Deformed random walk: Suppression of randomness and inhomogeneous diffusion

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(Received 5 September 2022; accepted 17 February 2023; published 8 March 2023)

We study a generalization of the random walk (RW) based on a deformed translation of the unitary step, inherited by the q algebra, a mathematical structure underlying nonextensive statistics. The RW with deformed step implies an associated deformed random walk (DRW) provided with a deformed Pascal triangle along with an inhomogeneous diffusion. The paths of the RW in deformed space are divergent, while those corresponding to the DRW converge to a fixed point. Standard random walk is recovered for $q \to 1$ and a suppression of randomness is manifested for the DRW with $-1 < \gamma_q < 1$ and $\gamma_q = 1 - q$. The passage to the continuum of the master equation associated to the DRW led to a van Kampen inhomogeneous diffusion equation when the mobility and the temperature are proportional to $1 + \gamma_q x$, and provided with an exponential hyperdiffusion that exhibits a localization of the particle at $x = -1/\gamma_q$ consistent with the fixed point of the DRW. Complementarily, a comparison with the Plastino-Plastino Fokker-Planck equation is discussed. The two-dimensional case is also studied, by obtaining a 2D deformed random walk and its associated deformed 2D Fokker-Planck equation, which give place to a convergence of the 2D paths for $-1 < \gamma_{q_1}, \gamma_{q_2} < 1$ and a diffusion with inhomogeneities controlled by two deformation parameters $\gamma_{q_1}, \gamma_{q_2}$ in the directions x and y. In both the one-dimensional and the two-dimensional cases, the transformation $\gamma_q \to -\gamma_q$ implies a change of sign of the corresponding limits of the random walk paths, as a property of the deformation employed.

DOI: 10.1103/PhysRevE.107.034113

Random walks represent one of the most illustrative models in statistical physics that contain fundamental aspects of randomness, diffusion, and dynamical features of master equations in a unified manner [1].

Let start by reviewing the basics of the one-dimensional RW. We assume that the walker starts at $x = 0$ and without restrictions (unrestricted RW) so he can continue along the infinite line with no obstacles. Thus, the RW is equivalent to a sequence of Bernoulli trials, so if X_i is the distance moved on the ith step then the probability of $X_i = \pm 1$ is

$$P(X_i = +1) = p, \quad P(X_i = -1) = 1 - p. \tag{1}$$

Since it is assumed the trials are independent and identically distributed as Bernoulli variables, we have the expectation and variance of X_i:

$$E(X_i) = 2p - 1, \quad \mathrm{Var}(X_i) = 4p(1 - p). \tag{2}$$

and the position after n steps is

$$S_n = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i \tag{3}$$

with $S_0 = 0$. In Fig. 1 it is shown a path of the walker after 100 steps with $x_0 = 0$ and $p_1 = p_2 = 1/2$. Some global predictions can be made about the position of the walker after n steps. More precisely, we have

$$E(S_n) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = n(2p - 1) \tag{4}$$

and

$$\mathrm{Var}(S_n) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = 4np(1 - p) \tag{5}$$

for the expectation and the variance of S_n. Also, we can construct the Pascal triangle associated to the random walk that describes all the positions with their occurrences and probabilities. In Table I we see that the probability $P(k, n)$ that the position of the walker is k after n steps obeys the binomial distribution

$$P(k, n) = \binom{n}{m} p^m (1 - p)^{n-m} \tag{6}$$

with $m = (1/2)(n + k)$. In addition, when $n \to \infty$ with $\lambda = np$ and small p, we have $P(k, n) \to \mathrm{Pois}_n(k) = \lambda^k e^{-\lambda}/k!$ so the number of arrivals of the RW in a fixed period of time of length k is described by the Poisson distribution.

Given a real number x, let

$$x_q = \frac{1}{\gamma_q} \ln(1 + \gamma_q x) \tag{7}$$

be the deformation of x [2,3] with $\gamma_q = 1 - q$, and q the entropic parameter associated to the Tsallis entropy [4,5]. $S_q = k_B \sum_{i=1}^{N} \frac{p_i^q - 1}{1-q}$ (with $\{p_i\}$ a discrete probability distribution, $\sum_i p_i = 1$, $p_i \geqslant 0$). In other words, $x_q = \ln(e_q^x)$ is a

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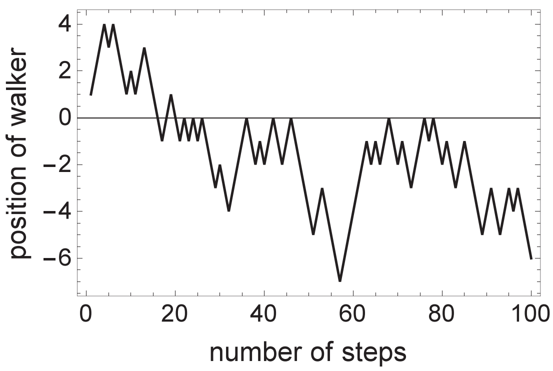

FIG. 1. Path of the walker after $n = 100$ steps starting at $x = 0$ with $p = 1 - p = 1/2$.

transformation of x inherited by the nonextensive entropy S_q, satisfying $x_q \to x$ in the limit $q \to 1$. Here $e_q^x = [1 + (1-q)x]^{1/(1-q)}$ stands for the q exponential, a nonextensive generalization of the exponential e^x. In the limit $q \to 1$, the Boltzmann-Gibbs entropy $S = -k_B \sum_{i=1}^N p_i \ln(p_i)$ and e^x are recovered. It should be noted that Tsallis entropy S_q is an important object of applications and theoretical developments in which the standard statistical mechanics shows some limitations to describe phenomena like driven anomalous diffusion [6], dynamics at the onset of chaos [7], and nonergodic regimes [8], among others.

Our strategy is to consider the deformation of the distance X_i (or equivalently, the description of the RW in the deformed space x_q) in order to study the resultant dynamics in the standard space x. At this point we warn that the RW in the transformed space x_q is simply a mathematical artifice with the aim to map a dynamics in the standard x space. This will be clarified in the next steps. We postulate the probability of the deformed position $(X_i)_q$ being the same of the RW in standard space; that is

$$P((X_i)_q = (+1)_q) = p \quad P((X_i)_q = (-1)_q) = 1 - p, \tag{8}$$

and using (7)

$$(+1)_q = \frac{1}{\gamma_q}\ln(1+\gamma_q) \quad (-1)_q = \frac{1}{\gamma_q}\ln(1-\gamma_q). \tag{9}$$

Equation (8) means that we are assuming a RW dynamics in the transformed space x_q. To prevent eventual divergences in the arguments of the logarithms of (9), in what follows we restrict the range of values of γ_q to the interval $-1 < \gamma_q < 1$. The deformed position $(S_n)_q$ after n steps is

$$(S_n)_q = (X_1 + X_2 + \ldots + X_n)_q = \sum_{i=1}^n (X_i)_q. \tag{10}$$

Identical predictions of the RW in x space can be exported for the RW in x_q space by making the substitutions $+1 \to (+1)_q$ and $-1 \to (-1)_q$. For instance, Eqs. (2), (4), and (5) turn out

$$\begin{aligned} E((X_i)_q) &= p(+1)_q + (1-p)(-1)_q \\ V((X_i)_q) &= ((+1)_q - (-1)_q)^2 p(1-p) \\ E((S_n)_q) &= n(p(+1)_q + (1-p)(-1)_q) \\ \mathrm{Var}((S_n)_q) &= n((+1)_q - (-1)_q)^2 p(1-p), \end{aligned} \tag{11}$$

TABLE I. Pascal triangle for the first five steps of the RW with $p = 1/2$. As the number of steps increases the probability is concentrated around the values $-1, 0, 1$, as expected due to the randomness and $E(X_i) = 0$ for all $i = 1, \ldots, n$. The probability distribution follows a binomial distribution.

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(S_0 = k)$						0					
$2P(S_1 = k)$					1		1				
$2^2P(S_2 = k)$				1		2		1			
$2^3P(S_3 = k)$			1		3		3		1		
$2^4P(S_4 = k)$		1		4		6		4		1	
$2^5P(S_5 = k)$	1		5		10		10		5		1

which reduces to (2), (4), and (5) in the limit $q \to 1$. Since $(+1)_q \neq (-1)_q$ we see that (11) is equivalent to a RW with two different steps $(+1)_q$ and $(-1)_q$. Now a natural question arises: What kind of RW results in standard space from the RW in x_q space given by Eqs. (10) and (11)? Here a crucial property of the q deformation appears to respond to the question. The sum of two q-deformed numbers x_q and y_q can be expressed by means of the q-deformed sum $\oplus_q$ of the q algebra [3]

$$x \oplus_q y = x + y + \gamma_q xy \tag{12}$$

that represents a polynomial deformation of the usual sum $x + y$ that is recovered for $\gamma_q = 1 - q \to 0$. It is worth noting that polynomial deformed sums like (12) have been generalized within the group entropy theory [9]. Using (7) and (12) it follows that the q sum satisfies

$$x_q + y_q = (x \oplus_q y)_q. \tag{13}$$

By applying the inverse of the q deformation (7) in (13) we obtain

$$x \oplus_q y = \frac{e^{\gamma_q(x_q + y_q)} - 1}{\gamma_q} \tag{14}$$

which applied repeatedly in the deformed position $(S_n)_q$ allows to find out the expression of $S_n(S_n)_q$ after n steps is

$$S_n = X_1 \oplus_q X_2 \oplus_q \ldots \oplus_q X_n = \frac{e^{\gamma_q(\sum_{i=1}^n (X_i)_q)} - 1}{\gamma_q}. \tag{15}$$

Equation (15) contains the dynamics of a deformed random walk (DRW) inherited by the RW in deformed space x_q given by (10). It is instructive to explicit some terms of (15) in order to visualize the dynamics generated. We have

$$\begin{aligned} S_1 &= X_1, \\ S_2 &= X_1 + X_2 + \gamma_q X_1 X_2, \quad \ldots = \ldots \\ S_n &= \sum_{i=1}^n X_i + \gamma_q \sum_{i<j}^n X_i X_j + \gamma_q^2 \sum_{i<j<k}^n X_i X_j X_k + \ldots, \end{aligned} \tag{16}$$

which expresses that the deformed position $S_n = \oplus_{i=1}^n X_i$ represents a generalization of the nondeformed position $\sum_{i=1}^n X_i$ provided with corrections of order γ_q^i $(i = 1, \ldots, n)$ in terms of all the sums of nonrepeated products $X_1 X_2 \cdots X_i$. Moreover, due to the independence of the variables $X_1, \ldots, X_n$ each sum of the right hand of (16) does not add correlations between the

TABLE II. Deformed Pascal triangle for the first five steps (counting down and being each file in step, zero step $X_0 = 0$) of the DRW with $p = 1/2$ and $\gamma_q = 1/2$. The value of the possible steps are indicated with their corresponding frequencies in parenthesis. The tendency is a rapid convergence to $x = -2$, in accordance with Fig. 2. For simplicity the values were truncated up the second decimal place.

			0			
		−1		1		
		−1.5 (1)	−0.5 (2)	2.5 (1)		
	−1.7 (1)	−1.2 (3)		0.2 (3)	5 (1)	
	−1.9 (1)	−1.6 (4)	−0.9 (6)	1.4 (4)	8.1 (1)	
−1.9 (1)	−1.8 (5)	−1.4 (10)		−0.3 (10)	3.1 (5)	13.2 (1)

X_i to the RW but only corrections of all the orders less than or equal to n. In addition, for small deformations $|\gamma_q| << 1$ the formula (16) can be interpreted as a series perturbation with γ_q the perturbation parameter. In Table II we see the asymmetrical behavior of the DRW in virtue of the deformation parameter $\gamma_q = 1/2$ for the first five steps. The convergence to the value $x = -2 = -\gamma_q^{-1}$ is observed in accordance with Fig. 2 for a few steps, which can be interpreted as a suppression of randomness due to the effect of the deformed sum $\oplus_q$ that introduce corrections in the form of a perturbation expansion in function of the parameter γ_q. Qualitatively, we can argue that the suppression of randomness is performed exponentially. By definition of the DRW we have $X_{n+1} - X_n = \pm 1 \pm \gamma_q X_n$ so when passing to a continuous time step $X(t)$ this relation turns out $\frac{dX}{dt} \approx \pm 1 \pm \gamma_q X$. In order to have a convergence for $-1 < \gamma_q < 1$ we require

$$\frac{dX}{dt} \approx 1 - |\gamma_q| X, \quad X(0) = 0, \tag{17}$$

whose solution is $X(t) = (e^{-|\gamma_q|t} - 1)/\gamma_q$. This exponential asymptotic behavior is observed in Fig. 2 for $\gamma_q = \pm 0.5$ from $n \geqslant 100$, which expresses a rapid convergence of the DRW.

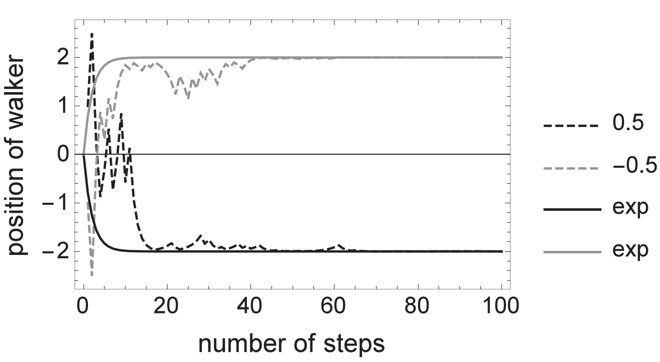

FIG. 2. Path of the walker after $n = 100$ steps starting at $x = 0$ with $p = 1 - p = 1/2$ for the deformed random walk with $\gamma_q = 0.5$ (black dashed line) and $\gamma_q = -0.5$ (gray dashed line). We see that the convergence to $-\gamma_q^{-1}$ is very fast, as a consequence of the corrections up to the order γ_q^{100} expressed by (16) for $n = 100$. Solid lines indicate their corresponding exponential asymptotic behavior $X(t) = (e^{-|\gamma_q|t} - 1)/\gamma_q$.

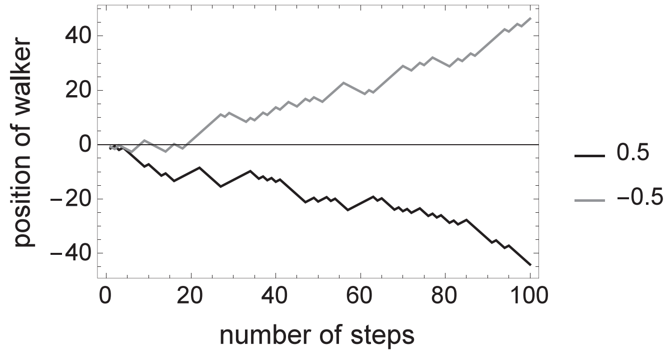

FIG. 3. Path of the walker after $n = 100$ steps starting at $x = 0$ provided with deformed steps $\{(-1)_q, (+1)_q\}$ and with $p = 1 - p = 1/2$. We see the divergence to $-\infty$ for $\gamma_q = 0.5$ (black) and to $+\infty$ for $\gamma_q = -0.5$ (gray).

More generally, we can demonstrate general conditions for the convergence of the deformed random walk for $|\gamma_q| < 1$. This is the content of the following result.

Theorem 1: Let $f_+, f_- : \mathbb{R} \to \mathbb{R}$ be the functions $f_+(x) = x \oplus_q 1 = x(1 + \gamma_q) + 1$ and $f_-(x) = x \oplus_q (-1) = x(1 - \gamma_q) - 1$. Let X_n be the position of the walker at instant $t = n$. Then,

(i) $X_{n+1} = f_+(X_n)$ or $X_{n+1} = f_-(X_n)$ for all $n \in \mathbb{N}$.

(ii) If $|\gamma_q| < 1$ then $\lim_{n\to\infty} X_n = -\gamma_q^{-1}$.

Proof. (i) By definition, we have $X_{n+1} = X_n \oplus_q (\pm 1) = f_\pm(X_n)$ being $X_{n+1} = f_+(X_n)$ if $+1$ or $X_{n+1} = f_-(X_n)$ if -1.

(ii) From the definition of X_n we have $X_{n+1} - X_n = \pm 1 \pm \gamma_q X_n$ so it follows $|X_{n+1} - X_n| = |1 + \gamma_q X_n|$ for all $n \in \mathbb{N}$. Also, X_n is obtained from $n - k$ and k applications of f_+ and f_- over $x = 0$ respectively (with $1 \leqslant k \leqslant n$). More precisely, we have $X_n = f_+^{n-k}(f_-^k(0))$ and then $X_{n+1} - X_n = f_+^{n+1-k}(f_-^k(0)) - f_+^{n-k}(f_-^k(0))$ or $X_{n+1} - X_n = f_+^{n-k}(f_-^{k+1}(0)) - f_+^{n-k}(f_-^k(0))$. Since $f_\pm^m(x) - f_\pm^m(y) = \pm(x - y)\gamma_q^m$ for all $m \in \mathbb{N}_0$ this implies that $X_{n+1} - X_n = \pm(-1)^k \gamma_q^n$, where we used $f_\pm(0) = \pm 1$. Hence, if $|\gamma_q| < 1$ and using $|X_{n+1} - X_n| = |1 + \gamma_q X_n|$, it follows that $\lim_{n\to\infty} X_{n+1} - X_n = 0$ and then $\lim_{n\to\infty} |X_{n+1} - X_n| = \lim_{n\to\infty} |1 + \gamma_q X_n| = 0$, which proves (ii).

It is also instructive to study the RW in the deformed space provided with the deformed steps $\{(-1)_q, (+1)_q\}$. Just note that the randomness is the same as in the standard case so we expect, using Eqs. (9), (11), and $p = 1/2$, that $(X_n)_q \approx E((S_n)_q) = (n/2)(1/\gamma_q)\ln(1 - \gamma_q^2)$ for $n \gg 1$. Thus, considering $|\gamma_q| < 1$ we see that when the number of steps n goes to infinity $(X_n)_q$ tends to $-\infty$ ($+\infty$ respectively) if $\gamma_q > 0$ ($\gamma_q < 0$ respectively). This behavior is shown in Fig. 3, in which RW in the deformed space is illustrated for $\gamma_q = \pm 1/2$.

Now we analyze the dynamics of the DRW in terms of the master equation framework [10,11]. We follow the same prescription than in the DRW, i.e., by considering the master equation in the deformed space x_q for the deformed probability distribution $\mathcal{P}(x_q, t)$ we obtain the motion equation of the probability distribution $P(x, t)$ in standard space of the walker at the position x and time t. The relation between $\mathcal{P}(x_q, t)$ and $P(x, t)$ is established from the conservation of probability for a volume dx in standard space, or equivalently, having a volume $dx_q = dx/(1+\gamma_q x)$ in the deformed space

$$P(x,t)dx = \mathcal{P}(x_q,t)dx_q \Longrightarrow P(x,t) = \frac{\mathcal{P}(x_q(x),t)}{1+\gamma_q x}. \quad (18)$$

It should be noted that the derivatives and integrals in the deformed space x_q results affected by the deformation. The q derivative D_q and the q integral $\int_q$ are the corresponding analogues of the usual derivative and integral, given by [3,12]

$$D_q f(x) = \frac{df}{dx_q} = (1+\gamma_q x)\frac{df}{dx} \quad (19)$$

$$\int_q f = \int f(x)dx_q = \int_{\{x>-\gamma_q^{-1}\}} f(x)\frac{dx}{1+\gamma_q x}. \quad (20)$$

In other words, the q integral is nothing but the Riemann-Stieltjes integral with the weight function $w(x) = 1/(1+\gamma_q x)$. Let us consider a random walk on a one-dimensional deformed lattice where the jumps occur in regular intervals of time Δt, being p and $1-p$ the probability of a right jump $(+a)_q$ and of a left jump $(-a)_q$ with a the parameter of the nondeformed lattice. Thus, the master equation for $\mathcal{P}(x_q,t)$ is given by

$$\mathcal{P}(x_q, t+\Delta t) = p\mathcal{P}(x_q+(+a)_q, t) + (1-p)\mathcal{P}(x_q+(-a)_q, t). \quad (21)$$

As usual, the admit to the continuum of (21) is performed by letting $a \to 0$. For comparing with the DRW (16) we take $p = 1/2$ in (21). In the limit $a \to 0$ we have the approximations

$$(\pm a)_q = \frac{1}{\gamma_q}\ln(1 \pm \gamma_q a) \approx \pm a,$$

$$\mathcal{P}(x_q+(+a)_q,t) \approx \mathcal{P} + (+a)_q D_q\mathcal{P} + \frac{1}{2}((+a)_q)^2 D_q^2\mathcal{P},$$

$$\mathcal{P}(x_q+(-a)_q,t) \approx \mathcal{P} + (-a)_q D_q\mathcal{P} + \frac{1}{2}((-a)_q)^2 D_q^2\mathcal{P},$$

$$\mathcal{P}(x_q, t+\Delta t) \approx \mathcal{P} + \Delta t\frac{\partial \mathcal{P}}{\partial t}. \quad (22)$$

Substituting (22) in (21) we arrive at the deformed Fokker Planck equation (DFPE), employed recently in [13], with a null drift term

$$\frac{\partial \mathcal{P}(x_q,t)}{\partial t} = \Gamma D_q^2 \mathcal{P}(x_q,t), \quad (23)$$

where $\Gamma = a^2/(2\Delta t)$ is the diffusion coefficient of the nondeformed standard case. Alternatively, the DFPE can be written in standard space

$$\frac{\partial P(x,t)}{\partial t} = \Gamma\frac{\partial}{\partial x}(1+\gamma_q x)\frac{\partial}{\partial x}(1+\gamma_q x)P(x,t) \quad (24)$$

which is a particular case of the van Kampen diffusion equationtype (equation (5) of [14])

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}[\mu(x)V'(x)P(x,t)] + \frac{\partial}{\partial x}\mu(x)\frac{\partial}{\partial x}T(x)P(x,t) \quad (25)$$

with $V'(x) = 0$ and the conditions

$$\frac{\mu(x)}{\mu_0} = \frac{T(x)}{T_0} = 1+\gamma_q x, \quad \Gamma = \mu_0 T_0 \quad (26)$$

for a mobility and temperature that are position-dependent. Here μ_0, T_0 are the values of the mobility and the temperature of the homogeneous case. A detailed discussion of the DFPE (23) and its relationships with the van Kampen diffusion [14], the superstatistics FPE [15,16], and the nonlinear Langevin equation [17] is included in [13]. Other generalized Fokker-Planck equations have been studied highlighting the role of the nonextensive statistics [18–20] and employing fractional derivatives [21]. We also stress that, recently, inhomogeneities in general have been characterized in the context of superstatistics by means of an effective position-dependent mass [22]. For studying the stationary regime in this case [with $V(x) = 0$] we cannot take advantage of the formula of the stationary solution of the van Kampen equation (25)

$$P_s(x) = \frac{C}{T(x)}\exp\left(-\int \frac{V'(x)}{T(x)}dx\right) \quad (27)$$

due to the fact that $1/T(x) = 1/(T_0(1+\gamma_q x))$ is not normalized in $(-\infty,\infty)$. Noting that the deformed solution $\mathcal{P}(x_q,t)$ of the DFPE (23) corresponds to the free particle in the deformed space x_q

$$\mathcal{P}(x_q,t) = \frac{1}{\sqrt{2\pi\Gamma t}}\exp\left(-\frac{x_q^2}{2\Gamma t}\right), \quad (28)$$

then using (18) it follows the solution $P(x,t)$

$$P(x,t) = \frac{1}{1+\gamma_q x}\frac{1}{\sqrt{2\pi\Gamma t}}\exp\left(-\frac{\ln^2(1+\gamma_q x)}{(2\Gamma t)\gamma_q^2}\right) \quad (29)$$

for all t. From the normalization condition (18), the DFPE solution (29), and Fig. 4 it follows that the stationary solution is

$$P_s(x) = \delta(x+1/\gamma_q) \quad \text{with} \quad x > -1/\gamma_q \ \text{if}\ \gamma_q > 0 \quad \text{or} \quad x < -1/\gamma_q \ \text{if}\ \gamma_q < 0. \quad (30)$$

Moreover, the divergence of the DFPE solution (29) at $x = -1/\gamma_q$ is a consequence of the deformed space $dx_q = dx/(1+\gamma_q x)$ and can be physically interpreted as a region of localization of the particle due to the asymmetric inhomogeneous diffusion. By letting l_0 a constant with dimensions of length, $\tau = 1/(\gamma_q^2\Gamma)$ and with the aim of showing the asymptotic behavior of $P(x,t)$, in Fig. 4 it is illustrated $P(x,t)$ for times $t/\tau = 0.01, 1, 4, 8$ and for $\gamma_q = \pm 0.5$ with the initial condition $P(x, t=0) = \delta(x)$. The domain of the diffusion is restricted to the interval $x > -|\gamma_q|^{-1}$ if $\gamma_q > 0$ or $x < |\gamma_q|^{-1}$ if $\gamma_q < 0$. The probability distribution $P(x,t)$ converges rapidly to the stationary solution $P_s(x) = \delta(x+1/\gamma_q)$ as we can see in Fig. 4, where the dashed curves in gray show more clearly the limit $\lim_{t\to\infty} P(x,t) = P_s(x)$. In order to classify the type of diffusion we calculate the first and second moments of the

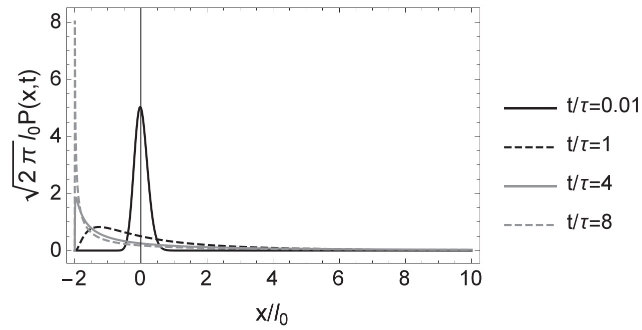

FIG. 4. Profiles of the probability distribution $P(x, t)$ of the DFPE [Eq. (29)] for times $t/\tau = 0.01, 1, 4, 8$ with $\gamma_q = 0.5$ (left panel) and $\gamma_q = -0.5$ (right panel) with the initial condition $P(x, t = 0) = \delta(x)$. The gray dashed lines illustrate the asymptotic behavior of $P(x, t) \to \delta(x + 1/\gamma_q)$, exhibiting asymmetry and a localization at $x = -1/\gamma_q$ due to the deformation.

solution $P(x, t)$ [by calling $\mathrm{sg}(\gamma_q)$ the sign of γ_q]

$$\begin{aligned}\langle x(t)\rangle &= \int_{\{x>-\mathrm{sg}(\gamma_q)|\gamma_q|^{-1}\}} xP(x,t)dx\\ &= \int_{\{x>-\mathrm{sg}(\gamma_q)|\gamma_q|^{-1}\}} \frac{1}{1+\gamma_q x}\frac{x}{\sqrt{2\pi\Gamma t}}\\ &\quad\times \exp\left(-\frac{\ln^2(1+\gamma_q x)}{(2\Gamma t)\gamma_q^2}\right)dx\\ &= \frac{e^{\Gamma t\gamma_q^2/2}-1}{\gamma_q},\end{aligned} \tag{31}$$

$$\begin{aligned}\langle x(t)^2\rangle &= \int_{\{x>-\mathrm{sg}(\gamma_q)|\gamma_q|^{-1}\}} x^2P(x,t)dx\\ &= \int_{\{x>-\mathrm{sg}(\gamma_q)|\gamma_q|^{-1}\}} \frac{1}{1+\gamma_q x}\frac{x^2}{\sqrt{2\pi\Gamma t}}\\ &\quad\times \exp\left(-\frac{\ln^2(1+\gamma_q x)}{(2\Gamma t)\gamma_q^2}\right)dx\\ &= \frac{e^{2\Gamma t\gamma_q^2}-2e^{\Gamma t\gamma_q^2/2}+1}{\gamma_q^2}.\end{aligned} \tag{32}$$

Joining Eqs. (31) and (32) we obtain the mean standard deviation (MSD)

$$\langle(\Delta x)^2(t)\rangle = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = \frac{e^{2\Gamma t\gamma_q^2}-e^{\Gamma t\gamma_q^2}}{\gamma_q^2}, \tag{33}$$

which exhibits two regimes regarding the characteristic time $\tau = 1/(\gamma_q^2\Gamma)$: (i) normal diffusion behavior $\langle(\Delta x)^2\rangle \approx \Gamma t$ for $t \ll \tau$ and (ii) exponential hyper diffusion $\langle(\Delta x)^2\rangle \propto e^{2t/\tau}$ for $t \gg \tau$. In Fig. 5 we illustrate behavior of the MSD with the values $\gamma_q l_0 = 0, 0.2, 0.4, 0.8$. Since the characteristic time τ is symmetric in γ_q, it is enough with illustrating for positive values of $\gamma_q l_0$. We can see that the effect of the deformation is to increase the exponential hyper-diffusion, while for a short time interval $t \ll \tau$ the diffusion is indistinguishable from the normal diffusion. It is instructive to show that the linear diffusion in the deformed space

$$\langle(\Delta x_q)^2(t)\rangle = \langle x_q^2(t)\rangle - \langle x_q(t)\rangle^2 = \langle x_q^2(t)\rangle = \Gamma t \tag{34}$$

leads to the relation in standard space

$$\left\langle \frac{\ln^2(1+\gamma_q x)}{\gamma_q^2}\right\rangle = \Gamma t, \tag{35}$$

where the mean values $\langle\ldots\rangle$ are taken over the solution $P(x, t)$. Equations (33) and (35) show the relationship between the exponential hyper-diffusion in standard space x and the linear diffusion in deformed space x_q.

We end the analysis of the one-dimensional case with a comparison between the deformed Fokker Planck equation (24) and the Plastino-Plastino Fokker-Planck equation for nonadditive systems with null drift term, given by [18,19,23]

$$\frac{\partial P(x,t)}{\partial t} = \Gamma(2-q)\frac{\partial}{\partial x}\left\{[P(x,t)]^{1-q}\frac{\partial P(x,t)}{\partial x}\right\}, \tag{36}$$

with q the entropic parameter that allows to recover the standard FPE when $q \to 1$. By means of the nonlinear Nobre derivative [24]

$$\widetilde{\mathcal{D}}_q f(x) = [f(x)]^{1-q}\frac{df}{dx} \tag{37}$$

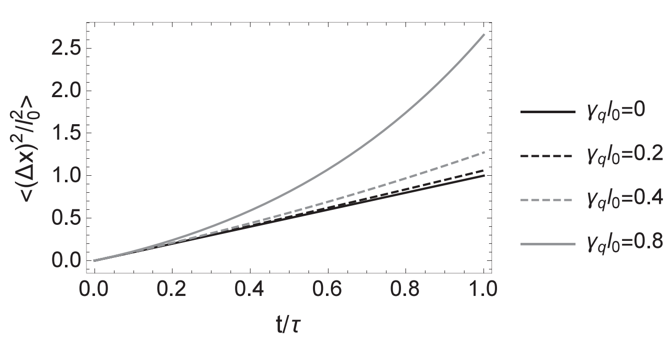

FIG. 5. Time evolution of the mean standard deviation for $\gamma_q l_0 = 0, 0.2, 0.4, 0.8$ of the deformed Fokker-Planck eqaution (24). The black straight line indicates the normal diffusion case $\langle(\Delta x)^2(t)\rangle = \Gamma t$, corresponding to null deformation $\gamma_q l_0 = 0$. As the deformation parameter increases the exponential hyper-diffusion is more pronounced, which is consistent with the localization of the particle at $x = -1/\gamma_q$ and with the deformed random walk (Fig. 2).

TABLE III. Some properties of the deformed FPE and the Plastino-Plastino FPE reflecting the interplay between the character of the FPE (linear/nonlinear), the type of diffusion, the entropy associated and the generalize derivative employed.

Property	Deformed FPE	Plastino-Plastino FPE
Equation	$\partial_t \mathcal{P} = \Gamma D_q^2 \mathcal{P}$	$\partial_t P^q = \Gamma(2-q)\widetilde{\mathcal{D}}_q^2 P$
Character of the FPE	Linear	Nonlinear
Generalized derivative associated	Deformed derivative $D_q f = (1+\gamma_q x)\frac{df}{dx}$	Nobre derivative $\widetilde{\mathcal{D}}_q f = [f(x)]^{1-q}\frac{df}{dx}$
Asymptotic MSD behavior	$\propto \exp(2\Gamma t \gamma_q)$ exponential hyper-diffusion	$\propto t^{2/(1+q)}$ power-law diffusion [23]
Entropy associated to H theorem	Deformed Boltzmann Gibbs entropy [13] $\mathcal{S} = -\int \mathcal{P}\ln\mathcal{P}dx_q$	Tsallis entropy [18,19,23] $S_q = \frac{\int P(x)^q dx - 1}{1-q}$

we can recast the Plastino-Plastino FPE (36) in the compact form

$$\frac{\partial P(x,t)^q}{\partial t} = \Gamma(2-q)\widetilde{\mathcal{D}}_q^2 P(x,t) \tag{38}$$

where we have used the definition

$$\widetilde{\mathcal{D}}_q^2 f(x) \equiv [f(x)]^{1-q}\frac{d}{dx}\left\{[f(x)]^{1-q}\frac{df}{dx}\right\}, \tag{39}$$

i.e., $\widetilde{\mathcal{D}}_q^2 f(x) \neq \widetilde{\mathcal{D}}_q[\widetilde{\mathcal{D}}_q f(x)]$ and higher order derivatives are defined analogously. In Table III are listed some characteristics of the DFPE (24) and the Plastino-Plastino FPE (36), that result as a consequence of the underlying derivative structures given by the linear deformed derivative D_q and the nonlinear Nobre derivative $\widetilde{\mathcal{D}}_q$.

We generalize our previous results for the two-dimensional case. The novelty of the two-dimensional case is that we can choose different deformations for each one of the directions x and y. More precisely, we postulate the probability distribution of the deformed vector position $((X_i)_{q_1},(Y_i)_{q_2})$, i.e.,

$$\begin{aligned}
&P\big(((X_i)_{q_1},(Y_i)_{q_2}) = ((+1)_{q_1},(+1)_{q_2})\big) = p_1\\
&P\big(((X_i)_{q_1},(Y_i)_{q_2}) = ((+1)_{q_1},(-1)_{q_2})\big) = p_2\\
&P\big(((X_i)_{q_1},(Y_i)_{q_2}) = ((-1)_{q_1},(+1)_{q_2})\big) = p_3\\
&P\big(((X_i)_{q_1},(Y_i)_{q_2}) = ((-1)_{q_1},(+1)_{q_2})\big) = 1-p_1-p_2-p_3.
\end{aligned} \tag{40}$$

By repeating the same arguments as in the one-dimensional case it is straightforward to show that the deformed vector position after n steps

$$((S_n^x)_{q_1},(S_n^y)_{q_2}) = \left(\sum_{i=1}^n (X_i)_{q_1}, \sum_{i=1}^n (Y_i)_{q_2}\right) \tag{41}$$

which can be recasted in standard space (x,y) as

$$\begin{aligned}
\big(S_n^x,S_n^y\big) &= \big(X_1 \oplus_{q_1} \dots \oplus_{q_1} X_n, Y_1 \oplus_{q_2} \dots \oplus_{q_2} Y_n\big)\\
&= \left(\frac{e^{\gamma_{q_1}\left(\sum_{i=1}^n (X_i)_q\right)}-1}{\gamma_{q_1}}, \frac{e^{\gamma_{q_2}\left(\sum_{i=1}^n (Y_i)_q\right)}-1}{\gamma_{q_2}}\right).
\end{aligned} \tag{42}$$

Here $\gamma_{q_i} = 1-q_i$ is the deformation parameter in the i direction with $i=1,2$. Also, we have for S_n^x and S_n^y

$$\begin{aligned}
S_1^x &= X_1\\
S_2^x &= X_1 + X_2 + \gamma_{q_1}X_1X_2 \dots = \dots\\
S_n^x &= \sum_{i=1}^n X_i + \gamma_{q_1}\sum_{i<j}^n X_iX_j + \gamma_{q_1}^2 \sum_{i<j<k}^n X_iX_jX_k + \dots
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
S_1^y &= Y_1\\
S_2^y &= Y_1 + Y_2 + \gamma_{q_2}Y_1Y_2 \dots = \dots\\
S_n^y &= \sum_{i=1}^n Y_i + \gamma_{q_2}\sum_{i<j}^n Y_iY_j + \gamma_{q_2}^2 \sum_{i<j<k}^n Y_iY_jY_k + \dots.
\end{aligned} \tag{44}$$

Since the coordinates X_n and Y_n of the two-dimensional DRW are uncorrelated the same argument of the one-dimensional DRW applies so we have

$$\begin{aligned}
\frac{dX}{dt} &\approx 1 - |\gamma_{q_1}|X, \quad X(0)=0,\\
\frac{dY}{dt} &\approx 1 - |\gamma_{q_2}|Y, \quad Y(0)=0,
\end{aligned} \tag{45}$$

whose solution is $(X(t),Y(t)) = (e^{-|\gamma_{q_1}|t}-1)/\gamma_{q_1}, e^{-|\gamma_{q_2}|t}-1)/\gamma_{q_2})$. This exponential asymptotic behavior is observed for $\gamma_{q_1}=0.5$ and $\gamma_{q_2}=-0.5$. Analogously, Theorem 1 for the one-dimensional DRW remains valid in the two-dimensional case.

Theorem 2: Let $f_+, f_- : \mathbb{R}\to\mathbb{R}$ be the functions $f_+(x) = x\oplus_{q_1} 1 = x(1+\gamma_{q_1})+1$, $f_-(x) = x\oplus_{q_1}(-1) = x(1-\gamma_{q_1})-1$, $g_+(x) = y\oplus_{q_2}(-1) = y(1+\gamma_{q_2})-1$, and $g_-(y) = y\oplus_{q_2}(-1) = y(1-\gamma_{q_2})-1$. Let (X_n,Y_n) be the vector position of the walker at instant $t=n$. Then,

(i) $X_{n+1} = f_+(X_n)$ or $X_{n+1} = f_-(X_n)$ and $Y_{n+1} = g_+(Y_n)$ or $Y_{n+1} = g_-(Y_n)$ for all $n\in\mathbb{N}$.

(ii) If $|\gamma_{q_1}|, |\gamma_{q_2}| < 1$ then $\lim_{n\to\infty}(X_n,Y_n) = (-\gamma_{q_1}^{-1}, -\gamma_{q_2}^{-1})$.

Proof. Repeat the steps of the demonstration of Theorem 1 for X_n and Y_n separately.

At this point some kind of generalization of the DFPE to the two-dimensional case is also expected. We begin with the master equation of the 2D case for the deformed probability

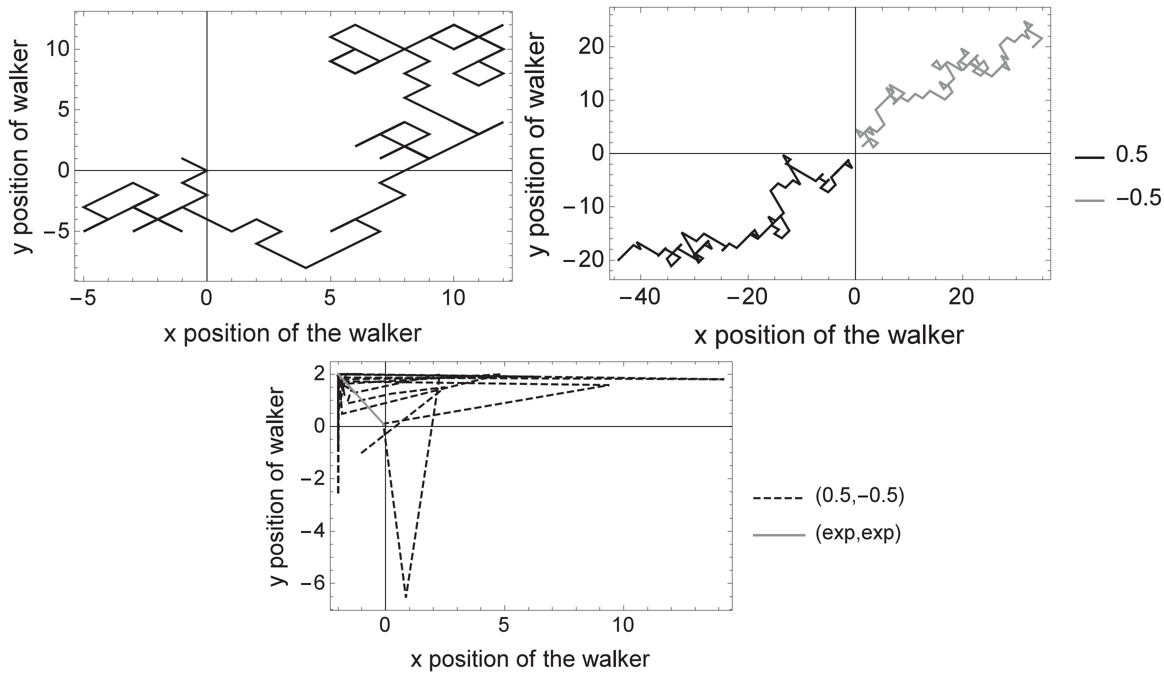

FIG. 6. Paths of the walker after $n = 100$ steps starting at $(x, y) = (0, 0)$ provided $p_1 = p_2 = p_3 = 1/4$. Top left shows the two-dimensional RW and top right shows the RW provided with deformed steps $\{((+1)_{q_1}, (+1)_{q_2}), ((+1)_{q_1}, (-1)_{q_2}), ((-1)_{q_1}, (+1)_{q_2}), ((-1)_{q_1}, (-1)_{q_2})\}$ for $\gamma_{q_1} = \gamma_{q_2} = 0.5$ (black) and $\gamma_{q_1} = \gamma_{q_2} = 0.5$ (gray). Again, divergent paths are manifested. The plot below illustrates the two-dimensional DRW with a mixture of deformations $\gamma_{q_1} = 0.5$ and $\gamma_{q_2} = -0.5$, whose convergence to the fixed point $(-2, 2)$ is observed and the gray straight line indicates the exponential asymptotic behavior $(X(t), Y(t)) = ((e^{-|\gamma_{q_1}|t} - 1)/\gamma_{q_1}, (e^{-|\gamma_{q_2}|t} - 1)/\gamma_{q_2})$.

distribution $\mathcal{P}(x_{q_1}, y_{q_2}, t)$:

$$\begin{aligned}\mathcal{P}(x_{q_1}, y_{q_2}, t + \Delta t) = {} & p_1\mathcal{P}\big(x_{q_1} + (+a)_{q_1}, y_{q_2}, t\big) \\ & + p_2\mathcal{P}\big(x_{q_1} + (-a)_{q_1}, y_{q_2}, t\big) + p_3 \\ & \times \mathcal{P}\big(x_{q_1}, y_{q_2} + (+b)_{q_2}, t\big) \\ & + (1 - p_1 - p_2 - p_3) \\ & \times \mathcal{P}\big(x_{q_1}, y_{q_2} + (-b)_{q_2}, t\big). \end{aligned} \tag{46}$$

Here we are considering that a and b are the parameters of the nondeformed lattice. In the limit $a, b \to 0$ we have the approximations

$$\begin{aligned}(\pm a)_{q_1} &= \frac{1}{\gamma_{q_1}}\ln(1 \pm \gamma_{q_1}a) \approx \pm a \\ (\pm b)_{q_2} &= \frac{1}{\gamma_{q_2}}\ln(1 \pm \gamma_{q_2}b) \approx \pm b \\ & \times \mathcal{P}\big(x_{q_1} + (+a)_{q_1}, y_{q_2}, t\big) \approx \mathcal{P} + (+a)_{q_1}D_{x,q_1}\mathcal{P} \\ & + \frac{1}{2}((+a)_{q_1})^2D^2_{x,q_1}\mathcal{P} \\ & \times \mathcal{P}(x_{q_1} + (-a)_{q_1}, y_{q_2}, t) \approx \mathcal{P} + (-a)_{q_1}D_{x,q_1}\mathcal{P} \\ & + \frac{1}{2}\big((-a)_{q_1}\big)^2D^2_{x,q_1}\mathcal{P} \\ & \times \mathcal{P}(x_{q_1}, y_{q_2} + (+b)_{q_2}, t) \approx \mathcal{P} + (+b)_{q_2}D_{y,q_2}\mathcal{P} \\ & + \frac{1}{2}((+b)_{q_2})^2D^2_{x,q_2}\mathcal{P} \\ & \times \mathcal{P}\big(x_{q_1}, y_{q_2} + (-b)_{q_2}, t\big) \approx \mathcal{P} + (-b)_{q_2}D_{y,q_2}\mathcal{P} \\ & + \frac{1}{2}\big((-b)_{q_2}\big)^2D^2_{y,q_2}\mathcal{P} \\ & \times \mathcal{P}\big(x_{q_1}, y_{q_2}, t + \Delta t\big) \approx \mathcal{P} + \Delta t\frac{\partial\mathcal{P}}{\partial t}, \end{aligned} \tag{47}$$

with $D_{x,q_1} = (1 + \gamma_{q_1}x)\partial/\partial x$ and $D_{y,q_2} = (1 + \gamma_{q_2}y)\partial/\partial y$ the $q_{1,2}$-partial derivatives. Substituting (47) in (46) we obtain the two-dimensional DFPE

$$\frac{\partial\mathcal{P}(x_{q_1}, y_{q_2}, t)}{\partial t} = \Gamma_1D^2_{x,q_1}\mathcal{P}\big(x_{q_1}, y_{q_2}, t\big) + \Gamma_2D^2_{y,q_2}\mathcal{P}\big(x_{q_1}, y_{q_2}, t\big) \tag{48}$$

with $\vec{\Gamma} = (\Gamma_1, \Gamma_2) = (a^2/(2\Delta t), b^2/(2\Delta t))$ the vector diffusion. Again, we have an expression of the two-dimensional DFPE (48) in standard space (x, y)

$$\begin{aligned}\frac{\partial P(x, y, t)}{\partial t} = {} & \Gamma_1\frac{\partial}{\partial x}\big(1 + \gamma_{q_1}x\big)\frac{\partial}{\partial x}(1 + \gamma_{q_1}x)P(x, y, t) \\ & + \Gamma_2\frac{\partial}{\partial y}\big(1 + \gamma_{q_2}y\big)\frac{\partial}{\partial y}(1 + \gamma_{q_2}y)P(x, y, t)\end{aligned} \tag{49}$$

giving place to a inhomogeneous two-dimensional diffusion equation, which in principle does not admit a correspondence with any two-dimensional version of the van Kampen equation (25). Noting that (49) is separable, the solution is simply

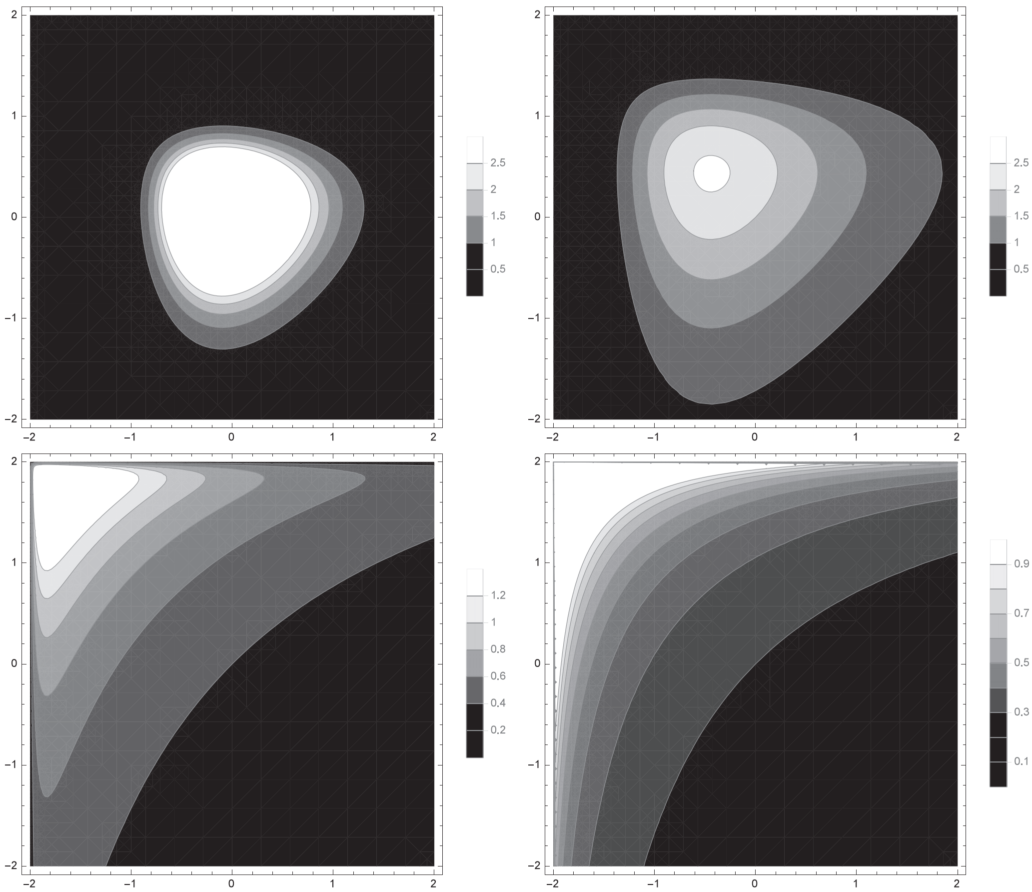

FIG. 7. Contours of the probability distribution $P(x, y, t)$ of the two-dimensional DFPE [Eq. (50)] for times $t/\tau = 0.1, 0.5, 5, 10$ with $\gamma_{q_1} = 0.5$ and $\gamma_{q_2} = -0.5$ with the initial condition $P(x, y, t = 0) = \delta(x)\delta(y)$. In white and black are indicated the regions with low and high probability density.

the product of (29) in x and y

$$P(x, y, t) = \frac{1}{1+\gamma_{q_1}x}\frac{1}{\sqrt{2\pi\Gamma_1 t}}\exp\left(-\frac{\ln^2(1+\gamma_{q_1}x)}{(2\Gamma_1 t)\gamma_{q_1}^2}\right) \times \frac{1}{1+\gamma_{q_2}y}\frac{1}{\sqrt{2\pi\Gamma_2 t}}\exp\left(-\frac{\ln^2(1+\gamma_{q_2}y)}{(2\Gamma_2 t)\gamma_{q_2}^2}\right), \tag{50}$$

with the stationary solution

$$P_s(x, y) = \delta(x + 1/\gamma_{q_1})\delta(y + 1/\gamma_{q_2}), \quad \text{with}$$
$$x > -1/\gamma_{q_1} \text{ if } \gamma_{q_1} > 0 \text{ or } x < -1/\gamma_{q_1} \text{ if } \gamma_{q_1} < 0,$$
$$y > -1/\gamma_{q_2} \text{ if } \gamma_{q_2} > 0 \text{ or } y < -1/\gamma_{q_2} \text{ if } \gamma_{q_2} < 0. \tag{51}$$

For illustrating we set $\Gamma_1\tau/l_1^2 = \Gamma_2\tau/l_2^2 = 2$, $\gamma_{q_1}l_1 = 0.5$, and $\gamma_{q_2}l_2 = -0.5$, being l_1, l_2 constants with dimension of length and $\tau = 1/(\gamma_{q_1}^2\Gamma) = 1/(\gamma_{q_2}^2\Gamma)$ the same characteristic time in both directions x and y due to the choice $\gamma_{q_1} = -\gamma_{q_2} = 0.5$. In Fig. 7 it is shown $P(x, y, t)$ for times $t/\tau = 0.1, 0.5, 5, 10$ with the initial condition $P(x, y, t = 0) = \delta(x)\delta(y)$. The domain of the diffusion is restricted to the rectangle $x > -|\gamma_{q_1}|^{-1}$ if $\gamma_{q_1} > 0$ or $x < |\gamma_{q_1}|^{-1}$ if $\gamma_{q_1} < 0$ for the coordinate x and $y > -|\gamma_{q_2}|^{-1}$ if $\gamma_{q_2} > 0$ or $y < |\gamma_{q_2}|^{-1}$ if $\gamma_{q_2} < 0$ for the coordinate y. The probability distribution $P(x, y, t)$ converges rapidly to the stationary solution $P_s(x, y) = \delta(x + 1/\gamma_{q_1})\delta(y + 1/\gamma_{q_2})$, as we can see from Fig. 7, where it is shown the behavior $\lim_{t\to\infty} P(x, y, t) = P_s(x, y)$.

We have presented a generalization of the random walk in the presence of a deformed translation of the unitary step based on the deformed sum of the q-algebra, a mathematical structure inherited by nonextensive statistics. We enumerate the contributions of this work as follows.

(a) A suppression of the randomness is observed for $-1 < \gamma_q < 1$, with $\gamma_q = 1 - q$ and q the entropic index, provided with a convergence to $x = -\gamma_q^{-1}$ for any path X_n starting at $x = 0$ (Fig. 2). In addition, this behavior has been shown mathematically by means of Theorem 1.

(b) The deformed sum $\oplus_{\gamma_q}$ does not affect the randomness of the deformed position S_n but only adds contribution terms of the order γ_q^{n-1} [Eq. (16)], thus resulting a series expansion being the deformation γ_q the perturbation parameter.

(c) A deformed Pascal triangle arises for $-1 < \gamma_q < 1$ showing the convergence $X_n \to -\gamma_q^{-1}$, in compatibility with Theorem 1, by means of the corresponding frequencies (Table II).

(d) The master equation of the deformed random walk (21) led to a deformed Fokker-Planck equation (24) that is a special case of the van Kampen equation (25), when the mobility and the temperature are proportional to $1 + \gamma_q x$, that describes inhomogeneous diffusion controlled by the deformation parameter γ_q. Alternatively, the deformed Fokker Planck (24) can be expressed as a linear homogeneous FPE (23) provided with deformed coordinates and derivatives.

(e) The exponential convergence $X_n \to -\gamma_q^{-1}$ for the paths of the deformed random walk (Fig. 2) is represented in the continuous limit by the asymptotic behavior of the probability distribution of the deformed Fokker-Planck equation (Fig. 4).

(f) An exponential hyper-diffusion emerges as a result of the q-deformation $dx_q = dx/(1+\gamma_q x)$.

(g) By comparing the deformed FPE (24) and the Plastino-Plastino FPE (36), an interplay between the linear/nonlinear character of the FPE, the type of diffusion, the entropy associated to H theorem, and the generalized derivative structure is evidenced (Table III).

(h) The main results of the one-dimensional case are generalized to the two-dimensional case, thus obtaining 2D convergent paths $(X_n, Y_n) \to (-\gamma_{q_1}^{-1}, -\gamma_{q_2}^{-1})$ for $-1 < \gamma_{q_1}, \gamma_{q_2} < 1$ (Fig. 6) along with a 2D deformed Fokker-Planck equation [deduced by the 2D master equation (46)] that currently does not find an analog with van Kampen diffusion, but only exhibiting an inhomogeneous diffusion with the asymptotic behavior illustrated in Fig. 7. The real and continuous parameters$\gamma_{q_1}, \gamma_{q_2}$ control the inhomogeneities in the directions x and y and due to the uncorrelation of X_n and Y_n Theorem 1 is generalized in a straightforward way (Theorem 2).

(i) For both the one-dimensional and two-dimensional cases, the deformed calculus (q calculus) is shown to be compatible with inhomogeneous diffusion, more specifically of the van Kampen type.

(j) As a property of the deformation employed, in both the one-dimensional and the two-dimensional cases, the transformation $\gamma_q \to -\gamma_q$ implies a change of sign of the corresponding limits of the random walk paths (Figs. 1–5).

(k) Random walks in multiple contexts have been studied in the literature [25–27] and the formalism presented in this work shows a simple generalization deduced from the property of the q-sum belonging to the q-algebra [Eq. (13)].

The results of this work are potentially generalizable to other types of deformations, for instance the deformation inherited by the κ statistics [28]. Moreover, the structure of the deformed position S_n [Eq. (16)] could be used to model Hamiltonians with interacting terms like Ising models or perturbed Hamiltonians. We hope these ideas will be useful for further developments and other researches.

I.S.G. acknowledges support from the Department of Exact and Natural Sciences of the State University of Southwest Bahia (UESB), Itapetinga, Bahia, Brazil.

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