Relaxation dynamics of phase oscillators with generic heterogeneous coupling

Can Xu,[1,*] Xin Jin,[2] and Yonggang Wu[3]

[1]*Institute of Systems Science and College of Information Science and Engineering, Huaqiao University, Xiamen 361021, China*
[2]*School of Physics and Electronic Engineering, Jiangsu Normal University, Xuzhou 221116, China*
[3]*School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China*

(Received 9 November 2022; accepted 1 February 2023; published 13 February 2023)

The coupled phase oscillator model serves as a paradigm that has been successfully used to shed light on the collective dynamics occurring in large ensembles of interacting units. It was widely known that the system experiences a continuous (second-order) phase transition to synchronization by gradually increasing the homogeneous coupling among the oscillators. As the interest in exploring synchronized dynamics continues to grow, the heterogeneous patterns between phase oscillators have received ample attention during the past years. Here, we consider a variant of the Kuramoto model with quenched disorder in their natural frequencies and coupling. Correlating these two types of heterogeneity via a generic weighted function, we systematically investigate the impacts of the heterogeneous strategies, the correlation function, and the natural frequency distribution on the emergent dynamics. Importantly, we develop an analytical treatment for capturing the essential dynamical properties of the equilibrium states. In particular, we uncover that the critical threshold corresponding to the onset of synchronization is unaffected by the location of the inhomogeneity, which, however, does depend crucially on the value of the correlation function at its center. Furthermore, we reveal that the relaxation dynamics of the incoherent state featuring the responses to external perturbations is significantly shaped by all the considered effects, thereby leading to various decaying mechanisms of the order parameters in the subcritical region. Moreover, we untangle that synchronization is facilitated by the out-coupling strategy in the supercritical region. Our study is a step forward in highlighting the potential importance of the inhomogeneous patterns involved in the complex systems, and could thus provide theoretical insights for profoundly understanding the generic statistical mechanical properties of the steady states toward synchronization.

DOI: 10.1103/PhysRevE.107.024206

I. INTRODUCTION

Synchronization refers to an emergent phenomenon that occurs in large ensembles of interacting oscillators. Such a collective behavior has wide applications in diverse complex systems ranging from physics and biology to human society [1,2]. Unravelling the intrinsic mechanism behind such self-organized behaviors has been a major subject of research in the fields of nonlinear dynamics and network science [3].

The Kuramoto model introduced in 1975 has become a classic for investigating synchronization transitions and other related collective dynamics [4]. The model elucidates synchronization at the onset of a nonequilibrium phase transition unveiling the interplay between the tendency that each individual phase oscillator has to oscillate at its native frequency and the dissipative phase difference coupling attempting to synchronize the system.

In the classical Kuramoto model, the coupling among the phase oscillators is assumed to be uniform, and the system undergoes a continuous (second-order) phase transition towards synchronization. Correspondingly, the population may aggregate to form a crowd of oscillators by either increasing the overall coupling strength or by decreasing the width (diversity) of natural frequencies of the phase oscillators [5–8].

Recently, there has been a great interest in exploring the synchronized dynamics induced by the heterogeneous interactions, in which the classical Kuramoto model was generalized by considering the added effects. In contrast to the homogeneous interactions, there will inevitably be some variations among oscillators in many realistic setups. Typical examples include the intrinsic frequencies of the oscillators themselves and the network topology encoding the connection structure, which account for a sort of inhomogeneity of the coupled system. The most notable dynamical consequence arising from the correlation between these two types of heterogeneity is the emergence of explosive synchronization [9,10], which is characterized by an abrupt transition between the disordered and ordered states. Investigating such an issue, therefore, represents an important topic of research that provides a theoretical underpinning for understanding the relation between dynamical structures and functional fittings of the complex system [11–23].

In this work, we consider a variant of the Kuramoto model of globally coupled phase oscillators incorporating the correlation between the quenched disorder of the natural frequencies and the coupling, in which the correlation is described by a generic weighted function. We reveal that the coupling strategy, the correlation function, as well as the

*xucan@hqu.edu.cn

frequency distribution of the oscillators can greatly alter the synchronized dynamics. In particular, we develop an analytical treatment that is capable of capturing the critical behaviors for the onset of synchronization. Specifically, the critical threshold corresponding to the emergence of synchronization is expressed as an analytical formula that is unaffected by the location of the heterogeneous correlation function. Moreover, we demonstrate that the stability of the incoherent state, as well as the associated eigenspectrum structure, remain the same for both the in-coupling and out-coupling strategies. Nevertheless, the relaxation dynamics of the disordered state is significantly shaped by the coupling strategies. We develop a framework for systematically exploring the relaxation dynamics of the equilibrium states induced by various weighted functions. More importantly, we clarify the intrinsic decaying mechanism for the macroscopic order parameters that are at the heart of the Landau damping effects observed in coupled oscillator systems. Using the methodology of the resonant poles theory, the classical Kuramoto order parameter together with the weighted order parameter are established in an analytical formalism in the subcritical region. Furthermore, in the supercritical regime, we untangle that synchronization is facilitated by the out-coupling strategy even for the same settings of the system.

The remainder of this paper is organized as follows. In Sec. II, we briefly introduce the dynamical model incorporating the correlation into the coupling via a generic weighted function. In Sec. III, we carry out a detailed linear stability analysis of the incoherent state to obtain the critical point for the onset of synchronization. In Sec. IV, we further explore the relaxation dynamics of the incoherent state in the subcritical region. The two order parameters describing the macroscopic dephasing effects are obtained in the analytical form. In Sec. V, we explore the asymptotical behaviors of the order parameters in the neighborhood of critical point. Finally, we conclude with a discussion of our results in Sec. VI.

II. DYNAMICAL MODEL

To begin with, the generalized Kuramoto model consisting of a population of globally coupled phase oscillators is ruled by

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\sin(\theta_j - \theta_i), \quad i = 1, \dots, N. \tag{1}$$

Here, $\theta_i(t)$ is the instantaneous phase of the oscillator i, and $N > 0$ is the size of the system (oscillator number). $\{\omega_i\}$ are the natural frequencies chosen randomly from a prescribed distribution function $g(\omega)$. K_{ij} accounts for the coupling scheme underlying the system that encodes the connectivity patterns of the system.

The classical Kuramoto model implies that $K_{ij} = K$, with $K > 0$ being the global (uniform) coupling strength, thereby defining an all to all connected topology of the coupled system. However, given a network structure, the matrix $\mathbf{K} = \{K_{ij}\}$ describes the connectivity patterns between each pair of phase oscillators encoding the underlying topology of the connections. For example, in the case of an undirected and unweighted graph, $K_{ij} = K_{ji} = 1$, if the pair of oscillators θ_i and θ_j are linked. $K_{ij} = K_{ji} = 0$, otherwise.

Besides the network structures affecting the synchronized dynamics, there has been a great interest in exploring the emergent dynamics by incorporating the additional inhomogeneity into the coupling. For instance, the element K_{ij} may be endowed with a random variable representing the quenched disorder of the system. The concrete example is the conformists-contrarians model reported in [24], in which K_{ij} is a random variable with mixed signs. Specifically, if $K_{ij} > 0$, the interaction between each pair of oscillators is excitatory that tends to attract each other. On the contrary, if $K_{ij} < 0$, the associated interaction turns out to be inhibitory that is prone to repulse each other. For that matter, such an excitatory-inhibitory model is massively investigated shedding light on various rhythmic dynamics observed in social and neural networks [25–27].

It should be pointed out that the intrinsic frequencies themselves are random variables featuring a kind of heterogeneous quenched disorder of the population. In this regard, it is reasonable to assume an internal relation between the $\{\omega_{i,j}\}$ and $\{K_{ij}\}$. To achieve this, one may choose the matrix $\mathbf{K}$ to be the form $K_{ij} \propto f(\omega_i, \omega_j)$, where $f(\cdot)$ is an arbitrary function of the natural frequencies described by either an explicit or an implicit relation. Notably, this setting establishes a correlation between the natural frequencies and the coupling strength. For instance, the frequency-weighted coupling, e.g., $f(\omega_i, \omega_j) = |\omega_i|$ or $f(\omega_i, \omega_j) = |\omega_i - \omega_j|$, was used to mimic the frequency-degree correlation in a networked oscillator system giving birth to the explosive synchronization transition route [28,29]. It unveils that the frequency-weighted coupling scheme displays a suppressive rule for the formation of small clusters in the neighborhood of the critical point for phase transition. Subsequently, such a consideration was further extended to a wealth of different systems by taking into account various weighted schemes, thereby leading to a host of fascinating dynamical phenomena [30–39].

In order to advance the existing studies and highlight the correlation between the coupling and frequency, we consider a general form of the correlation function. Particularly, we focus on a correlation function that is decomposable with respect to the arguments, i.e., $f(\omega_i, \omega_j) = Kf(\omega_i)$ and $f(\omega_i, \omega_j) = Kf(\omega_j)$. $K > 0$ parametrizes the attracting global coupling strength. $f(\omega_i)$ and $f(\omega_j)$ are arbitrary weighted functions characterizing correlations between the intrinsic properties of the oscillators and the coupling. Here, and in the following, the first and the second scenarios are termed as the in-coupling and out-coupling, respectively. The goal of this work is to systematically investigate how the heterogeneous structures and the correlation functions influence the onset of synchronization, as well as the generic properties of the relaxation dynamics of the equilibrium states.

Before proceeding with the analysis, it is convenient to introduce two order parameters to characterize the level of synchronization, which are

$$Z(t) = R(t)e^{i\Theta(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} \tag{2}$$

and

$$Z_w(t) = R_w(t)e^{i\Theta_w(t)} = \frac{1}{N}\sum_{j=1}^{N} f(\omega_j)e^{i\theta_j(t)}. \tag{3}$$

Remarkably, $Z(t)$ denotes the classical Kuramoto order parameter corresponding to the centroid of the configuration $\{e^{i\theta_j}\}$ placed on the complex unit circle. $R(t) \in [0, 1]$ measures the coherence of the system with $R = 0$ and $R = 1$ representing a totally disordered state and completely synchronized state, respectively. $\Theta(t)$ gives the average phase of the population. Likewise, $Z_w(t)$ is a sort of weighted order parameter corresponding to the centroid of the configuration $\{f(\omega_j)e^{i\theta_j}\}$. Note that $R_w(t) \in [0, \langle|f(\omega)|\rangle]$ with $\langle\cdot\rangle$ being the ensemble average, and $\Theta_w(t) \in S^1$ locates the average phase of the associated configuration.

Until otherwise stated, the size N is assumed to be infinity, i.e., $N \to \infty$ (thermodynamic limit). Throughout this paper, we assume that the natural frequency distribution $g(\omega)$ and the correlation function $f(\omega)$ are unimodal and symmetric functions with respect to ω, i.e., $g(\omega) = g(-\omega)$, $f(\omega) = f(-\omega)$, and $g'(\omega) < 0$, $f'(\omega) < 0$ for $\omega > 0$. As we shall see below, these assumptions are essential for the emergence of collective behaviors as well as their dynamical properties. In particular, the special form of $g(\omega)$ and $f(\omega)$ implies that the dynamical system is unchanged under the reflection $(\theta_i, \omega_i) \to (-\theta_i, -\omega_i)$ and phase shift $\theta_i \to \theta_i + \alpha$ (with α being a constant phase shift) actions. Remarkably, the reflection and phase shift symmetry rolls out the occurrence of Hopf bifurcation of the incoherent state that leads to the time-dependent (standing wave) states in the supercritical region. In the following, we will discuss in detail the model defined above by investigating several aspects of its generic dynamical properties both theoretically and numerically.

III. CRITICAL POINT FOR SYNCHRONIZATION TRANSITION

In this section, we pay particular attention to determining the critical coupling for the phase transition, beyond which the stability of the incoherent state is inverted and the system gets synchronized. As we shall see below, the generic correlation function $f(\omega)$ plays an important role for the onset of synchronization, as well as the emergence of long term dynamics.

Passing to the thermodynamic limit, a smooth single oscillator distribution $\rho(\theta, \omega, t)$ is needed to characterize the dynamics of Eq. (1), in which $\rho(\theta, \omega, t)\mathrm{d}\theta$ denotes the fraction of oscillators with their phases lying in the interval $(\theta, \theta + \mathrm{d}\theta)$ at a fixed time t and a given natural frequency ω. The distribution $\rho(\theta, \omega, t)$ is 2π-periodic function of θ that satisfies the normalization condition on S^1,

$$\int_0^{2\pi} \rho(\theta, \omega, t)\mathrm{d}\theta = 1. \tag{4}$$

Correspondingly, the two order parameters $Z(t)$ and $Z_w(t)$ in the limit $N \to \infty$ become

$$Z(t) = \int_{-\infty}^{+\infty}\int_0^{2\pi} e^{i\theta}\rho(\theta, \omega, t)g(\omega)\mathrm{d}\theta\mathrm{d}\omega \tag{5}$$

and

$$Z_w(t) = \int_{-\infty}^{+\infty}\int_0^{2\pi} e^{i\theta}\rho(\theta, \omega, t)f(\omega)g(\omega)\mathrm{d}\theta\mathrm{d}\omega. \tag{6}$$

Since the dynamics of Eq. (1) is deterministic, the number of oscillators is conservative under the dynamics, which implies a continuity equation of the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v)}{\partial\theta} = 0. \tag{7}$$

For the in-coupling case, the velocity field $v(\theta, \omega, t)$ is given by

$$v_{\text{in}}(\theta, \omega, t) = \omega + \frac{Kf(\omega)}{2i}(Z(t)e^{-i\theta} - \bar{Z}(t)e^{i\theta}). \tag{8}$$

For the out-coupling, the velocity is

$$v_{\text{out}}(\theta, \omega, t) = \omega + \frac{K}{2i}(Z_w(t)e^{-i\theta} - \bar{Z}_w(t)e^{i\theta}). \tag{9}$$

Throughout the paper, the bar denotes the complex conjugate.

For the incoherent state, the system is totally disordered, at which the two order parameters $Z(t) = Z_w(t) = 0$. The stationary distribution corresponds to $\rho_0(\theta) = \frac{1}{2\pi}$, which is a trivial solution of Eq. (7). Next, we carry out a linear stability analysis of Eq. (7) around the fixed point ρ_0, which allows for obtaining the critical point for the onset of synchronization.

Let $\rho(\theta, \omega, t) = \rho_0 + \varepsilon\eta(\theta, \omega, t)$, with ε $(0 < \varepsilon \ll 1)$ and $\eta(\theta, \omega, t)$ being the perturbative magnitude and function, respectively. Inserting the perturbation into Eq. (7) up to the linear order of ε, the linearized dynamics is obtained as

$$\frac{\partial\eta}{\partial t} = -\omega\frac{\partial\eta}{\partial\theta} + \frac{Kf(\omega)}{2\pi}\mathrm{Re}(e^{-i\theta}Z[\eta]). \tag{10}$$

For convenience, we here take the in-coupling as an example to showcase the program, and the out-coupling case can be discussed in a similar way. $\mathrm{Re}(\cdot)$ represents the real part, and the perturbed order parameter becomes

$$Z[\eta] = \int_{-\infty}^{+\infty}\int_0^{2\pi} e^{i\theta}\eta(\theta, \omega, t)g(\omega)\mathrm{d}\theta\mathrm{d}\omega. \tag{11}$$

The perturbation function $\eta(\theta, \omega, t)$ is 2π-periodic with respect to θ, which implies a Fourier expansion of the form

$$\eta(\theta, \omega, t) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\eta_n(\omega, t)e^{in\theta}. \tag{12}$$

Here, $\eta_0(\omega, t) = 0$ that is due to the normalization condition of $\rho(\theta, \omega, t)$. Because the function $\eta(\theta, \omega, t)$ is real, we thus have $\eta_{-n}(\omega, t) = \bar{\eta}_n(\omega, t)$. Consequently, the order parameter $Z[\eta]$ is reduced to

$$Z[\eta] = \int_{-\infty}^{+\infty}\eta_{-1}(\omega, t)g(\omega)\mathrm{d}\omega = \hat{g}\eta_{-1}(\omega, t), \tag{13}$$

where $\hat{g}$ denotes the integral operator. Obviously, Eq. (13) implies that only the $1-th$ Fourier modes $(\eta_{\pm1})$ have contribution to the order parameter. In this sense, the evolution of the high-order Fourier modes $(|n| > 1)$ can be disregarded, since they do not contribute to the order parameter.

The linearized dynamics Eq. (10) can be proceeded in each independent Fourier subspace. In the $1-th$ Fourier subspace, it yields

$$\frac{d\eta_1}{dt} = -i\omega\eta_1 + \frac{Kf(\omega)}{2}\hat{g}\eta_1. \quad (14)$$

The stability information of the incoherent state is totally determined by $1-th$ Fourier modes. To this end, let $\frac{d\eta_1}{dt} = \lambda\eta_1$, with $\lambda \in \mathbb{C}$ being the eigenvalue of the linearized dynamics. Accordingly, the eigenfunction $\eta_1(\omega)$ is solved as

$$\eta_1(\omega) = \frac{Kf(\omega)}{2}\frac{\hat{g}\eta_1}{\lambda + i\omega}. \quad (15)$$

Applying the integral operator $\hat{g}$ to both sides of Eq. (15), and using the fact that $\hat{g}\eta_1$ is a constant, we thus obtain the eigenvalue equation describing the linear growth of the perturbation, which yields

$$\frac{1}{K} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{f(\omega)g(\omega)}{\lambda + i\omega}d\omega. \quad (16)$$

We emphasize that the eigenvalue equation (16) remains true for the out-coupling scenario. This is because the weighted order parameter $Z_w[\eta]$ now becomes

$$Z_w[\eta] = \int_{-\infty}^{+\infty}\eta_{-1}(\omega,t)f(\omega)g(\omega)d\omega = \hat{g}_w\eta_{-1}(\omega,t), \quad (17)$$

where $\hat{g}_w$ stands for the weighted integral operator. The associated perturbation function $\eta(\theta,\omega,t)$ evolves according to

$$\frac{\partial\eta}{\partial t} = -\omega\frac{\partial\eta}{\partial\theta} + \frac{K}{2\pi}Re(e^{-i\theta}Z_w[\eta]). \quad (18)$$

Similarly, the analysis can be performed in each Fourier subspace. Therefore, the $1-th$ Fourier mode η_1 is governed by

$$\frac{d\eta_1}{dt} = -i\omega\eta_1 + \frac{K}{2}\hat{g}_w\eta_1. \quad (19)$$

Eventually, the same eigenvalue equation as described by Eq. (16) can be reobtained. On that basis, it can be concluded that the stability property of the incoherent state is unaffected by the possession of the coupling weight. In other words, the in-coupling and out-coupling have the same critical point for phase transition provided that the correlation function has the same form.

To get analytical insights for the eigenvalue λ and the critical point, we first note that, for a unimodal $g(\omega)$ and $f(\omega)$, it can be proven rigorously that $\lambda \in \mathbb{R}$ [40]. The corresponding eigenvalue function begets

$$\frac{1}{K} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\lambda}{\lambda^2+\omega^2}f(\omega)g(\omega)d\omega. \quad (20)$$

Second, we remark that $\lambda > 0$ so long as $K > 0$. Third, the right-hand side of Eq. (20) is a strictly decreasing function with respect to λ, which indicates that the roots to Eq. (20) do not exist for a sufficiently small value of K. Specifically, for $K < K_c$, the eigenvalues are absent. Therefore, the incoherent state remains neutrally stable to perturbation characterized by the continuous spectrum $\lambda = i\omega$ sitting on the whole imaginary axis. Conversely, the eigenvalue λ will across through the imaginary axis as the coupling K increases, and the incoherent state becomes linearly unstable once $K > K_c$. As a result, the critical coupling K_c is obtained by imposing the condition $\lambda \to 0^+$,

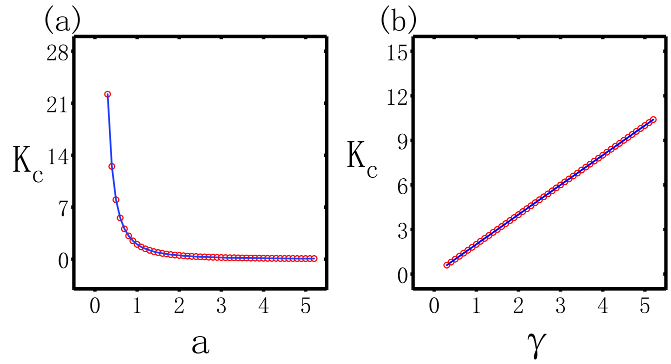

FIG. 1. Critical point K_c vs. the control parameters with $g(\omega) = \frac{\gamma}{\pi(\omega^2+\gamma^2)}$ and $f(\omega) = a^2e^{-\omega^2}$. The size $N = 10000$ is used to simulate the system. (a) $\gamma = 1.0$. (b) $a = 1.0$. The solid lines correspond to the theoretical predictions and the circles are numerical simulations.

$$K_c = \frac{2}{\pi g(0)f(0)}, \quad (21)$$

which uses the fact that the following identity holds:

$$\lim_{\lambda\to 0^+}\frac{\lambda}{\lambda^2+\omega^2} = \pi\delta(\omega). \quad (22)$$

The formula above is a straightforward generalization of the result obtained by Kuramoto. This analytical formulism permits us to dissect the mathematical consequence of the generic correlation function that gives rise to the occurrence of synchronization. It reveals that the value of the correlation function at its origin plays a critical role for the onset of synchronization. On the one hand, for a small value of $f(0)$, the correlation hinders the occurrence of synchronization, thereby leading to the increase of K_c. On the other hand, for a large value of $f(0)$, the correlation makes synchronization transition easier to happen characterized by a decreasing effect of K_c.

To make sense of it, we make specific choices of $g(\omega)$ and $f(\omega)$, e.g., $g(\omega) = \frac{\gamma}{\pi(\omega^2+\gamma^2)}$ and $f(\omega) = a^2e^{-\omega^2}$. Then, $K_c = \frac{2\gamma}{a^2}$. Figure 1 plots the critical coupling K_c as a function of the control parameters. In both panels, the solid lines and the circles are, respectively, the theoretical predictions and numerical simulations, which match well. The increasing and decreasing behaviors of K_c indicate that the critical point for the onset of synchronization can either be postponed or accelerated by adjusting the width of frequency distribution and the magnitude of correlation function at its center.

IV. SUBCRITICAL REGIME: RELAXATION DYNAMICS

As we have already shown that the critical point for phase transition and the stability properties of the incoherent state do not depend on the specific location of the correlation function, and both coupling schemes yield the same eigenspectrum structure. More importantly, it demonstrates that the microscopic incoherent state remains neutrally stable to perturbation in the subcritical region $K < K_c$. Nevertheless, as a well-known result, the macroscopic order parameters actually decay to zero in the long time limit before the synchronization transition [41]. In this section, we continue to explore the generic properties of the relaxation dynamics of the incoherent state by taking into account both the in-coupling and out-coupling circumstances. As we shall see below, such a decaying effect is far more generic, which is significantly shaped by the coupling strategies [42,43].

Addressing such an issue may be helpful in giving insights into the intrinsic properties of the statistical mechanics of the equilibrium states. It has been argued that the relaxation dynamics of the equilibrium states is intimately related to the critical slowing phenomenon and the susceptibility in response to the external fields [44–48]. Remarkably, such a dephasing mechanism is akin to the Landau damping effects in plasma physics [49]. Below, we provide a framework for clarifying the inner decaying mechanism in both the in-coupling and out-coupling schemes.

A. In-coupling

As the first step, we start with the in-coupling situation. For the sake of notation, we set $\eta_{-1}(\omega, t) = z(\omega, t)$, which is governed by the following linear differential equation:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = i\omega z + \frac{Kf(\omega)}{2}\hat{g}z = \mathcal{L}z, \tag{23}$$

where $\mathcal{L}$ denotes the linear operator. It becomes apparent that the formal solution of $z(\omega, t)$ with its initial condition $z(\omega, 0)$ is expressed as

$$z(\omega, t) = e^{\mathcal{L}t}z(\omega, 0). \tag{24}$$

The operator $e^{\mathcal{L}t}$ is calculated in terms of the inverse Laplace transform, which reads

$$e^{\mathcal{L}t} = \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}(s-\mathcal{L})^{-1}\mathrm{d}s \tag{25}$$

with $t > 0$ and $x > 0$.

The key task herein is to derive the explicit expression for the resolvent $(s-\mathcal{L})^{-1}$. For this purpose, let $\phi(\omega)$ be an arbitrary smooth function, we define

$$\begin{aligned}\mathbf{B}(s)\phi(\omega) &= (s-\mathcal{L})^{-1}\phi(\omega)\\ &= \left(s - i\omega - \frac{Kf(\omega)}{2}\hat{g}\right)^{-1}\phi(\omega).\end{aligned} \tag{26}$$

Multiplying both sides of Eq. (26) by the operator $(s-\mathcal{L})$, we have that

$$\mathbf{B}(s)\phi(\omega) = (s-i\omega)^{-1}\phi(\omega) + \frac{Kf(\omega)}{2}(s-i\omega)^{-1}\hat{g}[\mathbf{B}(s)\phi(\omega)]. \tag{27}$$

Once again, applying the integral operator $\hat{g}$ to both sides of Eq. (27), we get

$$\hat{g}[\mathbf{B}(s)\phi] = \hat{g}\left[\frac{\phi}{s-i\omega}\right] + \frac{K}{2}\hat{g}\left[\frac{f(\omega)}{s-i\omega}\right]\hat{g}[\mathbf{B}(s)\phi], \tag{28}$$

which is rearranged as

$$\hat{g}[\mathbf{B}(s)\phi] = \frac{\hat{g}\left[\frac{\phi}{s-i\omega}\right]}{1-\frac{K}{2}\hat{g}\left[\frac{f(\omega)}{s-i\omega}\right]}. \tag{29}$$

Turning to the order parameter $Z(t)$, from the definition Eq. (13), we have

$$\begin{aligned}Z(t) &= \hat{g}z(\omega, t)\\ &= \hat{g}[e^{\mathcal{L}t}z(\omega, 0)]\\ &= \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}\hat{g}[(s-\mathcal{L})^{-1}z(\omega, 0)]\mathrm{d}s\\ &= \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}\hat{g}[\mathbf{B}(s)z(\omega, 0)]\mathrm{d}s.\end{aligned} \tag{30}$$

Using the results above, the order parameter is expressed as a simple form yielding

$$Z(t) = \mathbf{L}^{-1}\left[\frac{D(s)}{1-Q_w(s)}\right] \tag{31}$$

with $\mathbf{L}^{-1}$ being the inverse Laplace transform and the two characteristic functions are respectively defined by

$$D(s) = \hat{g}\left[\frac{z(\omega, 0)}{s-i\omega}\right] \tag{32}$$

and

$$Q_w(s) = \frac{K}{2}\hat{g}_w\left[\frac{1}{s-i\omega}\right]. \tag{33}$$

B. Out-coupling

The discussion of the out-coupling case becomes slightly intricate. As before, the governing equation of the $1-th$ Fourier mode $\eta_{-1}(\omega, t) = z(\omega, t)$ now reduces to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = i\omega z + \frac{K}{2}\hat{g}_w z = \mathcal{L}_w z \tag{34}$$

with $\mathcal{L}_w$ being the weighted linear operator. Similarly, the formal solution of $z(\omega, t)$ with the initial value $z(\omega, 0)$ is given by

$$z(\omega, t) = e^{\mathcal{L}_w t}z(\omega, 0), \tag{35}$$

and associated operator $e^{\mathcal{L}_w t}$ is defined by means of the inverse Laplace transform,

$$e^{\mathcal{L}_w t} = \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}(s-\mathcal{L}_w)^{-1}\mathrm{d}s \tag{36}$$

with $t > 0$ and $x > 0$.

Analogously, the key step is to obtain the explicit expression for the resolvent $(s-\mathcal{L}_w)^{-1}$. To this aim, let

$$\begin{aligned}\mathbf{A}(s)\phi(\omega) &= (s-\mathcal{L}_w)^{-1}\phi(\omega)\\ &= \left(s - i\omega - \frac{K}{2}\hat{g}_w\right)^{-1}\phi(\omega).\end{aligned} \tag{37}$$

Multiplying both sides of Eq. (37) by the operator $(s-\mathcal{L}_w)$, we then have

$$\mathbf{A}(s)\phi(\omega) = \frac{\phi(\omega)}{s-i\omega} + \frac{\frac{K}{2}\hat{g}_w[\mathbf{A}(s)\phi(\omega)]}{s-i\omega}. \tag{38}$$

Applying the weighted integral operator $\hat{g}_w$ to both sides of Eq. (38) and using the fact that $\hat{g}_w[\mathbf{A}(s)\phi(\omega)]$ is a constant, we obtain

$$\hat{g}_w[\mathbf{A}(s)\phi] = \frac{\hat{g}_w\left[\frac{\phi}{s-i\omega}\right]}{1-\frac{K}{2}\hat{g}_w\left[\frac{1}{s-i\omega}\right]}. \tag{39}$$

With respect to the order parameters, we note that the weighted order parameter $Z_w(t)$ can be calculated in a similar way that was done in the proceeding subsection. Straightforward calculations yield

$$Z_w(t) = \mathbf{L}^{-1}\left[\frac{D_w(s)}{1-Q_w(s)}\right] \tag{40}$$

with the characteristic function $D_w(s)$ defined by

$$D_w(s) = \hat{g}_w\left[\frac{z(\omega,0)}{s-i\omega}\right]. \tag{41}$$

As for the classical order parameter $Z(t)$, it is controlled by the weighted order parameter. To see it, $Z(t)$ is calculated based on the original definition, we have that

$$\begin{aligned} Z(t) &= \hat{g}z(\omega,t) \\ &= \hat{g}[e^{\mathcal{L}_w t}z(\omega,0)] \\ &= \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}\,\hat{g}[(s-\mathcal{L}_w)^{-1}z(\omega,0)]\mathrm{d}s \\ &= \frac{1}{2\pi i}\lim_{y\to+\infty}\int_{x-iy}^{x+iy} e^{st}\,\hat{g}[\mathbf{A}(s)z(\omega,0)]\mathrm{d}s. \end{aligned} \tag{42}$$

Clearly, $Z(t)$ can be obtained by invoking Eq. (38), which is simplified as

$$Z(t) = \mathbf{L}^{-1}\left[D(s) + \frac{D_w(s)}{1-Q_w(s)}Q(s)\right]. \tag{43}$$

The corresponding characteristic function $Q(s)$ is defined by

$$Q(s) = \frac{K}{2}\hat{g}\left[\frac{1}{s-i\omega}\right]. \tag{44}$$

C. Discussion

Up to now, we have finished the derivations of the order parameters described by Eqs. (31), (40), and (43). Remarkably, the inverse Laplace transforms are totally controlled by the poles of the denominator appearing in each characteristic function, which are precisely the resonant poles causing the order parameters to decay. In order to invert the Laplace transform, the calculations should be analytically continued to the left complex plane with $\mathrm{Re}(s) < 0$. It reveals that both the natural frequency distribution $g(\omega)$ and the correlation function $f(\omega)$ play a central role for determining the resonant poles and the decaying mechanisms of the order parameters. In fact, $f(\omega) \equiv 1$, then $D_w(s) = D(s)$, and $Q_w(s) = Q(s)$, so the three order parameters described by Eqs. (31), (40), and (43) are exactly the same. This is an evident result since the weight does vanish.

In the case of $K = 0$, the original order parameters for the in-coupling and out-coupling scenarios are the same, i.e., $Z(t) = \mathbf{L}^{-1}[D(s)] = \hat{g}[e^{i\omega t}z(\omega,0)]$. Correspondingly, the weighted order parameter for the out-coupling is $Z_w(t) = \mathbf{L}^{-1}[D_w(s)] = \hat{g}_w[e^{i\omega t}z(\omega,0)]$. These order parameters are respectively the Fourier transform of $g(\omega)z(\omega,0)$ and $g(\omega)f(\omega)z(\omega,0)$. It implies that each phase oscillator rotates at its own angular frequency. Furthermore, the Riemann-Lebesgue lemma ensures that the order parameters tend to zero in the long time limit provided that $z(\omega,0)$ is a smooth bounded function. Namely, the order parameters of the incoherent state always relax to zero regardless of the distribution $g(\omega)$ and correlation function $f(\omega)$.

It is worth mentioning that the three order parameters described above are significantly different from each other, even the correlation function $f(\omega)$ keeps the same form in both cases. Nonetheless, we can prove that the classical order parameters defined by Eqs. (31) and (43) are exactly the same for a particular case. To see it, we assume that the initial perturbed distribution of the incoherent state is independent of the natural frequencies, i.e., $\rho(\theta,\omega,0) = \rho(\theta,0)$. Based on this assumption, the $1-th$ Fourier mode $z(\omega,0) = \int_0^{2\pi} e^{i\theta}\rho(\theta,0)\mathrm{d}\theta = z_0$, which is a constant, we then have

$$D(s) = \frac{2z_0}{K}Q(s) \tag{45}$$

and

$$D_w(s) = \frac{2z_0}{K}Q_w(s). \tag{46}$$

Hence

$$\begin{aligned} Z^{\mathrm{out}}(t) &= \mathbf{L}^{-1}\left[\frac{D(s)-D(s)Q_w(s)+D_w(s)Q(s)}{1-Q_w(s)}\right] \\ &= \mathbf{L}^{-1}\left[\frac{D(s)}{1-Q_w(s)}\right] = Z^{\mathrm{in}}(t), \end{aligned} \tag{47}$$

where superscripts 'in' and 'out' denote the in-coupling and out-coupling, respectively.

To gain some analytical intuition of the order parameters, we choose the frequency distribution $g(\omega)$ and the correlation function $f(\omega)$ to be of the rational types, e.g., $g(\omega) = \frac{1}{\pi(\omega^2+1)}$ and $f(\omega) = \frac{1}{\omega^2+1}$. Without loss of generality, we set $z_0 = 1$ implying an identical initial distribution $\rho(\theta,0) = \delta(\theta)$. Straightforward calculations yield

$$D(s) = \frac{1}{1+s} = \frac{2}{K}Q(s) \tag{48}$$

and

$$D_w(s) = \frac{2+s}{2(1+s)^2} = \frac{2}{K}Q_w(s). \tag{49}$$

After some tedious computations, we eventually arrive at the order parameters yielding

$$\begin{aligned} Z^{\mathrm{in}}(t) = Z^{\mathrm{out}}(t) &= e^{\frac{(-8+K)t}{8}}\cosh\left[\frac{\sqrt{K(16+K)}}{8}t\right] \\ &+ \frac{e^{\frac{(-8+K)t}{8}}\sqrt{K}\sinh\left[\frac{\sqrt{K(16+K)}}{8}t\right]}{\sqrt{16+K}} \end{aligned} \tag{50}$$

and

$$\begin{aligned} Z_w^{\mathrm{out}}(t) &= \frac{1}{2}e^{\frac{(-8+K)t}{8}}\cosh\left[\frac{\sqrt{K(16+K)}}{8}t\right] \\ &+ \frac{e^{\frac{(-8+K)t}{8}}(8+K)\sinh\left[\frac{\sqrt{K(16+K)}}{8}t\right]}{2\sqrt{K(16+K)}}. \end{aligned} \tag{51}$$

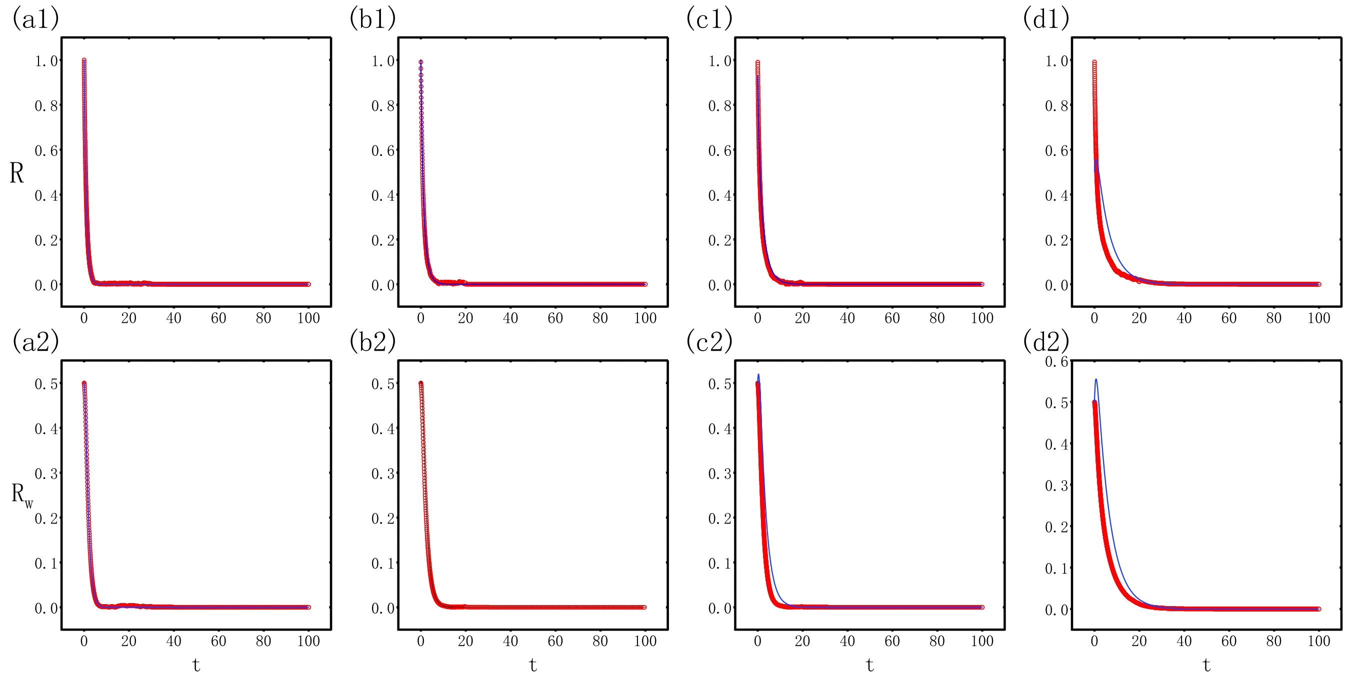

FIG. 2. Relaxation behaviors of the two order parameters in the subcritical region with $g(\omega) = \frac{1}{\pi(\omega^2+1)}$ and $f(\omega) = \frac{1}{\omega^2+1}$. (a) $K = 0$. (b) $K = 0.5$. (c) $K = 1.0$. (d) $K = 1.5$. The size $N = 100000$. The solid lines are theoretical predictions given by Eqs. (50) and (51), and the circles are numerical simulations. The initial phases of the oscillators are set to zero, it is shown that the theoretical predictions, Eqs. (50) and (51), are asymptotically valid for $K > 0$.

As shown in Fig. 2, we choose four typical values of K to confirm these asymptotical results. Although the specific expressions for the order parameters are complicated, the resonant poles can be obtained analytically, which are

$$s_{0,\pm} = -1 + \frac{K}{8} \pm \frac{\sqrt{16K + K^2}}{8}. \quad (52)$$

Notably, the first pole $s_{0,-}$ represents a stable mode that is always negative no matter how large the K is. Whereas the second pole $s_{0,+}$ goes through the negative real axis at the critical point $K_c = 2$, which is consistent with the result obtained by Eq. (21). All the results suggest that the natural frequency distribution $g(\omega)$ together with the correlation function $f(\omega)$ have significant influences on the Landau damping rooted in the occurrence of the resonant poles, as well as the decaying form of the order parameters.

It should be pointed out that there exists a discrepancy between the theory and numerics for the initial time with $K > 0$ (see Figs 2(c) and 2(d). This can be understood as follows. Theoretically, we remark that the relaxation dynamics of the macroscopic order parameters are valid only in the linear region. It implies that the corresponding microscopic perturbed distribution should be near the incoherent state. In other words, all the analytical expressions for the time-varying order parameters [see Eqs. (31), (40), and (43)] are asymptotically valid. This is because all the discussions about the relaxation dynamics are based on the framework of the linear stability analysis. Numerically, the initial phases of the oscillators are chosen to be identical to obtain the explicit formula of the order parameters. However, this initial perturbation is far away from the linear region. Nevertheless, the associated perturbation is getting closer and closer to the linear region as the evolution of time. Consequently, the corresponding deviation between the theoretical predictions and the numerical simulations will be smaller and smaller. This explains why there exists a deviation between the theory and the simulation during the initial transient time. Particularly, this feature becomes more evident when K is slightly larger than 0.

V. SUPERCRITICAL REGIME: SCALING BEHAVIORS

To have a far down insight into the synchronized dynamics induced by the frequency-coupling correlations, in this section, we discuss how the heterogeneous schemes under consideration affect the asymptotical behaviors of synchronization near the critical threshold K_c, i.e., the scaling behaviors of the order parameters.

For convenience, we start with the out-coupling scenario. According to the definition of the weighted order parameter Eq. (3), the governing equation (1) can be rewritten into the mean-field form, which is

$$\dot{\theta}_i = \omega_i + KR_w \sin(\Theta_w - \theta_i). \quad (53)$$

As stated, the special forms of $g(\omega)$ and $f(\omega)$ imply that the system possesses reflection symmetry, i.e., the dynamics remains invariant under the transformation $(\theta_i, \omega_i) \to (-\theta_i, -\omega_i)$. In turn, the mean-field phase $\Theta_w(t)$ can be set to zero by shifting initial conditions. Therefore, Eq. (53) is simplified as

$$\dot{\theta} = \omega - p \sin \theta, \quad (54)$$

where $p = KR_w$, and the index is dropped in the thermodynamic limit $N \to \infty$.

For the unimodal and symmetric functions of $g(\omega)$ and $f(\omega)$ considered in the paper, the equilibrium states require that the macroscopic order parameters approach constants in the limit $t \to \infty$, or the auxiliary parameter p is independent of time. From Eq. (54), the phase-locked condition satisfies $|\omega| < p$, which yields

$$\sin\theta_\omega = \frac{\omega}{p}, \cos\theta_\omega = \sqrt{1 - \frac{\omega^2}{p^2}}. \tag{55}$$

Otherwise, the oscillators are drifting with $|\omega| > p$ that can never be entrained by the mean field. The self-consistent equation for the weighted order parameter is expressed as

$$Z_w = R_w = \hat{g}_w \cos\theta_\omega. \tag{56}$$

Using the definitions above, Eq. (56) is reformulated as a simple form

$$\frac{1}{K} = F_w(p) = \int_{-1}^{1} \sqrt{1 - x^2} G_w(px) \mathrm{d}x, \tag{57}$$

where $x = \frac{\omega}{p}$ and the virtual frequency distribution is defined as $G_w(px) = g(px)f(px)$. As for the classical Kuramoto order parameter, it becomes

$$Z = R = \hat{g} \cos\theta_\omega = p \int_{-1}^{1} \sqrt{1 - x^2} g(px) \mathrm{d}x. \tag{58}$$

Equation (57), together with Eq. (58), describes the stationary behaviors of the order parameters for a given $g(\omega)$ and $f(\omega)$. Remarkably, the incoherent state corresponds to $p = 0$ yielding $K_c = 1/F_w(0) = 2/\pi g(0)f(0)$, which recovers Eq. (21) obtained from the linear stability analysis.

To proceed with the scaling analysis, let $K = K_c + \delta K$, with $0 < \delta K \ll 1$ being the small perturbation of the coupling strength, the resulting perturbations of other parameters are, respectively, given by $p = 0 + \delta p$, $R_w = 0 + \delta R_w$, and $R = 0 + \delta R$. Substituting these perturbations into Eq. (57) and using Taylor series expansion for both sides, we have that

$$\frac{1}{K_c} - \frac{\delta K}{K_c^2} = F_w(0) + F_w'(0)\delta p + \frac{F_w''(0)}{2}\delta p^2. \tag{59}$$

Obviously, the first term appearing in both sides of Eq. (59) cancels each other out due to Eq. (21), while the second term on the right-hand side of Eq. (59) vanishes because of the symmetry condition of $g(\omega)$ and $f(\omega)$. Thus, we have

$$\delta K = -\tfrac{1}{2} F_w''(0) K_c^2 \delta p^2. \tag{60}$$

Note that $\delta p^2 = K_c^2 \delta R_w^2$, we thus have the weighted order parameter at criticality as

$$\delta R_w = \sqrt{\frac{-2\delta K}{K_c^4 F_w''(0)}}. \tag{61}$$

Regarding the Kuramoto order parameter, we obtain

$$\delta R = \delta\left[p \int_{-1}^{1} \sqrt{1 - x^2} g(px) \mathrm{d}x\right] = \frac{\pi g(0)}{2}\delta p. \tag{62}$$

Using Eq. (61) and $\delta p = K_c \delta R_w$, we eventually arrive at the asymptotical behavior of the order parameter near K_c yielding

$$\delta R = C_{\text{out}}\sqrt{\delta K} = \sqrt{\frac{-16\delta K}{\pi K_c^4 [g''(0) + g(0)f''(0)]}}, \tag{63}$$

which we have used the fact that

$$\begin{aligned} F_w''(0) &= \int_{-1}^{1} x^2 \sqrt{1 - x^2} G_w(0) \mathrm{d}x \\ &= \frac{\pi}{8}[g''(0) + g(0)f''(0)] \end{aligned} \tag{64}$$

and $f(0)$ is set to be 1 that can be always achieved by rescaling the coupling strength.

As for the in-coupling scenario, the discussion can be performed in a similar way. To this end, the mean-field equation reads

$$\dot{\theta}_i = \omega_i + K f(\omega_i) R \sin(\Theta - \theta_i). \tag{65}$$

Similarly, by assuming $\Theta = 0$ and defining $q = KR$, the self-consistent equation is established as

$$\frac{1}{K} = \int_{-1}^{1} \sqrt{1 - y^2} G(qy) \mathrm{d}y, \tag{66}$$

where $y = \mathcal{F}(\omega)/q$, and $\mathcal{F}(\omega)$ is defined by $\mathcal{F}(\omega) = \omega/f(\omega)$. The virtual frequency distribution now becomes

$$G(qy) = \frac{g[\mathcal{F}^{-1}(qy)] f[\mathcal{F}^{-1}(qy)]}{1 - qy f'[\mathcal{F}^{-1}(qy)]}, \tag{67}$$

where $\mathcal{F}^{-1}$ denotes the inverse function. Likewise, imposing small perturbation away from the critical coupling K_c and using Taylor series expansion about Eq. (66), we finally obtain the asymptotical behavior of the order parameter in the neighborhood of K_c yielding

$$\delta R = C_{\text{in}}\sqrt{\delta K} = \sqrt{\frac{-16\delta K}{\pi K_c^4 [g''(0) + 3g(0)f''(0)]}}. \tag{68}$$

Taken together, we conclude that the square-root scaling law of the order parameters maintains for both in- and out-coupling correlations manifesting a mean-field character. However, we reveal that the asymptotical coefficient for the Kuramoto order parameter above K_c differs significantly. For the considered model, $g''(0) < 0$ and $f''(0) < 0$, we thus have $C_{\text{out}} > C_{\text{in}}$ (as shown in Fig. 3). This indicates that the ability of synchronization for the out-coupling correlation is profoundly enhanced compared with the in-coupling correlation near the critical point K_c, i.e., synchronization is facilitated by the out-coupling strategy even for the same correlation function.

VI. CONCLUSIONS

To summarize, we considered a generalized Kuramoto model of globally coupled phase oscillators, in which the intrinsic frequencies and the global coupling are chosen deterministically in a way such that these two types of disorder are correlated via a generic weighted function. We explored the

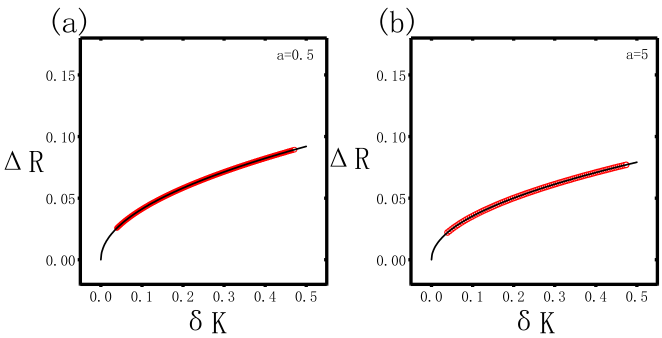

FIG. 3. The difference between two Kuramoto order parameters considered in the in-coupling and out-coupling scenarios vs. the perturbed coupling strength $\delta K = K - K_c$, i.e., $\triangle R = R_{\mathrm{out}} - R_{\mathrm{in}}$. $g(\omega) = \frac{1}{\pi(\omega^2+1)}$, $f(\omega) = e^{-a\omega^2}$. (a) and (b) correspond to different parameters a. The solid lines are theoretical predictions with $\triangle R = (C_{\mathrm{out}} - C_{\mathrm{in}})\sqrt{\delta K}$. The circles are numerical simulations with $N = 100\,000$ and $K_c = 2$.

impacts of the correlation function on the emergent dynamics by taking into account both the in-coupling and out-coupling schemes. We revealed that the coupling strategies of the heterogeneity, the correlation function and natural frequency distribution play the central role in system's collective dynamics, as well as the critical behaviors for the onset of synchrony. In particular, we developed a general framework allowing for grasping the main mechanism of the dynamical phenomena underlying the coupled system.

With the aid of the linear stability analysis, we deduced the analytical formula of the critical point for the onset of synchronization. Interestingly, we found that the expression for the critical point K_c does not change by the location of heterogeneity. However, its value is strongly influenced by the values of the correlation function and frequency distribution at their centers. Specifically, we showed that both the in-coupling and out-coupling schemes yield the same eigenspectrum structure of the incoherent state, thereby leading to the same stability property of the system in $K < K_c$. Consequently, the critical point K_c can be obtained by imposing the condition corresponding to the instability of the incoherent state.

In terms of the methodology of the analytical continuation, we further explored the relaxation dynamics of the incoherent state in the subcritical region. Furthermore, we developed a general framework for systematically capturing the time-varying behaviors of the order parameters, the two types of the order parameter are established in the analytical form described by Eqs. (31), (40), and (43). We proved that the microscopic incoherent state remains neutrally stable to perturbation, whereas the macroscopic order parameters relax to zero in the long time limit. We uncovered that such a decaying mechanism arises from the resonant poles on the left half complex plane, which are remarkably influenced by the specific forms of the correlation function and its locations. In particular, for the case of the rational $g(\omega)$ and $f(\omega)$, the order parameters are expressed as a formula containing several exponential components, in which each ingredient locates a resonant pole that goes through the negative real axis at a critical point manifesting the instability of the asynchronous state.

Lastly, we explored the asymptotical behaviors of the order parameters in the supercritical regime. Using the self-consistent equations and perturbed approach in the vicinity of K_c, we identified that the square-root scaling law of both order parameters remains owing to the mean-field character. Nevertheless, we untangled that synchronization is significantly improved by the out-coupling scheme even for the same correlation function.

We hope that our work may provide remarkable insights for better understanding the generic statistical mechanical properties of the steady states to synchronization in complex systems.

ACKNOWLEDGMENT

This work is supported by the National Natural Science Foundation of China (Grant No. 11905068) and the Scientific Research Funds of Huaqiao University (Grant No. ZQN-810).

[1] S. H. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hypernion, New York, 2003).

[2] D. Witthaut, F. Hellmann, and J. Kurths, Collective nonlinear dynamics and self-organization in decentralized power grids, Rev. Mod. Phys. **94**, 015005 (2022).

[3] J. Wu and X. Li, Collective synchronization of Kuramoto-oscillator networks, IEEE Circuits and Systems Magazine (2020).

[4] Y. Kuramoto, in *International Symposium on Mathematical problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics No. 30 (Springer, New York, 1975), p. 420.

[5] S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, Physica D **143**, 1 (2000).

[6] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys. **77**, 137 (2005).

[7] A. Pikovsky and M. Rosenblum, Dynamics of globally coupled oscillators: Progress and perspectives, Chaos **25**, 097616 (2015).

[8] C. Xu, X. Wang, and P. S. Skardal, Universal scaling and phase transitions of coupled phase oscillator populations, Phys. Rev. E **102**, 042310 (2020).

[9] H. Wang and X. Li, Synchronization and chimera states of frequency-weighted Kuramoto-oscillator networks, Phys. Rev. E **83**, 066214 (2011).

[10] J. Gómez-Gardeñes, S. Gómez, and A. Arenas, Explosive Synchronization Transitions in Scale-Free Networks, Phys. Rev. Lett. **106**, 128701 (2011).

[11] S. Boccaletti, J. A. Almendral, and S. Guan, Explosive transitions in complex networks structure and dynamics: Percolation and synchronization, Phys. Rep. **660**, 1 (2016).

[12] A. R. Francisco, P. Thomas, and K. Juergen, Kuramoto model in complex networks, Phys. Rep. **610**, 1 (2016).

[13] R. D'Souza, J. Gómez-Gardeñes, and J. Nagler, Explosive phenomena in complex networks, Adv. Phys. **68**, 123 (2019).
[14] C. Xu, X. Wang, and P. S. Skardal. Generic criterion for explosive synchronization in heterogeneous phase oscillator populations, Phys. Rev. Res. **4**, L032033 (2022).
[15] X. Zhang, S. Boccaletti, S. Guan, and Z. Liu, Explosive Synchronization in Adaptive and Multilayer Networks, Phys. Rev. Lett. **114**, 038701 (2015).
[16] C. Kuehn and C. Bick, A universal route to explosive phenomena, Sci. Adv. **7**, eabe3824 (2021).
[17] T. Wu, S. Huo, K. Alfaro-Bittner, S. Boccaletti, and Z. Liu, Double explosive transition in the synchronization of multilayer networks, Phys. Rev. Res. **4**, 033009 (2022).
[18] S. Chandra, M. Girvan, and E. Ott, Continuous versus Discontinuous Transitions in the D-Dimensional Generalized Kuramoto Model: Odd D is Different, Phys. Rev. X **9**, 011002 (2019).
[19] W. Zou, M. Zhan, and J. Kurths, Phase transition to synchronization in generalized Kuramoto model with low-pass filter, Phys. Rev. E **100**, 012209 (2019).
[20] A. D. Kachhvah and S. Jalan, Explosive synchronization and chimera in interpinned multilayer networks, Phys. Rev. E **104**, L042301 (2021).
[21] A. Navas, J. A. Villacorta-Atienza, I. Leyva, J. A. Almendral, I. Sendiña-Nadal, and S. Boccaletti, Effective centrality and explosive synchronization in complex networks, Phys. Rev. E **92**, 062820 (2015).
[22] S. Yang, J. Park, and B. Kim, Discontinuous phase transition in the Kuramoto model with asymmetric dynamic interaction, Phys. Rev. E **102**, 052207 (2020).
[23] A. Kumar, S. Jalan, and A. Deep Kachhvah, Interlayer adaptation-induced explosive synchronization in multiplex networks, Phys. Rev. Res. **2**, 023259 (2020).
[24] H. Hong and S. H. Strogatz, Kuramoto Model of Coupled Oscillators with Positive and Negative Coupling Parameters: An Example of Conformist and Contrarian Oscillators, Phys. Rev. Lett. **106**, 054102 (2011).
[25] C. Bick, M. Goodfellow, C. R. Laing, and E. A. Martens, Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: A review, J. Math. Neurosci. **10**, 9 (2020).
[26] D. Iatsenko, S. Petkoski, and P. V. E. McClintock, Stationary and Traveling Wave States of the Kuramoto Model with an Arbitrary Distribution of Frequencies and Coupling Strengths, Phys. Rev. Lett. **110**, 064101 (2013).
[27] D. Iatsenko, P. V. E. McClintock, and A. Stefanovska, Glassy states and super-relaxation in populations of coupled phase oscillators, Nat. Commun. **5**, 4118 (2014).
[28] X. Zhang, X. Hu, J. Kurths, and Z. Liu, Explosive synchronization in a general complex network, Phys. Rev. E **88**, 010802(R) (2013).
[29] I. Leyva, J. A. Almendral, and A. Navas, Explosive synchronization in weighted complex networks, Phys. Rev. E **88**, 042808 (2013).
[30] H. Bi, X. Hu, and S. Boccaletti, Coexistence of Quantized, Time Dependent, Clusters in Globally Coupled Oscillators, Phys. Rev. Lett. **117**, 204101 (2016).
[31] C. Xu, J. Gao, and H. Xiang, Dynamics of phase oscillators with generalized frequency-weighted coupling, Phys. Rev. E **94**, 062204 (2016).
[32] C. Xu, S. Boccaletti, S. Guan, and Z. Zheng, Origin of Bellerophon states in globally coupled phase oscillators, Phys. Rev. E **98**, 050202(R) (2018).
[33] C. Xu, S. Boccaletti, and Z. Zheng, Universal phase transitions to synchronization in Kuramoto-like models with heterogeneous coupling, New J. Phys. **21**, 113018 (2019).
[34] Y. Xiao, W. Jia, C. Xu, and Z. Zheng, Synchronization of phase oscillators in the generalized Sakaguchi–Kuramoto model, Europhys. Lett. **118**, 60005 (2017).
[35] X. Tang, H. Lü, and C. Xu, Exact solutions of the abrupt synchronization transitions and extensive multistability in globally coupled phase oscillator populations, J. Phys. A: Math. Theor. **54**, 285702 (2021).
[36] C. Xu, Y. Wu, and Z. Zheng, Partial locking in phase-oscillator populations with heterogenous coupling, Chaos **32**, 063106 (2022).
[37] Y. Wu, Z. Zheng, and C. Xu, Synchronization dynamics of phase oscillator populations with generalized heterogeneous coupling, Chaos Solitons Fractals **164**, 112680 (2022).
[38] C. Xu, X. Tang, K. Alfaro-Bittner, S. Boccaletti, and S. Guan, Collective dynamics of heterogeneously and nonlinearly coupled phase oscillators, Phys. Rev. Res. **3**, 043004 (2021).
[39] N. Lotfi, F. A. Rodrigues, and A. H. Darooneh, The role of community structure on the nature of explosive synchronization, Chaos **28**, 033102 (2018).
[40] S. H. Strogatz and R. E. Mirollo, Stability of incoherence in a population of coupled oscillators, J. Stat. Phys. **63**, 613 (1991).
[41] S. H. Strogatz, R. E. Mirollo, and P. C. Matthews, Coupled Nonlinear Oscillators below the Synchronization Threshold: Relaxation by Generalized Landau Damping, Phys. Rev. Lett. **68**, 2730 (1992).
[42] T. Qiu, X. Zhang, and J. Liu, Landau damping effects in the synchronization of conformist and contrarian oscillators, Sci. Rep. **5**, 18235 (2015).
[43] T. Pan, X. Huang, and C. Xu, Relaxation dynamics of Kuramoto model with heterogeneous coupling, Chin. Phys. B **28**, 120503 (2019).
[44] H. Chiba and I. Nishikawa, Center manifold reduction for large populations of globally coupled phase oscillators, Chaos **21**, 043103 (2011).
[45] H. Chiba, A proof of the Kuramoto conjecture for a bifurcation structure of the infinite-dimensional Kuramoto model, Ergod. Theory Dyn. Syst. **75**, 3 (2015).
[46] B. C. Coutinho, A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Kuramoto model with frequency-degree correlations on complex networks, Phys. Rev. E **87**, 032106 (2013).
[47] S. Yoon, M. S. Sindaci, and A. V. Goltsev, Critical behavior of the relaxation rate, the susceptibility, and a pair correlation function in the Kuramoto model on scale-free networks, Phys. Rev. E **91**, 032814 (2015).
[48] H. Daido, Susceptibility of large populations of coupled oscillators, Phys. Rev. E **91**, 012925 (2015).
[49] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena*, International Series of Monographs on Physics (Oxford University Press, New York, 1987).