Universal scaling of higher-order spacing ratios in Gaussian random matrices

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Higher-order spacing ratios are investigated analytically using a Wigner-like surmise for Gaussian ensembles of random matrices. For a kth order spacing ratio ($r^{(k)}, k > 1$) the matrix of dimension $2k + 1$ is considered. A universal scaling relation for this ratio, known from earlier numerical studies, is proved in the asymptotic limits of $r^{(k)} \to 0$ and $r^{(k)} \to \infty$.

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I. INTRODUCTION

Random matrix theory (RMT), introduced more than 50 years ago, has been applied successfully in various fields [1–3]. Originally it was introduced to explain the intricate spectra of heavy nuclei [4]. Later, it has found applications in complex networks [5,6], many-body physics [7–11], wireless communications [12], etc. One of the main objectives of RMT is to study the spectral fluctuations in these systems. These fluctuations can be used to characterize the different types of phases of these complex systems, for example, integrable to chaotic limits of the underlying classical systems [13–15], thermal or localized phases of condensed matter systems [9–11,16], etc. Bohigas, Giannoni, and Schmit conjectured that the eigenvalue fluctuations in a quantum chaotic system can be modeled by one of the three classical ensembles of RMT depending on the underlying symmetry. These ensembles having Dyson indices as $\beta = 1$, 2, and 4 respectively correspond to Hermitian random matrices whose entries are chosen/distributed independently, respectively, as real (GOE), complex (GUE), or quaternionic (GSE) random variables [1].

The most popular measure to model the spectral fluctuations is the nearest-neighbor (NN) level spacings, $s_i = E_{i+1} - E_i$, where E_i, $i = 1, 2, \ldots$ are the eigenvalues of the given Hamiltonian H. A surmise by Wigner states that in a time-reversal invariant system ($\beta = 1$) does not have a spin degree of freedom, these spacings are distributed as $P(s) = (\pi/2)s \exp(-\pi s^2/4)$, which indicates the level repulsion. This result is very close to the exact one which has been obtained later on [1,3,17]. For such systems, the Gaussian orthogonal ensemble (GOE) is well suited to study the statistical properties of their spectra. There are other ensembles also commonly used in RMT, namely, the Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE) having a Dyson index of $\beta = 2$ and 4, respectively. The GUE is applicable to systems without time reversal whereas GSE to spin-1/2 systems having time reversal respectively but no rotational symmetry [1,3]. The member matrices of these families are real symmetric, complex Hermitian, and quaternion self-dual, respectively [18]. These ensembles have been implemented successfully in various fields [2,19]. In this paper, the Gaussian ensembles are studied in detail and various analytical results are obtained.

When the fluctuations are studied using the spacing distribution, one needs to carry out the procedure called unfolding the spectra which takes off the system-dependent spectral properties, i.e., the average part of the density of states (DOS) [1,4,20–24]. Thus, the comprehension of the system's DOS is required. This procedure is nontrivial and cumbersome especially in many-body physics where not enough eigenvalues are available to get a good fit of the DOS [20,21,25–27]. It can reduce the accuracy of statistical tests in such systems. It is also shown that short-range correlations are not sensitive to the unfolding method whereas the long-range level correlations are strongly dependent on the unfolding procedure employed (see Ref. [27] for more details).

This challenging problem can be resolved by using the NN spacing ratio [16], because it is independent of the local DOS which implies that unfolding is not required. It is defined as $r_i = s_{i+1}/s_i$, $i = 1, 2, \ldots$. For the case of Gaussian ensembles, a Wigner-like surmise for the distribution of r_i, $P(r)$, has been obtained as follows [28,29],

$$P(r, \beta) = \frac{1}{Z_\beta} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{(1+3\beta/2)}}, \quad \beta = 1, 2, 4, \tag{1}$$

where Z_β is the normalization constant. It must be noted here that this distribution has been derived using *only* three eigenvalues with the Gaussian weight. The expression will change with the matrix dimensions N (as observed in Ref. [29] for $N = 4$) as well as the weight. Although small deviations for smaller N are observed and pointed out in Ref. [28], this works as a very good approximation for large N and in the bulk of the spectrum. The exact analytical expression for any N still remains an open question.

This distribution has found many applications to study the eigenvalue statistics in spin systems [11,16,26,30–35], in triangular billiards [36], in the Hessians of artificial neural networks [37], in the Sachdev-Ye-Kitaev model [38–42], in quantum field theory [43], and to quantify symmetries in various complex systems [44,45].

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As pointed in Refs. [28,29], the distributions in Eq. (1) are universal, i.e., they can be applied without any unfolding or renormalization to the eigenvalues of complex physical systems. It also shows an interesting behavior (thus, universal) as follows:

$$P(r,\beta) \to r^{\beta} \quad \text{for } r \to 0,$$
$$P(r,\beta) \to r^{-2-\beta} \quad \text{for } r \to \infty. \tag{2}$$

A correction ansatz $\delta P_{\text{fit}}(r)$ was given to Eq. (1) such that $P(r) + \delta P_{\text{fit}}(r)$ fits very well for all values of N [28], where

$$\delta P_{\text{fit}}(r) = \frac{C}{(1+r)^2}\left[\left(r+\frac{1}{r}\right)^{-\beta} - c_\beta\left(r+\frac{1}{r}\right)^{-1-\beta}\right]. \tag{3}$$

Here, C and c_β are some constants. It can be seen that despite this correction term the universal behavior remains unchanged in Eq. (2).

Variants of these spacings are proposed and applied to various systems [29,46–48] which includes the generalization to the complex eigenvalues [49–54].

In this paper, we study the nonoverlapping kth order spacing ratios, which are defined such that no eigenvalue is common between the spacings in the numerator and denominator. It is defined as follows:

$$r_i^{(k)} = \frac{s_{i+k}^{(k)}}{s_i^{(k)}} = \frac{E_{i+2k} - E_{i+k}}{E_{i+k} - E_i}, \quad i, k = 1, 2, 3, \ldots. \tag{4}$$

The case $k = 1$ corresponds to the earlier solved case from Ref. [28]. Its distribution has found applications to study higher-order fluctuation statistics in the Gaussian [55], Wishart [56], and circular ensembles [55]. An important scaling relation in these cases, in the asymptotic limit of $N \to \infty$ and in bulk of the spectra, by extensive numerical computations is given as follows [55,56]:

$$P^k(r,\beta) = P(r,\beta'), \quad \beta \geqslant 1,$$
$$\beta' = \frac{k(k+1)}{2}\beta + (k-1), \quad k \geqslant 1. \tag{5}$$

It means the distribution of the kth order spacing ratio for a given β ensemble is the same as that of NN spacing ratios of some other ensemble with a Dyson index $\beta'(> \beta)$. It should be noted that the exact analytical expression for any $k \geqslant 2$ and any N is not known yet but the numerics suggest that Eq. (5) works very well for large enough N and in the bulk of the spectra. For given k, the effect of increasing N is studied numerically in Ref. [55]. There it is shown that for given k, however large, the fitted β' converges to the value given in Eq. (5) as N is increased. For smaller N, we expect the same expression in Eq. (3) can be used as the correction term but with a modified index β'. For this, we have assumed that for large N the asymptotic behavior for small and large r is the same for both $P(r,\beta')$ and $\delta P_{\text{fit}}(r,\beta')$ for $k \geqslant 2$ [28]. It should be noted that Eqs. (1) and (3) taken together still represent an approximation. Thus, it is more likely that the exact (currently unknown) expression for the kth spacing distribution also shares the same asymptotics of Eq. (5).

This relation has been employed successfully to various physical systems such as chaotic billiards, Floquet systems, circular ensembles, spin chains, observed stock market, etc. [9,30,55–59], to estimate the number of symmetries in complex physical systems [44,45]. It should be noted that a similar scaling relation between the higher-order and NN *spacing distributions* has been proposed earlier in Refs. [60,61], and later proved partly in Ref. [56] and completely in Ref. [9] using a Wigner-like surmise for the Gaussian ensembles. It is shown numerically in Ref. [9] using random spin systems, nontrivial zeros of the Riemann ζ function, and a Gaussian ensemble, that as N is increased, the deviations from the surmise become smaller and smaller. Although the bulk statistics, for given β, is the same in these three ensembles (Gaussian+Wishart+circular) in the large-N limit, the physical systems described by them are very different from each other [2].

It should be noted that the result in Eq. (5) for the spacing ratios is a purely numerical one except for few special cases [62–64]. Thus, a complete analytical understanding of this result is lacking. In this paper, we give partial analytical support to it since proving the entire result is mathematically challenging. If this result is correct, then by using the universality aspect as per Eq. (2) one can conclude that, for the higher-order spacing ratios, the following must be true,

$$P^k(r,\beta) = P(r,\beta') \to r^{\beta'} \quad \text{for } r \to 0,$$
$$P^k(r,\beta) = P(r,\beta') \to r^{-2-\beta'} \quad \text{for } r \to \infty, \tag{6}$$

with β' as per Eq. (5). In this paper, we derive analytically Eq. (6) using a Wigner-like surmise for the Gaussian ensembles.

The structure of the paper is as follows: In Sec. II we present the results for the case $k = 2$. In Sec. III (Sec. IV) the general result for any k is provided in the limit $r \to 0$ ($r \to \infty$). In Sec. V the case of uncorrelated spectra is studied in the asymptotic limits and related to the results from the Gaussian ensembles. Finally, in Sec. VI a summary of the results and conclusions are presented.

II. RESULTS: $k = 2$ CASE

Before entering into the main results we would like to mention that throughout the paper our calculations are restricted to the simplest and lowest matrix dimensions N such that for a given order k, $N = 2k + 1$ (which is our Wigner-like surmise). One should note that, in order to study k-order spacing ratios one should have at least $2k + 1$ levels to be in the RMT regime. In a Hamiltonian system these levels E_i become eigenenergies. Earlier studies from Ref. [65] indicate that the difference $E_{2k+1} - E_1$ should be less than the system's Thouless energy (E_c) for RMT to hold true. This is an important point to be noted when applying our results to various physical systems [2]. Let us first start with the joint probability distribution function (joint pdf) of the Gaussian ensemble which is given as follows,

$$f(\{E_l\}) \propto \prod_{1\leqslant i<j\leqslant N} |E_i - E_j|^\beta \exp\left(-A\sum_{i=1}^{N} E_i^2\right), \quad (7)$$

where $\beta = 1, 2$, and 4 for GOE, GUE, and GSE, respectively [1,3]. Without loss of generalities, we will be assuming $E_1 \leqslant E_2 \leqslant \cdots \leqslant E_N$ throughout this paper. First, consider the case of $k = 2$ and general β. Here, for a Wigner-like surmise, we need to have five eigenvalues [9]. Then we get

$$r^{(2)} = \frac{E_5 - E_3}{E_3 - E_1}. \quad (8)$$

Then the distribution $P(r^{(2)})$ becomes [9]

$$P(r^{(2)}) \propto \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{1\leqslant i<j\leqslant 5} |E_i - E_j|^\beta \times \exp\left(-A\sum_{i=1}^{5} E_i^2\right)\delta\left(r^{(2)} - \frac{E_5 - E_3}{E_3 - E_1}\right)\prod_{i=1}^{5} dE_i. \quad (9)$$

We first change the variables to $x_i = E_{i+1} - E_i$ for $i = 1$–4 and $x_5 = \sum_{i=1}^{5} E_i$ [9]. Then $P(r^{(2)})$ simplifies to [9]

$$P(r^{(2)}) \propto \int_0^\infty\cdots\int_0^\infty \frac{\partial(E_1,\ldots,E_5)}{\partial(x_1,\ldots,x_5)}\left(\prod_{i=1}^{4}\prod_{j=i}^{4}\left|\sum_{l=i}^{j} x_l\right|^\beta\right)\exp\left\{-\frac{A}{5}\left[\sum_{i=1}^{4}\sum_{j=i}^{4}\left(\sum_{l=i}^{j} x_l\right)^2 + x_5^2\right]\right\}\delta\left(r^{(2)} - \frac{x_3 + x_4}{x_1 + x_2}\right)\prod_{i=1}^{5} dx_i.$$

Here, the Jacobian $\frac{\partial(E_1,\ldots,E_5)}{\partial(x_1,\ldots,x_5)}$ and integral for x_5 are constants that can be absorbed into the normalization factor, and using the property of the delta function we obtain

$$P(r^{(2)}) \propto \int_0^\infty\cdots\int_0^\infty (x_1 + x_2)\left(\prod_{i=1}^{4}\prod_{j=i}^{4}\left|\sum_{l=i}^{j} x_l\right|^\beta\right)\exp\left\{-\frac{A}{5}\left[\sum_{i=1}^{4}\sum_{j=i}^{4}\left(\sum_{l=i}^{j} x_l\right)^2\right]\right\}\delta(r^{(2)}(x_1 + x_2) - (x_3 + x_4))\prod_{i=1}^{4} dx_i. \quad (10)$$

First, the integral over x_4 in Eq. (10) is carried out. Then, x_4 will be replaced by $r(x_1 + x_2) - x_3$ due to the delta function in Eq. (10) (here, we define $r^{(2)} = r$ for simplicity of notation). Then the limits of integration of x_3 will be from 0 to $r(x_1 + x_2)$. Thus, our strategy is first to find the lowest degree polynomial in x_3 and x_4, since in the limit $r \to 0$ the integration over both x_3 and x_4 will give us the leading-order term in r.

Thus, first consider the following term from the integrand of Eq. (10),

$$\prod_{i=1}^{4}\prod_{j=i}^{4}\left|\sum_{l=i}^{j} x_l\right|^\beta, \quad (11)$$

which can be expanded to

$$\{x_1x_2x_3x_4(x_1 + x_2)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4) \times (x_2 + x_3)(x_2 + x_3 + x_4)(x_3 + x_4)\}^\beta \quad (12)$$

$$= \{x_3x_4(x_3 + x_4) \times x_1x_2(x_1 + x_2)(x_1 + x_2 + x_3) \times (x_1 + x_2 + x_3 + x_4)(x_2 + x_3)(x_2 + x_3 + x_4)\}^\beta. \quad (13)$$

This can be written as

$$\{x_3x_4(x_3 + x_4)[f_1(x_1, x_2) + f_2(x_1, x_2, x_3, x_4)]\}^\beta, \quad (14)$$

where f_1 and f_2 are polynomial functions of the respective arguments. This kind of split is possible because apart from the term $x_3x_4(x_3 + x_4)$ all the terms contain at least one variable from x_1 and x_2. The exact forms of f_i's can be found easily but are not required for our purpose here. That is because if we see Eq. (14) carefully, after expanding it, the lowest-order polynomial in x_3 and x_4 turns out to be $\{x_3x_4(x_3 + x_4)\}^\beta f_1^\beta(x_1, x_2)$ with order 3β in r. It is this term which will give us the required lowest power of r in the limit $r \to 0$. This will be clear in the subsequent calculations. Thus, Eq. (10) becomes

$$P(r^{(2)}) \propto \int_0^\infty\cdots\int_0^\infty (x_1 + x_2)\{x_3x_4(x_3 + x_4)[f_1(x_1, x_2) + f_2(x_1, x_2, x_3, x_4)]\}^\beta \times \exp\left\{-\frac{A}{5}\left[\sum_{i=1}^{4}\sum_{j=i}^{4}\left(\sum_{l=i}^{j} x_l\right)^2\right]\right\}\delta(r^{(2)}(x_1 + x_2) - (x_3 + x_4))\prod_{i=1}^{4} dx_i. \quad (15)$$

Integrating over x_4 and simplifying further we get

$$P(r^{(2)}) \propto \iint_{x_1,x_2=0}^{\infty} \int_{x_3=0}^{r(x_1+x_2)} (x_1+x_2)\{x_3(rx_1+rx_2-x_3)r(x_1+x_2)[f_1(x_1,x_2)+f_2(x_1,x_2,x_3,rx_1+rx_2-x_3)]\}^{\beta} \times \exp\left\{-\frac{A}{5}\left[2(2+r+2r^2){x_1}^2+(6+4r+4r^2){x_2}^2+2x_1x_2(3+3r+4r^2)\right]\right\} \times \exp\left\{-\frac{A}{5}\left[4{x_3}^2+x_3((4-2r)x_2+(2-2r)x_1)\right]\right\}\prod_{i=1}^{3}dx_i. \tag{16}$$

This we write as follows,

$$P(r^{(2)}) \propto \iint_{x_1,x_2=0}^{\infty} I_{x_3}(x_1,x_2,r)\exp\left\{-\frac{A}{5}[2(2+r+2r^2){x_1}^2+(6+4r+4r^2){x_2}^2+2x_1x_2(3+3r+4r^2)]\right\}(x_1+x_2)^{1+\beta}\prod_{i=1}^{2}dx_i, \tag{17}$$

where the x_3 integral is given as follows:

$$I_{x_3}(x_1,x_2,r)=\int_{x_3=0}^{r(x_1+x_2)}\{x_3(rx_1+rx_2-x_3)r[f_1(x_1,x_2)+f_2(x_1,x_2,x_3,rx_1+rx_2-x_3)]\}^{\beta} \times \exp\left\{-\frac{A}{5}\left[4{x_3}^2+x_3((4-2r)x_2+(2-2r)x_1)\right]\right\}dx_3. \tag{18}$$

Here, we are interested only to find the leading-order term in r in the limit $r \to 0$ of $P(r^{(2)})$ and thus to find the dominant term in r. It can be seen from Eq. (17) that the leading order will come only from that of $I_{x_3}(x_1,x_2,r)$, whereas integration over x_1 and x_2 are converging and will give another constant, keeping the exponent of r unchanged. Thus, we need to find only the lowest power of r. Using the fact that the limit and the integral can be interchanged [66], and the limit of the product is the product of the limits, let us first consider the term $I_{x_3}(x_1,x_2,r)$, and then the term $(rx_1+rx_2-x_3)^{\beta}$. It can be simplified as follows:

$$(rx_1+rx_2-x_3)^{\beta}=\sum_{q=0}^{\beta}\binom{\beta}{q}r^q(x_1+x_2)^q(-x_3)^{\beta-q}. \tag{19}$$

Thus,

$$I_{x_3}(x_1,x_2,r)=\int_{x_3=0}^{r(x_1+x_2)}[x_3]^{\beta}\left[\sum_{q=0}^{\beta}\binom{\beta}{q}r^q(x_1+x_2)^q(-x_3)^{\beta-q}\right]^{\beta}[r][f_1(x_1,x_2)+f_2(x_1,x_2,x_3,rx_1+rx_2-x_3)]^{\beta} \times \exp\left\{-\frac{A}{5}\left[4{x_3}^2+x_3((4-2r)x_2+(2-2r)x_1)\right]\right\}dx_3. \tag{20}$$

The square brackets around various terms are put in order to address them individually. Now, our strategy is to find the lowest order of the polynomial in x_3 and r. Then we will use Eq. (21) given as follows,

$$\int_{y=0}^{a} y^p dy \propto a^{p+1}, \tag{21}$$

and evaluate the integral. The first square bracket will give an exponent of β in r for x_3, the second square bracket will give $\beta - q$, and the fourth square bracket and the exponential term will give 0 as the lowest exponent of x_3. The second and third square brackets together will give $q+\beta$ as an exponent of r. Thus, using Eq. (21) in Eq. (20), the leading term in r in the $I_{x_3}(x_1,x_2,r)$ and eventually in $P(r^{(2)})$ is $r^{3\beta+1}$. The extra "+1" factor in the exponent comes from the integration measure dx_3. Thus, we obtain that $P(r^{(2)}) \to r^{3\beta+1}$ as $r \to 0$ supporting Eqs. (5) and (6).

III. RESULTS: GENERAL *K* CASE

In the case of general k, for the Wigner-like surmise, we need to have $2k+1$ eigenvalues [9]. Then the kth order spacing ratio is defined as

$$r^{(k)}=\frac{E_{2k+1}-E_{k+1}}{E_{k+1}-E_1}. \tag{22}$$

Considering the Gaussian ensemble with $N = 2k+1$ eigenvalues, the distribution of $r^{(k)}$ is given by

$$P(r^{(k)}) \propto \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\leqslant i<j\leqslant 2k+1} |E_i - E_j|^{\beta} \exp\left(-A\sum_{i=1}^{2k+1} E_i^2\right)\delta\left(r^{(k)} - \frac{E_{2k+1}-E_{k+1}}{E_{k+1}-E_1}\right)\prod_{i=1}^{2k+1} dE_i. \tag{23}$$

After changing the variables as $x_i = E_{i+1} - E_i$ for $i = 1$ to $2k$ and $x_{2k+1} = \sum_{i=1}^{2k+1} E_i$, we get [9]

$$P(r^{(k)}) \propto \int_0^{\infty} \cdots \int_0^{\infty} \frac{\partial(E_1,\ldots,E_{2k+1})}{\partial(x_1,\ldots,x_{2k+1})}\left(\prod_{i=1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta}\right)\exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j} x_l\right)^2 + x_{2k+1}^2\right]\right\}$$
$$\times\ \delta\left(r^{(k)} - \frac{\sum_{i=k+1}^{2k} x_i}{\sum_{i=1}^{k} x_i}\right)\prod_{i=1}^{2k+1} dx_i. \tag{24}$$

Here, the Jacobian $\frac{\partial(E_1,\ldots,E_{2k+1})}{\partial(x_1,\ldots,x_{2k+1})}$ and the integral for x_{2k+1} are constants that can be absorbed into the normalization factor. Using the property of the delta function we obtain

$$P(r^{(k)}) \propto \int_0^{\infty} \cdots \int_0^{\infty} \left(\sum_{i=1}^{k} x_i\right)\left(\prod_{i=1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta}\right)\exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j} x_l\right)^2\right]\right\}\delta\left(r^{(k)}\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{2k} x_i\right)\prod_{i=1}^{2k} dx_i. \tag{25}$$

Here, the integration is over $2k$ variables. First, the integration over x_{2k} is carried out. In that case the delta function goes away, replacing x_{2k} by the following:

$$r^{(k)} = \frac{\sum_{i=k+1}^{2k} x_i}{\sum_{i=1}^{k} x_i} \Rightarrow x_{2k} = r^{(k)}\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{2k-1} x_i. \tag{26}$$

This will put a constraint on the other variables x_i for $i = k+1$ to $2k-1$, such that $0 \leqslant \sum_{i=k+1}^{2k-1} x_i \leqslant r^{(k)} \sum_{i=1}^{k} x_i$. Thus, we need to find the polynomial that depends only on x_i for all $i = k+1$ to $2k$. Following on the lines of previous sections we can write

$$\prod_{i=1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta} = \prod_{i=1}^{k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta} \times \prod_{i=k+1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta} \tag{27}$$

$$= \left|\prod_{i=1}^{k}\prod_{j=i}^{2k}\left(\sum_{l=i}^{j} x_l\right)\right|^{\beta} \times \prod_{i=k+1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta} \tag{28}$$

$$= [\tilde{f}_1(x_1,\ldots,x_k) + \tilde{f}_2(x_1,\ldots,x_{2k})]^{\beta} \times \prod_{i=k+1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta}. \tag{29}$$

In the first step we have split the product from $i = 1$ to $2k$ in two terms such that the first one has the range of $i = 1$ to k while second has $i = k+1$ to $2k$. This gives the right-hand side of Eq. (27). The first multinomial term in Eq. (28) is fully expanded such that it is a sum of $\tilde{f}_1$ and $\tilde{f}_2$, where $\tilde{f}_1$ and $\tilde{f}_2$ are polynomial functions of the respective arguments only. The whole purpose of this split is to separate out terms containing the variables $x_1,\ldots,x_k$ only. This is possible because every product term in $|\prod_{i=1}^{k}\prod_{j=i}^{2k}(\sum_{l=i}^{j} x_l)|^{\beta}$ contains at least one variable from the set $\{x_1,\ldots,x_k\}$. This will show that when $(\tilde{f}_1+\tilde{f}_2)^{\beta}$ is fully expanded using the binomial theorem, will imply that the lowest degree of polynomial terms containing $x_{k+1},\ldots,x_{2k}$ is zero. Thus, we get the following:

$$P(r^{(k)}) \propto \int_0^{\infty} \cdots \int_0^{\infty} \left(\sum_{i=1}^{k} x_i\right) \times \prod_{i=k+1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j} x_l\right|^{\beta} \times [\tilde{f}_1(x_1,\ldots,x_k) + \tilde{f}_2(x_1,\ldots,x_{2k})]^{\beta}$$
$$\times \exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j} x_l\right)^2\right]\right\}\delta\left(r^{(k)}\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{2k} x_l\right)\prod_{i=1}^{2k} dx_i. \tag{30}$$

We will now further split the term $\prod_{i=k+1}^{2k}\prod_{j=i}^{2k}|\sum_{l=i}^{j}x_l|^\beta$ such that the terms containing x_{2k} are separated out as follows:

$$\prod_{i=k+1}^{2k}\prod_{j=i}^{2k}\left|\sum_{l=i}^{j}x_l\right|^\beta=\left(\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}\left|\sum_{l=i}^{j}x_l\right|^\beta\right)\left(\prod_{i=k+1}^{2k}\left|\sum_{l=i}^{2k}x_l\right|^\beta\right). \tag{31}$$

This is done because we will be first integrating over the variable x_{2k}. Thus, combining Eqs. (30) and (31) we get

$$P(r^{(k)})\propto\int_0^\infty\cdots\int_0^\infty\left(\sum_{i=1}^{k}x_i\right)\left(\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}\left|\sum_{l=i}^{j}x_l\right|^\beta\right)\left(\prod_{i=k+1}^{2k}\left|\sum_{l=i}^{2k}x_l\right|^\beta\right)[\tilde{f}_1(x_1,\dots,x_k)+\tilde{f}_2(x_1,\dots,x_{2k})]^\beta$$
$$\times\exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j}x_l\right)^2\right]\right\}\delta\left(r^{(k)}\sum_{l=1}^{k}x_l-\sum_{l=k+1}^{2k}x_l\right)\prod_{i=1}^{2k}dx_i. \tag{32}$$

Now, solving for the x_{2k} integral will remove the delta function and replace x_{2k} by $r^{(k)}\sum_{l=1}^{k}x_i-\sum_{l=k+1}^{2k-1}x_i$ at all the places in the integral as discussed in Eq. (26). First, consider the term $\prod_{i=k+1}^{2k}|\sum_{l=i}^{2k}x_l|^\beta$ from Eq. (32). It can be written and simplified further using $x_{2k}=r\sum_{l=1}^{k}x_l-\sum_{l=k+1}^{2k-1}x_l$ as follows (here, $r^{(k)}=r$ is defined for simplicity of notation):

$$\prod_{i=k+1}^{2k}\left|\sum_{l=i}^{2k}x_l\right|^\beta=\left|\sum_{l=k+1}^{2k}x_l\right|^\beta\times\prod_{i=k+2}^{2k}\left|\sum_{l=i}^{2k}x_l\right|^\beta \tag{33}$$

$$=\left(r\sum_{l=1}^{k}x_l\right)^\beta\times\left[\prod_{i=k+2}^{2k}\left(r\sum_{l=1}^{k}x_l-\sum_{l=k+1}^{i-1}x_l\right)\right]^\beta. \tag{34}$$

Similarly, using the constraint in Eq. (26), one obtains

$$[\tilde{f}_1(x_1,\dots,x_k)+\tilde{f}_2(x_1,\dots,x_{2k})]^\beta\to\left[\tilde{f}_1(x_1,\dots,x_k)+\tilde{f}_2\left(x_1,\dots,r^{(k)}\sum_{l=1}^{k}x_i-\sum_{l=k+1}^{2k-1}x_i\right)\right]^\beta$$
$$\text{and}\quad\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j}x_l\right)^2\to\sum_{j>i=1}^{2k-1}x_ix_jh'_{ij}(r), \tag{35}$$

where h'_{ij} are polynomials in r. Since $x_i\geqslant 0$ for all i, in order to have all integrals converging it is sufficient to show that $h'_{ii}>0$ for all i. This will be shown now. Considering the following term from the exponent of Eq. (32) and simplifying it, we get (see the text following for the steps on the simplifications done at each stage)

$$\sum_{i=1}^{2k}\sum_{j=i}^{2k}\left(\sum_{l=i}^{j}x_l\right)^2=\sum_{i=1}^{2k-1}\sum_{j=i}^{2k-1}\left(\sum_{l=i}^{j}x_l\right)^2+\sum_{i=1}^{2k-1}\left(\sum_{l=i}^{2k}x_l\right)^2+(x_k)^2 \tag{36}$$

$$=\sum_{i=1}^{2k-1}\sum_{j=i}^{2k-1}\left(\sum_{l=i}^{j}x_l\right)^2+\sum_{i=1}^{k}\left(\sum_{l=i}^{2k}x_l\right)^2+\left(\sum_{l=k+1}^{2k}x_l\right)^2+\sum_{i=k+2}^{2k-1}\left(\sum_{l=i}^{2k}x_l\right)^2+(x_k)^2 \tag{37}$$

$$=\sum_{i=1}^{2k-1}\sum_{j=i}^{2k-1}\left(\sum_{l=i}^{j}x_l\right)^2+\sum_{i=1}^{k}\left(\sum_{l=i}^{k}x_l+\sum_{l=k+1}^{2k}x_l\right)^2+\left(\sum_{l=k+1}^{2k}x_l\right)^2+\sum_{i=k+2}^{2k-1}\left(\sum_{l=k+1}^{2k}x_l-\sum_{l=k+1}^{i-1}x_l\right)^2+(x_k)^2 \tag{38}$$

$$=\sum_{i=1}^{2k-1}\sum_{j=i}^{2k-1}\left(\sum_{l=i}^{j}x_l\right)^2+\sum_{i=1}^{k}\left(\sum_{l=i}^{k}x_l+r\sum_{l=1}^{k}x_l\right)^2+\left(r\sum_{l=1}^{k}x_l\right)^2+\sum_{i=k+2}^{2k-1}\left(r\sum_{l=1}^{k}x_l-\sum_{l=k+1}^{i-1}x_l\right)^2+(x_k)^2. \tag{39}$$

Here, Eq. (36) is obtained by splitting the summation such that the term x_{2k} is separated out. The summation $i=1$ to $2k-1$ in the second term of Eq. (36) is further split into three parts: summation $i=1$ to k, single term $i=k+1$, and summation $i=k+2$

to $2k-1$ to get Eq. (37). The summation $l=i$ to $2k$ in the second and fourth term in Eq. (37) is further split depending on the range of i, so that we can use the constraint from Eq. (26). This will give us Eq. (38). Equation (39) is obtained by using the same constraint in Eq. (38). We can see from Eq. (39) that after expansion each term in the coefficient of x_i^2 (for all i) is either a positive number [at least one such number exists and is ensured by the first term in Eq. (39)] or a function of r and can be seen to be always non-negative. The only terms with a negative sign come from the second to last term in Eq. (39), which only contains mixed terms such as x_ix_j with $i \neq j$. Denoting the coefficient of x_ix_j by h'_{ij} we have $h'_{ii} > 0$ for all i, thus proving our claim. The exact expressions for h'_{ij} is not required for our purpose here. Thus, the h'_{ii} are polynomials in r such that in the limit $r \to 0$ they are all nonzero, which makes the integral converging. Thus, combining Eqs. (26) and (32)–(35) we get

$$P(r^{(k)}) \propto \int \cdots \int_{x_1,\ldots,x_k=0}^{\infty} \quad \int \cdots \int_{0\leqslant \sum_{i=k+1}^{2k-1} x_i \leqslant r(\sum_{i=1}^{k} x_i)} \left(\sum_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}\left|\sum_{l=i}^{j} x_l\right|^{\beta}\right)\left(r\sum_{l=1}^{k} x_l\right)^{\beta}$$
$$\times \left[\prod_{i=k+2}^{2k}\left(r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{i-1} x_l\right)\right]^{\beta}\left[\tilde{f}_1(x_1,\ldots,x_k) + \tilde{f}_2\left(x_1,\ldots,x_{2k-1}, r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{2k-1} x_l\right)\right]^{\beta}$$
$$\times \exp\left\{-\frac{A}{2k+1}\left[\sum_{j>i=1}^{2k-1} x_ix_jh'_{ij}(r)\right]\right\}\prod_{i=1}^{2k-1} dx_i. \tag{40}$$

Next, we rewrite the integral such that the summation term in the exponential term gets divided into parts. One part contains variables only from x_1 to x_k and the other term containing all of them, i.e., from x_1 to x_{2k-1}. Thus, we get

$$P(r^{(k)}) \propto \int \cdots \int_{x_1,\ldots,x_k=0}^{\infty} I_{x_{k+1}\cdots x_{2k-1}}\left(\sum_{l=1}^{k} x_l\right)^{1+\beta} \exp\left\{-\frac{A}{2k+1}\left[\sum_{j>i=1}^{k} x_ix_jh'_{ij}(r)\right]\right\}\prod_{i=1}^{k} dx_i, \tag{41}$$

where

$$I_{x_{k+1}\cdots x_{2k-1}} = \int \cdots \int_{0\leqslant \sum_{i=k+1}^{2k-1} x_i \leqslant r(\sum_{i=1}^{k} x_i)} r^{\beta}\left(\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}\left|\sum_{l=i}^{j} x_l\right|^{\beta}\right)\left[\prod_{i=k+2}^{2k}\left(r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{i-1} x_l\right)^{\beta}\right]$$
$$\times \left[\tilde{f}_1(x_1,\ldots,x_k) + \tilde{f}_2\left(x_1,\ldots,x_{2k-1}, r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{2k-1} x_l\right)\right]^{\beta} \exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1,j=k+1}^{2k-1} x_ix_jh'_{ij}(r)\right]\right\}\prod_{i=k+1}^{2k-1} dx_i. \tag{42}$$

It can be seen from Eq. (41) that in the limit $r \to 0$ the leading order of r will only come from evaluating that for $I_{x_{k+1}\cdots x_{2k-1}}$. In the subsequent part of the paper we will derive the latter. Now, consider the term $\prod_{i=k+2}^{2k}(r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{i-1} x_l)^{\beta}$ from Eq. (42). This can be simplified as follows (assuming that β is a natural number):

$$\prod_{i=k+2}^{2k}\left(r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{i-1} x_l\right)^{\beta} = \prod_{i=k+2}^{2k}\sum_{q=0}^{\beta}\binom{\beta}{q}\left(r\sum_{l=1}^{k} x_l\right)^{q}\left(-\sum_{l=k+1}^{i-1} x_l\right)^{\beta-q}$$
$$= \prod_{i=k+2}^{2k}\sum_{q=0}^{\beta}\binom{\beta}{q}\left(\sum_{l=1}^{k} x_l\right)^{q} r^{q}\left(-\sum_{l=k+1}^{i-1} x_l\right)^{\beta-q}. \tag{43}$$

Thus, Eq. (42) simplifies to

$$I_{x_{k+1}\cdots x_{2k-1}} = \int \cdots \int_{0\leqslant \sum_{i=k+1}^{2k-1} x_i \leqslant r(\sum_{i=1}^{k} x_i)} [r^{\beta}] \times \left[\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}\left|\sum_{l=i}^{j} x_l\right|^{\beta}\right]\left[\prod_{i=k+2}^{2k}\sum_{q=0}^{\beta}\binom{\beta}{q}\left(\sum_{l=1}^{k} x_l\right)^{q} r^{q}\left(-\sum_{l=k+1}^{i-1} x_l\right)^{\beta-q}\right]$$
$$\times \left[\tilde{f}_1(x_1,\ldots,x_k) + \tilde{f}_2\left(x_1,\ldots,x_{2k-1}, r\sum_{l=1}^{k} x_l - \sum_{l=k+1}^{2k-1} x_l\right)\right]^{\beta} \exp\left\{-\frac{A}{2k+1}\left[\sum_{i=1,j=k+1}^{2k-1} x_ix_jh'_{ij}(r)\right]\right\}\prod_{i=k+1}^{2k-1} dx_i. \tag{44}$$

The square brackets around various terms are put in order to address them individually. Here, we will be using the following integral identity [the generalization of Eq. (21)]:

$$\int\cdots\int_{0\leqslant y_1,\ldots,y_N,\sum_{i=1}^N y_i\leqslant a}\prod_{i=1}^N y_i^{p_i}dy_i \propto a^{\sum_{i=1}^N p_i+N}. \quad (45)$$

In Eq. (45), it should be noted that the exponent on the right-hand side is a function only of the order of the integrand polynomial ($\sum_{i=1}^N p_i$) and the number of variables (N) on the left-hand side. Here, we are interested only in the limit $r\to 0$. Thus, we need to find the lowest order of r in $I_{x_{k+1}\cdots x_{2k-1}}$. For that we need to first find the lowest order of the polynomial in x_{k+1} to x_{2k-1} in Eq. (44) and then use Eq. (45). This can be achieved by doing the same for each term in the Eq. (44), multiplying them together, and then use Eq. (45). This is now explained below.

The first square bracket in Eq. (44) will give us an exponent of β for r. The term $\prod_{i=k+1}^{2k-1}\prod_{j=i}^{2k-1}|\sum_{l=i}^{j}x_l|^\beta$ from the second bracket is a multinomial term and can be expanded fully. It will lead to a homogeneous polynomial of degree $(k-1)k\beta/2$. In the third square bracket, the term $(\sum_{l=1}^k x_l)^q$ does not have any of the variables from the set $\{x_{k+1},\ldots,x_{2k}\}$. Thus, it is not going to give any r-dependent factor in the limit $r\to 0$. Thus, we are left with two terms, namely r^q and $(-\sum_{l=k+1}^{i-1}x_l)^{\beta-q}$. Here, it can be seen that the term $(-\sum_{l=k+1}^{i-1}x_l)^{\beta-q}$ when expanded will give a homogeneous polynomial of order $\beta-q$. Both of them appear $(k-1)$ times due to the operation $\prod_{i=k+2}^{2k}$ on them. The range of the summation in $(-\sum_{l=k+1}^{i-1}x_l)^{\beta-q}$ does change with i but the order of the homogeneous polynomial remains the same. Thus, using Eq. (45) and $r\to 0$ we can say that the third square bracket will result in an exponent of $(k-1)q+(k-1)(\beta-q)$. The exponent of the lowest-order polynomial in $x_{k+1},\ldots,x_{2k-1}$ which can be obtained from the term in the fourth square bracket, namely $(\tilde{f}_1+\tilde{f}_2)^\beta$ is 0. This is because $\tilde{f}_1$ is a function of $x_1\cdots x_k$ *only* and use of the binomial theorem (assuming β is natural number) we get at least one term with variables $x_1\cdots x_k$ only. It means that the lowest order of the polynomial containing $x_{k+1}\cdots x_{2k}$ variables will be zero, while that from the exponential term (fifth term), using its Taylor expansion, is also 0. Finally, the integration measure $\prod_{i=k+1}^{2k-1}dx_i$ has $k-1$ variables. Thus, the exponent of $r=\beta'$ where $\beta'=[\beta]+[k(k-1)\beta/2]+[(k-1)q+(k-1)(\beta-q)]+[0]+[0]+[(k-1)]=\beta k(k+1)/2+k-1$.

Now, using the identity from Eq. (45) it can be seen that in the limit $r\to 0$ the dominant term will be proportional to $r^{\beta'}$ where $\beta'=\beta+k(k-1)\beta/2+(k-1)[q+(\beta-q)]+(k-1)=\beta k(k+1)/2+k-1$. Thus, the leading term in $I_{x_{k+1}\cdots x_{2k-1}}$ in the limit $r\to 0$ is $r^{\beta'}$ which will also be the same for $P(r^{(k)})$ as discussed earlier. Thus, we can write

$$P(r^{(k)})\to (r^{(k)})^{\beta'} \quad \text{for} \quad r^{(k)}\to 0. \quad (46)$$

With this, we have proved the first part of the most general and main result in Eq. (6) supporting Eq. (5).

IV. CASE OF $r\to\infty$

In order to find the limiting behavior in this case we use the property of the joint pdf in Eq. (7). For this we show that $P(s_1,s_2)=P(s_2,s_1)$, i.e., $P(s_1,s_2)$ is a symmetric function, where $s_1=E_{k+1}-E_1$, $s_2=E_{2k+1}-E_{k+1}$, $P(s_1,s_2)$ is a joint pdf of s_1 and s_2. We will show this for the Wigner-surmise setting, as per Eq. (23), i.e., for a given k we have $N=2k+1$. Using the change of variables as per Sec. III we get $s_1=\sum_{i=1}^k x_i$ and $s_2=\sum_{i=k+1}^{2k}x_i$ [9]. Now, using a property of the joint pdf in Eq. (7) it can be seen that it is invariant under the transformation $x_i\leftrightarrow x_{2k+1-i}$, where $i=1$ to k. This corresponds to a reflection symmetry about the eigenvalue E_{k+1}. It results in $s_1\leftrightarrow s_2$. Thus, the joint pdf is invariant, as is the $P(s_1,s_2)$ under the said transformation, i.e., $P(s_1,s_2)=P(s_2,s_1)$. Due to this left-right symmetry the distribution of $r^{(k)}=s_1/s_2$ is the same as that of $1/r^{(k)}$ so that the following duality relation holds true,

$$P(r^{(k)})=\frac{1}{(r^{(k)})^2}P\left(\frac{1}{r^{(k)}}\right), \quad (47)$$

where $P(x)$ is the probability distribution of x. The same relation corresponding to $k=1$ was presented earlier in Ref. [28]. Thus, we can find the asymptotic behavior of $r\to\infty$ using the solved case of $r\to 0$ in Eq. (46). Thus,

$$\begin{aligned}\lim_{r^{(k)}\to\infty}P(r)&=\lim_{r^{(k)}\to\infty}\frac{1}{(r^{(k)})^2}P\left(\frac{1}{r^{(k)}}\right)\\&=\lim_{t\to 0}t^2P(t)\quad\text{where}\quad t=\frac{1}{r^{(k)}}\\&=t^{2+\beta'}\\&=(r^{(k)})^{-2-\beta'}.\end{aligned} \quad (48)$$

Thus, we get the following result:

$$P(r^{(k)})\to (r^{(k)})^{-2-\beta'} \quad \text{for} \quad r^{(k)}\to\infty. \quad (49)$$

With this, the second part of Eq. (6) is proved. It must be noted that we have shown the $r\to\infty$ behavior using the Wigner-like surmise, i.e., for given order k matrix dimension is $2k+1$. For cases otherwise, the symmetry of $P(s_1,s_2)$ holds only in the bulk of the spectrum and in the limit $N\to\infty$. This symmetry will break down at the soft or hard edge of the spectrum, and deviations can be expected.

V. CASE OF UNCORRELATED SPECTRA

Let us now consider the case of uncorrelated spectra. The NN spacing ratio of such spectra shows Poissonian behavior which is shown by integrable systems [16,28]. Higher-order spacing ratios, in this case, are known as follows [44]:

$$P_P^k(r)=\frac{(2k-1)!}{[(k-1)!]^2}\frac{r^{k-1}}{(1+r)^{2k}}. \quad (50)$$

It is important to note that this is an exact result in the limit of $N\to\infty$ only, in contrast to many other equations in this paper. It can be shown easily that

$$P_P^k(r)\to r^{k-1} \quad \text{for} \quad r\to 0 \quad (51)$$

and

$$P_P^k(r)\to r^{-k-1} \quad \text{for} \quad r\to\infty. \quad (52)$$

This is a special case of our result above for β' evaluated at $\beta=0$.

VI. SUMMARY AND CONCLUSIONS

In recent times, higher-order spacing ratios have become a popular and important measure to study fluctuations in random matrices and complex physical systems. This is due to their computationally simple nature as no unfolding is required, compared to that of the *spacings* alone. Very few analytical results for the spacing ratios are available. This paper has analytically studied the asymptotic behavior of higher-order spacing ratios ($r^{(k)}$) in the Gaussian ensembles with a Dyson index β. Most of the results on it were numerical [9,11,55,56,67,68]. We have now proved a universal behavior of its distribution i.e., $P^k(r,\beta) \rightarrow r^{\beta'}$ ($r^{-2-\beta'}$) in the limit $r \rightarrow 0$ (∞), where $\beta' = \beta k(k+1)/2 + (k-1)$ based on the very good approximation Eq. (5). We also expect the same behavior by the exact expression (currently unknown) for $P^k(r,\beta)$. We have used the Wigner-like surmise [Eq. (5)] which becomes a good fit for the large-N scenario. Here, universality is referred to in the sense that the ratios can be studied without the procedure of unfolding or renormalization of the eigenvalues which is very much required in the case of the spacings [1,3]. In fact, from our study of uncorrelated eigenvalues, our results hold true for any $\beta \geqslant 0$. These results have given analytical support to the numerical results from various random matrix ensembles and complex physical systems, which was absent earlier [9,11,55,56,67,68]. Moreover, our analytical approach can be extended to other ensembles, for example, Laguerre ensemble [2,3,56,69–71], chiral ensembles [72–81], etc. Though Laguerre and chiral ensembles are related to each other mathematically they have different applications. Wishart ensembles are used in the study of entanglement [2], and wireless communication systems [3], whereas chiral ensembles are used to model Dirac operators in quantum chromodynamics [72–74]. Recently, it is shown that the NN level spacing distribution is insensitive to the position in RMT spectra at the edges or in the bulk despite the fact that fluctuations there are described by different limiting kernels [82]. We would like to investigate the same with the spacing ratios numerically as well as analytically.

It should be noted that we have given the asymptotic behavior of higher-order spacing ratios but finding an exact expression for the corresponding Wigner-like surmise still remains open. This is left for a future study.

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