parameters: (top) $\beta = -1,\ p = 1,\ L = 6, N = 6$; (bottom) $\beta = 1, p = 1, L = 6, N = 6$.

holds by the definition. As shown in Fig. 5, we compute the overlap centrality O_i computed from the stationary distribution, showing that the overlap centrality has a plus slope at attractive interaction of $\beta = 1$ and has a minus slope at repulsive interaction of $\beta = -1$.

In order to quantify the class of the overlap centrality in terms of the slope, we measure the correlation coefficient ϕ of the overlap centrality with respect to agents' rank. This is defined as

$$\phi := \frac{1}{N}\sum_{i=1}^{N}\frac{\left(O_i - \frac{1}{N}\sum_{j=1}^{N} O_j\right)\left(i - \frac{1}{N}\sum_{j=1}^{N} j\right)}{s_O s_I}, \tag{46}$$

where we define

$$s_O^2 := \frac{1}{N}\sum_i\left(O_i - \frac{1}{N}\sum_{j=1}^{N} O_j\right)^2, \tag{47}$$

$$s_I^2 := \frac{1}{N}\sum_i\left(i - \frac{1}{N}\sum_{j=1}^{N} j\right)^2 = \frac{(N-1)(N+1)}{12}. \tag{48}$$

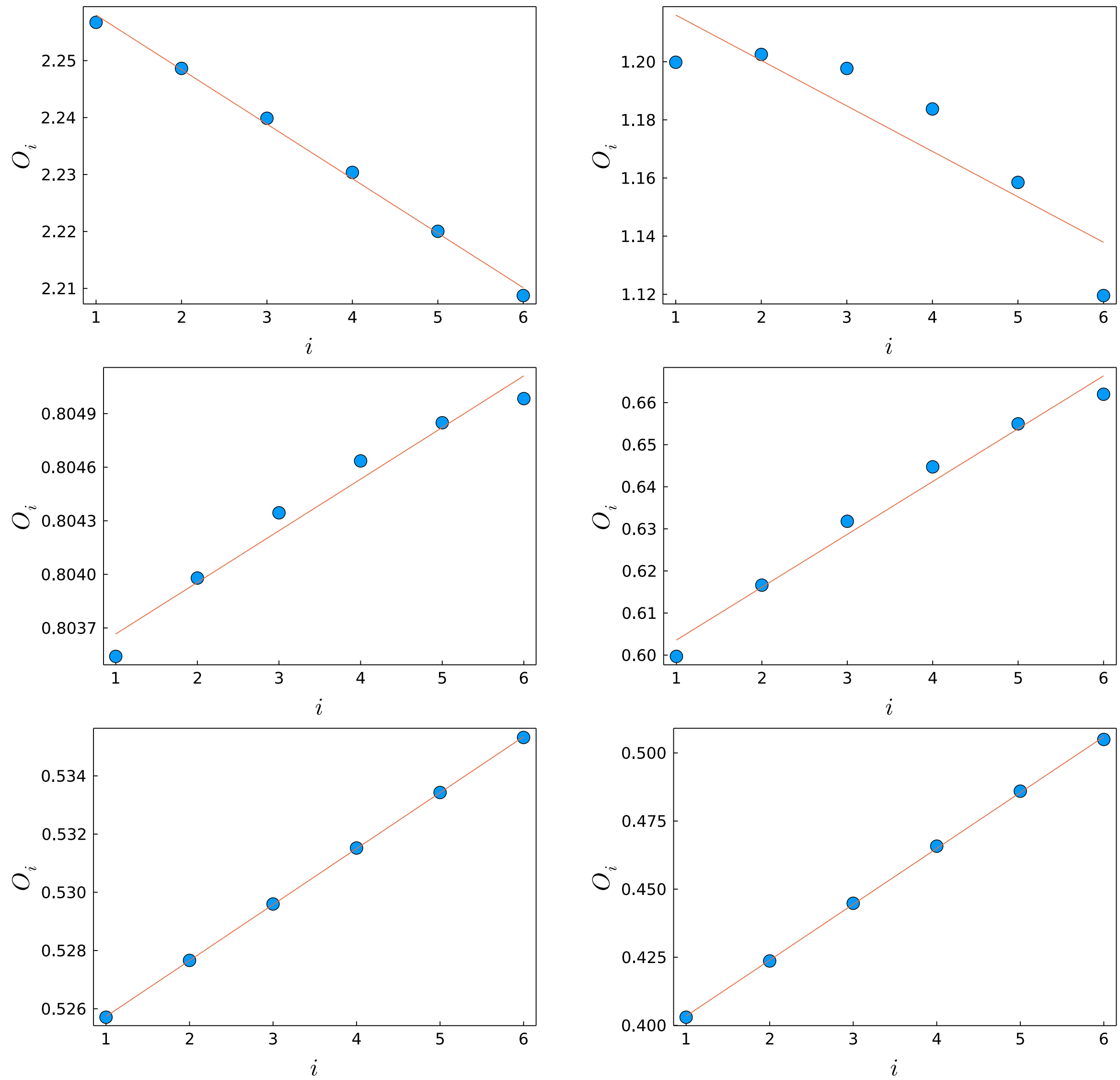

FIG. 5. The overlap centrality O_i determined by the stationary distribution obtained by the exact diagonalization as a function of agent i for $L = N = 6$. The solid lines are linear regressions for O_i and i. The parameters are set as follows; $p = 0.1$ (left) and $p = 10.0$ (right); $\beta = 1$ (top), $\beta = 0$ (center), and $\beta = -1$ (bottom), respectively. For negative β, ϕ is close to 1 regardless of the value of p.

By definition, when $|\phi| = 1$, O_i is a linear function of i. We check the condition when this quantity ϕ is not defined. Since we set $N \geqslant 2$, the denominator of ϕ is zero exactly when $s_O^2 = 0$. This corresponds to the case when O_i is constant as a function of i. In this case, we say that the quantity ϕ is *singular*.

In Fig. 6 we show the β dependence of the correlation coefficient ϕ. In the weak supplanting condition with $p \ll 1$ such as $p = 0.1$ or $p = 0.01$, ϕ is close to $+1$ for negative β. As β increases, ϕ sharply changes its sign around $\beta = 0$, and turns out to be -1 for positive β. We will discuss this behavior in a more general setting in Sec. IV.

IV. ANALYTIC RESULTS FOR OVERLAP CENTRALITY

Let us discuss the weak-supplanting limit of $p \ll 1$, where general results in terms of overlap centrality are available. Although we have focused on only the Potts energy as $E(\boldsymbol{x})$ in Sec. III, hereafter we consider all of the models which belong to the general class of energy functions satisfying the permutation symmetry condition (1). The neighbor matrix $\mathcal{R}$ (43), the overlap centrality O_i $(1 \leqslant i \leqslant N)$ (44), and the correlation coefficient ϕ of the overlap centrality with agents' rank (46) can also be defined for the general cases in the same manner. These are indeed the main subject in this section and this paper. Hereafter, we use the state vector description, fix parameters β, L, N as arbitrary values, and consider p dependence of the dynamics unless otherwise specified.

The main goal in this section is to derive perfect correlation, which means that the correlation coefficient (46) satisfies $\phi = \pm 1$ in the weak-supplanting limit of $p \to 0$ as long as ϕ is not singular in the sense mentioned after the definition of ϕ. In order to take a step forward, we start with introducing auxiliary stochastic processes (49), by which the transition matrix of the model can be completely reconstructed. Then, using this decomposition property (50), we construct another

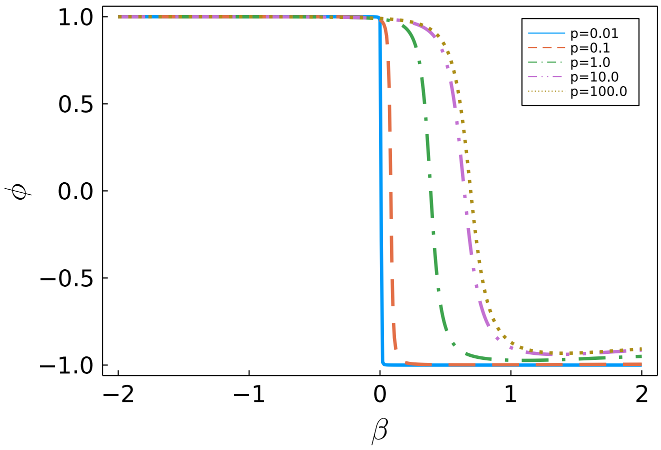

FIG. 6. Correlation coefficient ϕ as a function of β for $L = N = 6$.

decomposition form (53) of the transition matrix, which we call *beta decomposition*, where the asymptotic behaviors in terms of $p \to 0$ can be rigorously estimated. Note that the assumption of (1) is essential in the derivation of key properties (78) and (80).

A. Decompositions of transition matrix

In this subsection we would like to introduce *beta decomposition* (53) of the operator $\widehat{T}$. This decomposition enables us to investigate an asymptotic behavior of $\widehat{T}$ for $p \ll 1$ because of an asymptotic property (55). To describe it, first we introduce another decomposition, called *supplanting decomposition*. This is relatively easy to describe and simplifies the description of beta decomposition. For details, see Appendix A 4 and A 5.

Let us define the partial sum in (35). For an integer $1 \leqslant n \leqslant N-1$, we define

$$\widehat{T}_n := \sum_{\substack{1\leqslant i\leqslant N \\ d=\pm}} \sum_{\boldsymbol{x}\in X^N} \delta(\#S(\boldsymbol{x},i,d),n) \times \left[\sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} T\big(\boldsymbol{x}\to f_j^{d'} f_i^{d}\boldsymbol{x}\big)\big(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\big)\right]\widehat{\Delta}_i^d\,\widehat{\Xi}^{\boldsymbol{x}}. \tag{49}$$

This operator $\widehat{T}_n$ is the second term in (36) restricting indices $\boldsymbol{x}, i, d$ to those with $\#S(\boldsymbol{x}, i, d) = n$. For any subset $\mathcal{S} \subseteq \{1, 2, \dots, N-1\}$, the matrix $\widehat{T}_0 + \sum_{i\in\mathcal{S}} \widehat{T}_i$ is also a stochastic matrix. For example, the matrix $\widehat{T}_0 + \widehat{T}_n$ represents the supplanting process only when $\#S(\boldsymbol{x}, i, d) = n$. By definition, we have

$$\widehat{T} = \widehat{T}_0 + \widehat{T}_1 + \widehat{T}_2 + \cdots + \widehat{T}_{N-1}. \tag{50}$$

This is a decomposition of the operator $\widehat{T}$, which we call the *supplanting decomposition*. Note that the coefficients $\langle \boldsymbol{y}|\widehat{T}_n|\boldsymbol{x}\rangle$ of nth term $\widehat{T}_n$ for $n \geqslant 1$ are estimated as

$$|\langle \boldsymbol{y}|\widehat{T}_n|\boldsymbol{x}\rangle| \leqslant \frac{p}{2}. \tag{51}$$

For details, see Appendix A 4 b. In particular, the coefficients of $\widehat{T}_n$ are estimated by $O(p)$ when $p \to +0$. Moreover, since at most one of $\langle \boldsymbol{y}|\widehat{T}_n|\boldsymbol{x}\rangle$ $(1 \leqslant n \leqslant N-1)$ is nonzero for fixed $\boldsymbol{x}$ and $\boldsymbol{y}$, we also have

$$|\langle \boldsymbol{y}|(\widehat{T} - \widehat{T}_0)|\boldsymbol{x}\rangle| \leqslant \frac{p}{2}. \tag{52}$$

Though we explicitly estimate the coefficients of $\widehat{T}_n$ in (51), the supplanting decomposition does not give effective truncation of $\widehat{T}$ in terms of small p. Thus, we look for another decomposition of $\widehat{T}$

$$\widehat{T} = \widehat{T}_0 + \widehat{U}_1 + \cdots + \widehat{U}_{N-1} \tag{53}$$

satisfying

$$\langle \boldsymbol{y}|\widehat{U}_m|\boldsymbol{x}\rangle = O(p^m) \text{ as } p \to +0. \tag{54}$$

If we find such an expansion, we have

$$\widehat{T} = \widehat{T}_0 + \widehat{U}_1 + \cdots + \widehat{U}_m + O(p^{m+1}) \tag{55}$$

for $1 \leqslant m \leqslant N-1$.

It would not be straightforward to practically find such an expansion. Nevertheless, in this case, through a rather tricky procedure as shown in Appendix A 5 b, one can prove that (53) and (54) are satisfied by the following definition of $\widehat{U}_m$:

$$\widehat{U}_m := \frac{(-1)^{m+1}B(m, 1+1/p)}{p} \sum_{m\leqslant n\leqslant N} \binom{n-1}{m-1}(1+np)\widehat{T}_n, \tag{56}$$

for $1 \leqslant m \leqslant N-1$. Here $B(a, b)$ is the beta function and $\binom{n-1}{m-1}$ is the binomial coefficient. See (A96) and (A113) of Appendix A for the detail of the derivation; see also Remark A.2 for the motivation of this decomposition. We call this expansion *beta decomposition*. Note that, for an integer $m \geqslant 1$,

$$B\left(m, 1+\frac{1}{p}\right) = \frac{(m-1)!p^m}{(1+p)(1+2p)\cdots(1+mp)}. \tag{57}$$

By substituting (49) and (57) into (56) with certain sets of transformations in Appendix A 4 c and A 5 d, we reach another representation of $\widehat{U}_m$ as in (A87) and (A123):

$$\widehat{U}_m = \frac{(-1)^{m-1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)} \times \sum_{1\leqslant i<i_1<\cdots<i_m\leqslant N}\left[\sum_{1\leqslant k\leqslant m}\big(\widehat{\Delta}_{i_k}^{+} + \widehat{\Delta}_{i_k}^{-} - 2\,\widehat{id}_H^{\otimes N}\big)\right] \times \widehat{\Xi}_{i,i_1,\dots,i_m}\widehat{T}_{0,\text{move}}^{i}, \tag{58}$$

where we define

$$\widehat{T}_{0,\text{move}}^{i} := \sum_{d=\pm}\sum_{\boldsymbol{x}\in X^N} T_0\big(\boldsymbol{x}\to f_i^d\boldsymbol{x}\big)\widehat{\Delta}_i^d\,\widehat{\Xi}^{\boldsymbol{x}}, \tag{59}$$

$$\widehat{\Xi}_{i,i_1,\dots,i_m} := \sum_{x\in X}\widehat{\Xi}_i^x\widehat{\Xi}_{i_1}^x\cdots\widehat{\Xi}_{i_m}^x. \tag{60}$$

This representation of $\widehat{U}_m$ is suitable for the further calculation related to permutation symmetry in Sec. IV B. In the following sections, $\widehat{U}_1$ is the one we mainly consider in the weak supplanting limit $p \to +0$.

Recall that $\widehat{\Pi}_\sigma$ defined in (38) denotes the permutation operator corresponding to a permutation $\sigma \in \mathfrak{S}_N$. In Appendix A 3 e and A 3 g, we obtain commutation relations between permutation operators and other operators:

$$\widehat{\Pi}_\sigma \widehat{\Delta}^{\pm}_{i_1} = \widehat{\Delta}^{\pm}_{\sigma(i_1)} \widehat{\Pi}_\sigma, \tag{61}$$

$$\widehat{\Pi}_\sigma \widehat{\Xi}_{i_0,i_1} = \widehat{\Xi}_{\sigma(i_0),\sigma(i_1)} \widehat{\Pi}_\sigma, \tag{62}$$

$$\widehat{\Pi}_\sigma \widehat{T}^{i_0}_{0,\text{move}} = \widehat{T}^{\sigma(i_0)}_{0,\text{move}} \widehat{\Pi}_\sigma. \tag{63}$$

These relations will be used in (80).

B. Existence of perfect correlation

By using beta decomposition of the transition matrix obtained above, we are going to show that the correlation coefficient ϕ exhibits perfect correlation $|\phi| = 1$ in the limit of $p \to 0$.

One can use the Brillouin-Wigner-type perturbation theory [27] to rewrite the stationary state $|P(\beta, p)\rangle$ of $\widehat{T}(\beta, p)$ as a perturbation expansion from the stationary state of $\widehat{T}_0$. For that purpose, we introduce some symbols. Let $|P_{\text{can}}(\beta)\rangle$ be the stationary state of $\widehat{T}_0(\beta)$, i.e., the state corresponding to the Gibbs distribution (6):

$$|P_{\text{can}}(\beta)\rangle := \sum_{\boldsymbol{x}} \frac{\exp[-\beta E(\boldsymbol{x})]}{Z(\beta)} |\boldsymbol{x}\rangle, \tag{64}$$

which satisfies

$$\widehat{T}_0(\beta)|P_{\text{can}}(\beta)\rangle = |P_{\text{can}}(\beta)\rangle \tag{65}$$

and

$$\sum_{\boldsymbol{x}} \langle \boldsymbol{x}|P_{\text{can}}(\beta)\rangle = 1. \tag{66}$$

Note that $|P_{\text{can}}(\beta)\rangle$ is invariant under a permutation, that is, it holds that

$$\widehat{\Pi}_\sigma |P_{\text{can}}(\beta)\rangle = |P_{\text{can}}(\beta)\rangle \tag{67}$$

for any $\sigma \in \mathfrak{S}_N$, because of the permutation symmetry condition (1) for energy function. Let $\widehat{\text{pr}}(\beta)$ be a projection operator on $H_X^{\otimes N}$ to the orthogonal complement of the subspace $\mathbb{C}|P_{\text{can}}(\beta)\rangle$:

$$\widehat{\text{pr}}(\beta) := \widehat{id}_H^{\otimes N} - \frac{|P_{\text{can}}(\beta)\rangle\langle P_{\text{can}}(\beta)|}{\langle P_{\text{can}}(\beta)|P_{\text{can}}(\beta)\rangle}, \tag{68}$$

and $\widehat{G}_0(\beta)$ be a linear operator from $H_X^{\otimes N}$ to itself:

$$\widehat{G}_0(\beta) := \left(\widehat{id}_H^{\otimes N} - \widehat{T}_0(\beta)\right)^{-1} \widehat{\text{pr}}(\beta). \tag{69}$$

Here we need to set the coefficient of the term $\widehat{id}_H^{\otimes N}$ in $\widehat{G}_0(\beta)$ as the eigenvalue 1 corresponding to the eigenvector $|P(\beta, p)\rangle$ of $\widehat{T}(\beta, p)$ [see (37)]. With the above notations, $|P(\beta, p)\rangle$ can be written as follows:

$$|P(\beta, p)\rangle = C(\beta, p)\left[|P_{\text{can}}(\beta)\rangle + \sum_{n=1}^{\infty} (\widehat{G}_0(\widehat{T} - \widehat{T}_0))^n |P_{\text{can}}(\beta)\rangle\right], \tag{70}$$

where $C(\beta, p)$ is the positive normalization factor of $|P(\beta, p)\rangle$ such that $\sum_{\boldsymbol{x}\in X^N} \langle \boldsymbol{x}|P(\beta, p)\rangle = 1$. Since $\widehat{T} - \widehat{T}_0 = O(p)$, one can estimate that $C(\beta, p) = 1 + O(p)$.

As obtained in (53) and (58), within the asymptotic regime of small p, $\widehat{T} - \widehat{T}_0$ is can be expanded with powers of p, and then we have

$$\widehat{T} - \widehat{T}_0 = \widehat{U}_1 + O(p^2), \tag{71}$$

where

$$\widehat{U}_1 = \frac{p/2}{1+p} \sum_{1 \leqslant i_0 < i_1 \leqslant N} \left(\widehat{\Delta}^+_{i_1} + \widehat{\Delta}^-_{i_1} - 2\,\widehat{id}_H^{\otimes N}\right) \widehat{\Xi}_{i_0,i_1} \widehat{T}^{i_0}_{0,\text{move}}. \tag{72}$$

By substituting (72) into (70), we have

$$\begin{aligned} |P(\beta, p)\rangle &= C(\beta, p)[|P_{\text{can}}(\beta)\rangle + \widehat{G}_0 \widehat{U}_1 |P_{\text{can}}(\beta)\rangle + O(p^2)] \\ &= C(\beta, p)\left[|P_{\text{can}}(\beta)\rangle + \frac{p}{1+p} \sum_{1 \leqslant i_0 < i_1 \leqslant N} \widehat{B}_{i_0,i_1} |P_{\text{can}}(\beta)\rangle\right] \\ &\quad + O(p^2), \end{aligned} \tag{73}$$

where we define

$$\widehat{B}_{i_0,i_1} := \tfrac{1}{2} \widehat{G}_0 \left(\widehat{\Delta}^+_{i_1} + \widehat{\Delta}^-_{i_1} - 2\,\widehat{id}_H^{\otimes N}\right) \widehat{\Xi}_{i_0,i_1} \widehat{T}^{i_0}_{0,\text{move}}. \tag{74}$$

This operator $\widehat{B}_{i_0,i_1}$ is dependent on β but independent of p. Thus, using $r_{ii} = 1$ for any i, the overlap centrality can be written as follows:

$$\begin{aligned} O_i &= \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_i, x_j) \langle \boldsymbol{x}|P(\beta, p)\rangle - 1 \\ &= C(\beta, p)\left[\mathcal{A}_0(i) + \frac{p}{1+p} \sum_{1 \leqslant i_0 < i_1 \leqslant N} \mathcal{A}_1(i, i_0, i_1)\right] - 1 \\ &\quad + O(p^2), \end{aligned} \tag{75}$$

where

$$\mathcal{A}_0(i) = \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_i, x_j) \langle \boldsymbol{x}|P_{\text{can}}(\beta)\rangle, \tag{76}$$

$$\mathcal{A}_1(i, i_0, i_1) = \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_i, x_j) \langle \boldsymbol{x}|\widehat{B}_{i_0,i_1}|P_{\text{can}}(\beta)\rangle. \tag{77}$$

Here $\mathcal{A}_0(i)$ and $\mathcal{A}_1(i, i_0, i_1)$ are constant as a function of p.

Recall that $\widehat{\Pi}_\sigma$ is the permutation operator corresponding to a permutation $\sigma \in \mathfrak{S}_N$ [see (38) for a definition]. By using (67), we have

$$\begin{aligned} \mathcal{A}_0(i) &= \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_i, x_j) \langle \boldsymbol{x}|\widehat{\Pi}_\sigma|P_{\text{can}}(\beta)\rangle \\ &= \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_{\sigma(i)}, x_{\sigma(j)}) \langle \boldsymbol{x}|P_{\text{can}}(\beta)\rangle \\ &= \sum_{1 \leqslant j \leqslant N} \sum_{\boldsymbol{x}\in X^N} \delta(x_{\sigma(i)}, x_j) \langle \boldsymbol{x}|P_{\text{can}}(\beta)\rangle \\ &= \mathcal{A}_0(\sigma(i)). \end{aligned} \tag{78}$$

This means that $\mathcal{A}_0(i)$ does not depend on i:

$$\mathcal{A}_0(i) = \mathcal{A}_0(1) =: \mathcal{B}_0. \tag{79}$$

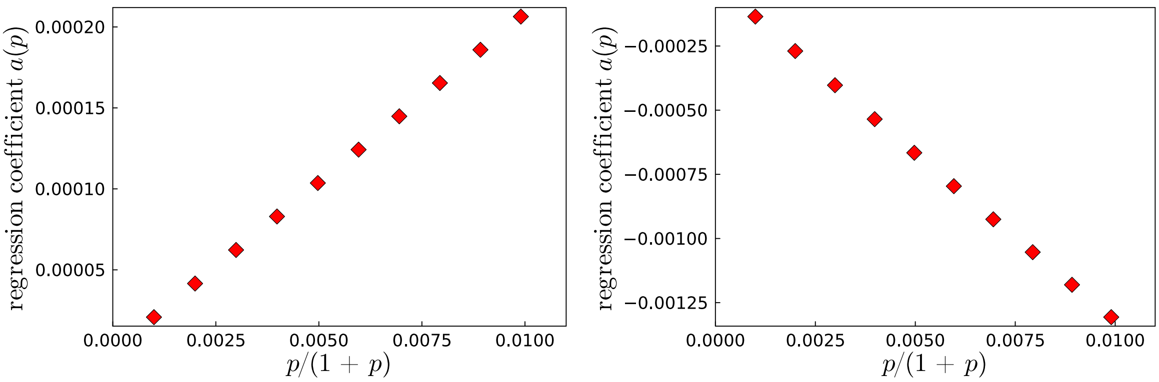

FIG. 7. The overlap centrality O_j computed by the exact diagonalization for $L = N = 6$ is expressed by $O_j \simeq a(p)j + b(p)$, where coefficient $a(p)$ and $b(p)$ are determined by linear regression. These two figures plot the regression coefficient $a(p)$ vs $p/(1+p)$ for $p = 0.001, 0.002, \ldots, 0.01$; $\beta = -1$ (left) and $\beta = 1$ (right). The correlation coefficient between $a(p)$ and $p/(1+p)$ is 0.9999998... for $\beta = -1$ and $-0.99997\ldots$ for $\beta = 1$. Note that p dependence of the first term in (85) corresponds to that the correlation coefficient is equal to 1 for $\beta = -1$ and -1 for $\beta = 1$, respectively.

Moreover, by using (61), (62), (63), and (67), we have $\widehat{\Pi}_\sigma \widehat{B}_{i_0,i_1} = \widehat{B}_{\sigma(i_0),\sigma(i_1)}\widehat{\Pi}_\sigma$. Therefore, it follows that

$$\begin{aligned}
&\mathcal{A}_1(i, i_0, i_1)\\
&= \sum_{1\leqslant j\leqslant N}\sum_{\boldsymbol{x}\in X^N} \delta(x_i, x_j)\langle \boldsymbol{x}|\widehat{\Pi}_\sigma^{-1}\widehat{B}_{\sigma(i_0),\sigma(i_1)}\widehat{\Pi}_\sigma|P_{\mathrm{can}}(\beta)\rangle\\
&= \sum_{1\leqslant j\leqslant N}\sum_{\boldsymbol{x}\in X^N} \delta(x_{\sigma(i)}, x_{\sigma(j)})\langle \boldsymbol{x}|\widehat{B}_{\sigma(i_0),\sigma(i_1)}|P_{\mathrm{can}}(\beta)\rangle\\
&= \sum_{1\leqslant j\leqslant N}\sum_{\boldsymbol{x}\in X^N} \delta(x_{\sigma(i)}, x_j)\langle \boldsymbol{x}|\widehat{B}_{\sigma(i_0),\sigma(i_1)}|P_{\mathrm{can}}(\beta)\rangle\\
&= \mathcal{A}_1(\sigma(i), \sigma(i_0), \sigma(i_1)) \qquad (80)
\end{aligned}$$

for any $\sigma \in \mathfrak{S}_N$. In this sense, $\mathcal{A}_1$ preserves permutation symmetry in spite of the permutation-symmetry breaking of $\widehat{T}$. By this equation, we obtain

$$\mathcal{A}_1(i, i_0, i_1) = \begin{cases} \mathcal{A}_1(1,1,2) =: \mathcal{B}_1 & (\text{if } i = i_0)\\ \mathcal{A}_1(2,1,2) =: \mathcal{B}_2 & (\text{if } i = i_1)\\ \mathcal{A}_1(3,1,2) =: \mathcal{B}_3 & (\text{if } i \neq i_0, i_1)\end{cases} \qquad (81)$$

for any i and any pair $i_0 < i_1$. These quantities $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are independent of p.

Note that for a given agent i, the number of pairs (i_0, i_1) which satisfies each of the above conditions corresponding with $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ is $N - i$, $i - 1$, and $(N-1)(N-2)/2$, respectively. Therefore, we obtain

$$\begin{aligned}
&\sum_{1\leqslant i_0<i_1\leqslant N} \mathcal{A}_1(i, i_0, i_1)\\
&= (N-i)\mathcal{B}_1 + (i-1)\mathcal{B}_2 + \frac{(N-1)(N-2)}{2}\mathcal{B}_3. \qquad (82)
\end{aligned}$$

Substituting (82) into (75), we find that

$$O_i = \frac{p}{1+p}C(\beta, p)(\mathcal{B}_2 - \mathcal{B}_1)i + c_0 + O(p^2), \qquad (83)$$

where c_0 is a real number independent of rank i:

$$\begin{aligned}
c_0 = &\frac{p}{1+p}C(\beta, p)\left(N\mathcal{B}_1 - \mathcal{B}_2 + \frac{(N-1)(N-2)}{2}\mathcal{B}_3\right)\\
&+ C(\beta, p)\mathcal{B}_0 - 1. \qquad (84)
\end{aligned}$$

With the estimation of $C(\beta, p) = 1 + O(p)$, we can rewrite (83) as

$$O_i = \frac{p}{1+p}(\mathcal{B}_2 - \mathcal{B}_1)i + c_0 + O(p^2). \qquad (85)$$

Note that, in (75), (83), and (85), the terms in $O(p^2)$ could depend on i. As shown in Fig. 7, we have confirmed that p dependence in the first term of (85) is consistent to that computed by the exact diagonalization. Moreover, as shown in Fig. 8, we can see good agreement of the value $\mathcal{B}_2 - \mathcal{B}_1$ by two distinct methods: one method is by definition (81). The other is by comparing O_i computed by the exact diagonalization with rank dependence in (85).

We write the remaining term in (85) as

$$\varepsilon_i(\beta, p) := O_i - \frac{p}{1+p}(\mathcal{B}_2 - \mathcal{B}_1)i - c_0 = O(p^2). \qquad (86)$$

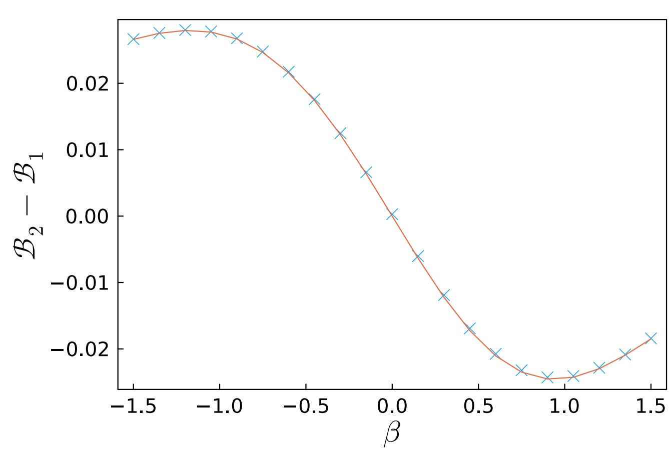

FIG. 8. $\mathcal{B}_2 - \mathcal{B}_1$ as a function of β for $L = N = 3$. The line corresponds to the computation through its definition (81). The blue crosses correspond to the estimation through (85) where O_i is calculated by the exact diagonalization. Practically, the estimated value is calculated as $\tilde{a}(\beta)$ where $a(p) \simeq \tilde{a}(\beta)\frac{p}{1+p} + \tilde{b}(\beta)$ through linear regression.

Let us suppose $\mathcal{B}_2 \neq \mathcal{B}_1$. The remainder term ε_i can be ignored when $p \ll 1$ compared to the term $\frac{p}{1+p}(\mathcal{B}_2 - \mathcal{B}_1)i$. After ignoring ε_i, the overlap centrality O_i is a linear function with respect to rank i. Therefore the perfect correlation holds:

$$\phi \to \begin{cases} +1 & (\text{if } \mathcal{B}_1 < \mathcal{B}_2) \\ -1 & (\text{if } \mathcal{B}_1 > \mathcal{B}_2) \end{cases} \quad \text{as } p \to +0. \tag{87}$$

This is consistent with the observation in Fig. 6 as long as p is small because the value of ϕ approaches $+1$ or -1 very closely.

The above discussion gives another indication. Ignoring ε_i, the linear dependency of the overlap centrality O_i with respect to rank i comes from permutation symmetry (80) in $\mathcal{A}_1$. Note that, as mentioned by (40), the permutation symmetry is broken in the transition matrix if $p \neq 0$, but this symmetry is partially recovered in the quantity $\mathcal{A}_1$. Note that one can also derive

$$M = \frac{p}{1+p}\left(\frac{N-1}{2N}(\mathcal{B}_1 + \mathcal{B}_2) + \frac{(N-1)(N-2)}{2N}\mathcal{B}_3\right) + C(\beta, p)\frac{\mathcal{B}_0}{N} + O(p^2) \tag{88}$$

by using (45).

Evaluating the sign of ϕ requires concrete calculation of both $\mathcal{B}_1$ and $\mathcal{B}_2$, but it gets complicated to obtain their analytic expressions for a given energy function such as the Potts energy (2). Nevertheless, it is still feasible to perform such a calculation in the case of $\beta = 0$ as discussed in the next subsection. It is because at $\beta = 0$, transition matrix $\widehat{T}_0(\beta)$ gets independent of the form of energy function, resulting in getting close to that of the free random walk process.

C. Singularity in ϕ at $\beta = 0$

One can calculate the overlap centrality at $\beta = 0$ for the asymptotic regime of small p concretely. As a result, we show that at $\beta = 0$, $\mathcal{B}_1 = \mathcal{B}_2$ holds, which means that the correlation coefficient ϕ is singular at $\beta = 0$.

From the definitions of $\mathcal{B}_1$, $\mathcal{B}_2$ in (81) and $\mathcal{A}_1$ in (77), we have

$$\mathcal{B}_1 = \sum_{1\leqslant j\leqslant N}\sum_{\boldsymbol{x}\in X^N} \delta(x_1, x_j)\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(\beta)\rangle, \tag{89}$$

$$\mathcal{B}_2 = \sum_{1\leqslant j\leqslant N}\sum_{\boldsymbol{x}\in X^N} \delta(x_2, x_j)\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(\beta)\rangle. \tag{90}$$

In order to calculate $\mathcal{B}_1$ and $\mathcal{B}_2$ at $\beta = 0$, we need to derive an explicit expression of $\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(0)\rangle$. First, let us define

$$|k) := \sum_{x\in X}\frac{e^{ikx}}{\sqrt{L}}|x\rangle \in H_X, \tag{91}$$

for $k = 2\pi n/L$ and $n \in \mathbb{Z}/L\mathbb{Z}$. The orthonormal system $\{|k)\}_k$ spans the whole space H_X over $\mathbb{C}$. In particular

$$|P_{\rm can}(0)\rangle = \left(\frac{|0)}{\sqrt{L}}\right)^{\otimes N} \tag{92}$$

holds. Note that, for $\boldsymbol{k} = (k_1, k_2, \dots, k_N)$ with $k_i = 2\pi n_i/L$ and $n_i \in \mathbb{Z}/L\mathbb{Z}$ with $1 \leqslant i \leqslant N$,

$$|\boldsymbol{k}) := |k_1) \otimes \cdots \otimes |k_N) \in H_X^{\otimes N} \tag{93}$$

is an eigenvector of $\widehat{T}_0(\beta = 0)$ with an eigenvalue

$$\frac{1}{N}\left(\cos^2\frac{k_1}{2} + \cdots + \cos^2\frac{k_N}{2}\right). \tag{94}$$

The vector $|\boldsymbol{k})$ is also an eigenvector of $\widehat{G}_0(\beta = 0)$, and the eigenvalue is

$$N\left(\sin^2\frac{k_1}{2} + \cdots + \sin^2\frac{k_N}{2}\right)^{-1} \tag{95}$$

for any $\boldsymbol{k} \neq \boldsymbol{0} = (0, 0, \dots, 0)$.

Next, substituting the definition (74) of $\widehat{B}_{1,2}$ into the term $\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(0)\rangle$, we obtain

$$\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(0)\rangle = \tfrac{1}{2}\langle \boldsymbol{x}|\widehat{G}_0(0)\big(\widehat{\Delta}_2^+ + \widehat{\Delta}_2^- - 2\,\widehat{id}_H^{\otimes N}\big) \times \widehat{\Xi}_{1,2}\widehat{T}^1_{0,\rm move}(\beta = 0)|P_{\rm can}(0)\rangle. \tag{96}$$

In order to obtain more explicit expression, let us multiply $|P_{\rm can}(0)\rangle$ by $\widehat{T}^1_{0,\rm move}$, $\widehat{\Xi}_{1,2}$, $(\widehat{\Delta}_2^+ + \widehat{\Delta}_2^- - 2\,\widehat{id}_H^{\otimes N})$, and $\widehat{G}_0(0)$ from the left, successively. Reminding of the definition (59) of $\widehat{T}^1_{0,\rm move}$, we have

$$\widehat{T}^1_{0,\rm move}(0)|P_{\rm can}(0)\rangle = \frac{1}{2N}|P_{\rm can}(0)\rangle \tag{97}$$

and can show that

$$\widehat{\Xi}_{1,2}|P_{\rm can}(0)\rangle = \sum_{x\in X}\frac{|x\rangle}{L}\otimes\frac{|x\rangle}{L}\otimes\left(\frac{|0)}{\sqrt{L}}\right)^{\otimes(N-2)} = \frac{1}{L^2}\sum_{k_1}|k_1)\otimes|-k_1)\otimes\left(\frac{|0)}{\sqrt{L}}\right)^{\otimes(N-2)}. \tag{98}$$

By using (97) and (98), we obtain

$$\big(\widehat{\Delta}_2^+ + \widehat{\Delta}_2^- - 2\,\widehat{id}_H^{\otimes N}\big)\widehat{\Xi}_{1,2}\widehat{T}^1_{0,\rm move}(0)|P_{\rm can}(0)\rangle = -\frac{1}{2N}\frac{1}{L^2}\sum_{k_1}4\sin^2\frac{k_1}{2}|k_1)\otimes|-k_1)\otimes\left(\frac{|0)}{\sqrt{L}}\right)^{\otimes(N-2)}. \tag{99}$$

Substituting (99) into (96), and using (95), we find

$$\langle \boldsymbol{x}|\widehat{B}_{1,2}|P_{\rm can}(0)\rangle = -\frac{1}{2L^2}\sum_{k_1\neq 0}\frac{e^{ik_1x_1}}{\sqrt{L}}\frac{e^{-ik_1x_2}}{\sqrt{L}}\left(\frac{1}{L}\right)^{N-2} = -\frac{1}{2L^{N+1}}[L\delta(x_1, x_2) - 1]. \tag{100}$$

Recalling (89) and (90) with

$$\sum_{\boldsymbol{x}\in X^N}\delta(x_i, x_j)[L\delta(x_1, x_2) - 1] = \begin{cases} L^{N-1}(L-1) & [\text{if } (i,j) = (1,2) \text{ or } (2,1)] \\ 0 & (\text{otherwise}), \end{cases} \tag{101}$$

one can calculate

$$\mathcal{B}_1 = \mathcal{B}_2 = -\frac{L-1}{2L^2}. \tag{102}$$

Thus, it turns out that ϕ is singular at $\beta = 0$.

One can also calculate $\mathcal{B}_0 = N/L$, and using (101), $\mathcal{B}_3 = 0$. By substituting the value of $\mathcal{B}_\ell$ ($\ell = 0, 1, 2, 3$) into (84) and

(85), the overlap centrality O_i is $O(N/L)$. This is consistent with the uniform distribution corresponding to the case of $\beta = 0$. Therefore, it is reasonable that $\mathcal{B}_0 = O(N/L)$ and $\mathcal{B}_1 = \mathcal{B}_2 = O(L^{-1})$ as functions of N and L. Note that one can also evaluate

$$M = \frac{1}{L}\left(C(0,p) - \frac{p}{2(1+p)}\frac{N-1}{N}\frac{L-1}{L}\right) + O(p^2). \tag{103}$$

D. Comparison between exact diagonalization and analytic result

In this subsection, we shall illustrate the behavior of the correlation coefficient ϕ computed by exact diagonalization in comparison with analytic discussion in Sec. IV.

In order to consider β dependence of ϕ, let us fix p as a small but nonzero value and change the value of β. When the parameter β varies with satisfying the condition

$$\mathcal{B}_2(\beta,p) - \mathcal{B}_1(\beta,p) = O(\varepsilon_i(\beta,p)), \tag{104}$$

the linear term $(\mathcal{B}_2 - \mathcal{B}_1)i$ in O_i is no longer dominant in the rank dependence. As a result, ϕ could change continuously from $\phi = -1$ to $\phi = 1$. In this case, (87) does not necessarily hold. Exact diagonalization indicates that the range of β, where ϕ takes a value close to ± 1, is wider as p is smaller.

Let us recall the observation in the exact diagonalization in the case of the Potts energy that the value of ϕ sharply changes around $\beta = 0$ for small p as shown in Fig. 6. Assuming that this observation is universal for sufficiently small p, by combining the existence of perfect correlation and the singularity of ϕ at $\beta = 0$, it may be a reasonable conjecture that, at least, in the case of the Potts energy, ϕ becomes discontinuous at $\beta = 0$ as a function of β in the limit of $p \to +0$.

E. Correspondence between overlap centrality and eigenvector centrality

Let us discuss the relation between the overlap centrality $\boldsymbol{O} = (O_i)_{i=1}^N$ and the other existing ways to define centrality.

First, the overlap centrality has a connection to another existing centrality in the following sense. Let us consider a weighted complete graph, where each agent is regarded as a vertex, and an element r_{ij} of the neighbor matrix $\mathcal{R}$ for the pair of agents i, j is regarded as the weight of the edge (i, j). Then the overlap centrality defined above is equivalent to the strength centrality of the complete graph constructed above, which has been introduced in the field of network science [14,16].

Second, one can also define the eigenvector centrality of the complete graph mentioned above as the eigenvector $\boldsymbol{V} = (V_i)_{1\leqslant i\leqslant N}$ of $\mathcal{R}$ with the maximum eigenvalue. Indeed, one may show that the eigenvector centrality is directly related to the overlap centrality when $p \ll 1$ in the following manner:

$$\boldsymbol{V} \propto \frac{1}{N^{3/2}c}\boldsymbol{O} - \gamma \times (1,1,\dots,1)^T + O(p^2), \tag{105}$$

where c and γ are constants with $O(1)$. In particular, combining with (85), we have

$$V_i \propto \frac{p}{1+p}\frac{\mathcal{B}_2 - \mathcal{B}_1}{N^{3/2}c}i + \left(\frac{c_0}{N^{3/2}c} - \gamma\right) + O(p^2), \tag{106}$$

where the coefficient of proportionality is independent of i. Thus, the eigenvector centrality as well as the overlap centrality depends on the rank i linearly if the term of $O(p^2)$ is ignored in (105). Remarkably, that relation (105) holds for general probability distribution which breaks the permutation symmetry of agents weakly. See Appendix B for the explicit two conditions to hold the relation (105) in a more general form. As shown in Fig. 9, one can see better agreement of both centralities for smaller p.

V. CONCLUDING REMARKS

In this paper, we have proposed a stochastic process without both detailed balance and permutation symmetry, which is inspired by the supplanting phenomenon of Japanese macaques. We have introduced a type of centrality, which we call overlap centrality, to characterize a rank-dependent correlation of agents' positions. Then we have found that this model of interacting ranked agents exhibits the unexpected linearity (85) with respect to the agent's rank i in overlap centrality O_i at a small supplanting condition, which could be regarded as a type of collective phenomenon. Precisely speaking, when $\mathcal{B}_1 \neq \mathcal{B}_2$, the perfect correlation corresponding to the linearity mentioned above appears between O_i and rank i in the regime of weak supplanting limit $p \to +0$. Even for small but nonzero p, concrete analysis by exact diagonalization shows that ϕ is very close to the perfect correlation in the case of the Potts energy if β is far from 0.

One might ask about the meaning of perfect correlation. So far, we do not have a clear answer to this question. However, one might have speculation that the perfect correlation implies divergence of characteristic scale in ranking. From this viewpoint, it could be intriguing to quantify such a characteristic scale in future studies.

Another problem on the singular behaviors of ϕ around $\beta = 0$ is to identify the effects which essentially cause those singular behaviors. Compared to the equilibrium Potts model, the model with supplanting process does not have permutation symmetry in terms of agents and also does not have detailed balance. In our derivation of perfect correlation, broken permutation symmetry is one of essential parts, but we are not aware of the effects from broken detailed balance. Concerning this point, one can consider an equilibrium model keeping with detailed balance without permutation symmetry by, for example, an energy function $\sum_{i,j} J_{i,j}\delta(x_i, x_j)$, where $J_{i,j}$ is a function of agents i, j such as $i \times j$. If one could prove that there does not exist the singularity of ϕ for such general equilibrium models, one could presumably expect that both permutation symmetry breaking and broken detailed balance are essential for causing the singularity. Such a motivation has been raised for broken rotational symmetry observed in active matter [28]. Indeed, it has been proven for a general class of systems having potentials dependent on position and velocity that rotational symmetry cannot be broken in equilibrium. This implies that the observed phase transitions associating with broken rotational symmetry in active matters are caused purely by nonequilibrium effects such as broken detailed balance. We will need somewhat similar ideas. We remark that the term *permutation symmetry* has been used in this paper in

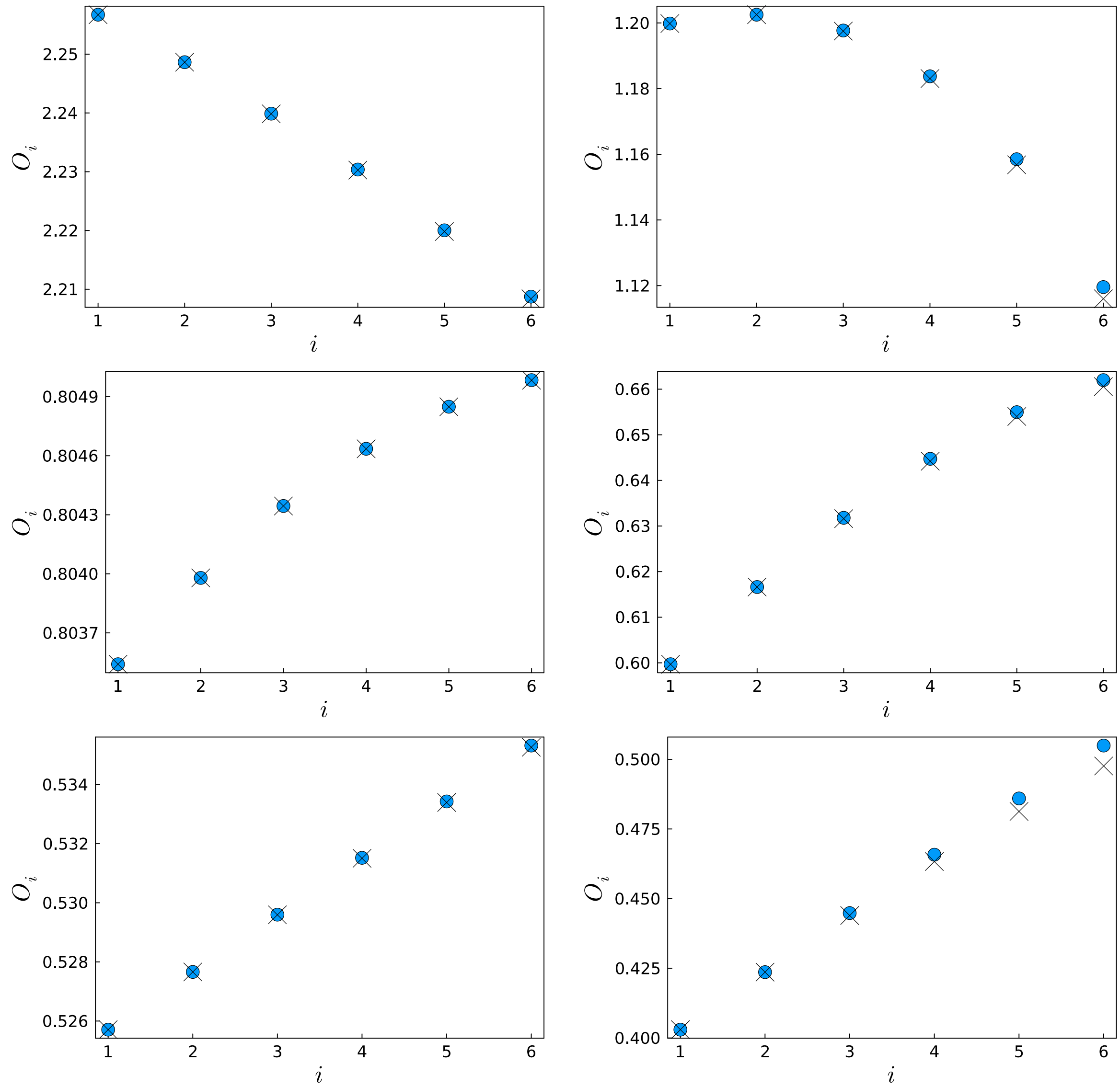

FIG. 9. The overlap centrality (blue circles) O_i and the normalized eigenvector centrality (black crosses) determined by the stationary distribution obtained by the exact diagonalization as a function of agent i for $L = N = 6$. The parameters are set as follows; $p = 0.1$ (left) and $p = 10.0$ (right); $\beta = 1$ (top), $\beta = 0$ (center), and $\beta = -1$ (bottom), respectively. The eigenvector centrality is normalized so that the two data points of $i = 1$ and $i = 2$ in the ranking axis are perfectly matched. The overlap centrality gets closer to the eigenvector centrality for $p = 0.1$ compared to the case of $p = 10.0$.

various manners depending on the quantity to which the term is applied. See the list of the various ways the term is applied in Table I.

Let us briefly discuss the obtained results in the context of linear response theory. If one defines a susceptibility of M as $\chi(p) := dM/dp$ in terms of p, we can obtain

$$\chi(0) = \frac{N-1}{2N}(\mathcal{B}_1 + \mathcal{B}_2) + \frac{(N-1)(N-2)}{2N}\mathcal{B}_3 + \partial_p C(\beta, 0)\mathcal{B}_0. \tag{107}$$

TABLE I. How to use the term *permutation symmetry* depending on the classes of quantity. For example of each class, the case (a) is applied to (80), (b) to (1), (c) to (67), and (d) to (39).

	Class of quantity	Definition for the quantity to be *permutation symmetric*	
(a)	Function $f(i_1, i_2, \dots, i_n)$ depending on ranks $i_1, i_2, \dots, i_n$	$f(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_n)) = f(i_1, i_2, \dots, i_n)$	
(b)	Function $f(\boldsymbol{x})$ depending on agents' configuration $\boldsymbol{x} \in X^N$	$f(\sigma(\boldsymbol{x})) = f(\boldsymbol{x})$	
(c)	State vector $\lvert\psi\rangle \in H_X^{\otimes N}$	$\widehat{\Pi}_\sigma \lvert\psi\rangle = \lvert\psi\rangle$	for any $\sigma \in \mathfrak{S}_N$
(d)	Linear operator $\widehat{A}$ from $H_X^{\otimes N}$ to itself	$\widehat{\Pi}_\sigma \widehat{A} \widehat{\Pi}_\sigma^{-1} = \widehat{A}$	

Note that the coefficient $\partial_p C(\beta, 0)$ of $\mathcal{B}_0$ is described as

$$-\binom{N}{2} \sum_{x \in X^N} \langle x | \widehat{B}_{i_0, i_1} | P_{\mathrm{can}}(\beta) \rangle \tag{108}$$

for some $1 \leqslant i_0 < i_1 \leqslant N$. By a similar computation to (80), we can find that (108) does not depend on i_0, i_1, and p. So far, it is not obvious for us to connect those quantities with the other known quantities, which remains an open problem.

Whereas our above discussions focus on the case of small $p \ll 1$, we move to the case of any p. The result of exact diagonalization implies that perfect correlation with $\phi = 1$ would hold for negative β sufficiently far from 0. This behavior is of interest in that the strong correlation effect originating from hard core repulsion may stabilize perfect correlation with $\phi = 1$. However, our strategy of the perturbation with respect to p is unavailable for not small $p \not\ll 1$. For this reason, it is not clear whether the perfect correlation is derived by use of permutation symmetry in a similar way to the case of $p \ll 1$. In order to tackle this problem, the situation with restricted values of L and N is to be considered. As an example, let us take the case satisfying $N = L + 1$ with the Potts energy. In this case, if the repulsion is sufficiently strong, then each site is occupied by at least an agent, and there is a single site occupied by two agents. Then one of the two agents occupying the site can hop in accordance with the equilibrium dynamics, while all of the other agents cannot hop effectively. This leads to reduction of the transition of states and could help us to analyze the overlap centrality of this model. Note that the above discussion is based on the Potts energy and is not necessarily applied to the other case with a general energy form. It remains for future work to perform further numerical calculations for energy functions other than the Potts energy as well as to explore analytic methods for general p.

Let us briefly discuss the possibility of phase transition lines in parameter space (β, p) for nonzero p. First, the equilibrium phase transition point $\beta = \beta_c$ with $p = 0$ for the case of Potts energy might extend toward nonzero p as a nonequilibrium phase transition line $\beta = \beta_c(p)$ where M shows a singular jump. Second, the point $\beta = 0$ with $p \to +0$ where the correlation coefficient shows singularity might also extend toward nonzero p as another nonequilibrium phase transition line $\beta = \beta_0(p)$ where ϕ shows a singular jump. However, since our analysis is limited to the parameter region close to $p = 0$, it is necessary to perform large-scale numerical simulations or develop other analytical methods in order to capture the true limiting behavior of large system size for nonzero p. This remains to be an intriguing future study.

From a mathematical viewpoint, *beta decomposition* of the transition matrix could be useful in a more general context. This expression gives the exact lowest order of the transition matrix in terms of p. Nevertheless, the concrete expression of it looks complicated, and then it would not be straightforward to use the expression for a given purpose. Further, it should be noted that the minor modification of the supplanting process prevents us from obtaining the beta decomposition. In this sense, the present version of the supplanting process can be regarded as a specially tractable case. It remains an open question how beta decomposition can be applied to calculation of the other quantities and whether one can find other tractable cases in this direction.

Let us remark on the direct relation between eigenvector centrality and strength centrality. Unifying two relations among centralities discussed in Sec. IV E, one can also have another conclusion in the context of network theory. Consider a complete graph whose edges are weighted by the same value with small fluctuation. Then the neighbor matrix $\mathcal{R}$ whose entries are weights of edges has a decomposition $\tilde{\mathcal{R}}^{(1)} + \tilde{\mathcal{R}}^{(2)}$ similar to (B5), and the graph holds a linear relation similar to (105) between the strength centrality and the eigenvector centrality.

Last, we mention the results in this paper relevant to behaviors of members in a group of primate species. Ranking has been widely known to be one characteristic structure which primates species have when they live as a group [18]. Indeed, it has been found that ranking affects the spatial location of the members in a group [19,20,29,30] and the distance between the members [31,32]. The overlap centrality observed in the model proposed in this paper might shed light on how one could estimate the ranking structures of such a group and its environmental conditions.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF TWO DECOMPOSITIONS OF TRANSITION MATRIX

In this section we would like to explain, in detail, the derivations of two decompositions of transition matrix $\widehat{T}$, which we call *supplanting decomposition* and *beta decomposition* as mentioned in Sec. IV A. To do this, we give a detailed account of our models in terms of operators. Hereafter, in order to make this Appendix self-contained, many definitions in the main text are repeated.

1. Notations

For a set $\boldsymbol{S}$, the number of elements is denoted as $\#\boldsymbol{S}$. The Kronecker's delta $\delta(i, j)$ is defined as

$$\delta(i, j) = \begin{cases} 1 & (\text{if } i = j) \\ 0 & (\text{if } i \neq j). \end{cases} \tag{A1}$$

We consider any vector space appearing in this section as complex vector space. We denote the set of complex numbers as $\mathbb{C}$, and the set of real numbers as $\mathbb{R}$.

2. Describing the model

In this subsection, as a preliminary, we introduce basic concepts which are necessary to explain the model.

a. Agents and states

We consider $N \geqslant 2$ agents labeled by $1, 2, \ldots, N$. They have a total ordering (i.e., linear dominance) meaning that if $1 \leqslant i < j \leqslant N$, agent i is higher than agent j. The set of all agents is written as

$$[N] := \{1, 2, \dots, N\}. \tag{A2}$$

The agents lie in the lattice $X = \mathbb{Z}/L\mathbb{Z} = \{0, 1, \dots, L-1\}$ with $L \geqslant 3$. For an agent i, the position is written as $x_i \in X$. Thus, the configuration space of agents is described by

$$X^N = \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \mid x_i \in X\}. \tag{A3}$$

b. Hopping map

We define a map $f_i^+ : X^N \to X^N$ (resp. $f_i^- : X^N \to X^N$) of configurations as the increment (resp. decrement) of position of agent i under a periodic boundary condition. Explicitly, we have

$$f_i^+(x_1, x_2, \dots, x_i, \dots, x_N) := (x_1, x_2, \dots, x_i + 1, \dots, x_N), \tag{A4}$$

$$f_i^-(x_1, x_2, \dots, x_i, \dots, x_N) := (x_1, x_2, \dots, x_i - 1, \dots, x_N). \tag{A5}$$

c. The permutation-invariant energy function

We take an energy function $E(\boldsymbol{x}) = E(x_1, x_2, \dots, x_N)$ which is symmetric in the following sense:

$$E(x_1, x_2, \dots, x_N) = E(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}) \tag{A6}$$

for any permutation $\sigma \in \mathfrak{S}_N$ of N elements in $[N]$. The symmetry will be used in Appendix A 3 e and A 5 e and is essential in (78) and (80). For example, in (2), we take the *normalized Potts model*

$$E(\boldsymbol{x}) = -\frac{(L-1)\log(L-1)}{N(L-2)} \sum_{1 \leqslant i,j \leqslant N} \delta(x_i, x_j) \tag{A7}$$

as an energy function.

To simplify the notation, we introduce the (i, d)-th difference of the energy function E for $1 \leqslant i \leqslant N$ and $d \in \{+, -\}$:

$$\mathcal{D}_i^d E(\boldsymbol{x}) := E\left(f_i^d \boldsymbol{x}\right) - E(\boldsymbol{x}). \tag{A8}$$

d. Describing the hopping by equilibrium dynamics

First, we describe the hopping by equilibrium dynamics as follows.

(a) Determine an agent $1 \leqslant i \leqslant N$ with equal probability $1/N$.

(b) Choose a direction $d = +$ or $-$ with equal probability $1/2$.

(c) Decide whether the agent i stays or hops to the direction d:

The probability that the agent i hops to direction d is

$$\frac{1}{1 + \exp\left[\beta \mathcal{D}_i^d E(\boldsymbol{x})\right]}. \tag{A9}$$

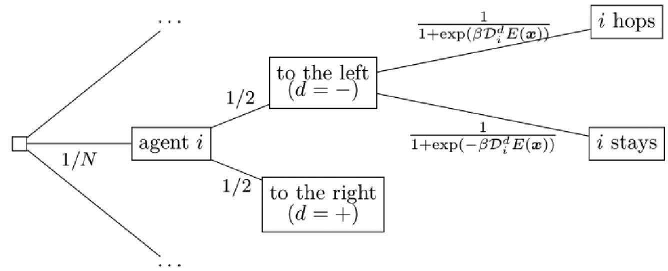

FIG. 10. Probability tree of hopping by equilibrium dynamics.

The probability that the agent i stays is

$$1 - \frac{1}{1 + \exp\left[\beta \mathcal{D}_i^d E(\boldsymbol{x})\right]} = \frac{1}{1 + \exp\left[-\beta \mathcal{D}_i^d E(\boldsymbol{x})\right]}. \tag{A10}$$

For a graphical explanation, see Fig. 10.

e. Describing the hopping by supplanting process

For a given configuration $\boldsymbol{x} \in X^N$, an agent $i \in [N]$, and a direction $d \in \{+, -\}$, we define the set $S(\boldsymbol{x}, i, d) \subseteq [N]$ as follows:

$$S(\boldsymbol{x}, i, \pm) := \{i < j \leqslant N \mid x_j = x_i \pm 1\}. \tag{A11}$$

In other words, $S(\boldsymbol{x}, i, d)$ is defined as a set of agents $j \in [N]$ satisfying the two conditions: (1) $j \in S(\boldsymbol{x}, i, d)$ is lower than i (in other words $j > i$) and (2) sits on the site $x_i + 1$ (when $d = +$) or $x_i - 1$ (when $d = -$).

Let us fix the supplanting rate $0 \leqslant p < \infty$. After step (c) of the first hopping, we define the *supplanting process* as described below.

(d) One of the following events occurs in the probabilities described below:

The agent $j \in S(\boldsymbol{x}, i, d)$ hops with the probability

$$\frac{p}{1 + p\#S(\boldsymbol{x}, i, d)}. \tag{A12}$$

In a probability

$$\frac{1}{1 + p\#S(\boldsymbol{x}, i, d)}, \tag{A13}$$

no one hops.

(e) If a hopping occurs in (d), choose a direction $d' = +$ or $-$ of the hopping of the agent j with uniform probability $1/2$.

Note that if $p = 0$ or $S(\boldsymbol{x}, i, d)$ is empty, then no supplanting occurs. The diagram in Fig. 11 describes the probability tree after the first hopping of the agent i.

3. Transition matrices

In this subsection, we write the transition matrices as linear operators on a state space.

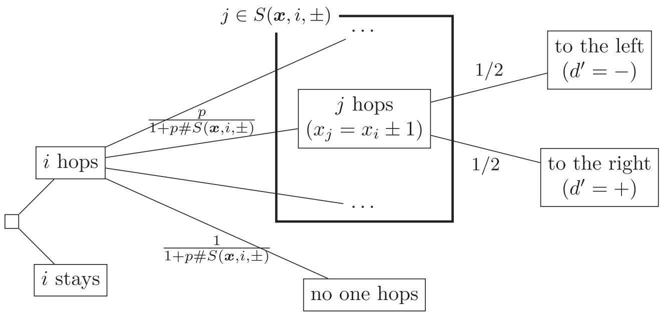

FIG. 11. Probability tree of supplanting process after the first hopping.

a. State vector spaces

The state space H_X is the complex vector space with a basis

$$|0\rangle, |1\rangle, \dots, |L-1\rangle \tag{A14}$$

corresponding to sites $0, 1, \dots, L-1 \in X$, respectively.

Similarly the multistate space H_{X^N} is the complex vector space with a basis $|\boldsymbol{x}\rangle$ corresponding to configurations $\boldsymbol{x} \in X^N$. It is identified with the tensor space $H_X^{\otimes N}$ with a map

$$\begin{aligned} H_{X^N} &\to H_X^{\otimes N} := \underbrace{H_X \otimes H_X \otimes \cdots \otimes H_X}_{N}; \\ |\boldsymbol{x}\rangle &\mapsto |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle. \end{aligned} \tag{A15}$$

With this identification, we freely use tensor notation to notate operators.

We introduce the inner product $\langle \cdot | \cdot \rangle$ on H_X such that the basis $(|x\rangle)_{x \in X}$ is orthonormal. This induces an inner product on $H_X^{\otimes N}$, and the basis $(|\boldsymbol{x}\rangle)_{\boldsymbol{x} \in X^N}$ on $H_X^{\otimes N}$ is orthonormal. We use the same symbol $\langle \cdot | \cdot \rangle$ to write this induced inner product on $H_X^{\otimes N}$.

b. Coefficients of operators

An operator $\widehat{A} : H_X^{\otimes N} \to H_X^{\otimes N}$ is described by its coefficients. The $(\boldsymbol{x}, \boldsymbol{y})$-th coefficient of $\widehat{A}$ is written by $\langle \boldsymbol{y} | \widehat{A} | \boldsymbol{x} \rangle$. In other words, we can write

$$\widehat{A} |\boldsymbol{x}\rangle = \sum_{\boldsymbol{y} \in X^N} |\boldsymbol{y}\rangle \langle \boldsymbol{y} | \widehat{A} | \boldsymbol{x} \rangle. \tag{A16}$$

c. Shift operators

As pieces of transition matrices, we define notations of shift operators.

The shift operators $\widehat{\Delta}_i^+$ and $\widehat{\Delta}_i^-$ on $H_X^{\otimes N}$ for $1 \leqslant i \leqslant N$ are defined as

$$\widehat{\Delta}_i^+ |\boldsymbol{x}\rangle := |f_i^+ \boldsymbol{x}\rangle \tag{A17}$$

$$= |x_1\rangle \otimes |x_2\rangle \cdots \otimes |x_i + 1\rangle \otimes \cdots \otimes |x_N\rangle, \tag{A18}$$

$$\widehat{\Delta}_i^- |\boldsymbol{x}\rangle := |f_i^- \boldsymbol{x}\rangle \tag{A19}$$

$$= |x_1\rangle \otimes |x_2\rangle \cdots \otimes |x_i - 1\rangle \otimes \cdots \otimes |x_N\rangle, \tag{A20}$$

for $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in X^N$. In terms of coefficients, the following hold:

$$\langle \boldsymbol{y} | \widehat{\Delta}_i^+ | \boldsymbol{x} \rangle = \begin{cases} 1 & (\text{if } \boldsymbol{y} = f_i^+ \boldsymbol{x}) \\ 0 & (\text{otherwise}), \end{cases} \tag{A21}$$

$$\langle \boldsymbol{y} | \widehat{\Delta}_i^- | \boldsymbol{x} \rangle = \begin{cases} 1 & (\text{if } \boldsymbol{y} = f_i^- \boldsymbol{x}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A22}$$

d. Projection operators

We denote the identity operator on the state space H_X as $\widehat{id}_H$. Then we can write the identity operator on the multistate space $H_X^{\otimes N}$ as $\widehat{id}_H^{\otimes N}$.

The projection operator $\widehat{\Xi}_i^y$ on $H_X^{\otimes N}$ for $1 \leqslant i \leqslant N$ and $y \in X$ is defined as

$$\widehat{\Xi}_i^y |\boldsymbol{x}\rangle := \begin{cases} |\boldsymbol{x}\rangle & (\text{if } x_i = y) \\ 0 & (\text{otherwise}). \end{cases} \tag{A23}$$

This projection operator checks whether $x_i = y$ or not; if $x_i = y$ it returns $|\boldsymbol{x}\rangle$, otherwise it returns the zero vector. For a configuration $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in X^N$, we define

$$\widehat{\Xi}^{\boldsymbol{x}} := \prod_{1 \leqslant j \leqslant N} \widehat{\Xi}_j^{x_j}. \tag{A24}$$

It satisfies

$$\widehat{\Xi}^{\boldsymbol{y}} |\boldsymbol{x}\rangle = \begin{cases} |\boldsymbol{x}\rangle & (\text{if } \boldsymbol{x} = \boldsymbol{y}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A25}$$

and, in terms of coefficients,

$$\langle \boldsymbol{y} | \widehat{\Xi}^{\boldsymbol{z}} | \boldsymbol{x} \rangle = \begin{cases} 1 & (\text{if } \boldsymbol{y} = \boldsymbol{x} = \boldsymbol{z}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A26}$$

The operator $\widehat{\Xi}^{\boldsymbol{z}}$ checks whether $\boldsymbol{x} = \boldsymbol{z}$.

We define some other projection operators: first, for $x \in X$ and $\emptyset \neq \boldsymbol{S} \subseteq [N]$,

$$\widehat{\Xi}_{\boldsymbol{S}} := \begin{cases} \sum_{x \in X} \prod_{i \in \boldsymbol{S}} \widehat{\Xi}_i^x & (\text{if } \boldsymbol{S} \neq \emptyset) \\ \widehat{id}_H^{\otimes N} & (\text{if } \boldsymbol{S} = \emptyset). \end{cases} \tag{A27}$$

The operator $\widehat{\Xi}_{\boldsymbol{S}}$ checks whether all of the agents in $\boldsymbol{S}$ sit on the same site. Explicitly,

$$\widehat{\Xi}_{\boldsymbol{S}} |\boldsymbol{x}\rangle = \begin{cases} |\boldsymbol{x}\rangle & (\text{if } x_i = x_j \text{ for any } i, j \in \boldsymbol{S}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A28}$$

If $\boldsymbol{S}$ is empty, it checks nothing; it is the identity operator.

Another projection operator $\widehat{\Upsilon}_{i;\boldsymbol{S}}$ for $1 \leqslant i \leqslant N$ and $\boldsymbol{S} \subseteq \{i+1, i+2, \dots, N\}$ checks two conditions: (1) $x_j = x_i$ if $j \in \boldsymbol{S}$, and (2) $x_{j'} \neq x_i$ if $j' \notin \boldsymbol{S}$ and $j' > i$. Explicitly, it is defined as

$$\widehat{\Upsilon}_{i;\boldsymbol{S}} |\boldsymbol{x}\rangle := \begin{cases} |\boldsymbol{x}\rangle & (\text{if } x_j = x_i \text{ for } j \in \boldsymbol{S}, \text{ and } x_i \notin \{x_{j'}\}_{\substack{j' > i \\ j' \notin \boldsymbol{S}}}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A29}$$

In terms of $\widehat{\Xi}_i^x$ and $\widehat{\Xi}_{\boldsymbol{S}}$, this operator is written as

$$\widehat{\Upsilon}_{i;\boldsymbol{S}} = \begin{cases} \sum_{x\in X} \widehat{\Xi}_i^x \cdot \prod_{j\in \boldsymbol{S}} \widehat{\Xi}_j^x \cdot \prod_{\substack{j'>i \\ j'\notin \boldsymbol{S}}} \big(\widehat{id}_H^{\otimes N} - \widehat{\Xi}_{j'}^x\big) & (\text{if } \boldsymbol{S} \neq \emptyset \text{ and } \boldsymbol{S} \neq \{i+1, i+2, \dots, N\}) \\ \sum_{x\in X} \widehat{\Xi}_i^x \cdot \prod_{j'>i} \big(\widehat{id}_H^{\otimes N} - \widehat{\Xi}_{j'}^x\big) & (\text{if } \boldsymbol{S} = \emptyset \text{ and } i \leqslant N-1) \\ \sum_{x\in X} \widehat{\Xi}_i^x \cdot \prod_{j\in \boldsymbol{S}} \widehat{\Xi}_j^x & (\text{if } \boldsymbol{S} \neq \emptyset \text{ and } \boldsymbol{S} = \{i+1, i+2, \dots, N\}) \\ \sum_{x\in X} \widehat{\Xi}_i^x & (\text{if } \boldsymbol{S} = \emptyset \text{ and } i = N) \end{cases} \tag{A30}$$

$$= \sum_{\boldsymbol{S}\subseteq \boldsymbol{S}'\subseteq\{i+1,\dots,N\}} (-1)^{\#\boldsymbol{S}'-\#\boldsymbol{S}} \widehat{\Xi}_{\boldsymbol{S}'}. \tag{A31}$$

A relation between these operators is

$$\widehat{\Xi}_{\{i\}\cup \boldsymbol{S}} = \sum_{\boldsymbol{S}\subseteq \boldsymbol{S}'\subseteq\{i+1,\dots,N\}} \widehat{\Upsilon}_{i;\boldsymbol{S}}. \tag{A32}$$

e. Permutation operators

For a permutation $\sigma \in \mathfrak{S}_N$ of $[N] = \{1, 2, \dots, N\}$, we define

$$\sigma(\boldsymbol{x}) := (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \tag{A33}$$

for $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in X^N$. Let $\widehat{\Pi}_\sigma$ be the corresponding matrix to σ^{-1}. Explicitly,

$$\widehat{\Pi}_\sigma |(x_1, x_2, \dots, x_n)\rangle\rangle = |\sigma^{-1}(x_1, x_2, \dots, x_n)\rangle\rangle \tag{A34}$$

$$= |(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(N)})\rangle\rangle. \tag{A35}$$

For an operator $\widehat{A}$ on $H_X^{\otimes N}$, the daggered symbol $\widehat{A}^\dagger$ denotes the Hermitian conjugate of $\widehat{A}$. Then we have $\widehat{\Pi}_\sigma^\dagger = \widehat{\Pi}_\sigma^{-1} = \widehat{\Pi}_{\sigma^{-1}}$. We also have some commutation relations as follows:

$$\widehat{\Delta}_j^\pm \widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma \widehat{\Delta}_{\sigma(j)}^\pm, \tag{A36}$$

$$\widehat{\Xi}_j^x \widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma \widehat{\Xi}_{\sigma(j)}^x, \tag{A37}$$

$$\widehat{\Xi}_{\boldsymbol{S}} \widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma \widehat{\Xi}_{\sigma(\boldsymbol{S})}. \tag{A38}$$

We remark that $\widehat{\Upsilon}_{i;\boldsymbol{S}}$ has a factor $\prod_{\substack{j>i \\ j\notin \boldsymbol{S}}} (\widehat{id}_H^{\otimes N} - \widehat{\Xi}_x^j)$. Since this factor involves an ordering $j > i$, it does not satisfy a simple relation in terms of permutation operators. A key procedure in our calculation is to rewrite an operator $\widehat{\Upsilon}_{i;\boldsymbol{S}}$ in terms of $\widehat{\Xi}_{\{i\}\cup \boldsymbol{S}}$. This enables us to use the commutation relations in terms of permutations.

Besides, since the energy function depends only on the numbers of agents on each site, the energy function does not depend on the labeling of the agents. In other terms,

$$E\big[f_{\sigma(i)}^d(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(N)})\big] = E\big[f_i^d(x_1, x_2, \dots, x_N)\big]. \tag{A39}$$

f. Other commutation relations of operators

The shift operators (resp. the projection operators) are commutative at each other, while a shift operator and a projection operator does not necessarily commute. Explicitly, it holds that

$$\widehat{\Xi}_i^x \widehat{\Delta}_j^+ = \begin{cases} \widehat{\Delta}_j^+ \widehat{\Xi}_i^x & (\text{if } i \neq j) \\ \widehat{\Delta}_j^+ \widehat{\Xi}_i^{x-1} & (\text{if } i = j), \end{cases} \tag{A40}$$

$$\widehat{\Xi}_i^x \widehat{\Delta}_j^- = \begin{cases} \widehat{\Delta}_j^- \widehat{\Xi}_i^x & (\text{if } i \neq j) \\ \widehat{\Delta}_j^- \widehat{\Xi}_i^{x+1} & (\text{if } i = j). \end{cases} \tag{A41}$$

g. Transition matrix for the first hopping

According to the description in Appendix A 2 d, the transition matrix $\widehat{T}_0$ for the first hopping can be described as

$$\widehat{T}_0 := \sum_{1\leqslant i\leqslant N} \frac{1}{N} \sum_{d=\pm} \frac{1}{2} \sum_{\boldsymbol{x}\in X^N} \left(\frac{\widehat{\Delta}_i^d}{1+\exp\big[\beta \mathcal{D}_i^d E(\boldsymbol{x})\big]} + \frac{\widehat{id}_H^{\otimes N}}{1+\exp\big[-\beta \mathcal{D}_i^d E(\boldsymbol{x})\big]} \right) \widehat{\Xi}^{\boldsymbol{x}}. \tag{A42}$$

For the latter use, we introduce some symbols. First, for $\boldsymbol{x} \in X^N$, $1 \leqslant i \leqslant N$, and $d \in \{+, -\}$, we write the coefficient of $\widehat{T}_0$ as

$$c_\beta(\boldsymbol{x}, i, d) := \frac{1}{2N\big\{1+\exp\big[\beta \mathcal{D}_i^d E(\boldsymbol{x})\big]\big\}}. \tag{A43}$$

Then we have

$$\frac{1}{2N} - c_\beta(\boldsymbol{x}, i, d) = \frac{1}{2N\big\{1+\exp\big[-\beta \mathcal{D}_i^d E(\boldsymbol{x})\big]\big\}}. \tag{A44}$$

Since $\exp(x) > 0$ for any real $x \in \mathbb{R}$, we can estimate

$$0 < c_\beta(\boldsymbol{x}, i, d) \leqslant \frac{1}{2N} \tag{A45}$$

for any $\beta, E, \boldsymbol{x}, i, d, N$, and L. We define a set of operators: for $d \in \{+, -\}$,

$$\widehat{T}_{0,\text{move}}^{i,d} := \sum_{\boldsymbol{x}\in X^N} c_\beta(\boldsymbol{x}, i, d) \widehat{\Delta}_i^d \widehat{\Xi}^{\boldsymbol{x}}, \tag{A46}$$

$$\widehat{T}_{0,\text{stay}}^{i,d} := \sum_{\boldsymbol{x}\in X^N} \left(\frac{1}{2N} - c_\beta(\boldsymbol{x}, i, d)\right) \widehat{\Xi}^{\boldsymbol{x}}, \tag{A47}$$

$$\widehat{T}_{0,\text{move}}^{i} := \widehat{T}_{0,\text{move}}^{i,+} + \widehat{T}_{0,\text{move}}^{i,-}, \tag{A48}$$

$$\widehat{T}_{0,\text{stay}}^{i} := \widehat{T}_{0,\text{stay}}^{i,+} + \widehat{T}_{0,\text{stay}}^{i,-}. \tag{A49}$$

Then we have

$$\widehat{T}_0 = \sum_{1\leqslant i\leqslant N} \left(\widehat{T}^i_{0,\mathrm{move}} + \widehat{T}^i_{0,\mathrm{stay}}\right) \tag{A50}$$

$$= \sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} \left(\widehat{T}^{i,d}_{0,\mathrm{move}} + \widehat{T}^{i,d}_{0,\mathrm{stay}}\right). \tag{A51}$$

Note that, using relations in Appendix A 3 e, we have

$$\widehat{\Pi}^\dagger_\sigma \widehat{T}^{i,d}_{0,\mathrm{move}} \widehat{\Pi}_\sigma = \widehat{T}^{\sigma(i),d}_{0,\mathrm{move}}, \tag{A52}$$

$$\widehat{\Pi}^\dagger_\sigma \widehat{T}^{i,d}_{0,\mathrm{stay}} \widehat{\Pi}_\sigma = \widehat{T}^{\sigma(i),d}_{0,\mathrm{stay}}, \tag{A53}$$

$$\widehat{\Pi}^\dagger_\sigma \widehat{T}_0 \widehat{\Pi}_\sigma = \widehat{T}_0. \tag{A54}$$

h. Transition matrix for supplanting process

According to Appendix A 2 e, in order to describe the transition matrix $\widehat{T}$ for the supplanting process, it suffices to replace $\widehat{\Delta}^d_i \widehat{\Xi}^{\boldsymbol{x}}$ appearing in $\widehat{T}^i_{0,\mathrm{move}}$ by

$$\sum_{j\in S(\boldsymbol{x},i,d)} \frac{1}{\#S(\boldsymbol{x},i,d)} \sum_{d'=\pm} \frac{1}{2}\left(\frac{p\#S(\boldsymbol{x},i,d)}{1+p\#S(\boldsymbol{x},i,d)}\widehat{\Delta}^{d'}_j + \frac{1}{1+p\#S(\boldsymbol{x},i,d)}\widehat{id}^{\otimes N}_H\right)\widehat{\Delta}^d_i\widehat{\Xi}^{\boldsymbol{x}} \tag{A55}$$

$$= \sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} \frac{1}{2\#S(\boldsymbol{x},i,d)}\left(\frac{p\#S(\boldsymbol{x},i,d)}{1+p\#S(\boldsymbol{x},i,d)}\widehat{\Delta}^{d'}_j + \frac{1}{1+p\#S(\boldsymbol{x},i,d)}\widehat{id}^{\otimes N}_H\right)\widehat{\Delta}^d_i\widehat{\Xi}^{\boldsymbol{x}} \tag{A56}$$

$$= \left(\widehat{\Delta}^d_i + \sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} \frac{p}{2(1+p\#S(\boldsymbol{x},i,d))}\left(\widehat{\Delta}^{d'}_j - \widehat{id}^{\otimes N}_H\right)\widehat{\Delta}^d_i\right)\widehat{\Xi}^{\boldsymbol{x}}. \tag{A57}$$

Thus, by (A51), the transition matrix $\widehat{T}$ is written as

$$\widehat{T} = \sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} \sum_{\boldsymbol{x}\in X^N}\left[c_\beta(\boldsymbol{x},i,d)\left(\widehat{\Delta}^d_i + \sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} \frac{p}{2(1+p\#S(\boldsymbol{x},i,d))}\left(\widehat{\Delta}^{d'}_j - \widehat{id}^{\otimes N}_H\right)\widehat{\Delta}^d_i\right)\widehat{\Xi}^{\boldsymbol{x}} + \widehat{T}^{i,d}_{0,\mathrm{stay}}\right] \tag{A58}$$

$$= \widehat{T}_0 + \sum_{\boldsymbol{x}\in X^N}\sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}}\sum_{\substack{j\in S(\boldsymbol{x},i,d)\\ d'=\pm}} \frac{pc_\beta(\boldsymbol{x},i,d)}{2(1+p\#S(\boldsymbol{x},i,d))}\left(\widehat{\Delta}^{d'}_j - \widehat{id}^{\otimes N}_H\right)\widehat{\Delta}^d_i\widehat{\Xi}^{\boldsymbol{x}}. \tag{A59}$$

i. Coefficients of two operators

From the descriptions of operators $\widehat{\Delta}^d_i$, $\widehat{\Xi}^{\boldsymbol{x}}$ in (A21), (A22), and (A26), we obtain

$$\langle \boldsymbol{y}|\widehat{\Delta}^d_i\widehat{\Xi}^{\boldsymbol{z}}|\boldsymbol{x}\rangle = \begin{cases} 1 & (\text{if } \boldsymbol{x}=\boldsymbol{z} \text{ and } \boldsymbol{y}=f^d_i\boldsymbol{z}) \\ 0 & (\text{otherwise}), \end{cases} \tag{A60}$$

$$\langle \boldsymbol{y}|\widehat{\Delta}^{d'}_j\widehat{\Delta}^d_i\widehat{\Xi}^{\boldsymbol{z}}|\boldsymbol{x}\rangle = \begin{cases} 1 & (\text{if } \boldsymbol{x}=\boldsymbol{z} \text{ and } \boldsymbol{y}=f^{d'}_j f^d_i\boldsymbol{z}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A61}$$

Using this and (A51), the coefficients of $\widehat{T}_0$ is

$$\langle \boldsymbol{y}|\widehat{T}_0|\boldsymbol{x}\rangle = \begin{cases} c_\beta(\boldsymbol{x},i,d) & (\text{if } \boldsymbol{y}=f^d_i\boldsymbol{x}) \\ 1-\sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} c_\beta(\boldsymbol{x},i,d) & (\text{if } \boldsymbol{y}=\boldsymbol{x}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A62}$$

Similarly by (A59), the coefficients of $\widehat{T}$ is

$$\langle \boldsymbol{y}|\widehat{T}|\boldsymbol{x}\rangle = \begin{cases} \frac{pc_\beta(\boldsymbol{x},i,d)}{2(1+np)} & (\text{if } \boldsymbol{y}=f^{d'}_j f^d_i\boldsymbol{x} \text{ and } j\in S(\boldsymbol{x},i,d)) \\ \frac{2pc_\beta(\boldsymbol{x},i,d)}{2(1+np)} & (\text{if } \boldsymbol{y}=f^d_i\boldsymbol{x}) \\ 1-\sum_{\substack{1\leqslant i\leqslant N\\ d=\pm}} c_\beta(\boldsymbol{x},i,d) & (\text{if } \boldsymbol{y}=\boldsymbol{x}) \\ 0 & (\text{otherwise}). \end{cases} \tag{A63}$$

j. The broken permutation symmetry of transition matrix

In this subsection, we explain an example of the broken permutation symmetry of transition matrix $\widehat{T}$ by using commutative relations. Recall that, for an operator $\widehat{A}$ on $H^{\otimes N}_X$, the daggered symbol $\widehat{A}^\dagger$ denotes the Hermitian conjugate of $\widehat{A}$.

By (A63), for $i\in[N]$ and $j\in S(\boldsymbol{x},i,d)$, we have

$$\left\langle f^{d'}_j f^d_i\boldsymbol{x}\middle|\widehat{T}|\boldsymbol{x}\right\rangle = \frac{pc_\beta(\boldsymbol{x},i,d)}{2[1+p\#S(\boldsymbol{x},i,d)]}. \tag{A64}$$

On the other hand, for a transposition $\sigma = (i\ j) \in \mathfrak{S}_N$, one has

$$\langle f_j^{d'} f_i^d \boldsymbol{x} | \widehat{\Pi}_\sigma^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle = \langle \boldsymbol{x} | \widehat{\Delta}_i^{d,\dagger} \widehat{\Delta}_j^{d',\dagger} \widehat{\Pi}_\sigma^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle \quad \text{(A65)}$$

$$= \langle \boldsymbol{x} | \big(\widehat{\Pi}_\sigma \widehat{\Delta}_j^{d'} \widehat{\Delta}_i^d \big)^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle. \quad \text{(A66)}$$

Using the commutative relation (A36), one can see

$$\langle \boldsymbol{x} | \big(\widehat{\Pi}_\sigma \widehat{\Delta}_j^{d'} \widehat{\Delta}_i^d \big)^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle = \langle \boldsymbol{x} | \big(\widehat{\Delta}_{\sigma(j)}^{d'} \widehat{\Delta}_{\sigma(i)}^d \widehat{\Pi}_\sigma \big)^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle \quad \text{(A67)}$$

$$= \langle \boldsymbol{x} | \big(\widehat{\Delta}_i^{d'} \widehat{\Delta}_j^d \widehat{\Pi}_\sigma \big)^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle \quad \text{(A68)}$$

$$= \langle \boldsymbol{x} | \widehat{\Pi}_\sigma^\dagger \widehat{\Delta}_j^{d,\dagger} \widehat{\Delta}_i^{d',\dagger} \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle. \quad \text{(A69)}$$

Proceeding with the transformation using (A33), it follows that

$$\langle \boldsymbol{x} | \widehat{\Pi}_\sigma^\dagger \widehat{\Delta}_j^{d,\dagger} \widehat{\Delta}_i^{d',\dagger} \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle = \langle \sigma^{-1}(\boldsymbol{x}) | \widehat{\Delta}_j^{d,\dagger} \widehat{\Delta}_i^{d',\dagger} \widehat{T} | \sigma^{-1}(\boldsymbol{x}) \rangle \quad \text{(A70)}$$

$$= \langle f_i^{d'} f_j^d \sigma^{-1}(\boldsymbol{x}) | \widehat{T} | \sigma^{-1}(\boldsymbol{x}) \rangle. \quad \text{(A71)}$$

Since $j \in S(\boldsymbol{x}, i, d)$, we have $j > i$, hence $i \notin S(\boldsymbol{x}, j, d)$. With (A63), we obtain

$$\langle f_i^{d'} f_j^d \sigma^{-1}(\boldsymbol{x}) | \widehat{T} | \sigma^{-1}(\boldsymbol{x}) \rangle = 0. \quad \text{(A72)}$$

Thus, since $c_\beta(\boldsymbol{x}, i, d) \neq 0$, we obtain the broken permutation symmetry of $\widehat{T}$ with respect to the permutation of components:

$$\langle f_j^{d'} f_i^d \boldsymbol{x} | \widehat{\Pi}_\sigma^\dagger \widehat{T} \widehat{\Pi}_\sigma | \boldsymbol{x} \rangle \neq \langle f_j^{d'} f_i^d \boldsymbol{x} | \widehat{T} | \boldsymbol{x} \rangle. \quad \text{(A73)}$$

This shows that, for any energy function $E(\boldsymbol{x})$ as in Appendix A 2 c and any transposition $\sigma = (i\ j) \in \mathfrak{S}_N$, we find that

$$\widehat{\Pi}_\sigma^\dagger \widehat{T} \widehat{\Pi}_\sigma \neq \widehat{T}. \quad \text{(A74)}$$

In fact, for any nontrivial permutation $\sigma \in \mathfrak{S}_N$, we can show (A74). Explicitly, for a nontrivial permutation $\sigma \in \mathfrak{S}_N$, we can find the following four data $(i, j, \boldsymbol{x}, d)$ satisfying the following two conditions: (1) $i < j$ with $\sigma(i) > \sigma(j)$ and (2) $\boldsymbol{x} \in X^N$ and $d = +$ or $-$ such that $j \in S(\boldsymbol{x}, i, d)$. Then, in parallel, the above argument works to show (A73) and thus (A74) for the permutation σ.

4. The supplanting decomposition

From here to the next section, we introduce two decompositions of the operator $\widehat{T}$. The idea of our first decomposition of $\widehat{T}$ is to split the sums by the number $\#S(\boldsymbol{x}, i, d)$.

a. The nth term

First we note that a coefficient in (A59),

$$\frac{pc_\beta(\boldsymbol{x}, i, d)}{2[1 + p\#S(\boldsymbol{x}, i, d)]}, \quad \text{(A75)}$$

TABLE II. A part of $(\boldsymbol{x}, \boldsymbol{y})$-th coefficients of $\widehat{T} - \widehat{T}_0$ and $\widehat{T}_n$ with $\boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x}$.

$\#S(\boldsymbol{x}, i, d)$	$\widehat{T} - \widehat{T}_0$	$\widehat{T}_1$	$\widehat{T}_2$	$\widehat{T}_3$	$\widehat{T}_4$	$\cdots$	$\widehat{T}_n$
1	$\frac{pc_\beta}{2(1+p)}$	$\frac{pc_\beta}{2(1+p)}$	0	0	0	$\cdots$	0
2	$\frac{pc_\beta}{2(1+2p)}$	0	$\frac{pc_\beta}{2(1+2p)}$	0	0	$\cdots$	0
3	$\frac{pc_\beta}{2(1+3p)}$	0	0	$\frac{pc_\beta}{2(1+3p)}$	0	$\cdots$	0
4	$\frac{pc_\beta}{2(1+4p)}$	0	0	0	$\frac{pc_\beta}{2(1+4p)}$	$\cdots$	0
n	$\frac{pc_\beta}{2(1+np)}$	0	0	0	0	$\cdots$	$\frac{pc_\beta}{2(1+np)}$

does not depend on $j \in S(\boldsymbol{x}, i, d)$ and $d' \in \{+, -\}$. Recalling (A59), equivalently,

$$\widehat{T} - \widehat{T}_0 = \sum_{\boldsymbol{x} \in X^N} \sum_{\substack{1 \leqslant i \leqslant N \\ d = \pm}} \sum_{\substack{j \in S(\boldsymbol{x}, i, d) \\ d' = \pm}} \frac{pc_\beta(\boldsymbol{x}, i, d)}{2[1 + p\#S(\boldsymbol{x}, i, d)]} \times \big(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N} \big) \widehat{\Delta}_i^d \widehat{\Xi}^{\boldsymbol{x}}, \quad \text{(A76)}$$

we define the operators

$$\widehat{T}_n := \sum_{\boldsymbol{x}, i, d} \delta(\#S(\boldsymbol{x}, i, d), n) \sum_{\substack{j \in S(\boldsymbol{x}, i, d) \\ d' = \pm}} \frac{pc_\beta(\boldsymbol{x}, i, d)}{2(1 + np)} \times \big(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N} \big) \widehat{\Delta}_i^d \widehat{\Xi}^{\boldsymbol{x}}, \quad \text{(A77)}$$

where $1 \leqslant n \leqslant N - 1$. Then we obtain

$$\widehat{T} = \widehat{T}_0 + \widehat{T}_1 + \widehat{T}_2 + \cdots + \widehat{T}_{N-1}. \quad \text{(A78)}$$

b. Coefficients of nth terms

Next we describe coefficients of $\widehat{T}_n$ as in (A16). Using (A60) and (A61), we have

$$\langle \boldsymbol{y} | \widehat{T}_n | \boldsymbol{x} \rangle = \begin{cases} \frac{pc_\beta(\boldsymbol{x}, i, d)}{2(1+np)} & (\text{if } \boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x}, i, d) = n) \\ \frac{-2npc_\beta(\boldsymbol{x}, i, d)}{2(1+np)} & (\text{if } \boldsymbol{y} = f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x}, i, d) = n) \\ 0 & (\text{otherwise}). \end{cases} \quad \text{(A79)}$$

By the estimate (A45), using $n < N$ and $1 + np \geqslant 1$, we see that

$$0 \leqslant |\langle \boldsymbol{y} | \widehat{T}_n | \boldsymbol{x} \rangle| \leqslant \frac{np}{2N(1 + np)} < \frac{p}{2}. \quad \text{(A80)}$$

This shows that the nonzero coefficients of $\widehat{T}_n$ are estimated as $O(p)$ as $p \to +0$, independent of n. See Table II for the coefficients $\langle \boldsymbol{y} | \widehat{T}_n | \boldsymbol{x} \rangle$ if $\boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x}$.

c. Another description of nth terms

Here we give another description (A88) of nth terms. We change the ordering of summations in (A77) as follows: we first choose $1 \leqslant i_0 \leqslant N$, an n-elements set $\boldsymbol{S} = \{i_1 < \cdots < i_n\} \subset [N]$ with $i_0 < i_1$, and then configurations $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ so that $S(\boldsymbol{x}, i_0, d) = \boldsymbol{S}$. Then explicitly we reach

$$\widehat{T}_n = \sum_{1\leqslant i_0\leqslant N} \sum_{S=\{i_0<i_1<\cdots<i_n\leqslant N\}} \sum_{\substack{\boldsymbol{x} \\ S(\boldsymbol{x},i_0,d)=\boldsymbol{S} \\ d=\pm}} \frac{pc_\beta(\boldsymbol{x}, i_0, d)}{2(1+np)} \sum_{\substack{j\in S(\boldsymbol{x},i_0,d) \\ d'=\pm}} \left(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\right)\widehat{\Delta}_{i_0}^d \widehat{\Xi}^{\boldsymbol{x}} \tag{A81}$$

$$= \frac{p}{2(1+np)} \sum_{1\leqslant i_0\leqslant N} \sum_{\boldsymbol{S}} \sum_{\substack{\boldsymbol{x} \\ S(\boldsymbol{x},i_0,d)=\boldsymbol{S} \\ d=\pm}} c_\beta(\boldsymbol{x}, i_0, d)\left[\sum_{\substack{j\in \boldsymbol{S} \\ d'=\pm}} \left(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\right)\right]\widehat{\Delta}_{i_0}^d \widehat{\Xi}^{\boldsymbol{x}} \tag{A82}$$

$$= \frac{p}{2(1+np)} \sum_{1\leqslant i_0\leqslant N} \sum_{\boldsymbol{S}} \left[\sum_{\substack{j\in \boldsymbol{S} \\ d'=\pm}} \left(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\right)\right] \sum_{\substack{\boldsymbol{x} \\ S(\boldsymbol{x},i_0,d)=\boldsymbol{S} \\ d=\pm}} c_\beta(\boldsymbol{x}, i_0, d)\widehat{\Delta}_{i_0}^d \widehat{\Xi}^{\boldsymbol{x}}. \tag{A83}$$

Using $\widehat{\Upsilon}_{i_0;\boldsymbol{S}}$ defined in (A29), we can write

$$\sum_{\substack{\boldsymbol{x} \\ S(\boldsymbol{x},i_0,d)=\boldsymbol{S} \\ d=\pm}} c_\beta(\boldsymbol{x}, i_0, d)\widehat{\Delta}_{i_0}^d \widehat{\Xi}^{\boldsymbol{x}} = \widehat{\Upsilon}_{i_0;\boldsymbol{S}} \sum_{d=\pm}\sum_{\boldsymbol{x}} c_\beta(\boldsymbol{x}, i_0, d)\widehat{\Delta}_{i_0}^d \widehat{\Xi}^{\boldsymbol{x}} \tag{A84}$$

$$= \widehat{\Upsilon}_{i_0;\boldsymbol{S}}\widehat{T}_{0,\mathrm{move}}^{i_0}. \tag{A85}$$

By putting $\widehat{P}_\emptyset = 0$ and

$$\widehat{P}_{\boldsymbol{S}} := \sum_{\substack{j\in \boldsymbol{S} \\ d'=\pm}} \left(\widehat{\Delta}_j^{d'} - \widehat{id}_H^{\otimes N}\right) \tag{A86}$$

$$= \sum_{j\in \boldsymbol{S}} \left(\widehat{\Delta}_j^{+} + \widehat{\Delta}_j^{-} - 2\widehat{id}_H^{\otimes N}\right), \tag{A87}$$

we can write

$$\widehat{T}_n = \frac{p}{2(1+np)} \sum_{1\leqslant i_0\leqslant N} \sum_{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}} \delta(\#\boldsymbol{S}, n)\widehat{P}_{\boldsymbol{S}}\widehat{\Upsilon}_{i_0;\boldsymbol{S}}\widehat{T}_{0,\mathrm{move}}^{i_0}. \tag{A88}$$

5. The beta decomposition

In this section, we introduce the second decomposition (A96) of $\widehat{T}$ which we call the *beta decomposition*. The coefficients of mth term in the beta decomposition is estimated as $O(p^m)$ [see (A113)].

a. Definition of beta terms

Here we give a definition of mth term $\widehat{U}_m$ of beta decomposition by using $\widehat{T}_n$ which appear in the supplanting decomposition [see (A77)]. For $1 \leqslant m \leqslant N-1$, we define

$$\widehat{U}_m := \frac{(-1)^{m-1}B(m, 1+1/p)}{p} \sum_{m\leqslant n\leqslant N-1} \binom{n-1}{m-1}(1+np)\widehat{T}_n. \tag{A89}$$

Here $B(a, b)$ is the beta function:

$$B(a, b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{A90}$$

$$= \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{A91}$$

In particular, we have

$$B\left(m, 1+\frac{1}{p}\right) = \frac{\Gamma(m)\Gamma(1+1/p)}{\Gamma(m+1+1/p)} \tag{A92}$$

$$= \frac{(m-1)!}{(1+1/p)(2+1/p)\cdots(m+1/p)} \tag{A93}$$

$$= \frac{(m-1)!p^m}{(1+p)(1+2p)\cdots(1+mp)}. \tag{A94}$$

By these descriptions, one can write

$$\widehat{U}_m = \frac{(-1)^{m-1}(m-1)!p^{m-1}}{(1+p)(1+2p)\cdots(1+mp)} \times \sum_{m\leqslant n\leqslant N-1} \binom{n-1}{m-1}(1+np)\widehat{T}_n. \tag{A95}$$

b. The beta decomposition

We can prove the following decomposition:

$$\widehat{T} = \widehat{T}_0 + \sum_{1\leqslant m\leqslant N-1} \widehat{U}_m. \tag{A96}$$

In fact,

$$\sum_{1\leqslant m\leqslant N-1} \widehat{U}_m = \sum_{1\leqslant m\leqslant N-1} \frac{(-1)^{m-1}B(1+1/p, m)}{p} \times \sum_{m\leqslant n\leqslant N-1} \binom{n-1}{m-1}(1+np)\widehat{T}_n \tag{A97}$$

$$= \sum_{1\leqslant n\leqslant N-1} \left[\sum_{1\leqslant m\leqslant n} (-1)^{m-1}\frac{B(1+1/p, m)}{p} \times \binom{n-1}{m-1}\right](1+np)\widehat{T}_n. \tag{A98}$$

Then the following lemma is enough to show the desired decomposition (A96).

Lemma A.1.

$$\sum_{1\leqslant m\leqslant n} \frac{(-1)^{m-1}B(1+1/p,m)}{p}\binom{n-1}{m-1} = \frac{1}{1+np}. \tag{A99}$$

Sketch of proof. We only sketch the proof. With two generating series of exponential type

$$F_0(t) := \sum_{m\geqslant 0} a_m \frac{t^m}{m!}, \tag{A100}$$

$$F_1(t) := F_0(t)e^t, \tag{A101}$$

where the coefficients of $F_0(t)$ is

$$a_m := \frac{(-1)^m B(1+1/p, m+1)}{p} \tag{A102}$$

$$= \frac{(-1)^m m! p^m}{(1+p)(1+2p)\cdots[1+(m+1)p]}, \tag{A103}$$

it is enough to prove

$$F_1(t) = \sum_{r\geqslant 0} \frac{1}{1+(r+1)p}\frac{t^r}{r!}. \tag{A104}$$

Since $a_0 = F_0(0) = (1+p)^{-1}$ and

$$(1+(m+2)p)a_{m+1} = (-1)(m+1)pa_m \tag{A105}$$

$$\iff (m+1)a_{m+1} + (m+1)a_m = -\left(1+\frac{1}{p}\right)a_{m+1} \tag{A106}$$

for $m \geqslant 0$, we have a differential equation for $F_0(t)$

$$t\left(\frac{dF_0}{dt}(t) + F_0(t)\right) = -\frac{1}{p}[(1+p)F_0(t) - 1]. \tag{A107}$$

Thus $F_1(t) = F_0(t)e^t$ satisfies

$$t\frac{dF_1}{dt}(t) = -\left(1+\frac{1}{p}\right)F_1(t) + \frac{1}{p}e^t. \tag{A108}$$

Now Eq. (A104) can be proved by comparison of coefficients. ■

c. Coefficients of beta terms

With our description (A79) of coefficients of $\widehat{T}_n$, we can write the coefficients of $\widehat{U}_m$.

For $\boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x}$ or $f_i^d \boldsymbol{x}$, the coefficients $\langle \boldsymbol{y}|\widehat{T}_n|\boldsymbol{x}\rangle$ of $\widehat{T}_n$ is zero when $n \neq \#S(\boldsymbol{x}, i, d)$. Hence we have

$$\langle \boldsymbol{y}|\widehat{U}_m|\boldsymbol{x}\rangle = \begin{cases} \binom{n-1}{m-1}\frac{(-1)^{m+1}B(1+1/p,m)}{2}c_\beta(\boldsymbol{x},i,d) & \left(\text{if } \boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x},i,d) = n \geqslant m\right) \\ -2n\binom{n-1}{m-1}\frac{(-1)^{m+1}B(1+1/p,m)}{2}c_\beta(\boldsymbol{x},i,d) & \left(\text{if } \boldsymbol{y} = f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x},i,d) = n \geqslant m\right) \\ 0 & (\text{otherwise}). \end{cases} \tag{A109}$$

Using the description

$$B(m, 1+1/p) = \frac{(m-1)!p^m}{(1+p)(1+2p)\cdots(1+mp)}, \tag{A110}$$

we can write those coefficients in another form:

$$\langle \boldsymbol{y}|\widehat{U}_m|\boldsymbol{x}\rangle = \begin{cases} \binom{n-1}{m-1}\frac{(-1)^{m+1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)}c_\beta(\boldsymbol{x},i,d) & \left(\text{if } \boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x},i,d) = n \geqslant m\right) \\ -2n\binom{n-1}{m-1}\frac{(-1)^{m+1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)}c_\beta(\boldsymbol{x},i,d) & \left(\text{if } \boldsymbol{y} = f_i^d \boldsymbol{x} \text{ and } \#S(\boldsymbol{x},i,d) = n \geqslant m\right) \\ 0 & (\text{otherwise}). \end{cases} \tag{A111}$$

This allows us to estimate the coefficients of $\widehat{U}_m$. With $p \to 0$ and fixing other parameters β, N, L, and E, we have

$$\langle \boldsymbol{y}|\widehat{U}_m|\boldsymbol{x}\rangle = O(p^m). \tag{A112}$$

In particular, we have

$$\widehat{T} = \widehat{T}_0 + \widehat{U}_1 + \cdots + \widehat{U}_m + O(p^{m+1}) \tag{A113}$$

for $1 \leqslant m \leqslant N-1$. See Table III for the coefficients $\langle \boldsymbol{y}|\widehat{U}_m|\boldsymbol{x}\rangle$ if $\boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x}$.

d. Another decomposition of beta terms

Substituting (A88) into (A95), we can give a combinatorial decomposition of beta terms: for $m \geqslant 1$,

$$\widehat{U}_m = \sum_{m\leqslant n\leqslant N-1} \frac{(-1)^{m+1}(m-1)!p^{m-1}(1+np)}{(1+p)(1+2p)\cdots(1+mp)}\binom{n-1}{m-1}\frac{p}{2(1+np)}\sum_{1\leqslant i_0\leqslant N}\sum_{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}}\delta(\#\boldsymbol{S},n)\widehat{P}_{\boldsymbol{S}}\widehat{\Upsilon}_{i_0;\boldsymbol{S}}\widehat{T}^{i_0}_{0,\text{move}} \tag{A114}$$

$$= \frac{(-1)^{m+1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)}\sum_{1\leqslant i_0\leqslant N}\left[\sum_{\substack{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}\\ \#\boldsymbol{S}\geqslant m}}\binom{\#\boldsymbol{S}-1}{m-1}\widehat{P}_{\boldsymbol{S}}\widehat{\Upsilon}_{i_0;\boldsymbol{S}}\right]\widehat{T}^{i_0}_{0,\text{move}}. \tag{A115}$$

TABLE III. A part of $(\boldsymbol{x}, \boldsymbol{y})$-th coefficients of $\widehat{T} - \widehat{T}_0$ and $\widehat{U}_m$ with $\boldsymbol{y} = f_j^{d'} f_i^d \boldsymbol{x}$.

$\#S(\boldsymbol{x}, i, d)$	$\widehat{T} - \widehat{T}_0$	$\widehat{U}_1$	$\widehat{U}_2$	$\widehat{U}_3$	$\widehat{U}_4$	$\cdots$	$\widehat{U}_m$
1	$\frac{pc_\beta}{2(1+p)}$	$\binom{0}{0}\frac{pc_\beta}{2(1+p)}$	0	0	0	$\cdots$	0
2	$\frac{pc_\beta}{2(1+2p)}$	$\binom{1}{0}\frac{pc_\beta}{2(1+p)}$	$\binom{1}{1}\frac{(-1)1!p^2c_\beta}{2(1+p)(1+2p)}$	0	0	$\cdots$	0
3	$\frac{pc_\beta}{2(1+3p)}$	$\binom{2}{0}\frac{pc_\beta}{2(1+p)}$	$\binom{2}{1}\frac{(-1)1!p^2c_\beta}{2(1+p)(1+2p)}$	$\binom{2}{2}\frac{(-1)^2 2!p^3c_\beta}{2(1+p)...(1+3p)}$	0	$\cdots$	0
4	$\frac{pc_\beta}{2(1+4p)}$	$\binom{3}{0}\frac{pc_\beta}{2(1+p)}$	$\binom{3}{1}\frac{(-1)1!p^2c_\beta}{2(1+p)(1+2p)}$	$\binom{3}{2}\frac{(-1)^2 2!p^3c_\beta}{2(1+p)...(1+3p)}$	$\binom{3}{3}\frac{(-1)^3 3!p^4c_\beta}{2(1+p)...(1+4p)}$	$\cdots$	0
m	$\frac{pc_\beta}{2(1+mp)}$	$\binom{m-1}{0}\frac{pc_\beta}{2(1+p)}$	$\binom{m-1}{1}\frac{(-1)1!p^2c_\beta}{2(1+p)(1+2p)}$	$\binom{m-1}{2}\frac{(-1)^2 2!p^3c_\beta}{2(1+p)...(1+3p)}$	$\binom{m-1}{3}\frac{(-1)^3 3!p^4c_\beta}{2(1+p)...(1+4p)}$	$\cdots$	$\binom{m-1}{m-1}\frac{(-1)^{m-1}p^mc_\beta}{2(1+p)...(1+mp)}$

To obtain a more convenient formula of (A115), we use the following combinatorial equation: for $\boldsymbol{S} \subseteq \{i_0+1, \ldots, N\}$ with $\#\boldsymbol{S} \geqslant m$:

$$\binom{\#\boldsymbol{S}-1}{m-1}\widehat{P}_{\boldsymbol{S}} = \sum_{\boldsymbol{S}'\subseteq\boldsymbol{S}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'}. \tag{A116}$$

Then we can perform the following transformation of the part surrounded by square brackets in (A115):

$$\sum_{\substack{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}\\ \#\boldsymbol{S}\geqslant m}} \binom{\#\boldsymbol{S}-1}{m-1}\widehat{P}_{\boldsymbol{S}}\widehat{\Upsilon}_{i_0;\boldsymbol{S}} \tag{A117}$$

$$= \sum_{\substack{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}\\ \#\boldsymbol{S}\geqslant m}} \sum_{\boldsymbol{S}'\subseteq\boldsymbol{S}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'}\widehat{\Upsilon}_{i_0;\boldsymbol{S}} \tag{A118}$$

$$= \sum_{\boldsymbol{S}'\subseteq\{i_0+1,\ldots,N\}} \sum_{\substack{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}\\ \boldsymbol{S}\supseteq\boldsymbol{S}'}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'}\widehat{\Upsilon}_{i_0;\boldsymbol{S}} \tag{A119}$$

$$= \sum_{\boldsymbol{S}'\subseteq\{i_0+1,\ldots,N\}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'} \sum_{\substack{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}\\ \boldsymbol{S}\supseteq\boldsymbol{S}'}} \widehat{\Upsilon}_{i_0;\boldsymbol{S}} \tag{A120}$$

$$= \sum_{\boldsymbol{S}'\subseteq\{i_0+1,\ldots,N\}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'}\widehat{\Xi}_{\{i_0\}\cup\boldsymbol{S}'}, \tag{A121}$$

where we used (A32) in the last equality. Hence, we obtain

$$\widehat{U}_m = \frac{(-1)^{m+1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)} \times \sum_{1\leqslant i_0\leqslant N}\left[\sum_{\boldsymbol{S}'\subseteq\{i_0+1,\ldots,N\}} \delta(\#\boldsymbol{S}', m)\widehat{P}_{\boldsymbol{S}'}\widehat{\Xi}_{\{i_0\}\cup\boldsymbol{S}'}\right]\widehat{T}^{i_0}_{0,\mathrm{move}} \tag{A122}$$

$$= \frac{(-1)^{m+1}(m-1)!p^m}{2(1+p)(1+2p)\cdots(1+mp)} \times \sum_{1\leqslant i_0<i_1<\cdots<i_m\leqslant N} \widehat{P}_{\{i_1,\ldots,i_m\}}\widehat{\Xi}_{\{i_0,i_1,\ldots,i_m\}}\widehat{T}^{i_0}_{0,\mathrm{move}}. \tag{A123}$$

For example when $m = 1$, we have

$$\widehat{U}_1 = \frac{p}{2(1+p)} \sum_{1\leqslant i_0<i_1\leqslant N} \widehat{P}_{\{i_1\}}\widehat{\Xi}_{\{i_0,i_1\}}\widehat{T}^{i_0}_{0,\mathrm{move}} \tag{A124}$$

$$= \frac{p}{2(1+p)} \sum_{1\leqslant i_0<i_1\leqslant N} \left(\widehat{\Delta}^+_{i_1} + \widehat{\Delta}^-_{i_1} - 2\,\widehat{id}_H^{\otimes N}\right)\widehat{\Xi}_{\{i_0,i_1\}}\widehat{T}^{i_0}_{0,\mathrm{move}}. \tag{A125}$$

When $m = N-1$, we have

$$\widehat{U}_{N-1} = \frac{(-1)^N N!p^{N-1}}{2(1+p)(1+2p)\cdots[1+(N-1)p]} \times \widehat{P}_{\{2,3,\ldots,N\}}\widehat{\Xi}_{\{1,2,\ldots,N\}}\widehat{T}^1_{0,\mathrm{move}}. \tag{A126}$$

For an agent i_0 and a subset $\boldsymbol{S} \subseteq [N] \setminus \{i_0\}$, let us define

$$\widehat{U}_{i_0;\boldsymbol{S}} := \frac{(-1)^{\#\boldsymbol{S}}\#\boldsymbol{S}!p^{\#\boldsymbol{S}-1}}{2(1+p)(1+2p)\cdots[1+(\#\boldsymbol{S}-1)p]} \times \widehat{P}_{\boldsymbol{S}}\widehat{\Xi}_{\{i_0\}\cup\boldsymbol{S}}\widehat{T}^{i_0}_{0,\mathrm{move}}. \tag{A127}$$

Then we can rewrite (A123) as

$$\widehat{U}_m = \sum_{1\leqslant i_0\leqslant N} \sum_{\boldsymbol{S}\subseteq\{i_0+1,\ldots,N\}} \delta(\#\boldsymbol{S}, m)\widehat{U}_{i_0;\boldsymbol{S}}. \tag{A128}$$

Note that we can define $\widehat{U}_{i_0;\boldsymbol{S}}$ even if $\boldsymbol{S} \not\subseteq \{i_0+1, \ldots, N\}$.

Remark A.1 (The origin of the beta decomposition). With the operators $\widehat{U}_{i_0;\boldsymbol{S}}$, we can define a variant model of supplanting process. For a subset $\boldsymbol{S}_0 \subseteq \{1, \ldots, N\}$, we define

$$\widehat{T}_{\boldsymbol{S}_0} := \sum_{i_0\in\boldsymbol{S}_0} \sum_{\boldsymbol{S}'\subseteq\{i_0+1,\ldots,N\}\cap\boldsymbol{S}_0} \widehat{U}_{i_0;\boldsymbol{S}'}. \tag{A129}$$

This operator $\widehat{T}_{\boldsymbol{S}_0}$ represents the model where supplanting process occurs only on pairs of agents i, j with $i, j \in \boldsymbol{S}_0$.

Conversely, we can define $\widehat{U}_{i_0;\boldsymbol{S}}$ from these variants $\widehat{T}_{\boldsymbol{S}_0}$ by the inclusion-exclusion principle:

$$\widehat{U}_{i_0;\boldsymbol{S}_0} = \sum_{\boldsymbol{S}'\subsetneq\{i_0+1,\ldots,N\}\cap\boldsymbol{S}_0} (-1)^{\#\boldsymbol{S}'-\#\boldsymbol{S}_0}\widehat{T}_{\{i_0\}\cup\boldsymbol{S}'}. \tag{A130}$$

The beta decomposition was originally derived from this point of view.

e. Commutation relation between beta terms and permutation operators

Though it is not used in the main text, here we write a commutation relation between beta terms and permutation operators.

Using relations in Appendix A 3 e, we have the following forms [the third equation is equivalent to (A52)]:

$$\widehat{P}_{\boldsymbol{S}}\widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma\widehat{P}_{\sigma(\boldsymbol{S})}, \tag{A131}$$

$$\widehat{\Xi}_{\{i_0\}\cup\boldsymbol{S}}\widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma\widehat{\Xi}_{\{\sigma(i_0)\}\cup\sigma(\boldsymbol{S})}, \tag{A132}$$

$$\widehat{T}^{i_0}_{0,\mathrm{move}}\widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma\widehat{T}^{\sigma(i_0)}_{0,\mathrm{move}}. \tag{A133}$$

With these three relations, we obtain that

$$\widehat{U}_{i_0;\boldsymbol{S}}\widehat{\Pi}_\sigma = \widehat{\Pi}_\sigma \widehat{U}_{\sigma(i_0);\sigma(\boldsymbol{S})}. \quad \text{(A134)}$$

This enables us to investigate the commutation relation between $\widehat{U}_m$ and permutation operators with using (A128) or between $\widehat{T}_{\boldsymbol{S}_0}$ and permutation operators using (A129).

APPENDIX B: EIGENVECTOR CENTRALITY AND OVERLAP CENTRALITY

In this Appendix we see a relation between the eigenvector centrality and the overlap centrality of the neighbor matrix. Let us recall the definitions of their centrality in our context. The eigenvector centrality is defined as a normalized eigenvector with the maximum eigenvalue of the neighbor matrix $\mathcal{R}$. Such an eigenvector exists uniquely, and its components can be taken to be real and positive because of the Perron-Frobenius theorem [15,17]. The overlap centrality (44) is the expectation value of how many agents are at the same site with a given agent.

First we describe our settings. We take a state $|P\rangle \in H_X^{\otimes N}$ corresponding to a probability distribution $P(\boldsymbol{x})$. $|P\rangle$ does not have to be a stationary state of a certain stochastic process. The neighbor matrix $\mathcal{R} = (r_{ij})_{1\leqslant i,j\leqslant N}$ of $|P\rangle$ is defined similar to (43), explicitly

$$r_{ij} = \sum_{\boldsymbol{x}} \delta(x_i, x_j)\langle \boldsymbol{x}|P\rangle. \quad \text{(B1)}$$

We consider a decomposition of the state $|P\rangle$

$$|P\rangle = |P_1\rangle + |P_2\rangle, \quad \text{(B2)}$$

and define matrices $\mathcal{R}^{(\ell)} = (r_{ij}^{(\ell)})$ as

$$r_{ij}^{(\ell)} = \sum_{\boldsymbol{x}} \delta(x_i, x_j)\langle \boldsymbol{x}|P_\ell\rangle \quad \text{(B3)}$$

for $\ell = 1, 2$. By definition, we also have a decomposition of the neighbor matrix $\mathcal{R}$:

$$\mathcal{R} = \mathcal{R}^{(1)} + \mathcal{R}^{(2)}. \quad \text{(B4)}$$

We assume that the decomposition (B2) satisfies the following two conditions:

(i) $|P_1\rangle$ is symmetric under permutations: that is, $\widehat{\Pi}_\sigma|P_1\rangle = |P_1\rangle$ for any $\sigma \in \mathfrak{S}_N$. From this assumption there is a constant c with $r_{ij}^{(1)} = c$ for any $i \neq j$. We assume that $c \neq 0$.

(ii) The off-diagonal entries $r_{ij}^{(2)}$ of $\mathcal{R}^{(2)}$ are sufficiently smaller than $|c|$: that is, $|r_{ij}^{(2)}| \ll |c|$ for any $i \neq j$.

We modify the matrices $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}$ to $\tilde{\mathcal{R}}^{(1)} = (\tilde{r}_{ij}^{(1)}), \tilde{\mathcal{R}}^{(2)} = (\tilde{r}_{ij}^{(2)})$ such that all of the diagonal entries of $\tilde{\mathcal{R}}^{(2)}$ are zero and the following decomposition of $\mathcal{R}$ holds:

$$\mathcal{R} = \tilde{\mathcal{R}}^{(1)} + \tilde{\mathcal{R}}^{(2)}. \quad \text{(B5)}$$

Considering all of the diagonal entries of $\mathcal{R}$ are one, we can take

$$\tilde{r}_{ij}^{(1)} = \begin{cases} 1 & (\text{if } i = j) \\ r_{ij}^{(1)} = c & (\text{if } i \neq j) \end{cases} \text{ and } \tilde{r}_{ij}^{(2)} = \begin{cases} 0 & (\text{if } i = j) \\ r_{ij}^{(2)} & (\text{if } i \neq j). \end{cases} \quad \text{(B6)}$$

From the assumption (ii), the matrix $\tilde{\mathcal{R}}^{(2)}$ can be regarded as a perturbative part in $\mathcal{R}$.

The eigenvalue problem of $\tilde{\mathcal{R}}^{(1)}$ can be solved easily: eigenvalues of $\tilde{\mathcal{R}}^{(1)}$ are $1 + (N-1)c$ and $1 - c$, and their corresponding eigenspaces are $\mathbb{C}\boldsymbol{v}_0$ and the orthogonal complement $(\mathbb{C}\boldsymbol{v}_0)^\perp$, respectively, where $\boldsymbol{v}_0 := \frac{1}{\sqrt{N}}(1, 1, \dots, 1)^T \in \mathbb{C}^N$. In particular, if $\tilde{\mathcal{R}}^{(2)}$ vanishes, then the eigenvector centrality of $\mathcal{R} = \tilde{\mathcal{R}}^{(1)}$ is the vector $\boldsymbol{v}_0$.

Let us use a first-order perturbation theory to calculate the eigenvector centrality $\boldsymbol{V}$ of $\mathcal{R}$. Here we introduce an orthonormal system $(\boldsymbol{u}_i)_{i=1}^{N-1}$ of the vector space $(\mathbb{C}\boldsymbol{v}_0)^\perp$. According to the Rayleigh-Schrödinger-type perturbation theory, we have

$$\boldsymbol{V} \propto \boldsymbol{v}_0 + \sum_{i=1}^{N-1} \frac{\boldsymbol{u}_i\boldsymbol{u}_i^\dagger \tilde{\mathcal{R}}^{(2)}\boldsymbol{v}_0}{1 + (N-1)c - (1-c)} + O(|\tilde{r}^{(2)}/c|^2) \quad \text{(B7)}$$

$$= \boldsymbol{v}_0 + \frac{(1 - \boldsymbol{v}_0\boldsymbol{v}_0^\dagger)\tilde{\mathcal{R}}^{(2)}\boldsymbol{v}_0}{Nc} + O(|\tilde{r}^{(2)}/c|^2) \quad \text{(B8)}$$

$$= \frac{1}{Nc}\mathcal{R}\boldsymbol{v}_0 - \left(\frac{\boldsymbol{v}_0^\dagger\tilde{\mathcal{R}}^{(2)}\boldsymbol{v}_0}{Nc} + \frac{1-c}{Nc}\right)\boldsymbol{v}_0 + O(|\tilde{r}^{(2)}/c|^2) \quad \text{(B9)}$$

up to the first order of $|\tilde{r}^{(2)}/c| := \max_{1\leqslant i,j\leqslant N} |\tilde{r}_{ij}^{(2)}/c|$. Since $(\mathcal{R}\boldsymbol{v}_0)_i = \frac{1}{\sqrt{N}}\sum_{j=1}^N r_{ij} = \frac{1}{\sqrt{N}}(O_i - 1)$, we obtain

$$\boldsymbol{V} \propto \frac{1}{N^{3/2}c}\boldsymbol{O} - \gamma \times (1, 1, \dots, 1)^T + O(|\tilde{r}^{(2)}/c|^2), \quad \text{(B10)}$$

where $\boldsymbol{O} = (O_i)_{i=1}^N$ is the vector consisting of the overlap centrality, and γ is a constant which is expressed by

$$\gamma = \frac{1}{\sqrt{N}}\left(\frac{\boldsymbol{v}_0^\dagger\tilde{\mathcal{R}}^{(2)}\boldsymbol{v}_0}{Nc} + \frac{2-c}{Nc}\right). \quad \text{(B11)}$$

Thus we find that the eigenvector centrality and the overlap centrality are equal up to multiplying by a constant and adding a vector in $\mathbb{C}(1, 1, \dots, 1)^T$. Note that the higher order terms of this perturbation can have a nontrivial $\boldsymbol{O}$ dependence, but are ignored in the approximation.

Let us apply this result to the case of the stationary state $|P(\beta, p)\rangle$ of the transition matrix $\widehat{T}(\beta, p)$. Suppose that p is small enough to be able to perform the perturbation expansion (70). Under this assumption, we give the decomposition of $|P(\beta, p)\rangle$ as follows:

$$|P(\beta, p)\rangle = |P_1(\beta, p)\rangle + |P_2(\beta, p)\rangle, \quad \text{(B12)}$$

$$|P_1(\beta, p)\rangle = C(\beta, p)|P_{\text{can}}(\beta)\rangle, \quad \text{(B13)}$$

$$|P_2(\beta, p)\rangle = C(\beta, p)\sum_{n=1}^{\infty}(\widehat{G}_0(\widehat{T} - \widehat{T}_0))^n|P_{\text{can}}(\beta)\rangle. \quad \text{(B14)}$$

Let us check that this decomposition satisfies the above conditions (i) and (ii).

(i) From (67), $|P_1(\beta, p)\rangle$ is symmetric under permutations. Since $\beta, E(\boldsymbol{x}) \in \mathbb{R}$, we have

$$c = \frac{C(\beta, p)}{Z_N(\beta)}\sum_{\boldsymbol{x}\in X^N}\delta(x_1, x_2)e^{-\beta E(\boldsymbol{x})} \neq 0. \quad \text{(B15)}$$

(ii) $\widehat{T} - \widehat{T}_0 = O(p)$ leads to $\tilde{r}_{ij}^{(2)} = O(p)$, and c is of order of unity in terms of p, which follows that $|\tilde{r}^{(2)}/c| = O(p)$.

Hence, the relation (B10) can be applied in the present case, and we obtain the expression (105), in which the term of $O(|\tilde{r}^{(2)}/c|^2)$ in (B10) is replaced with $O(p^2)$. Note that the manner of such a decomposition (B12) is not unique. In particular, one can choose the coefficient of $|P_{\rm can}(\beta)\rangle$ in $|P_1(\beta, p)\rangle$ as any form of $1 + O(p)$, though the values of c and γ depend on the manner.

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[2] G. Ódor, Universality classes in nonequilibrium lattice systems, Rev. Mod. Phys. **76**, 663 (2004).

[3] K.-t. Leung, B. Schmittmann, and R. K. P. Zia, Phase Transitions in a Driven Lattice Gas with Repulsive Interactions, Phys. Rev. Lett. **62**, 1772 (1989).

[4] G. Schötz and E. Domany, Phase transitions in an exactly soluble one-dimensional exclusion process, J. Stat. Phys. **72**, 277 (1993).

[5] M. Clincy, B. Derrida, and M. R. Evans, Phase transition in the ABC model, Phys. Rev. E **67**, 066115 (2003).

[6] F. Spitzer, Interaction of Markov processes, Adv. Math. **5**, 246 (1970).

[7] M. R. Evans and T. Hanney, Nonequilibrium statistical mechanics of the zero-range process and related models, J. Phys. A: Math. Gen. **38**, R195 (2005).

[8] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Novel Type of Phase Transition in a System of Self-Driven Particles, Phys. Rev. Lett. **75**, 1226 (1995).

[9] J. Dzubiella, G. P. Hoffmann, and H. Lowen, Lane formation in colloidal mixtures driven by an external field, Phys. Rev. E **65**, 021402 (2002).

[10] A. G. Thompson, J. Tailleur, M. E. Cates, and R. A. Blythe, Lattice models of nonequilibrium bacterial dynamics, J. Stat. Mech. (2011) P02029.

[11] A. P. Solon and J. Tailleur, Revisiting the Flocking Transition Using Active Spins, Phys. Rev. Lett. **111**, 078101 (2013).

[12] H. Ohta, Lane formation in a lattice model for oppositely driven binary particles, Europhys. Lett. **99**, 40006 (2012).

[13] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur, Exact Hydrodynamic Description of Active Lattice Gases, Phys. Rev. Lett. **120**, 268003 (2018).

[14] L. C. Freeman, Centrality in social networks: Conceptual clarification, Soc. Netw. **1**, 215 (1979).

[15] P. Bonacich, Power and centrality: A family of measures, Am. J. Sociol. **92**, 1170 (1987).

[16] A. Barrat, M. Bartélemy, R. Pastor-Satorras, and A. Vespignani, The architecture of complex weighted networks, Proc. Natl. Acad. Sci. USA **101**, 3747 (2004).

[17] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, Oxford, 2018).

[18] C. J. Campbell, A. Fuentes, K. C. MacKinnon, S. K. Bearder, and R. M. Stumpf, *Primates in Perspective* (Oxford University Press, Oxford, 2010).

[19] C. Sueur, O. Petit, A. De Marco, A. T. Jacobs, K. Watanabe, and B. Thierry, A comparative network analysis of social style in macaques, Anim. Behav. **82**, 845 (2011).

[20] L. J. Wooddell, S. S. K. Kaburu, and A. M. Dettmer, Dominance rank predicts social network position across developmental stages in rhesus monkeys, Am. J. Primatol. **82**, e23024 (2020).

[21] A. O'Neill, Correlates between ovarian cycle phase and mating season behaviour in female Japanese macaques (*Macaca fuscata*) in *The Monkeys of Stormy Mountain* (Cambridge University Press, Cambridge, 2012), Chap. 7, pp. 88–111.

[22] N. Nakagawa, M. Nakamichi, and H. Sugiura, *The Japanese Macaques* (Springer, Tokyo, 2010).

[23] F. Y. Wu, The Potts model, Rev. Mod. Phys. **54**, 235 (1982).

[24] M. Ostilli and F. Mukhamedov, Continuous- and discrete-time Glauber dynamics. First- and second-order phase transitions in mean-field Potts models, Europhys. Lett. **101**, 60008 (2013).

[25] M. Ostilli and F. Mukhamedov, 1D Three-state mean-field Potts model with first- and second-order phase transitions, Physica A **555**, 124415 (2020).

[26] R. S. Ellis and K. Wang, Limit theorems for the empirical vector of the Curie-Weiss-Potts model, Stoch. Process. Appl. **35**, 59 (1990).

[27] I. Hubač and S. Wilson, *Brillouin-Wigner Methods for Many-Body Systems*, Progress in Theoretical Chemistry and Physics, Vol. 21 (Springer, Dordrecht, 2010).

[28] H. Tasaki, Hohenberg-Mermin-Wagner-Type Theorems for Equilibrium Models of Flocking, Phys. Rev. Lett. **125**, 220601 (2020).

[29] C. H. Janson, Social correlates of individual spatial choice in foraging groups of brown capuchin monkeys, *Cebus apella*, Anim. Behav. **40**, 910 (1990).

[30] M. Heesen, S. Macdonald, J. Ostner, and O. Schülke, Ecological and social determinants of group cohesiveness and within-group spatial position in wild Assamese macaques, Ethology **121**, 270 (2015).

[31] H. Sugiura, Y. Shimooka, and Y. Tsuji, Variation in spatial cohesiveness in a group of Japanese macaques (*Macaca fuscata*), Int. J. Primatol. **32**, 1348 (2011).

[32] M. Nishikawa, M. Suzuki, and D. S. Sprague, Activity and social factors affect cohesion among individuals in female Japanese macaques: A simultaneous focal-follow study, Am. J. Primatol. **76**, 694 (2014).