Anomalous scaling in kinematic magnetohydrodynamic turbulence: Two-loop anomalous dimensions of leading composite operators

E. Jurčišinová, M. Jurčišin, and R. Remecký
Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 040 01 Košice, Slovakia

(Received 2 November 2022; accepted 7 February 2023; published 21 February 2023)

Using the field theoretic formulation of the kinematic magnetohydrodynamic turbulence, the explicit expressions for the anomalous dimensions of leading composite operators, which govern the inertial-range scaling properties of correlation functions of the weak magnetic field passively advected by the electrically conductive turbulent environment driven by the Navier-Stokes velocity field, are derived and analyzed in the second order of the corresponding perturbation expansion (in the two-loop approximation). Their properties are compared to the properties of the same anomalous dimensions obtained in the framework of the Kazantsev-Kraichnan model of the kinematic magnetohydrodynamics with the Gaussian statistics of the turbulent velocity field as well as to the analogous anomalous dimensions of the leading composite operators in the problem of the passive scalar advection by the Gaussian (the Kraichnan model) and non-Gaussian (driven by the Navier-Stokes equation) turbulent velocity field. It is shown that, regardless of the Gaussian or non-Gaussian statistics of the turbulent velocity field, the two-loop corrections to the leading anomalous dimensions are much more important in the case of the problem of the passive advection of the vector (magnetic) field than in the case of the problem of the passive advection of scalar fields. At the same time, it is also shown that, in phenomenologically the most interesting case with three spatial dimensions, higher velocity correlations of the turbulent environment given by the Navier-Stokes velocity field play a rather limited role in the anomalous scaling of passive scalar as well as passive vector quantities, i.e., that the two-loop corrections to the corresponding leading anomalous dimensions are rather close to those obtained in the framework of the Gaussian models, especially as for the problem of scalar field advection. On the other hand, the role of the non-Gaussian statistics of the turbulent velocity field becomes dominant for higher spatial dimensions in the case of the kinematic magnetohydrodynamic turbulence but remains negligible in the problem of the passive scalar advection.

DOI: 10.1103/PhysRevE.107.025106

I. INTRODUCTION

The presence of the anomalous scaling in fully developed turbulent environments [1–12], i.e., the existence of deviations from the scaling behavior predicted by simple dimensional analysis in the framework of the classical phenomenological Kolmogorov-Obukhov (KO) theory [13], belongs among most typical basic features of turbulent systems, the ultimate theoretical understanding of which on the fundamental level of well-defined microscopic models still remains an open problem. Let us recall that the KO theory is based on two fundamental assumptions formulated in two well-known hypotheses about the statistical properties of various random quantities (such as, e.g., correlation functions of turbulent velocity field) deep inside the so-called inertial interval $l \ll r \ll L$, where l represents the so-called dissipation scale (the scale where energy starts to dissipate intensively) and L is the so-called integral scale, i.e., a typical large scale at which the energy is pumped into the system in order to maintain the steady state. According to the KO theory the statistical properties of random quantities deep inside the inertial interval are independent of L (the assertion of the first Kolmogorov hypothesis) as well as of l (the subject of the second Kolmogorov hypothesis). For example, the assumption of validity of these two Kolmogorov hypothesis directly leads to the prediction of simple scale-invariant inertial-range behavior of phenomenologically interesting single-time two-point structure functions of the turbulent velocity field

$$S_N(r) = \langle [v_r(t,\mathbf{x}) - v_r(t,\mathbf{x}')]^N \rangle, \quad r = |\mathbf{x}-\mathbf{x}'| \tag{1}$$

in the following form:

$$S_N(r) = \text{const} \times (\bar{\epsilon} r)^{N/3}, \tag{2}$$

where v_r denotes the component of the velocity field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $\bar{\epsilon}$ is the mean dissipation rate. However, as real experiments, numerical simulations, and theoretical investigations show (see, e.g. Refs. [1–12, 14–18] as well as references cited therein), the scaling properties of turbulent systems remain dependent on the integral scale L even deep inside the inertial interval in contradiction with the first Kolmogorov hypothesis. This behavior is known as the anomalous or nondimensional scaling, and, as a result, e.g., the aforementioned structure functions of the velocity field (1) must be written in the following more general scaling form:

$$S_N(r) = (\bar{\epsilon} r)^{N/3} R_N(r/L), \tag{3}$$

with the explicit presence of the corresponding scaling function R_N for given value of N. At the same time, it is quite clear that the scaling functions R_N must depend singularly on dimensionless parameter r/L in the limit $L \to \infty$ to give nontrivial correction to the inertial-range scaling exponents of the structure functions S_N. In this respect, it is usually supposed that the asymptotic behavior of the scaling functions R_N for $r \ll L$ can be represented in the following power-like form:

$$R_N(r/L) \sim (r/L)^{q_N}, \tag{4}$$

where it is evident that the presence of the anomalous scaling of the structure functions (1), i.e., the deviation from the scaling behavior predicted by the simple KO theory with simple exponent $N/3$ in Eq. (2), requires negative values of the exponents q_N in the representation (4) of the scaling functions R_N. Physically, such a behavior of turbulent systems, i.e., the presence of the anomalous scaling, is usually explained by the existence of strong developed fluctuations of the dissipative rate, i.e., by the intermittency [1–4,6,11]. From a geometrical point of view, it means that the turbulent flows have fractal nature, i.e., that not the whole volume of the turbulent environment is filled by vortices (such a situation in fact corresponds to the pure dimensional scaling in the KO theory) but that there always exist dynamically changing places with pure laminar flows at all scales of the inertial range.

At the same time, it is also well known that the anomalous scaling is even more strongly pronounced in the inertia-range scaling behavior of various correlation functions of scalar or vector quantities passively advected by turbulent environments than in the behavior of turbulent velocity fields themselves [4,6,9,19–35]. In this respect, for a long time the central role has been played by simple models of passive advection of scalar (e.g., the temperature field or field of impurity) or vector (e.g., the weak magnetic field in an electrically conductive turbulent environment) fields by turbulent velocity fields with significantly simplified statistics, namely, by the so-called Kraichnan model [36] and the Kazantsev-Kraichnan model [37], in the framework of which scalar and vector fields are advected by turbulent velocity fields that obey Gaussian statistics. Note that, namely, in the framework of the Kraichnan model of the passive scalar advection a systematic theoretical analysis of anomalous scaling in turbulent systems was performed for the first time using the so-called zero-mode technique (see Ref. [6] as well as references cited therein).

However, although the Kraichnan model and the Kazantsev-Kraichnan model describe many features of the anomalous scaling of passively advected scalar and vector quantities in genuine turbulent systems, nevertheless it is desirable to have a fundamental description of the anomalous scaling of various passively advected quantities in more realistic turbulent environments driven by the stochastic Navier-Stokes equation. In this respect, the field theoretic renormalization group (RG) technique is invaluable since it allows systematic perturbative investigation of anomalous scaling using the operator product expansion (OPE) [38–40]. Moreover, the field theoretic approach also allows one to investigate systematically the influence of various symmetry breaking on the inertial-range anomalous scaling properties of random quantities in the Gaussian as well as non-Gaussian turbulent environments [9,31,41–59].

All these field theoretic investigations have also shown that although the presence of the anomalous scaling in turbulent systems is always visible already in the framework of the first-order approximation (the one-loop approximation in the field theoretic language), nevertheless many important features as well as differences in the anomalous scaling of various turbulent systems are invisible at this simplest level of perturbative approximation. Therefore, at least a two-loop level of approximation is usually needed for the correct description as well as deeper understanding of various peculiarities of anomalous scaling in different turbulent systems. For example, it is known that at the one-loop level of approximation the anomalous dimensions of the leading composite operators in the corresponding OPEs (that drive the scaling properties of correlation functions of the corresponding passive quantities) are completely the same for the Kraichnan model of the Gaussian passive scalar advection [41] and for the Kazantsev-Kraichnan model of the Gaussian passive vector advection [31], as well as for the advection of passive scalar and vector fields by the turbulent velocity field driven by the stochastic Navier-Stokes equation [47,49]. On the other hand, the two-loop calculations of the corresponding leading anomalous dimensions in the Kraichnan model [41], in the Kazantsev-Kraichnan model [53,54], and in the model of passive scalar advection in the Navier-Stokes turbulence [47] have shown that this universality of the leading anomalous dimensions is only an artifact of the one-loop approximation. First of all, it was shown in Ref. [47] that the two-loop corrections to the leading anomalous dimensions of passive scalar field remain rather restricted even in the case when the Navier-Stokes turbulent velocity field is considered. At the same time, it was shown in Refs. [53,54] that, at least in the case of the Kazantsev-Kraichnan model with the Gaussian statistics of the turbulent velocity field, the internal vector structure of a passively advected quantity [e.g., the weak magnetic field in the magnetohydrodynamic (MHD) turbulence] has significant impact on the anomalous scaling, namely, that the anomalous scaling is much more pronounced in the case of the passive vector advection than in the case of the passive scalar advection. Here the open question is whether this behavior is not related only to the Gaussian statistics of the velocity field in the Kazantsev-Kraichnan model. Of course, to find an answer to this question, it is necessary to perform the two-loop calculations in the framework of the corresponding model of the passively advected vector field by the Navier-Stokes turbulence, i.e., in the framework of the kinematic MHD turbulence.

In this respect, the aim of the present study is threefold. First of all, our aim is to calculate and present explicit analytic two-loop expressions (as functions of the spatial dimension) for the anomalous dimensions of the leading composite operators, which drive the asymptotic scaling behavior of various single-time two-point correlation functions of the weak magnetic field in the genuine kinematic MHD turbulence. The second aim is to compare the obtained results to the corresponding two-loop leading anomalous dimensions obtained in the framework of the Kazantsev-Kraichnann model of the kinematic MHD turbulence [53,54], where the turbulent velocity field is supposed to be Gaussian, to analyze in detail the importance of the presence of higher-order correlations of the turbulent velocity field (represented by the turbulent velocity field driven by the stochastic Navier-Stokes equation in the genuine kinematic MHD turbulence) for the values of aforementioned leading anomalous dimensions.

Finally, the third aim is to compare the obtained results for the two-loop leading anomalous dimensions to the analogous anomalous dimensions of the corresponding leading composite operators that drive the inertial-range scaling properties of various single-time two-point correlation functions of passively advected scalar quantities (fields) in the framework of the Kraichnan model [41] as well as in the framework of the model with advection by the Navier-Stokes turbulent velocity field [47] to understand the role and importance of the internal tensor structure of the advected quantity (scalar or vector) for the existence and intensity of the anomalous scaling of the corresponding correlation functions.

Moreover, having explicit expressions for the leading anomalous dimensions of all four models as functions of the spatial dimension, we will also investigate in detail theoretically interesting differences in properties of their leading anomalous dimensions in higher spatial dimensions related to the different tensor nature (scalar or vector) of the advected fields.

As will be shown, the internal vector nature of the magnetic field in the Gaussian (the Kazantsev-Kraichnan model) as well as non-Gaussian (with the Navier-Stokes turbulent velocity field) model of the kinematic MHD turbulence has a significant impact on the anomalous scaling properties of magnetic field correlation functions since the corresponding leading anomalous dimensions behave radically in a different way than the leading anomalous dimensions in the analogous Gaussian (the Kraichnan model) and non-Gaussian (the velocity field driven by the stochastic Navier-Stokes equation) models of the passive scalar advection. We will also show that, at least at the studied two-loop level of approximation, the higher correlations of the turbulent velocity field (represented by the velocity field driven by the stochastic Navier-Stokes equation) play a rather restricted role in the anomalous scaling in the most interesting three-dimensional turbulent systems. Nevertheless, it seems that higher correlations of the turbulent velocity field must play much a more important role in the case of passive advection of a vector (magnetic) field than in the case of the problem of passive scalar advection since, as we will see, the dependence of the leading anomalous dimensions in the two studied models of kinematic MHD turbulence on the spatial dimension is radically different than in the models of passive scalar advection, where the value of the spatial dimension plays rather limited role.

The paper is organized as follows. In Sec. II the model of the kinematic MHD turbulence is defined. In Sec. III the field theoretic formulation of the model is given and basic facts of its ultraviolet (UV) renormalization in the two-loop approximation are presented. In Sec. IV the two-loop explicit analytic expressions for the anomalous dimensions of leading composite operators of the studied model are determined. In Sec. V the behavior of the leading anomalous dimensions of the studied model is analyzed and compared to other three models of passive scalar and vector advection. Obtained results are briefly reviewed and discussed in Sec. VI.

II. THE KINEMATIC MHD TURBULENCE

As was discussed in the Introduction, we are intending to calculate and investigate in detail the properties of the anomalous dimensions of the leading composite operators that drive the anomalous scaling of the single-time two-point correlation functions of the magnetic field $\mathbf{b} \equiv \mathbf{b}(x)$ $[x \equiv (t, \mathbf{x})]$ in the MHD turbulence in the framework of the so-called kinematic limit with the weak magnetic field when the influence of the magnetic field on the turbulent velocity field of the electrically conductive environment can be neglected, i.e., when the magnetic field behaves as a passively advected solenoidal $(\partial \cdot \mathbf{b} = 0)$ vector field.

In the framework of the incompressible kinematic MHD turbulence, the behavior of the magnetic field and of the solenoidal (owing to the incompressibility) velocity field $\mathbf{v} \equiv \mathbf{v}(x)$ $(\partial \cdot \mathbf{v} = 0)$ is driven by the following system of stochastic equations

$$\partial_t \mathbf{b} = \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \partial)\mathbf{b} + (\mathbf{b} \cdot \partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \tag{5}$$

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v} \cdot \partial)\mathbf{v} - \partial P + \mathbf{f}^{\mathbf{v}}, \tag{6}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is the viscosity (the subscript 0 will always denote bare parameters of the unrenormalized theory), $\nu_0 u_0 = c^2/(4\pi\sigma)$ represents the magnetic diffusivity (where the dimensionless reciprocal magnetic Prandtl number u_0 is extracted for convenience), c is the speed of light, σ is the conductivity, and $P \equiv P(x)$ is the pressure.

The energy pumping into the dissipative stochastic system described by Eqs. (5) and (6) is realized through transverse random noises $\mathbf{f}^{\mathbf{b}}$ and $\mathbf{f}^{\mathbf{v}}$, the statistics of which is supposed to be Gaussian. The transverse random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ represents the source of fluctuations of the magnetic field $\mathbf{b}$ (the magnetic energy pumping) to maintain the steady state of the system. Its Gaussian statistics is assumed in the following form of the correlation function:

$$D^b_{ij}(x_1; x_2) \equiv \langle f^b_i(x_1) f^b_j(x_2)\rangle = \delta(t_1 - t_2) C_{ij}(\mathbf{r}/L), \tag{7}$$

where $\mathbf{r} = \mathbf{x_1} - \mathbf{x_2}$, L is an integral (large) scale related to the corresponding stirring, and C_{ij} is a tensor function that rapidly decreases for $|\mathbf{r}| \gg L$ and must be finite in the limit $L \to \infty$. At the same time, in what follows, the detailed form of the function C_{ij} is not important. However, let us note that through the form of the function C_{ij} a large-scale anisotropy can be introduced into the system (see, e.g., Ref. [31]).

On the other hand, the kinetic energy pumping into the system from large scales is realized through the random force $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ in Eq. (6). Its Gaussian statistics is defined through the correlation function

$$D^v_{ij}(x_1; x_2) \equiv \langle f^v_i(x_1) f^v_j(x_2)\rangle = \delta(t_1 - t_2) \int \frac{d^d\mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{8}$$

where d denotes the spatial dimension of the system, $\mathbf{k}$ is the wave-number vector (momentum), $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the ordinary isotropic transverse projector, and $D_0 \equiv g_0 \nu_0^3 > 0$ is the positive amplitude. The physical value of formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. Note that the parameter ε plays an analogous role as the parameter $\epsilon = 4 - d$ in the theory of critical behavior and the introduced parameter g_0 plays the role of the coupling constant of the model (a formal small parameter of the ordinary perturbation theory). The coupling constant g_0 is related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by the following relation:

$$g_0 \simeq \Lambda^{2\varepsilon}. \tag{9}$$

The needed infrared finiteness of the correlation function (8) (its infrared regularization) is realized by the restriction of the integration from below, i.e., it is suppose that $|\mathbf{k}| \geqslant m$, where m represents another integral scale. In what follows, it is supposed that $L \gg 1/m$.

For completeness, note that the correlation function (8) is chosen in the form which, on the one hand, describes the real infrared energy pumping into the system since the function $D_0 k^{4-d-2\varepsilon}$ is proportional to $\delta(\mathbf{k})$ when $\varepsilon \to 2$ for appropriate choice of the amplitude factor D_0 (it corresponds to the injection of energy to the system through interaction with the largest turbulent eddies), and, on the other hand, its power-like form gives the possibility to apply the RG technique for analysis of the problem [5,40,60].

Finally, let us also note that the stochastic model of kinematic MHD turbulence given in Eqs. (5)–(8) represents a simplification of real MHD turbulence problem since the so-called Lorentz force term is absent in Eq. (6) (the Navier-Stokes equation), i.e., there is no influence (feedback) of the magnetic field on the behavior of the velocity field of the conductive fluid. Therefore, as was already mentioned, the magnetic field $\mathbf{b}$ in the present model behaves like a passively advected vector field.

III. FIELD THEORETIC FORMULATION OF THE KINEMATIC MHD TURBULENCE AND THE MAIN RESULTS OF ITS TWO-LOOP RENORMALIZATION GROUP ANALYSIS

The stochastic model given by Eqs. (5)–(8) can be rewritten into the corresponding field theoretic model of the double set of fields $\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\}$ [61] described by the action functional

$$\begin{aligned} S(\Phi) = {} & \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\big[v_i'(x_1)D_{ij}^{v}(x_1;x_2)v_j'(x_2) \\ & + b_i'(x_1)D_{ij}^{b}(x_1;x_2)b_j'(x_2)\big] \\ & + \int dt\, d^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} \\ & + \mathbf{b}'[-\partial_t\mathbf{b} + \nu_0 u_0\triangle\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}]\}, \end{aligned} \tag{10}$$

where $\mathbf{v}'$ and $\mathbf{b}'$ are auxiliary transverse fields which have the same tensor properties as fields $\mathbf{v}(x)$ and $\mathbf{b}(x)$, D_{ij}^b and D_{ij}^v are correlation functions given in Eqs. (7) and (8), respectively, and summations over dummy indices are tacitly assumed.

Note that, due to the assumption of transversality of the auxiliary vector field $\mathbf{v}'(x)$, the pressure ∂P in Eq. (6) is omitted in action (10) since it vanishes when the corresponding integration by parts is performed:

$$\int dt\, d^d\mathbf{x}\;\; v_i'\partial_i P = -\int dt\, d^d\mathbf{x}\;\; P\,\partial_i v_i' = 0.$$

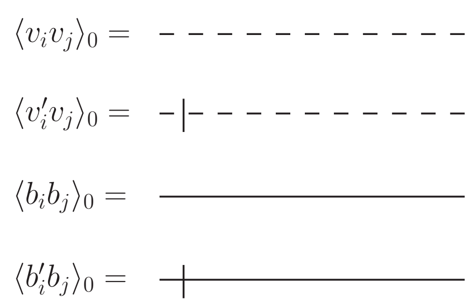

FIG. 1. Graphical representation of the propagators (11)–(14) of the model.

The field theoretic model (10) can be investigated perturbatively using the standard Feynman diagrammatic technique, in the framework of which we have the following set of four bare propagators [52]:

$$\langle v_i v_j\rangle_0 = \frac{g_0\nu_0^3 k^{4-d-2\varepsilon}P_{ij}(\mathbf{k})}{(-i\omega + \nu_0 k^2)(i\omega + \nu_0 k^2)}, \tag{11}$$

$$\langle v_i' v_j\rangle_0 = \langle v_i v_j'\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 k^2}, \tag{12}$$

$$\langle b_i b_j\rangle_0 = \frac{C_{ij}(\mathbf{k})}{(-i\omega + \nu_0 u_0 k^2)(i\omega + \nu_0 u_0 k^2)}, \tag{13}$$

$$\langle b_i' b_j\rangle_0 = \langle b_i b_j'\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 u_0 k^2}, \tag{14}$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of function $C_{ij}(\mathbf{r}/L)$ in Eq. (7). Since, in what follows, we will need their graphical representation, therefore it is shown explicitly in Fig. 1 (the end with a slash in the propagators $\langle b_i' b_j\rangle_0$ and $\langle v_i' v_j\rangle_0$ corresponds to the field $\mathbf{b}'$ and $\mathbf{v}'$, respectively, and the end without a slash corresponds to the field $\mathbf{b}$ and $\mathbf{v}$, respectively).

On the other hand, the model contains two interaction vertices (triple vertices) of the following analytic form: $b_i'(-v_j\partial_j b_i + b_j\partial_j v_i) = b_i' v_j V_{ijl} b_l$ and $-v_i' v_j \partial_j v_i = v_i' v_j U_{ijl} v_l/2$, where, in the momentum-frequency representation (in which all calculations are performed), $V_{ijl} = i(k_j\delta_{il} - k_l\delta_{ij})$ and $U_{ijl} = i(k_l\delta_{ij} + k_j\delta_{il})$. Their graphical representation is shown in Fig. 2, where the momentum $\mathbf{k}$ is flowing into the vertices via auxiliary fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively.

Note that, in the field theoretic formulation, the statistical averages of random quantities in the stochastic problem described by Eqs. (5)–(8) are replaced by the corresponding functional averages with weight $\exp S(\Phi)$ (see, e.g., Ref. [40] for details). At the same time, the main advantage of the field theoretic formulation of the studied stochastic problem

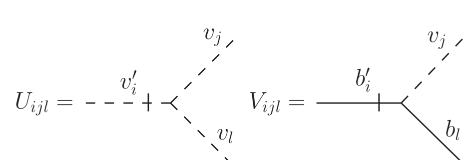

FIG. 2. The interaction vertices U and V of the model.

is given by the fact that the well-defined field theoretic means (such as the RG technique) can be used for its analysis.

Since the detailed two-loop RG analysis of the field-theoretic model (10) was already performed in Ref. [52], where the corresponding two-loop value of the turbulent magnetic Prandtl number was calculated, therefore, there is no need to repeat it here. In what follows, however, we present the main results of this analysis, which are important for the determination of the anomalous dimensions of the leading composite operators that drive the inertial-range scaling properties of correlation functions of the magnetic field.

As was shown in Ref. [52], the field theoretic model described by the action (10) is multiplicatively renormalizable and, in the framework of the minimal subtraction scheme [39], all UV divergences in correlation functions of the model have the form of poles in ε. The RG analysis performed in Ref. [52] has shown that the scaling properties of the studied stochastic system deep inside the inertial interval is driven by the infrared (IR) stable fixed point of the RG equations, the coordinates g_* and u_* of which have the following form in the two-loop approximation (the second-order approximation in the corresponding perturbation expansion):

$$g_* = g_*^{(1)}\varepsilon + g_*^{(2)}\varepsilon^2 + O(\varepsilon^3), \quad (15)$$

$$u_* = u_*^{(1)} + u_*^{(2)}\varepsilon + O(\varepsilon^2), \quad (16)$$

where the one-loop corrections ($g_*^{(1)}$ and $u_*^{(1)}$) and the two-loop corrections ($g_*^{(2)}$ and $u_*^{(2)}$) are given as follows:

$$g_*^{(1)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}, \quad (17)$$

$$g_*^{(2)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}\lambda, \quad (18)$$

$$u_*^{(1)} = \frac{1}{2}\left(-1 + \sqrt{\frac{9d+16}{d}}\right), \quad (19)$$

$$u_*^{(2)} = \frac{2(d+2)}{d[1+2u_*^{(1)}]}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right], \quad (20)$$

where $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the d-dimensional unit sphere, $\Gamma(x)$ is Euler's Gamma function, and the explicit expressions of the functions λ and $\mathcal{B}(u_*^{(1)})$ can be found in Refs. [62] and [52], respectively. Note that this fixed point is IR stable when $\varepsilon > 0$. Note also that, for clarity, in all expressions we preserve completely the same notation used in Refs. [52] and [62].

Existence of the stable IR fixed point means that the correlation functions of the model exhibit scaling behavior deep inside the inertial range with given critical dimensions. The issue of interest are usually various multiplicatively renormalizable equal-time two-point quantities $G(r)$. Among such quantities belong, e.g., various equal-time structure functions a general definition of which is

$$S_N(r) = \langle[\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^N\rangle, \quad r = |\mathbf{x}-\mathbf{x}'|, \quad (21)$$

where N is the order of the structure function and θ can represent a given component of the velocity field [see Eq. (1) in the Introduction], or of the advected scalar field, etc. In the case of the studied model of the passively advected magnetic field in the framework of the kinematic MHD turbulence, the important structure functions are defined as follows:

$$S_N(r) = \langle[b_r(t,\mathbf{x}) - b_r(t,\mathbf{x}')]^N\rangle, \quad r = |\mathbf{x}-\mathbf{x}'|, \quad (22)$$

where b_r denotes the component of the magnetic field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. However, it is also quite evident that the structure functions (22) represent a definite linear combinations of simpler quantities, namely, of the single-time two-point correlation functions [31,33]

$$B_{N-m,m}(r) \equiv \langle b_r^{N-m}(t,\mathbf{x}) b_r^m(t,\mathbf{x}')\rangle, \quad r = |\mathbf{x}-\mathbf{x}'|, \quad (23)$$

built of two composite operators $b_r^{N-m}(t,\mathbf{x})$ and $b_r^m(t,\mathbf{x}')$. The existence of the IR scaling driven by the corresponding IR stable fixed point means that the correlation functions (23) exhibit the following scaling behavior in the inertial range (for all general details see, e.g., Ref. [49], where the one-loop analysis of the studied problem was performed together with the needed dimensional analysis):

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2} l^N (r/l)^{N(1-\varepsilon/3)-\gamma_{N-m}^*-\gamma_m^*} R_{N,m}(r/L), \quad (24)$$

where γ_{N-m}^* and γ_m^* are the fixed point values of the anomalous dimensions of the composite operators b_r^{N-m} and b_r^m, respectively, and the scaling functions $R_{N,m}(r/L)$ remain unknown within the standard RG analysis.

At the same time, however, the asymptotic behavior of the scaling functions $R_{N,m}(r/L)$ in the limit $r/L \to 0$, i.e., deep inside the inertial interval, can be found using the OPE technique [39], in the framework of which they can be written in the following power form:

$$R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \quad r/L \to 0, \quad (25)$$

where the summation over all possible renormalized composite operators F_i allowed by the symmetry of the problem is performed, Δ_{F_i} are their critical dimensions, and all coefficient functions $C_{F_i}(r/L)$ are regular in r/L.

From the asymptotic representation (25) of the scaling functions, it is evident that they have a nontrivial impact on the inertial-range behavior of the correlation functions (23) only when there exist the so-called dangerous composite operators in the OPE, i.e., the operators with negative critical dimensions, since only such operators give singular contributions in the limit $r/L \to 0$ (see, e.g., Ref. [31] for details) and are therefore responsible for the anomalous scaling of the corresponding correlation or structure functions. Of course, if more different types of such composite operators exist, then the leading role is played by those with the smallest values of the critical dimensions. In this respect, when full isotropy is assumed in the studied model, the leading composite operators that drive the asymptotic inertial-range behavior of the correlation functions (23) have the following simple form:

$$F_N = (\mathbf{b}\cdot\mathbf{b})^{N/2}. \quad (26)$$

On the other hand, in the more realistic situation with the presence of the large-scale anisotropy [introduced, e.g., by the following specific form of the random noise $\mathbf{f^b} \propto (\mathbf{B}\cdot\partial)\mathbf{v}$, where $\mathbf{B} = |\mathbf{B}|\mathbf{n}$ represents a constant large-scale (macroscopic) magnetic field, the source of the aforementioned uniaxial large-scale anisotropy described by the unit vector

n], the central role is played by the following set of composite operators (see, e.g., Refs. [31,49]):

$$F_{N,p} = (\mathbf{n}\cdot\mathbf{b})^p(\mathbf{b}\cdot\mathbf{b})^l, \quad N = 2l + p. \tag{27}$$

Using all these facts, it can be shown [31,49] that the final asymptotic behavior of the correlation functions (23) deep inside the inertial interval has the following form:

$$B_{N-m,m}(r) \sim r^{-\gamma^*_{N-m}-\gamma^*_m+\gamma^*_N}, \tag{28}$$

where γ^*_X for $X = N - m, m$, and N are the corresponding fixed point values of the anomalous dimensions $\gamma_{X,p}$ of the composite operators $F_{X,p}$ for such value of p, for which $\gamma^*_{X,p}$ is minimal for given X.

Thus, to describe the scaling properties of the correlation functions (23), it is necessary to calculate the fixed-point values of the anomalous dimensions $\gamma_{N,p}$ of the composite operators (27). The corresponding one-loop calculations were performed in Ref. [49], and their two-loop expressions as functions of the spatial dimension $d > 2$ are calculated in the framework of the present study and are given in the next section. Moreover, in Sec. V, they will be also compared to the corresponding two-loop anomalous dimensions of the same leading composite operators (27) calculated in the framework of the Kazantsev-Kraichnan model of the kinematic MHD turbulence with the Gaussian statistics of the turbulent velocity field [53,54] as well as to the anomalous dimensions of the corresponding leading composite operators that drive the anomalous scaling of the correlation and structure functions of the passively advected scalar field in the Kraichnan model with the Gaussian statistics of the velocity field [41] and in the case when the scalar field is advected by the Navier-Stokes velocity field [47]. For completeness, let us note that, in the case of the passive scalar advection by the Gaussian or non-Gaussian turbulent velocity field, the leading composite operators that drive the scaling properties of various single-time two-point correlation functions of the advected scalar field ϕ in the presence of the large-scale uniaxial anisotropy have the form $(\mathbf{n}\cdot\partial\phi)^p(\partial\phi\cdot\partial\phi)^l$, i.e., they are built of gradients of the advected scalar field [41,47].

IV. TWO-LOOP EXPRESSIONS FOR THE ANOMALOUS DIMENSIONS OF THE COMPOSITE OPERATORS $F_{N,p}$

As was discussed in the previous section, the central role in the analysis of the inertial-range scaling properties of various correlation functions of the magnetic field in the framework of the kinematic MHD turbulence with the presence of the uniaxial large-scale anisotropy is played by the anomalous dimensions of the composite operators (27). These composite operators have two important features that significantly simplify the procedure of their renormalization (see, e.g., Ref. [49] for more details). The first of them is that the operators of different orders, i.e., with different values of N, are not mixed during the renormalization. The second important feature is that the corresponding matrix of renormalization constants $Z_{[N,p][N,p']}$ for a given value of N is triangular when the large-scale anisotropy of the turbulent environment is considered. It means that no diagonalization of the matrix of renormalization constants is needed and the anomalous dimensions $\gamma_{N,p}$ of operators $F_{N,p}$ are directly given by the diagonal elements of $Z_{[N,p][N,p']}$, i.e., by elements $Z_{N,p} \equiv Z_{[N,p][N,p]}$, through the standard relation

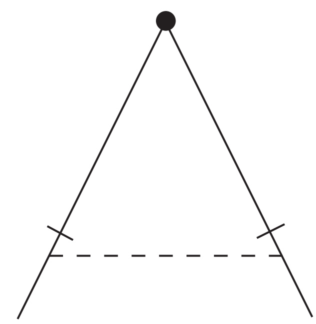

FIG. 3. Graphical representation of the one-loop contribution $\Gamma^{(1)}_{N,p}$ to the function $\Gamma_{N,p}(x;\mathbf{b})$ [see Eq. (32)].

$$\gamma_{N,p} = \mu\partial_\mu \ln Z_{N,p}, \tag{29}$$

where μ is the renormalization mass (a scale setting parameter), an artifact of the dimensional regularization of the model [49]. Thus, to find the two-loop expressions for the anomalous dimensions $\gamma_{N,p}$ it is necessary to calculate the corresponding two-loop renormalization constants $Z_{N,p}$ that connect the renormalized (finite) and the unrenormalized (infinite) composite operators:

$$F_{N,p} = Z_{N,p}F^R_{N,p}. \tag{30}$$

The renormalization constants $Z_{N,p}$ for the composite operators $F_{N,p}$ are determined analyzing the Nth-order term with respect to the magnetic field **b** of the expansion of the generating functional of one-irreducible Green's functions with the presence of one composite operator $F_{N,p}$ and any number of fields **b** (see, e.g., Ref. [31] for all details). It is given as follows:

$$\Gamma_{N,p}(x;\mathbf{b}) = \frac{1}{n!}\int dx_1\cdots\int dx_n\, b_{i_1}(x_1)\dots b_{i_N}(x_N) \times\langle F_{N,p}(x)\, b_{i_1}(x_1)\dots b_{i_N}(x_N)\rangle_{1-ir}. \tag{31}$$

Then, in the two-loop approximation, it can be written in the following series:

$$\Gamma_{N,p} = F_{N,p} + \Gamma^{(1)}_{N,p} + \Gamma^{(2)}_{N,p} + \cdots, \tag{32}$$

where $\Gamma^{(1)}_{N,p}$ and $\Gamma^{(2)}_{N,p}$ represent the corresponding one-loop and the two-loop contributions, which are determined by the calculation of the corresponding Feynman diagrams shown explicitly in Figs. 3 and 4, respectively (note that each diagram must be taken with the corresponding symmetry constant). The standard propagators and vertices of the model are given in Sec. II (see Figs. 1 and 2), and the black circle in each diagram represents vertex related to the composite operator $F_{N,p}$ defined as follows:

$$V_{i_1,\dots,i_k}(x;x_1,\dots,x_k) = \frac{\delta^k F_{N,p}}{\delta b_{i_1}(x_1)\dots\delta b_{i_k}(x_k)}, \tag{33}$$

where k denotes the number of attached lines.

Finally, the explicit expressions for the anomalous dimensions $\gamma_{N,p}$ are found from the renormalization constants $Z_{N,p}$ determined through the two-loop renormalization procedure

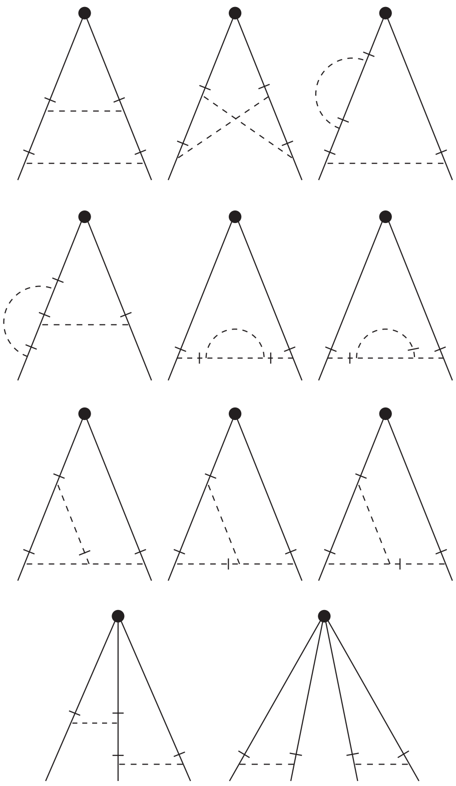

FIG. 4. The Feynman diagrams that determine the two-loop contribution $\Gamma^{(2)}_{N,p}$ to the function $\Gamma_{N,p}(x;\mathbf{b})$ [see Eq. (32)].

that removes all UV divergences presented in Feynman diagrams depicted in Figs. 3 and 4. Their final fixed-point expressions can be written in the following form:

$$\gamma^*_{N,p} = \gamma^{*(1)}_{N,p}\varepsilon + \gamma^{*(2)}_{N,p}\varepsilon^2 + O(\varepsilon^3), \tag{34}$$

where $\gamma^{*(1)}_{N,p}$ and $\gamma^{*(2)}_{N,p}$ represent the corresponding one-loop and the two-loop contributions.

The one-loop contribution $\gamma^{*(1)}_{N,p}$ was calculated in Ref. [49] and is given as follows:

$$\gamma^{*(1)}_{N,p} = \frac{2N(N-1) - (N-p)(d+N+p-2)(d+1)}{3(d+2)(d-1)}. \tag{35}$$

This one-loop result (of course, after the corresponding redefinition of the parameter ε) is completely the same as in the case of the Kazantsev-Kraichnan model of the kinematic MHD turbulence with pure Gaussian statistics of the turbulent velocity field (see, e.g., Ref. [31]). Moreover, the anomalous dimensions $\gamma^{*(1)}_{N,p}$ are also the same as the corresponding one-loop anomalous dimensions of the leading composite operators that drive the anomalous scaling of the correlation and structure functions of passively advected scalar field by the Gaussian turbulent velocity field (i.e., in the framework of the Kraichnan model) [41] as well as in the case when the scalar advection is realized in the turbulent environment driven by the stochastic Navier-Stokes equation [47]. It means that, at the one-loop level of approximation, there is no difference between the anomalous scaling properties of correlation (or structure) functions of passively advected scalar and vector quantities as well as there is no difference (at least as for these scaling properties) whether the advection is realized by the simple Gaussian turbulent velocity field or by the much more realistic turbulent velocity field driven by the stochastic Navier-Stokes equation. As was already discussed in the Introduction, this is in fact one of the main reasons why it is necessary to perform at least the second-order (two-loop) calculations that, as we will see in the next section, clearly distinguish scaling properties of all aforementioned different models.

The explicit expression for the two-loop contribution $\gamma^{*(2)}_{N,p}$ to the anomalous dimensions $\gamma^*_{N,p}$ (34) of the composite operators (27) in the kinematic MHD turbulence represents the main result of the present paper and can be written as follows:

$$\begin{aligned}\gamma^{*(2)}_{N,p} = &-\frac{128(d+2)}{9(d-1)^3}[(d+1)k_1 - 2k_2]\mathcal{B}(u^{(1)}_*)\\ &-\frac{256(d+2)}{9(d-1)^3}\frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)}\int_0^1 dx(1-x^2)^{\frac{d-1}{2}}\\ &\times\Bigg([(d+1)k_1 - 2k_2]X_1 + 2(dk_2 - k_1)(1-x^2)X_2\\ &+\frac{3}{d+4}\Bigg\{3[(d+1)k_3 - 2k_4]X_3\\ &+\frac{2x(1-x^2)[(d+2)k_4 - 3k_3]}{24u^2(u^2-1)^2}X_4\Bigg\}\Bigg),\end{aligned} \tag{36}$$

where the explicit expression for $\mathcal{B}(u^{(1)}_*)$ can be found in Ref. [52],

$$k_1 = (N-p)(d+N+p-2), \tag{37}$$

$$k_2 = N(N-1), \tag{38}$$

$$k_3 = (N-2)(N-p)(d+N+p-2), \tag{39}$$

$$k_4 = N(N-1)(N-2), \tag{40}$$

and the explicit form of the functions X_i, $i = 1, \dots, 4$ is given in the Appendix.

Before we will analyze the behavior of the fixed-point anomalous dimensions $\gamma^*_{N,p}$ of the composite operators $F_{N,p}$, it is necessary to bear in mind that, since (due to the presence of the large-scale anisotropy) there exists the corresponding set of anomalous dimensions with different values of p for given value of N, only the most negative of them is important for the determination of the anomalous scaling of various correlation functions of the magnetic field. Detailed analysis shows that, at least at the studied two-loop level of approximation, the anomalous dimensions (34) meet the following conditions:

$$\gamma^*_{N,p} < \gamma^*_{N,p'}, \quad p < p', \tag{41}$$

$$\gamma^*_{N,0} < \gamma^*_{N',0}, \quad N > N', \tag{42}$$

$$\gamma^*_{N,1} < \gamma^*_{N',1}, \quad N > N', \tag{43}$$

where relation (42) holds for even values of N and N' and relation (43) is valid for odd values of N and N', respectively. Therefore, only the anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) are important for the analysis of the scaling behavior of the correlation functions (23) since, using the hierarchy relations (41)–(43) together with the asymptotic representation (28), the final asymptotic behavior of the correlation functions (23) deep inside the inertial interval has the following form (which is valid at the one-loop level of approximation [49] as well as when the second-order corrections are taken into account):

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,0}-\gamma^*_{N-m,0}-\gamma^*_{m,0}}, \tag{44}$$

for even values of N and m,

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,0}-\gamma^*_{N-m,1}-\gamma^*_{m,1}}, \tag{45}$$

for even value of N and odd value of m, and

$$B_{N-m,m}(r) \sim r^{\gamma^*_{N,1}-\gamma^*_{N-m,0}-\gamma^*_{m,1}}, \tag{46}$$

for odd values of N and m. The fourth possibility with odd value of N and even value of m is in fact contained in the last case.

However, a detailed analysis of the scaling behavior of the correlation functions (23) will be given elsewhere. Instead, in the present paper, we will focus our attention on analysis of the properties of the leading anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N). In this respect, on the one hand, we will show the importance of the second-order corrections in comparison to the one-loop result, and, on the other hand, we will also compare the obtained two-loop values of the anomalous dimensions $\gamma^*_{N,p}$ to the corresponding anomalous dimensions obtained in the framework of the Kazantsev-Kraichnan model of the kinematic MHD turbulence [53,54] and to the corresponding anomalous dimensions of the leading composite operators that drive the scaling properties of the structure functions of the scalar field passively advected by the Gaussian velocity field in the framework of the Kraichnan model [41] as well as passively advected by the turbulent velocity field driven by the stochastic Navier-Stokes equation [47]. Note that such kind of comparison is possible since the same hierarchies (41)–(43) among the anomalous dimensions of the corresponding leading composite operators are valid in all aforementioned models.

V. COMPARISON OF ANOMALOUS DIMENSIONS $\gamma^*_{N,0}$ AND $\gamma^*_{N,1}$ OF LEADING COMPOSITE OPERATORS OF FOUR MODELS OF PASSIVE SCALAR AND VECTOR ADVECTION

Thus, as was already mentioned, our aim is, on the one hand, to show the importance of the two-loop corrections (36) to the anomalous dimensions $\gamma^*_{N,0}$ and $\gamma^*_{N,1}$ (34) of the composite operators $F_{N,p}$ defined in Eq. (27) and, on the other hand, to compare them to the analogous anomalous dimensions of the leading composite operators that drive the scaling properties of the corresponding passive quantities in the Kazantsev-Kraichnan model [53,54], in the Kraichnan model [41], and in the model of passive scalar advection by the stochastic Navier-Stokes equation [47], we seek to find answers at least to the following two fundamental questions. The first of them is: How important is different tensor structure (scalar or vector) of advected fields for the strength of the anomalous scaling, i.e., for the values of the anomalous dimensions of the leading composite operators that drive the scaling properties of various correlation functions of advected fields? The second question, the answer to which is not less important, is: How important is non-Gaussian statistics of the turbulent velocity field (represented by the turbulent velocity field driven by the stochastic Navier-Stokes equation) for the anomalous scaling of the passive magnetic field in comparison to the much simpler case when the magnetic field is advected by the Gaussian turbulent velocity field in the framework of the Kazantsev-Kraichnan model of the kinematic MHD turbulence?

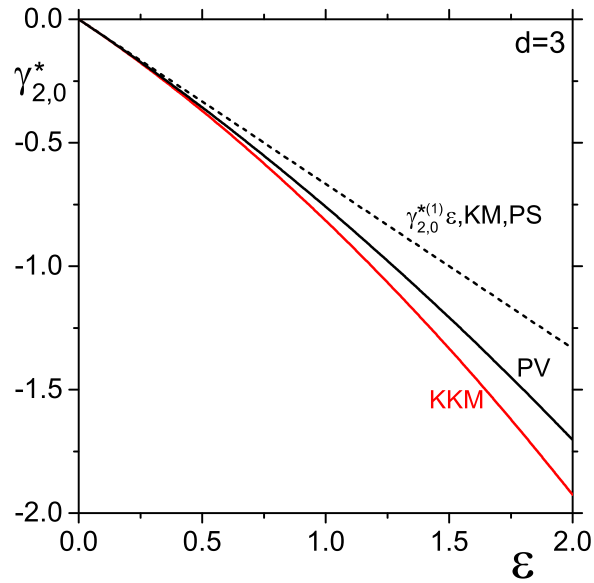

FIG. 5. Dependence of the two-loop anomalous dimensions $\gamma^*_{2,0}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the kinematic MHD turbulence studied in the present paper [the (black) curve denoted as PV], in the Kazantsev-Kraichnan model of passive vector advection [the (red) curve denoted as KKM], in the Kraichnan model of passive scalar advection [the dashed curve denoted as KM], and in the model of passive scalar advection by the Navier-Stokes velocity field [the same dashed curve denoted as PS], on the common, i.e., in the same way defined, parameter ε for spatial dimension $d = 3$. The one-loop result, which is the same for all four studied models [which is also the same as the two-loop result for the Kraichnan model (KM) as well as the two-loop result for the passive scalar advection by the Navier-Stokes turbulent velocity field (PS)], is denoted as $\gamma^{*(1)}_{2,0}\varepsilon$ (the same dashed curve).

In this respect, in Figs. 5–10, the dependence of the two-loop anomalous dimensions $\gamma^*_{N,0}$ for $N = 2, 4, 6$ and $\gamma^*_{N,1}$ for $N = 3, 5, 7$ of the corresponding leading composite operators of all four aforementioned models on the parameter ε is shown for the most interesting three-dimensional case. Here it is necessary to stress that, to be able to perform such a comparison, the corresponding redefining of the parameter ε in the Kraichnan model [41] and in the Kazantsev-Kraichnan model [53,54] is done, to have the same meaning of the parameter ε for all models with the same physical value $\varepsilon = 2$ that corresponds to the Kolmogorov scaling [63].

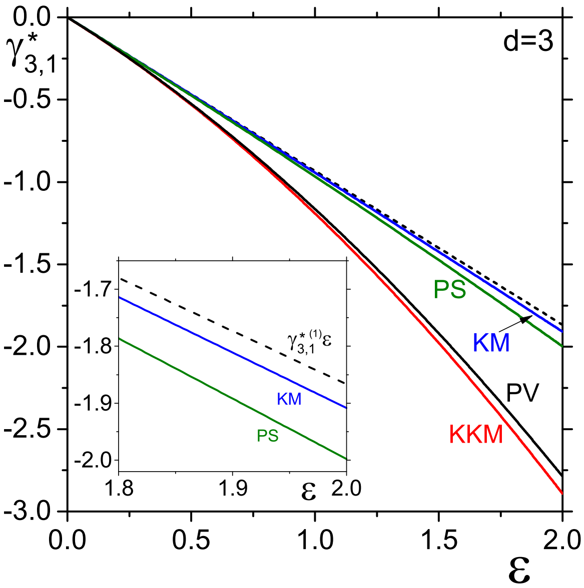

FIG. 6. Dependence of the two-loop anomalous dimensions $\gamma^*_{3,1}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the kinematic MHD turbulence studied in the present paper [the (black) curve denoted as PV], in the Kazantsev-Kraichnan model of passive vector advection [the (red) curve denoted as KKM], in the Kraichnan model of passive scalar advection [the (blue) curve denoted as KM], and in the model of passive scalar advection by the Navier-Stokes velocity field [the (green) curve denoted as PS], on the common parameter ε for spatial dimension $d = 3$. The corresponding one-loop result, which is common for all four studied models, is represented by dashed curve and is denoted as $\gamma^{*(1)}_{3,1}\varepsilon$ in the inset, which gives a detailed view near $\varepsilon = 2$.

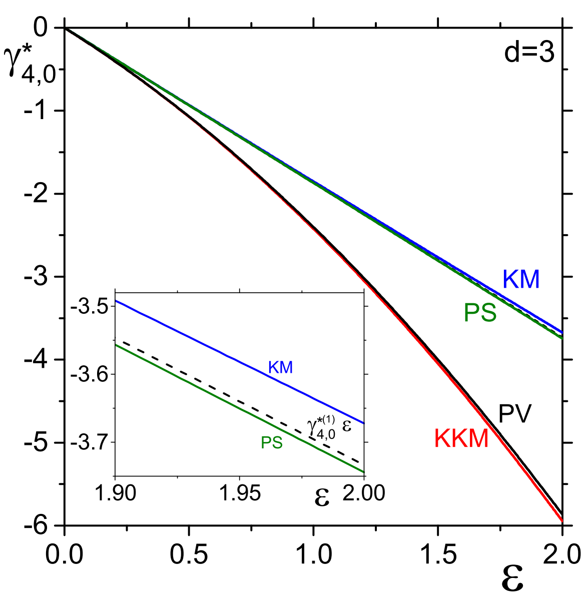

FIG. 7. Dependence of the two-loop anomalous dimensions $\gamma^*_{4,0}$ of the leading composite operators relevant in the discussed four models of passive advection on the parameter ε for spatial dimension $d = 3$ (see Fig. 6 for notation). The corresponding one-loop result, which is common for all four studied models, is represented by the dashed curve denoted as $\gamma^{*(1)}_{4,0}\varepsilon$ in the inset, which gives a detailed view near $\varepsilon = 2$.

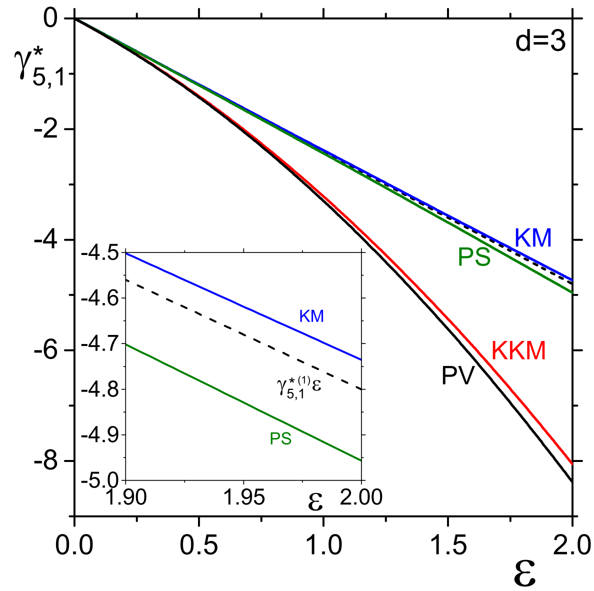

FIG. 8. Dependence of the two-loop anomalous dimensions $\gamma^*_{5,1}$ of the leading composite operators relevant in the discussed four models of passive advection on the parameter ε for spatial dimension $d = 3$ (see Fig. 6 for notation). The corresponding one-loop result, which is common for all four studied models, is represented by the dashed curve denoted as $\gamma^{*(1)}_{5,1}\varepsilon$ in the inset, which gives a detailed view near $\varepsilon = 2$.

As follows from Fig. 5, where the two-loop anomalous dimensions $\gamma^*_{2,0}$ for all four models are shown as the functions of the common parameter ε for $d = 3$, while there are no two-loop corrections to these anomalous dimensions in the case of the Gaussian model (the Kraichnan model) as well as non-Gaussian model (with the velocity field driven by the stochastic Navier-Stokes equation) of passive scalar advection (see the dashed curve in Fig. 5), the corresponding two-loop corrections in the case of the analogous models of the passive vector (weak magnetic field) advection are significant in both cases, i.e., in the case of the Gaussian advection (the Kazantsev-Kraichnan model) as well as in the case of genuine kinematic MHD turbulence. At the same time, the two-loop corrections to the anomalous dimension $\gamma^*_{2,0}$ are larger (more negative) in the case of the advection by the Gaussian turbulent velocity field, i.e., in the framework of the Kazantsev-Kraichnan model [the (red) curve denoted as KKM], than in the case of the kinematic MHD turbulence with velocity field driven by the stochastic Navier-Stokes equation [the (black) curve denoted as PV]. It means that the presence of higher velocity correlations of the turbulent conductive environment reduces the negative value of the anomalous dimension $\gamma^*_{2,0}$, i.e., as a result, their presence must also reduce the manifestation (visibility) of the anomalous scaling.

The same behavior of the leading anomalous dimensions, i.e., that the two-loop negative corrections are larger in the Gaussian model of the passive vector advection than in the non-Gaussian model, is also valid for anomalous dimensions $\gamma^*_{3,1}$ and $\gamma^*_{4,0}$ (see Figs. 6 and 7, respectively). It means that one can expect less pronounced anomalous behavior of the corresponding correlation functions of the magnetic field of

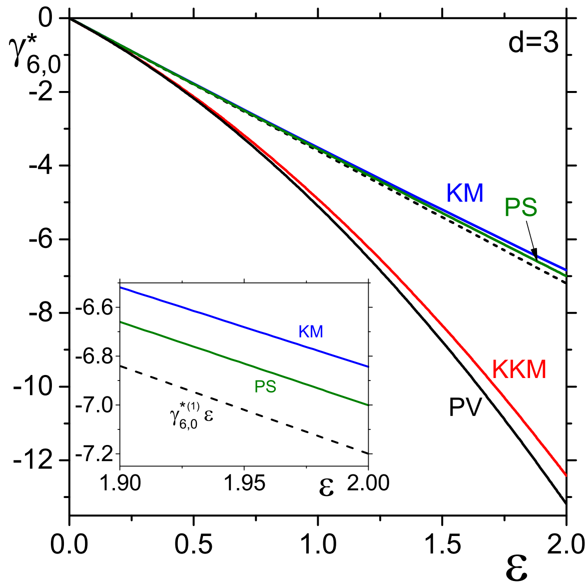

FIG. 9. Dependence of the two-loop anomalous dimensions $\gamma^*_{6,0}$ of the leading composite operators relevant in the discussed four models of passive advection on the parameter ε for spatial dimension $d=3$ (see Fig. 6 for notation). The corresponding one-loop result, which is common for all four studied models, is represented by the dashed curve denoted as $\gamma^{*(1)}_{6,0}\varepsilon$ in the inset, which gives a detailed view near $\varepsilon=2$.

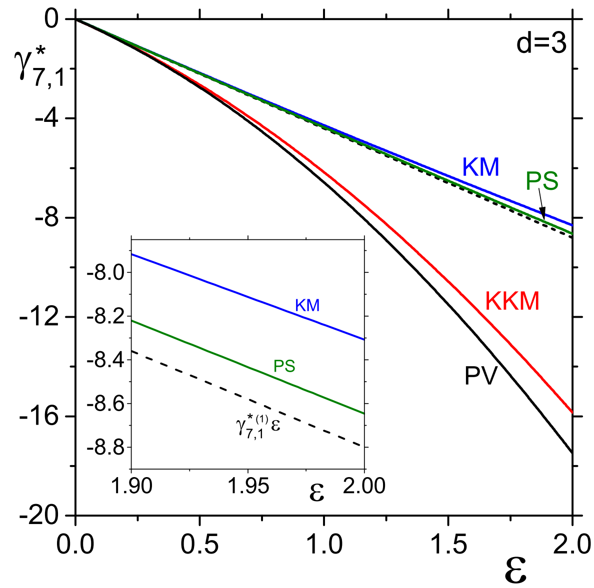

FIG. 10. Dependence of the two-loop anomalous dimensions $\gamma^*_{7,1}$ of the leading composite operators relevant in the discussed four models of passive advection on the parameter ε for spatial dimension $d=3$ (see Fig. 6 for notation). The corresponding one-loop result, which is common for all four studied models, is represented by the dashed curve denoted as $\gamma^{*(1)}_{7,1}\varepsilon$ in the inset, which gives a detailed view near $\varepsilon=2$.

the order $N=3$ and $N=4$ in the genuine kinematic MHD turbulence with the Navier-Stokes velocity field than in the framework of the simplified model of the kinematic MHD turbulence with Gaussian statistics of the velocity field (the Kazantsev-Kraichnan model).

On the other hand, the situation is opposite for higher anomalous dimensions with $N \geqslant 5$ (see Figs. 8–10, where the corresponding behavior of the anomalous dimensions $\gamma^*_{5,1}$, $\gamma^*_{6,0}$, and $\gamma^*_{7,1}$ is shown). It means that the presence of higher correlations of the turbulent velocity field described by the stochastic Navier-Stokes equation leads to more negative values of the anomalous dimensions of the composite operators of higher orders (with $N \geqslant 5$) that drive the scaling properties of the higher correlation functions (23) with $N \geqslant 5$ of the passive vector (magnetic) field through the corresponding asymptotic expressions (44)–(46).

Note also that the situation in the case of the two analogous models of passive scalar advection is different, namely, the two-loop results for anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) in the case with the Navier-Stokes velocity field are always smaller than the corresponding results for the Kraichnan model with the Gaussian turbulent velocity field (see Figs. 6–10). The only exception is the case with $N=2$, when there are no two-loop corrections at all (see Fig. 5).

As also follows from all Figs. 5–10, the two-loop corrections to the anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) are much more important in the problem of the passive vector advection than in the case of the passive scalar advection, regardless whether the turbulent velocity field is Gaussian or non-Gaussian (driven by the stochastic Navier-Stokes equation). At the same time, it is also evident that in three-dimensional turbulent environments, at least from the anomalous scaling point of view, the Gaussian or non-Gaussian character of the turbulent velocity field is much less important than the intrinsic tensor nature of the advected field (scalar or vector).

From a theoretical point of view, in order to understand deeper the difference between the scaling behavior of various correlation functions of passively advected scalar and vector fields deep inside the inertial interval of turbulent environments, it is also important to analyze and compare the dependence of the corresponding anomalous dimensions of leading composite operators in the aforementioned four models of passive advection on the value of the spatial dimension d. Such an analysis is also important for deeper theoretical understanding of the role of the non-Gaussian statistics of the turbulent velocity fields in the anomalous scaling of passively advected scalar and vector fields. In this respect, such a comparison is shown explicitly in Figs. 11–16, where the spatial dependence of the anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) of leading composite operators of four studied models is presented for the physical value $\varepsilon=2$ (let us note once more that the needed redefinition of the parameter ε in the Kraichnan model and the Kazantsev-Kraichnan model is performed).

As follows from Figs. 11–16, the spatial dimension plays much more important role in the case of passive vector advection than in the case of passive scalar advection. For example, while the two-loop anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) of the leading composite operators in the problem of passive scalar advection by turbulent velocity fields with the Gaussian (the

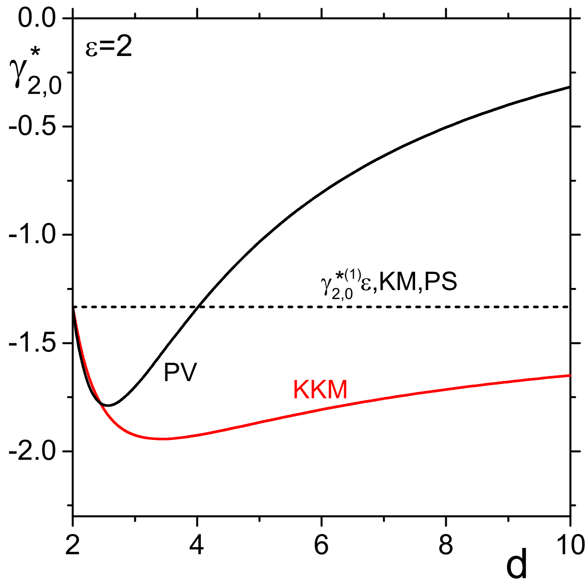

FIG. 11. Dependence of the two-loop anomalous dimensions $\gamma^*_{2,0}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 5. The one-loop result, which is the same for all four studied models [which is also the same as the two-loop result for the Kraichnan model (KM) as well as the two-loop result for the passive scalar advection by the Navier-Stokes turbulent velocity field (PS)], is denoted as $\gamma^{*(1)}_{2,0}\varepsilon$ (the same dashed curve).

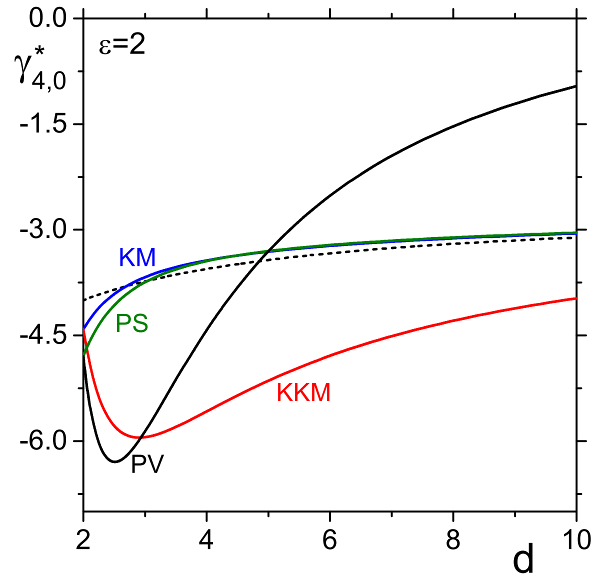

FIG. 13. Dependence of the two-loop anomalous dimensions $\gamma^*_{4,0}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 6. The corresponding one-loop result $\gamma^{*(1)}_{4,0}\varepsilon$, which is common for all four studied models, is represented by the dashed curve.

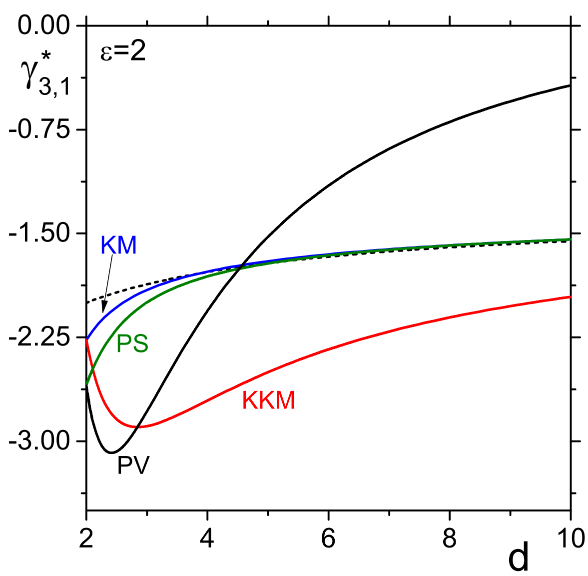

FIG. 12. Dependence of the two-loop anomalous dimensions $\gamma^*_{3,1}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 6. The corresponding one-loop result $\gamma^{*(1)}_{3,1}\varepsilon$, which is common for all four studied models, is represented by the dashed curve.

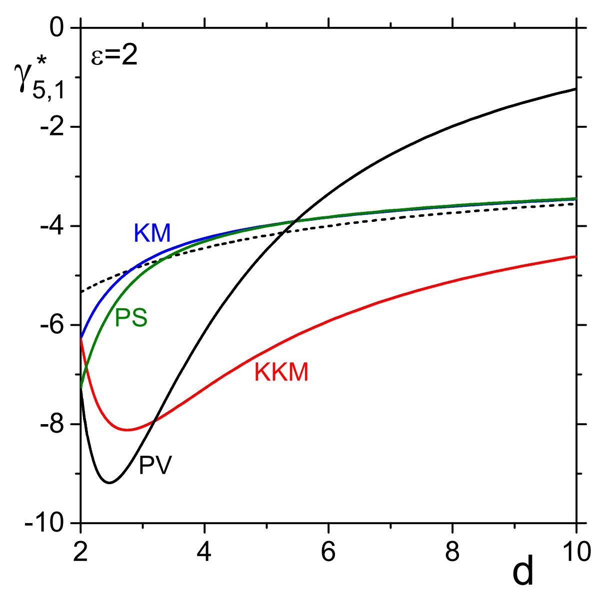

FIG. 14. Dependence of the two-loop anomalous dimensions $\gamma^*_{5,1}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 6. The corresponding one-loop result $\gamma^{*(1)}_{5,1}\varepsilon$, which is common for all four studied models, is represented by the dashed curve.

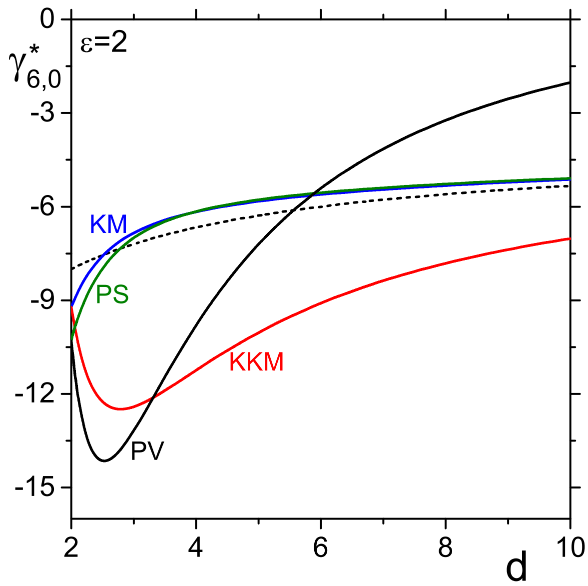

FIG. 15. Dependence of the two-loop anomalous dimensions $\gamma^*_{6,0}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 6. The corresponding one-loop result $\gamma^{*(1)}_{6,0}\varepsilon$, which is common for all four studied models, is represented by the dashed curve.

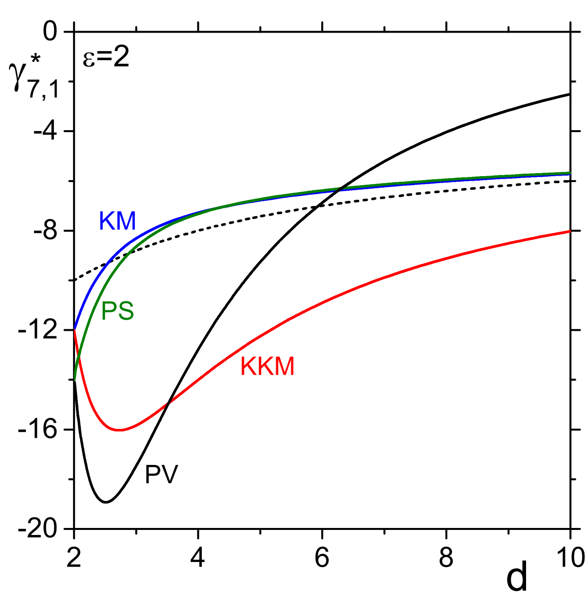

FIG. 16. Dependence of the two-loop anomalous dimensions $\gamma^*_{7,1}$ of the leading composite operators, which drive the scaling properties of the correlation functions of the corresponding passively advected quantities in the framework of four studied models, on the spatial dimension d for the common physical value of the parameter $\varepsilon = 2$. For notation see Fig. 6. The corresponding one-loop result $\gamma^{*(1)}_{7,1}\varepsilon$, which is common for all four studied models, is represented by the dashed curve.

Kraichnan model) and non-Gaussian (driven by the stochastic Navier-Stokes equation) statistics rapidly converge one to another with increasing value of the spatial dimension (see the corresponding curves denoted as KM and PS in Figs. 12–16), such a behavior is not observed in the case of the Gaussian (Kazantsev-Kraichnan model) and non-Gaussian models of the kinematic MHD turbulence (see the corresponding curves denoted as KKM and PV in Figs. 11–16). Moreover, as also follows from these figures, while the two-loop corrections to these anomalous dimensions in the problem of passive scalar advection become very small even for moderate values of the spatial dimension, the corresponding two-loop corrections remain dominant even for large values of spatial dimension in the problems of passive vector advection. These differences in the behavior of the anomalous dimensions of leading composite operators in the problems of passive scalar and vector advection show the importance of the inner tensor structure of the advected field for the inertial-range scaling properties of the corresponding correlation (or structure) functions. At the same time, this behavior of the leading anomalous dimensions also shows that one can expect that the scaling properties of the correlation functions of passively advected vector fields must be much more sensitive to the presence of higher correlations of the velocity field (here represented by the Navier-Stokes turbulent velocity field) than the analogous scaling properties of the correlation functions of passively advected scalar fields.

Note that the dependence of the two-loop anomalous dimensions of the leading composite operators on the spatial dimension (Figs. 11 and 12) clearly demonstrates the fact that they are smaller in the case of the Kazantsev-Kraichnan model than in the genuine kinematic MHD turbulence for $N = 2, 3,$ and 4 (Figs. 5–7) and larger for $N \geqslant 5$ (Figs. 8–10) in the spatial dimension $d = 3$.

Finally, let us also note that the anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) for given value of N of the corresponding leading composite operators in the Kraichan model and the Kazantsev-Kraichnan model tend to the same value in the limit $d \to 2$. The same is also true for the models of passive scalar and vector advection by the Navier-Stokes velocity field. They also tend to the same value (but different than in the case of Gaussian models) in the limit $d \to 2$. This behavior is related to the well-known fact that the magnetic field behaves as the scalar field (it has one component) in the two-dimensional case.

VI. CONCLUSION

In conclusion, let us briefly discuss the main results obtained in this paper.

In the present paper, the explicit analytic expressions for the anomalous dimensions of the leading composite operators, which are crucial for the determination of the anomalous scaling properties of the single-time two-point correlation functions of the magnetic field in the kinematic MHD turbulence, are derived using the field theoretic RG technique together with the OPE in the two-loop approximation. The anisotropy hierarchy between various anomalous dimensions of the same order taken at the fixed point is established, and it is shown that for the even values of their order N the most negative are the isotropic ones, i.e., $\gamma^*_{N,0}$, and for the odd values of N the most negative are the anomalous dimensions $\gamma^*_{N,1}$, i.e., those with the smallest presence of anisotropy, in accordance with the Kolmogorov's local isotropy restoration hypothesis. Their dependence on the parameter ε for $d = 3$ as well as on the spatial dimension d for the physical value $\varepsilon = 2$, which corresponds to the Kolmogorov scaling, is investigated for various values of N, namely, for $N = 2, \ldots, 7$. Moreover, their properties are compared to the corresponding properties of the anomalous dimensions of the same leading composite operators obtained in the framework of the simplified kinematic MHD turbulence with the Gaussian statistics of the turbulent velocity field, i.e., in the Kazantsev-Kraichnan model [53,54]. In addition, the behavior of these anomalous dimensions are also compared to the analogous anomalous dimensions of the leading composite operators that drive the inertial-range scaling properties of the correlation functions of passively advected scalar quantities by the Gaussian turbulent velocity field, i.e., in the framework of the Kraichnan model [41], as well as by the genuine turbulent velocity field driven by the Navier-Stokes equation [47].

Our analysis shows a few nontrivial facts as for the properties of the passive scalar and vector advection by turbulent environments. First, our analysis shows that for phenomenologically the most interesting spatial dimension $d = 3$ the two-loop corrections to the leading anomalous dimensions in the problem of passive magnetic field advected by the Gaussian as well as non-Gaussian turbulent velocity fields are much more important than in the analogous models of passively advected scalar field (Figs. 5–10). It means that the presence of the internal tensor structure of the passively advected field (in our case the vector structure) has a strong impact on the properties of the anomalous scaling. At the same time, it is also shown that, in the three-dimensional case, the presence of higher correlations of the turbulent velocity field, here represented by the velocity field driven by the Navier-Stokes equation, has much less impact on the anomalous scaling even in the case of the passive vector advection (for example, as follows from Fig. 7, these nonlinear corrections almost vanish for $N = 4$), where their presence can even lead to the reduction of the anomalous scaling. This can be seen in the behavior of the two-loop anomalous dimensions for $N = 2, 3$, and 4 (see Figs. 5–7). On the other hand, for higher anomalous dimensions with $N \geqslant 5$, the leading anomalous dimensions $\gamma^*_{N,0}$ or $\gamma^*_{N,1}$ become smaller in the genuine kinematic MHD turbulence than in the Kazantsev-Kraichnan model (see Figs. 8–10). Note also that the importance of the higher correlations of the velocity field is even smaller in the case of the passive scalar advection (see Figs. 5–10).

In addition, the dependence of the leading anomalous dimensions $\gamma^*_{N,0}$ (for even values of N) and $\gamma^*_{N,1}$ (for odd values of N) of the model on the spatial dimension $d > 2$ is investigated for $N = 2, \ldots, 7$ for physically the most interesting value $\varepsilon = 2$. At the same time, the corresponding comparison to the analogous results obtained in the framework of the aforementioned three other models of passive advection is performed. It is shown that the studied leading anomalous dimensions as functions of the spatial dimension behave significantly in a different way in the models of passive scalar advection in comparison to the models of the kinematic MHD turbulence (see Figs. 11–16). While the leading anomalous dimensions of the corresponding composite operators of the Gaussian model (the Kraichnan model) and the non-Gaussian model (velocity driven by the Navier-Stokes equation) of passive scalar advection are almost the same and tend to the same value with increasing d, the presence of the higher correlations of the velocity field in the genuine kinematic MHD turbulence has a strong impact on the two-loop values of the leading anomalous dimensions, which are significantly different than in the case of the Kazantsev-Kraichnan model (see Figs. 11–16).

In the end, the analysis performed in this paper can be used directly for the analysis of the inertial-range scaling properties of the single-time two-point correlation functions (23) through their asymptotic expressions (44)–(46). However, such kind of analysis as well as other questions (for instance, such as the question of the anisotropy persistence deep inside the inertial range) requires separate systematic investigation and will be performed elsewhere.

ACKNOWLEDGMENTS

This work was supported by the Slovak Research and Development Agency under the Contract No. APVV-20-0293. This work was also supported by the VEGA Grant No. 2/0081/21 of the Slovak Academy of Sciences.

APPENDIX

The explicit form of the functions $X_i, i = 1, \ldots, 4$ in Eq. (36) is the following:

$$X_1 = \frac{A_1}{B_1} + \frac{A_2 Y_1}{B_2} + \frac{A_3 Y_2}{B_3} + \frac{A_4(Y_3+Y_4)}{B_4} + \frac{A_5 Y_5}{B_5} + \frac{A_6 Y_6}{B_6} + \frac{A_7 Y_7}{B_7} + \frac{A_8 Y_8}{B_8} + \frac{A_9 Y_9}{B_9} + \frac{A_{10} Y_{10}}{B_{10}} + \frac{A_{11} Y_{11}}{B_{11}},$$

$$X_2 = \frac{A_{12} Y_1}{B_{12}} + \frac{A_{13} Y_6}{B_{13}} + \frac{u x Y_7}{B_{14}} + \frac{A_{13} Y_8}{B_{15}} + \frac{x^2 Y_{12}}{B_{16}},$$

$$X_3 = -\frac{x}{72u^2(u^2-1)^2} \times \left[\frac{(-9+5d-4x^2)Y_2}{\sqrt{4-x^2}} + \frac{A_{14}Y_6}{\sqrt{2-u(x^2-2)}} + \frac{A_{15}Y_7}{\sqrt{u^2(1-x^2)+2u+1}} + \frac{A_{16}Y_8}{\sqrt{u(x^2-2)-2}}\right],$$

$$X_4 = -\frac{Y_2}{\sqrt{4-x^2}} - \frac{u^{3/2}Y_6}{\sqrt{2-u(x^2-2)}} - \frac{u^3 Y_7}{\sqrt{u^2(1-x^2)+2u+1}} + \frac{u^{3/2}Y_8}{\sqrt{u(x^2-2)-2}},$$

where

$$A_1 = u^3[(5-2d)x^2-2] + u^2[-4(4d-7)x^4 + (10d-21)x^2 + 2] - u(4x^2-1)[8(d-1)x^4 + (3-6d)x^2 + 2] - 16(d-1)x^4 + (14d-11)x^2 - 2,$$

$$A_2 = x\{-d(u+1)[3u^4 + 2u^3(12x^2-5) + u^2(32x^4-20x^2+9) + 4u(4x^2-1) + 2] + 4u^6(x^2-1) + u^5(32x^4-40x^2+13) + u^4(64x^6-64x^4+40x^2-11) + u^3(128x^6-144x^4+116x^2-39) + u^2(144x^4-112x^2+27) + u(48x^2-22) + 4\},$$

$$A_3 = (1-x^2)\{u[-(d+5)x^2+2x^4-4] + (5d-3)x^2 - 2x^4 + 4\},$$

$$A_4 = x\{d[11u^5 + u^4(23-4x^2) + 2u^3(48x^4-62x^2+19) + u^2(32x^4+36x^2-58) + u(28x^2-17) + 3] + 2u^6 - u^5(6x^2+13) + u^4(-48x^4+42x^2-51) + u^3(32x^6-256x^4 + 264x^2-90) - 4u^2(8x^6-20x^4+48x^2-33) + u(-32x^4+30x^2+7) - 10x^2 + 13\},$$

$$A_5 = (2-d)x,$$

$$A_6 = -x([u^6(8x^6-30x^4+37x^2-18) + 2u^5(20x^6-75x^4+81x^2-35) + 2u^4(16x^6-81x^4+64x^2-10) + u^3(-64x^6+118x^4-94x^2+76) + u^2(96x^4-37x^2+30) + u(60x^2-6) + 8] \times \sqrt{u(x^2-2)-2} + \{2d[u^5(4x^4-10x^2+7) + u^4(8x^4-14x^2+9) + 2u^3(2x^4+x^2-4) - 2u^2(8x^4-7x^2+4) - 8ux^2 + u - 1] + u[u^4(-8x^4+26x^2-19) + 2u^3(4x^4+x^2-5) + 2u^2(8x^4-13x^2+8) - 2u(x^2-5) + 3]\}\sqrt{[u(x^2-2)-2]^3}),$$

$$A_7 = u^5(dx^2-4x^4+4) + u^4[12(d-2)x^4 - 2(d-6)x^2+5] + 2u^3x^2[2d(5x^2-3) - 14x^2 + 13] + u^2[-4(5d-7)x^4 - 7(2d-3)x^2 - 5] + u[-12(d-2)x^4 + (6-5d)x^2 - 4] + u^6[-(x^2-1)] + 4x^4 - 1,$$

$$A_8 = (2-d)x[u^5(4x^4-10x^2+7) + u^4(8x^4-14x^2 + 9) + 2u^3(2x^4+x^2-4) - 2u^2(8x^4-7x^2+4) - 8ux^2 + u - 1],$$

$$A_9 = d[u^5 + u^4(-16x^4+52x^2-19) + u^3(-32x^4 + 60x^2+18) + 2u^2(24x^4-10x^2+9) + u(36x^2 - 19) + 1] + u^5(6x^2-3) + u^4(48x^4-98x^2+37) - 2u^3(16x^6-64x^4+72x^2+15) + 2u^2(16x^6 - 88x^4+36x^2-19) + u(33-86x^2) - 6x^2 + 1,$$

$$A_{10} = (2-d)[u^3(2x^4-3x^2+1) + u^2(4x^4-5x^2-1) + u(-6x^4+3x^2-1) - 3x^2 + 1],$$

$$A_{11} = 2(d-2)x^4 + 2u(x^2-1)^2 + 3x^2 - 2,$$

$$A_{12} = -x[u^2 + u(4x^2-1) + 1],$$

$$A_{13} = x[1 + u(x^2-1)],$$

$$A_{14} = \sqrt{u}[4du + d - u(4x^2+7) - 2],$$

$$A_{15} = u^2[d(3u+2) - u(4x^2+5) - 4],$$

$$A_{16} = \sqrt{u}[-d(4u+1) + u(4x^2+7) + 2],$$

$$B_1 = 128u(u+1)^2x^2(x^2-1)[u^2+u(4x^2-2)+1],$$

$$B_2 = 16u^2(u+1)^2\sqrt{1-x^2}[u^2+u(4x^2-2)+1]^2,$$

$$B_3 = 32(u-1)^2(u+1)x^3\sqrt{4-x^2},$$

$$B_4 = 32(u-1)^2u(u+1)^2\sqrt{2u-x^2+2} \times [u^2+u(4x^2-2)+1]^2,$$

$$B_5 = 4u(u^2-1)^2\sqrt{u^2+2u-x^2+1},$$

$$B_6 = 16(u-1)^2u^{3/2}(u+1)^2\sqrt{2-u(x^2-2)} \times [u(x^2-2)-2]^{3/2}[u^2+u(4x^2-2)+1]^2,$$

$$B_7 = 64(u-1)^2u(u+1)^3x^3\sqrt{u^2(1-x^2)+2u+1},$$

$$B_8 = 8(u-1)^2u^{3/2}(u+1)^2\sqrt{u(x^2-2)-2} \times (u^2+u[4x^2-2)+1]^2,$$

$$B_9 = 32(u-1)^2u(u+1)^2[u^2+u(4x^2-2)+1]^2,$$

$$B_{10} = 4(u-1)^2(u+1)^2[u^2+u(4x^2-2)+1]^2,$$

$$B_{11} = 256u(u+1)x^3(x^2-1)^{3/2},$$

$$B_{12} = 16u^2(u+1)\sqrt{1-x^2}[u^2+u(4x^2-2)+1],$$

$$B_{13} = 8(u-1)\sqrt{u}(u+1)^2\sqrt{2-u(x^2-2)} \times [u^2+u(4x^2-2)+1],$$

$$B_{14} = 16(u-1)(u+1)^2\sqrt{u^2(1-x^2)+2u+1},$$

$$B_{15} = 8(1-u)\sqrt{u}(u+1)^2\sqrt{u(x^2-2)-2} \times [u^2+u(4x^2-2)+1],$$

$$B_{16} = 8(1-u)(u+1)^2[u^2+u(4x^2-2)+1],$$

and

$$Y_1 = \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{x+1}{\sqrt{1-x^2}}\right),$$

$$Y_2 = \arctan\left(\frac{2-x}{\sqrt{4-x^2}}\right) - \arctan\left(\frac{x+2}{\sqrt{4-x^2}}\right),$$

$$Y_3 = \arctan\left(\frac{2-x}{\sqrt{2u-x^2+2}}\right) - \arctan\left(\frac{x+2}{\sqrt{2u-x^2+2}}\right),$$

$$Y_4 = \arctan\left(\frac{u-x+1}{\sqrt{2u-x^2+2}}\right) - \arctan\left(\frac{u+x+1}{\sqrt{2u-x^2+2}}\right),$$

$$Y_5 = \arctan\left(\frac{u-x+1}{\sqrt{u^2+2u-x^2+1}}\right) - \arctan\left(\frac{u+x+1}{\sqrt{u^2+2u-x^2+1}}\right),$$

$$Y_6 = \arctan\left(\frac{\sqrt{u}(x-2)}{\sqrt{2-u(x^2-2)}}\right) + \arctan\left(\frac{\sqrt{u}(x+2)}{\sqrt{2-u(x^2-2)}}\right),$$

$$Y_7 = \arctan\left(\frac{-ux+u+1}{\sqrt{u^2(1-x^2)+2u+1}}\right) - \arctan\left(\frac{ux+u+1}{\sqrt{u^2(1-x^2)+2u+1}}\right),$$

$$Y_8 = \text{argtgh}\left(\frac{u(x-1)-1}{\sqrt{u}\sqrt{u(x^2-2)-2}}\right) + \text{argtgh}\left(\frac{ux+u+1}{\sqrt{u}\sqrt{u(x^2-2)-2}}\right),$$

$$Y_9 = \ln\frac{2}{1+u}, \quad Y_{10} = \ln u,$$

$$Y_{11} = \ln\frac{1-x^2-x\sqrt{x^2-1}}{1-x^2+x\sqrt{x^2-1}},$$

$$Y_{12} = \ln\frac{2u}{1+u}.$$

In all expressions u represents the one-loop fixed-point value of the turbulent magnetic Prandtl number $u_*^{(1)}$ given in Eq. (19), i.e.,

$$u = \frac{1}{2}\left(-1+\sqrt{\frac{9d+16}{d}}\right).$$

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. 2 (MIT Press, Cambridge, MA, 1975).

[2] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).

[3] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

[4] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).

[5] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).

[6] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).

[7] A. Yoshizawa, S.-I. Itoh, and K. Itoh, *Plasma and Fluid Turbulence: Theory and Modelling* (IoP, Bristol and Philadelphia, 2003).

[8] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, 2003).

[9] N. V. Antonov, J. Phys. A: Math. Gen. **39**, 7825 (2006).

[10] Y. Zhou, Phys. Rep. **488**, 1 (2010).

[11] P. A. Davidson, *Turbulence* (Oxford University Press, Oxford, 2015).

[12] Y. Zhou, Phys. Rep. **935**, 1 (2021).

[13] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941) [reprinted in Proc. R. Soc. London A **434**, 9 (1991)]; **31**, 538 (1941); **32**, 16 (1941) [reprinted in **434**, 15 (1991)].

[14] R. A. Antonia, B. R. Satyaprakash, and A. K. F. Hussain, J. Fluid Mech. **119**, 55 (1982).

[15] F. Anselmet, Y. Gagne, E. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).

[16] C. Meneveau and K. R. Sreenivasan, Phys. Rev. A **41**, 2246 (1990).

[17] V. R. Kuznetsov and V. A. Sabel'nikov, *Turbulence and Combustion* (Hemisphere, New York, 1990).

[18] M. S. Borgas, Phys. Fluids **4**, 2055 (1992).

[19] R. A. Antonia, E. J. Hopfinger, Y. Gagne, and F. Anselmet, Phys. Rev. A **30**, 2704 (1984); K. R. Sreenivasan, Proc. R. Soc. London A **434**, 165 (1991); M. Holzer and E. D. Siggia, Phys. Fluids **6**, 1820 (1994); A. Pumir, *ibid.* **6**, 2118 (1994); C. Tong and Z. Warhaft, *ibid.* **6**, 2165 (1994); T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **52**, 2617 (1995); Phys. Rev. Lett. **76**, 224 (1996); Phys. Rev. E **53**, 3431 (1996); Z. Warhaft, Annu. Rev. Fluid Mech. **32**, 203 (2000); B. I. Shraiman and E. Siggia, Nature (London) **405**, 639 (2000); F. Moisy, H. Willaime, J. S. Andersen, and P. Tabeling, Phys. Rev. Lett. **86**, 4827 (2001); A. Arnèodo, R. Benzi, J. Berg, L. Biferale, E. Bodenschatz, A. Busse, E. Calzavarini, B. Castaing, M. Cencini, L. Chevillard *et al.*, *ibid.* **100**, 254504 (2008).

[20] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).

[21] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995); D. Bernard, K. Gawędzki, and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996); M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, *ibid.* **52**, 4924 (1995); M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996).

[22] M. Avellaneda and A. Majda, Commun. Math. Phys. **131**, 381 (1990); **146**, 139 (1992).

[23] A. Majda, J. Stat. Phys. **73**, 515 (1993); D. Horntrop and A. Majda, J. Math. Sci. Univ. Tokyo **1**, 23 (1994).

[24] Q. Zhang and J. Glimm, Commun. Math. Phys. **146**, 217 (1992).

[25] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994); R. H. Kraichnan, V. Yakhot, and S. Chen, *ibid.* **75**, 240 (1995).

[26] B. I. Shraiman and E. D. Siggia, Phys. Rev. Lett. **77**, 2463 (1996); A. Pumir, B. I. Shraiman, and E. D. Siggia, Phys. Rev. E **55**, R1263 (1997).

[27] A. Pumir, Europhys. Lett. **34**, 25 (1996); **37**, 529 (1997); **57**, 2914 (1998).

[28] I. Rogachevskii and N. Kleeorin, Phys. Rev. E **56**, 417 (1997).

[29] A. Lanotte and A. Mazzino, Phys. Rev. E **60**, R3483 (1999).

[30] I. Arad, L. Biferale, and I. Procaccia, Phys. Rev. E **61**, 2654 (2000).

[31] N. V. Antonov, A. Lanotte, and A. Mazzino, Phys. Rev. E **61**, 6586 (2000).

[32] N. V. Antonov, J. Honkonen, A. Mazzino, and P. Muratore-Ginanneschi, Phys. Rev. E **62**, R5891 (2000).

[33] N. V. Antonov, M. Hnatich, J. Honkonen, and M. Jurčišin, Phys. Rev. E **68**, 046306 (2003).

[34] M. Chaves, K. Gawędzki, P. Horvai, A. Kupiainen, and M. Vergassola, J. Stat. Phys. **113**, 643 (2003).

[35] H. Arponen, Phys. Rev. E **81**, 036325 (2010).

[36] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[37] A. P. Kazantsev, Zh. Eksp. Teor. Fiz. **53**, 1806 (1968) [Sov. Phys. JETP **26**, 1031 (1968)].

[38] D. J. Amit, *Field Theory, Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).

[39] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).

[40] A. N. Vasil'ev, *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (Chapman & Hall/CRC, Boca Raton, FL, 2004).

[41] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Phys. Rev. E **58**, 1823 (1998).

[42] N. V. Antonov, Phys. Rev. E **60**, 6691 (1999); L. Ts. Adzhemyan, N. V. Antonov, and J. Honkonen, *ibid.* **66**, 036313 (2002).

[43] L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Y. S. Kabrits, and A. N. Vasil'ev, Phys. Rev. E **63**, 025303(R) (2001); **64**, 056306 (2001).

[44] L. Ts. Adzhemyan, N. V. Antonov, M. Hnatich, and S. V. Novikov, Phys. Rev. E **63**, 016309 (2000); E. Jurčišinová, M. Jurčišin, R. Remecký, and M. Scholtz, Int. J. Mod. Phys. B **22**, 3589 (2008); E. Jurčišinová and M. Jurčišin, Phys. Rev. E **77**, 016306 (2008).

[45] L. Ts. Adzhemyan, and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998); N. V. Antonov and J. Honkonen, *ibid.* **63**, 036302 (2001).

[46] O. G. Chkhetiani, M. Hnatich, E. Jurčišinová, M. Jurčišin, A. Mazzino, and M. Repašan, Phys. Rev. E **74**, 036310 (2006); J. Phys. A: Math. Gen. **39**, 7913 (2006); Czech. J. Phys. **56**, 827 (2006).

[47] L. Ts. Adzhemyan, N. V. Antonov, J. Honkonen, and T. L. Kim, Phys. Rev. E **71**, 016303 (2005).

[48] E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Rev. E **80**, 046302 (2009).

[49] E. Jurčišinová, M. Jurčišin, and R. Remecký, J. Phys. A: Math. Theor. **42**, 275501 (2009).

[50] L. Ts. Adzhemyan, N. V. Antonov, and A. V. Runov, Phys. Rev. E **64**, 046310 (2001); M. Hnatič, M. Jurčišin, A. Mazzino, and S. Šprinc, Acta Phys. Slov. **52**, 559 (2002); S. V. Novikov, J. Phys. A: Math. Gen. **39**, 8133 (2006); E. Jurčišinová, M. Jurčišin, R. Remecký, and M. Scholtz, Phys. Part. Nucl. Lett. **5**, 219 (2008); N. V. Antonov and N. M. Gulitskiy, Theor. Math Phys. **176**, 851 (2013); L. Ts. Adzhemyan, N. V. Antonov, P. B. Gol'din, and M. V. Kompaniets, J. Phys. A: Math. Theor. **46**, 135002 (2013); N. V. Antonov and N. M. Gulitskiy, Phys. Rev. E **91**, 013002 (2015); **92**, 043018 (2015); N. V. Antonov and M. M. Kostenko, *ibid.* **92**, 053013 (2015); N. V. Antonov and N. M. Gulitskiy, EPJ Web Conf. **108**, 02008 (2016).

[51] M. Hnatich, J. Honkonen, M. Jurčišin, A. Mazzino, and S. Šprinc, Phys. Rev. E **71**, 066312 (2005).

[52] E. Jurčišinová, M. Jurčišin, and R. Remecký, Phys. Rev. E **84**, 046311 (2011).

[53] E. Jurčišinová and M. Jurčišin, J. Phys. A: Math. Theor. **45**, 485501 (2012).

[54] N. V. Antonov and N. M. Gulitskiy, Phys. Rev. E **85**, 065301(R) (2012); Lecture Notes Comput. Sci. **7125**, 128 (2012).

[55] E. Jurčišinová and M. Jurčišin, Phys. Rev. E **88**, 011004(R) (2013); **91**, 063009 (2015).

[56] E. Jurčišinová, M. Jurčišin, and M. Menkyna, Phys. Rev. E **95**, 053210 (2017).

[57] N. V. Antonov, N. M. Gulitskiy, M. M. Kostenko, and A. V. Malyshev, Phys. Rev. E **97**, 033101 (2018).

[58] E. Jurčišinová, M. Jurčišin, and M. Menkyna, Eur. Phys. J. B **91**, 313 (2018).

[59] E. Jurčišinová, M. Jurčišin, M. Menkyna, and R. Remecký, Phys. Rev. E **104**, 015101 (2021).

[60] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Usp. Fiz. Nauk **166**, 1257 (1996) [Phys. Usp. **39**, 1193 (1996)].

[61] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973); C. De Dominicis, J. Phys. (Paris) Colloq. **37**, C1-247 (1976); H. K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976).

[62] L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, and A. N. Vasil'ev, Int. J. Mod. Phys. B **17**, 2137 (2003).

[63] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Theor. Math. Phys. **57**, 1131 (1983).