Seven-state rotation-symmetric number-conserving cellular automaton that is not isomorphic to any septenary one

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We consider two-dimensional cellular automata with the von Neumann neighborhood that satisfy two properties of interest from a modeling viewpoint: rotation symmetry (i.e., the local rule is invariant under rotation of the neighborhood by 90°) and number conservation (i.e., the sum of all the cell states is conserved upon every update). It is known that if the number of states k is smaller than or equal to six, then each rotation-symmetric number-conserving cellular automaton is isomorphic to some k-ary one, i.e., one with state set $\{0, 1, \dots, k-1\}$. In this paper, we exhibit an example of a seven-state rotation-symmetric number-conserving cellular automaton that is not isomorphic to any septenary one. This example strongly supports our plea that research into multistate cellular automata should not only focus on those that have $\{0, 1, \dots, k-1\}$ as a state set.

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I. INTRODUCTION

Cellular automata (CAs) are discrete dynamical systems that evolve according to a simple local rule, but can nevertheless produce very complex dynamics (see [1] or [2], for instance). For this reason, such systems for several decades are willingly used to build models of various phenomena, in particular physical ones (see, for example, [3] or [4]).

The definition of a CA requires the choice of a cellular space $\mathcal{C}$, a neighborhood $\mathcal{N}$ of every cell, a state set Q, and a local rule f. Although the specification of the quadruple $(\mathcal{C}, \mathcal{N}, Q, f)$ has a great influence on the resulting model, usually little attention is paid to what elements Q contains, but only to how many there are. Therefore, theoretical research—if it does not consider some general hypothetical state set Q—mostly concerns the so-called k-ary CAs (binary, ternary, quaternary, quinary, and so on), for which the state set equals $\{0, 1, \dots, k-1\}$ for some integer $k \geqslant 2$ (see, for instance, [5–8]).

The above approach seems justified for the following reason. Suppose that $\mathcal{C}$ and $\mathcal{N}$ are fixed and that two finite equinumerous sets Q and $\widetilde{Q}$ are given. Then for each CA $A = (\mathcal{C}, \mathcal{N}, Q, f)$, using any bijection $\phi : Q \to \widetilde{Q}$, one can define an isomorphic CA $\widetilde{A} = (\mathcal{C}, \mathcal{N}, \widetilde{Q}, \widetilde{f})$, which means that when identifying the states q and $\phi(q)$ (for any $q \in Q$), the CAs A and $\widetilde{A}$ are indistinguishable (for the strict meaning, see Definition 4). For example, in the one-dimensional case, where the dynamics of a given cellular automaton is usually represented by a space-time diagram, the fact that two CAs $A = (\mathcal{C}, \mathcal{N}, Q, f)$ and $\widetilde{A} = (\mathcal{C}, \mathcal{N}, \widetilde{Q}, \widetilde{f})$ are isomorphic simply means that if we choose some collection of colors to represent the states of $\widetilde{A}$ and then for every state $q \in Q$ we use the same color as for the state $\phi(q)$, then the corresponding space-time diagrams for A and $\widetilde{A}$ will be exactly the same. Perhaps the most common situation is when considering binary CAs and the set of states can be named $\{0, 1\}$ or $\{white, black\}$ or $\{dead, alive\}$ or $\{healthy, sick\}$ or $\{empty, occupied\}$, to name but a few.

However, in many situations, only CAs having some additional property P are considered (for instance, resulting from the properties of the model being built). Since isomorphy does not need to preserve P, for two equinumerous sets Q and $\widetilde{Q}$, the family of all Q-state CAs satisfying P may differ significantly from the family of all $\widetilde{Q}$-state CAs satisfying P. For example, when modeling various kinds of physical phenomena, very often some additional condition is imposed on the local rule f related to the symmetries of the neighborhood $\mathcal{N}$ (for instance, rotation invariance) or related to some conservation laws (of mass, energy, and so on). While in the former case we still do not have to worry about the choice of Q, in the latter case, it is known that selecting Q may be important.

We are encountering such a situation in the case of two-dimensional rotation-symmetric number-conserving CAs (RSNCCAs) with the von Neumann neighborhood (see Sec. II for definitions). Tanimoto *et al.* [9] proved that such CAs with at most four states are all trivial, regardless of Q, i.e., if just $|Q| \leqslant 4$, then the identity is the only RSNCCA with such a state set. Later, Imai *et al.* [10] showed that the case of five states is a bit more complicated: if the elements of a given five-element set Q form an arithmetic progression, then there are exactly four RSNCCAs with this state set Q; if the elements of Q do not form an arithmetic progression, then there is only one RSNCCA with this state set (the identity). Thus the choice of Q has a big influence on the number of such

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CAs. However, it was also proved that the set $\{0, 1, 2, 3, 4\}$ can be treated as a basic set of states in the sense that any five-state RSNCCA is isomorphic to some quinary RSNCCA, i.e., with state set $\{0, 1, 2, 3, 4\}$ (see [10]). An analogous result was proved in [11] for the more complicated case of six states: the authors showed that the set $\{0, 1, 2, 3, 4, 5\}$ can be treated as a basic set for all six-state RSNCCAs. Indeed, there are exactly 116 senary RSNCCAs (with the state set $\{0, 1, 2, 3, 4, 5\}$) and any six-state RSNCCA is isomorphic to some of those 116 RSNCCAs. Thus, knowing the dynamics of 116 senary RSNCCAs, we know the dynamics of any six-state RSNCCA.

The results of [9–11] lead to the open question whether or not the following general statement holds.

Question 1. Is every RSNCCA with k states isomorphic to some k-ary RSNCCA?

Although the proof technique used in [11] does not easily generalize to larger state sets, it was believed that the answer to Question 1 would be positive.

Unfortunately, reality turns out to be different. In this paper, we demonstrate that the answer to the above question is negative. More precisely, we provide an example of a seven-state RSNCCA that is not isomorphic to any septenary RSNCCA (i.e., with the state set $\{0, 1, 2, 3, 4, 5, 6\}$).

II. PRELIMINARIES

As cellular space we consider the two-dimensional infinite grid $\mathbb{Z}^2$. At every discrete time step, every cell (i, j) in $\mathbb{Z}^2$ is assigned a state from a given set Q. We assume the state set Q to be a finite set of real numbers containing 0.

A configuration is any mapping from the grid $\mathbb{Z}^2$ to Q. The set of all possible configurations $Q^{\mathbb{Z}^2}$ is denoted as X. For a given configuration $\mathbf{x} \in X$, the value of cell (i, j) is denoted by $x_{i,j}$. By X_{fin} we denote the set of all *finite configurations*, i.e., those configurations that are almost everywhere equal to zero:

$$X_{\text{fin}} = \{\mathbf{x} \in X \mid \{(i, j) \in \mathbb{Z}^2 \mid x_{i,j} \neq 0\} \text{ is finite}\}.$$

A configuration $\mathbf{x} \in X$ is called *totally spatially periodic* if there exist positive integers p_1 and p_2 such that for each $(i, j) \in \mathbb{Z}^2$ it holds that

$$x_{i,j} = x_{i+p_1,j} = x_{i,j+p_2}$$

(see, for instance, [12]). In other words, a totally spatially periodic configuration consists of some rectangular pattern (with sizes p_1 and p_2) that is periodically repeated in both dimensions of $\mathbb{Z}^2$. The couple (p_1, p_2) is referred to as the period of $\mathbf{x}$. The set of all totally spatially periodic configurations is denoted as X_{per}.

Generally, a neighborhood is understood as an m tuple $(v_1, v_2, \ldots, v_m)$ of distinct vectors of $\mathbb{Z}^2$ that specify for a given cell the relative positions of its neighbor cells (with respect to that cell). In this paper, we consider the so-called von Neumann neighborhood (with radius 1), for a given cell consisting of the cell itself and its four adjacent cells; hence, we use $\mathcal{N} = ((0, -1), (-1, 0), (0, 0), (1, 0), (0, 1))$. For the sake of convenience, for a given configuration $\mathbf{x} \in X$ we list the states of the cells in the neighborhood of a cell (i, j) as

$$\begin{matrix} & x_{i,j-1} & \\ x_{i-1,j} & x_{i,j} & x_{i+1,j}. \\ & x_{i,j+1} & \end{matrix}$$

The new state of a cell (i, j) depends only on the states of the cells in its neighborhood at the previous time step through a local rule f. Thus, in our case, the domain of f is given by

$$\mathcal{D} = \left\{ \begin{matrix} & u & \\ l & c & r \\ & d & \end{matrix} \;\middle|\; u, l, c, r, d \in Q \right\}.$$

A given local rule $f : \mathcal{D} \to Q$ generates a global rule $F : X \to X$ in the usual way: for any $\mathbf{x} \in X$, the configuration $F(\mathbf{x})$ is given by

$$F(\mathbf{x})_{i,j} = f\begin{pmatrix} & x_{i,j-1} & \\ x_{i-1,j} & x_{i,j} & x_{i+1,j} \\ & x_{i,j+1} & \end{pmatrix}.$$

By a CA we mean the quadruple $(\mathbb{Z}^2, \mathcal{N}, Q, f)$.

In our investigation, we focus on CAs that have two properties that are important from the point of view of applications. The first one is rotation symmetry, which means that the local rule of the CA is invariant under rotation of the neighborhood by 90°.

Definition 1. A CA is called *rotation symmetric* if its local rule f satisfies for any $u, l, c, r, d \in Q$,

$$f\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = f\begin{pmatrix} & r & \\ u & c & d \\ & l & \end{pmatrix}. \tag{1}$$

The second property of CAs we are interested in is number conservation, which means that the sum of all the states in any configuration remains constant throughout the evolution of the automaton. Since the cellular space $\mathbb{Z}^2$ is infinite, it is convenient to consider number conservation in the sense of *finite* or *periodic* number conservation.

Definition 2. Let F be the global rule of a CA. We say that this CA is *finite number conserving* if for any $\mathbf{x} \in X_{\text{fin}}$, it holds that

$$\sum_{(i,j)\in\mathbb{Z}^2} F(\mathbf{x})_{i,j} = \sum_{(i,j)\in\mathbb{Z}^2} x_{i,j}.$$

Definition 3. Let F be the global rule of a CA. We say that this CA is *periodic number conserving* if for any $\mathbf{x} \in X_{\text{per}}$, it holds that

$$\sum_{0\leqslant i<p_1, 0\leqslant j<p_2} F(\mathbf{x})_{i,j} = \sum_{0\leqslant i<p_1, 0\leqslant j<p_2} x_{i,j},$$

where (p_1, p_2) is the period of $\mathbf{x}$.

Durand *et al.* [13] proved that Definitions 2 and 3 are equivalent.

Theorem 1. A cellular automaton $(\mathbb{Z}^2, \mathcal{N}, Q, f)$ is finite number conserving if and only if it is periodic number conserving.

The above theorem allows us to use the term "number conserving" having in mind any of these two definitions. If a CA is number conserving, then we also call its local and global rules number conserving. In this paper, a two-dimensional

CA with the von Neumann neighborhood that is both rotation symmetric and number conserving will be abbreviated as an RSNCCA.

A detailed characterization of all RSNCCAs with up to five states, as mentioned in the Introduction, can be found in [9,10]. The case of six states was considered in [11] (on finite grids only, but this is the same as considering the infinite grid using Definition 3), where a very tedious and technical approach allowed one to pick out all senary RSNCCAs from the huge set of all two-dimensional rotation-symmetric CAs with the state set {0, 1, 2, 3, 4, 5} and the von Neumann neighborhood (note that the cardinality of this set is greater than $6^{6^5/4}$, a number with 1513 digits). It appeared that there are only 116 such RSNCCAs. Moreover, the authors gave a full characterization of the set of all RSNCCAs for any six-element state set, from which it resulted, *inter alia*, that any six-state RSNCCA is isomorphic to some senary RSNCCA (see [11] for details). When stating that two CAs are isomorphic (sometimes the term *equivalent* is also used), we have the following definition in mind.

Definition 4. We say that a CA $A = (\mathbb{Z}^2, \mathcal{N}, Q, f)$ is isomorphic to a CA $\widetilde{A} = (\mathbb{Z}^2, \mathcal{N}, \widetilde{Q}, \widetilde{f})$ if there exists a bijection $\phi : Q \to \widetilde{Q}$ such that for any $u, l, c, r, d, x \in Q$,

$$f\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = x \Leftrightarrow \widetilde{f}\begin{pmatrix} & \phi(u) & \\ \phi(l) & \phi(c) & \phi(r) \\ & \phi(d) & \end{pmatrix} = \phi(x). \tag{2}$$

The results of [9–11] can be summarized as follows.

Theorem 2. Let $k \leqslant 6$. Then every RSNCCA with k states is isomorphic to some k-ary RSNCCA.

In the next section, we present a counterexample showing that Theorem 2 cannot be generalized, even for $k = 7$.

III. COUNTEREXAMPLE

The method described in [11] allows one to enumerate all septenary RSNCCAs with the help of a computer. One can also use a much more general method introduced in [14] based on the characterization of number-conserving CAs with the von Neumann neighborhood. Either of these two ways results in a complete list of all 30 144 septenary RSNCCAs, which can be found in [15] (this dataset contains the full list, before reduction according to conjugation or reflection dependency).

Now, let us define the following RSNCCA $A_* = (\mathbb{Z}^2, \mathcal{N}, Q, f_*)$, by setting $Q = \{0, 2, 3, 4, 5, 6, 8\}$ and

$$f_*\begin{pmatrix} & u & \\ l & c & r \\ & d & \end{pmatrix} = \begin{cases} 2N_8(u, r, d, l) & \text{if } c = 0 \\ 8 - 2N_0(u, r, d, l) & \text{if } c = 8 \\ 2 + N_6(u, r, d, l) & \text{if } c = 2 \\ 6 - N_2(u, r, d, l) & \text{if } c = 6 \\ c & \text{otherwise,} \end{cases}$$

where $N_b(u, r, d, l)$ denotes the number of occurrences of b among u, r, d, l. The local rule f_* thus has a very simple interpretation in the language of particle representation: every cell with eight particles gives two particles to each empty cell in its neighborhood (therefore also every empty cell gets two particles from each cell in its neighborhood having eight particles) and every cell with six particles gives one particle to each cell in its neighborhood having two particles (therefore also every cell with two particles gets one particle from each cell in its neighborhood having six particles). The remaining cells do not change their states. Since the local rule f_* can be interpreted in such a simple way in terms of motion of particles, one can see that it is rotation symmetric and number conserving. We will argue that the RSNCCA $A_* = (\mathbb{Z}^2, \mathcal{N}, Q, f_*)$ is not isomorphic to any septenary RSNCCA.

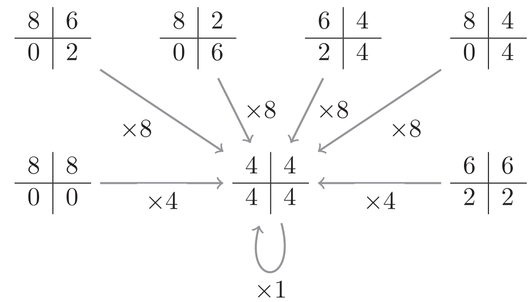

FIG. 1. Configurations with period (2,2) transformed by A_* into the homogeneous configuration with 4's.

Since there are as many as 7! possible bijections between two seven-element sets, a brute-force approach checking for all septenary RSNCCAs whether Eq. (2) holds or not requires $30\,144 \times 7! \times 7^5 = 2\,553\,416\,248\,320$ operations, which looks rather daunting. Instead, we propose to use the method of invariants: for each RSNCCA A we define some simple numerical parameter $\gamma(A)$, which proves invariant under isomorphy. Then we calculate the value of this parameter for all septenary RSNCCAs. If no value turns out to be equal to the value $\gamma(A_*)$, then this proves that A_* is not isomorphic to any septenary RSNCCAs.

Let $\begin{array}{c|c} a & b \\ \hline c & d \end{array}$ denote the configuration $\mathbf{x} \in X_{\text{per}}$ with period (2,2), for which $x_{0,0} = a$, $x_{1,0} = b$, $x_{0,1} = c$, and $x_{1,1} = d$. For a given $A = (\mathbb{Z}^2, \mathcal{N}, Q, f)$ and $q \in Q$, let $\gamma(A, q)$ denote the number of configurations $\mathbf{x} \in X_{\text{per}}$ with period (2,2) such that $F(\mathbf{x})$ is the homogeneous configuration in which all cells have state q, i.e.,

$$\gamma(A, q) = \left| \left\{ \begin{array}{c|c} a & b \\ \hline c & d \end{array} \in X_{\text{per}} \mid F\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) = \begin{array}{c|c} q & q \\ \hline q & q \end{array} \right\} \right|$$

and let $\gamma(A) = \max_{q \in Q} \gamma(A, q)$.

It is easy to see that if $A = (\mathbb{Z}^2, \mathcal{N}, Q, f)$ and $\widetilde{A} = (\mathbb{Z}^2, \mathcal{N}, \widetilde{Q}, \widetilde{f})$ are isomorphic by a bijection $\phi : Q \to \widetilde{Q}$, then for any $q \in Q$ it holds that $\gamma(A, q) = \gamma(\widetilde{A}, \phi(q))$, thus also $\gamma(A) = \gamma(\widetilde{A})$, which means that the parameter γ is invariant under isomorphy.

For the example local rule f_*, one can calculate that $\gamma(A_*) = \gamma(A_*, 4) = 41$. It is shown in Fig. 1 which configurations are transformed into the homogeneous configuration with 4's. To keep the schema readable, we grouped these configurations according to significantly different patterns, ignoring rotations and axial symmetries. For example, all eight configurations $\begin{array}{c|c} 6 & 4 \\ \hline 2 & 4 \end{array}$, $\begin{array}{c|c} 2 & 6 \\ \hline 4 & 4 \end{array}$, $\begin{array}{c|c} 4 & 2 \\ \hline 4 & 6 \end{array}$, $\begin{array}{c|c} 4 & 4 \\ \hline 6 & 2 \end{array}$, $\begin{array}{c|c} 2 & 4 \\ \hline 6 & 4 \end{array}$, $\begin{array}{c|c} 4 & 4 \\ \hline 2 & 6 \end{array}$, $\begin{array}{c|c} 4 & 6 \\ \hline 4 & 2 \end{array}$, and $\begin{array}{c|c} 6 & 2 \\ \hline 4 & 4 \end{array}$ are transformed into

TABLE I. Number of septenary RSNCCAs with a given value of $\gamma(A)$.

$\gamma(A)$	1	5	9	13	17
Number of septenary RSNCCAs	48	696	6048	9328	14024

$\frac{4}{4}\big|\frac{4}{4}$, but we present only the first one and give the information that it represents a group of eight configurations ($\times 8$). We have calculated $\gamma(A)$ for all septenary RSNCCAs and only the following values were obtained: 1, 5, 9, 13, and 17.

Table I presents the numbers of septenary RSNCCAs with a given $\gamma(A)$. As the largest possible value of $\gamma(A)$ for any septenary RSNCCA is 17, we conclude that A_* is not isomorphic to any septenary RSNCCA.

IV. CONCLUSIONS

There was a common belief among CA researchers that it suffices to consider CAs with the state set $\{0, 1, \ldots, k-1\}$. The authors of [11] were asked by an anonymous reviewer why they wrote down a complicated proof for the seemingly obvious fact that every RSNCCA with six states is isomorphic to some senary RSNCCA. The example in this paper shows that such effort is necessary, since Theorem 2 does not generalize to higher values of k. For general CAs, it is merely a matter of renaming states, so that one can always use the state set $\{0, 1, \ldots, k-1\}$. However, number conservation (for instance) imposes some additional additive structure on the state set, implying that some natural arguments regarding the renaming of states become invalid. One indeed has to be very careful when performing such operation.

The local rule f_* was inspired by some models in which particles carrying different amounts of energy coexist. These models are quite natural to consider in the context of number-conserving CAs and our example shows that limiting the attention to the most popular k-ary world is not always sufficient.

Finally, we would like to share a positive conclusion from our paper. Cellular automata are very often chosen by various researchers for modeling physical phenomena and are then usually assumed to be k-ary number-conserving CAs (if any principles of conservation come into play). The presented example shows that if we open ourselves to other sets of states, we can get new kinds of models with different physical properties.

[1] S. Wolfram, *Cellular Automata and Complexity: Collected Papers* (Basic Books, New York, 1994).

[2] J. Kari, Theory of cellular automata: A survey, Theor. Comput. Sci. **334**, 3 (2005).

[3] B. Chopard and M. Droz, *Cellular Automata Modeling of Physical Systems*, Collection Alea-Saclay: Monographs and Texts in Statistical Physics (Cambridge University Press, Cambridge, UK, 1998).

[4] G. 't Hooft, *The Cellular Automaton Interpretation of Quantum Mechanics*, Fundamental Theories of Physics (Springer International Publishing, New York, 2016), Vol. 185.

[5] F. G. Pazzona, G. Pireddu, and P. Demontis, Quasiequilibrium multistate cellular automata, Phys. Rev. E **105**, 014116 (2022).

[6] E. L. Ruivo, P. P. de Oliveira, F. Lobos, and E. Goles, Shift-equivalence of k-ary, one-dimensional cellular automata rules, Commun. Nonlinear Sci. Numer. Simul. **63**, 280 (2018).

[7] K. Bhattacharjee and S. Das, Reversibility of d-state finite cellular automata, J. Cell. Autom. **11**, 213 (2016).

[8] N. Boccara and H. Fukś, Number-conserving cellular automaton rules, Fund. Inf. **52**, 1 (2002).

[9] N. Tanimoto, K. Imai, C. Iwamoto, and K. Morita, On the non-existence of rotation-symmetric von Neumann neighbor number-conserving cellular automata of which the state number is less than four, IEICE Trans. Inf. Syst. **E92-D**, 255 (2009).

[10] K. Imai, H. Ishizaka, and V. Poupet, 5-state rotation-symmetric number-conserving cellular automata are not strongly universal, in *Cellular Automata and Discrete Complex Systems: 20th International Workshop, AUTOMATA 2014, Revised Selected Papers*, edited by T. Isokawa, K. Imai, N. Matsui, F. Peper, and H. Umeo (Springer International Publishing, Cham, 2015), pp. 31–43.

[11] A. Dzedzej, B. Wolnik, A. Nenca, J. M. Baetens, and B. De Baets, Two-dimensional rotation-symmetric number-conserving cellular automata, Inf. Sci. **577**, 599 (2021).

[12] A. Dennunzio, E. Formenti, and M. Weiss, Multidimensional cellular automata: Closing property, quasi-expansivity, and (un)decidability issues, Theor. Comput. Sci. **516**, 40 (2014).

[13] B. Durand, E. Formenti, and Z. Róka, Number-conserving cellular automata I: Decidability, Theor. Comput. Sci. **299**, 523 (2003).

[14] B. Wolnik, A. Nenca, J. M. Baetens, and B. De Baets, A split-and-perturb decomposition of number-conserving cellular automata, Phys. D (Amsterdam, Neth.) **413**, 132645 (2020).

[15] A. Nenca, A. Dzedzej, and B. Wolnik, The complete list of two-dimensional rotation-symmetric number-conserving septenary cellular automata (2022) [Data set]. Gdańsk University of Technology, https://doi.org/10.34808/jr67-r637.