Effects of correlation in an information ratchet with finite tape

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(Received 23 September 2022; revised 12 January 2023; accepted 13 January 2023; published 17 February 2023)

With the finite-tape autonomous information ratchet modeled by He *et al.* [Phys. Rev. E **105**, 054131 (2022)], we recast the information processing second law, giving a tighter bound on the work extracted, in terms of the *marginal bit-ratchet* distribution defined from the *joint tape-ratchet* distribution. The *marginal* distribution is further utilized to probe and elucidate the conditions that lead to the presence of *equilibrium* and *nonequilibrium* stationary states in general, which are related to the effects of correlation. Applying our analysis to two designs of this information ratchet, where correlations within manifest differently, we uncover the mathematical condition for equilibrium stationary states for information ratchets that harness correlation, to identify them for engine operation during the transient phase.

DOI: 10.1103/PhysRevE.107.024130

I. INTRODUCTION

It is well known that correlation has a subtle effect on the ratchet mechanism. The ratchet mechanism was introduced by Feynman through the metaphor of a ratchet and pawl where he demonstrated how the condition of nonequilibrium is necessary to drive directed motion [1]. It is a facet of a Maxwell demon which derives order out of a thermal environment [2,3]. It was exhibited in a class of system known as *Brownian motors* which undergoes dissipative dynamics in the presence of thermal noise with the system driven out of equilibrium [4]. The systems in this category appear in the form of various ratchet configurations which have been applied to depict the underlying mechanism of molecular motors and pumps [5]. One notable ingredient in these ratchet systems is the temporal correlation (within the environment), which acts through the nonequilibrium effect of symmetry breaking to convert thermal energy to useful work [6–8].

Recently, there is a new class of ratchet system whose operation is driven through information. One approach in this new class which is inspired by the Szilard's engine [9], executes a measurement-feedback formalism where work is extracted from heat in a thermal bath according to information acquired from the previous measurement [10–14]. However, another approach considers an autonomous information ratchet interacting with symbols or bits in an infinite tape, with no external agent manipulating it or thermodynamic force driving it [15–17]. In both approaches, correlation is found to enhance the efficiency of the ratchet systems. For the former approach, correlation leads to better flux performance [18] while the latter approach leverages correlations in the input bits to perform additional work [19,20].

In an earlier paper [21], we have considered an autonomous information ratchet that interacts with a finite tape (sequence) of bits in discrete time. We have shown in Ref. [21] that such a ratchet with memory is able to harness correlation to facilitate the work-energy transfer (amassing mechanical energy for future work) by taking a longer time to reach the stationary state. Moreover, we have uncovered a phenomenon of a possible negative asymptotic work from this ratchet with memory in its stationary behavior. We will show in this paper that this negative work is a consequence of the effect of correlation when the ratchet is being driven to a nonequilibrium stationary state where it manifests irreversible behavior with work expended as heat to the thermal reservoir. While there is an increase in order in the ratchet system, the entropy of the thermal environment increases such that the second law of thermodynamics is obeyed. However, a ratchet with memory can also exhibit equilibrium stationary behavior whose dynamics is reversible. It is at the transient phase of such a ratchet that the work-energy conversion confers it the potential to do the maximum mechanical work [21]. It is the purpose of this paper to illuminate the effects of correlation on these nonequilibrium and equilibrium stationary states and their associated work done. To delve into the effects of correlation in these cases, we shall perform our mathematical analysis using the *marginal bit-ratchet* distribution instead of the *joint tape-ratchet* distribution as was done in Ref. [21] previously.

The organization of our paper is as follows. We first introduce our finite-tape information ratchet system and its necessary preliminaries in Sec. II. The subsequent Secs. III and IV demonstrate the validity of the finite-tape information processing second law (IPSL) in terms of the *marginal* bit-ratchet distribution and show that it gives a tighter bound on the work extracted compared to the original version in Ref. [21] which used the *joint* tape-ratchet distribution. Through the marginal bit-ratchet distribution and the tightened IPSL, we then decipher the presence of *nonequilibrium* stationary states in our finite-tape information ratchet system (in addition to the *equilibrium* stationary states) and elucidate on the possibility of negative work in this nonequilibrium steady-state behavior in Sec. V. An account of the tape-ratchet dynamics in the *transient* phase, where the information ratchet is able to fulfill a variety of thermodynamic roles, is also

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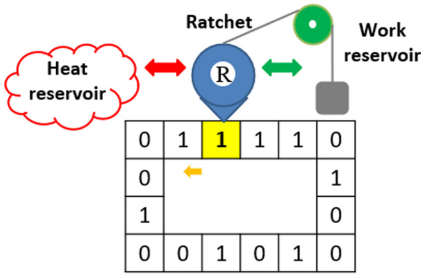

FIG. 1. Schematic representation of an information ratchet with a finite tape moving unidirectionally (indicated by arrow below the bit highlighted in yellow) as it sequentially interacts with each bit successively; the bit highlighted in yellow is the current bit B interacting with the ratchet. This bit and the ratchet constitute the present interacting subsystem $B \otimes R$. The circular tape here corresponds to the linear tape in Fig. 2 which is presented in a different but physically equivalent manner there; the interacting bit B highlighted in yellow in Figs. 1 and 2 is the same bit.

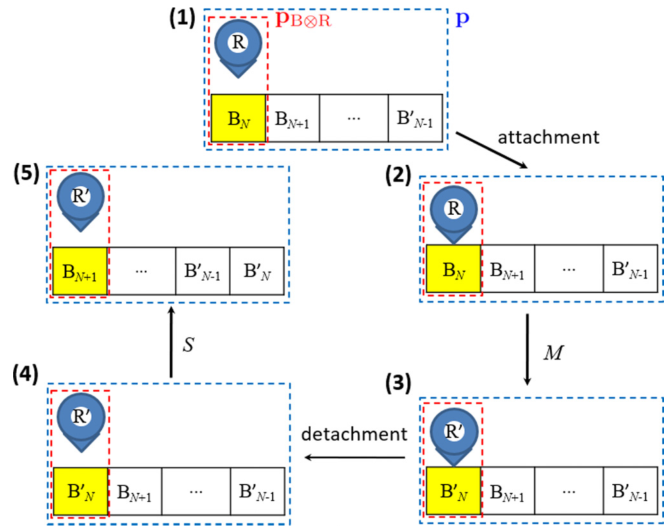

FIG. 2. Schematic representation of the information ratchet interacting with the finite tape (represented in a linear manner) of L bits. The (present) leftmost bit $B = B_N \in \{0, 1\}$ with the ratchet $R \in \{1, 2, \ldots, N_R\}$ form the interacting bit-ratchet state space $B \otimes R$, whose respective state probabilities are captured by the *marginal bit-ratchet* distribution $\mathbf{p}_{B\otimes R}$. In a single *bit* scan operation O, the intermediate stages of the coupling between this interacting bit $B = B_N$ and the ratchet R are detailed with the interacting subsystem $B \otimes R$ highlighted with a red dashed box at each stage, taking into account the position of B_N (and subsequently B'_N) relative to the other bits in the finite tape (of length L). The *joint tape-ratchet* distribution $\mathbf{p}$ accounts for the joint tape-ratchet states (blue dashed box), which was used in Ref. [21] to describe the same ratchet mechanism.

provided in Sec. VI. These results are the outcome when correlation is accounted for in our system, whose effects we then further illustrate through two designs of the finite-tape information ratchet with one unable to leverage on correlation (Sec. VII A) while the other can (Sec. VII B). Lastly, we work out the mathematical condition for equilibrium stationary states (with zero asymptotic work) of information ratchets that harness correlation in Sec. VII C. In such a state, the ratchet operates as an engine (in the transient phase) with the accumulated mechanical energy (and equivalently its capacity to do future work) maximized and are thus of practical interest in the realization of our finite-tape information ratchet system [21].

II. FINITE-TAPE INFORMATION RATCHET

We first provide an overview of the finite-tape information ratchet to give physical intuition of its inner workings before proceeding to the essential details of the ratchet mechanism and its mathematical formalism, with a focus on the subtleties involved. The IPSL (in terms of either the *joint* tape-ratchet distribution or *marginal* bit-ratchet distribution) obeyed by our information ratchet system will lastly be introduced at the end of this section.

A schematic illustrating a minimal representation of our information ratchet system is given in Fig. 1, with the finite tape presented in a circular manner. The motivation is the output bits will eventually recirculate back as the input bits in the next *tape* scan. One *tape* scan is completed when the ratchet has scanned all bits (of the L-bit tape) before the recirculation occurs, i.e., after L successive *bit* scans. We will denote a single *bit* scan operation O as in Ref. [21]. The physical attributes of the finite-tape information ratchet, i.e., its interactions with the heat and work reservoirs, are displayed abstractly in Fig. 1 to highlight its role as a channel mediating exchanges between these reservoirs and the finite tape. The heat and work reservoirs have their usual interpretations from classical thermodynamics, with the latter arbitrarily perceived as a mass-pulley system which stores (expends) mechanical energy (specifically gravitational potential energy) for future work production (expenditure) as the mass is raised (lowered). We note the finite tape here differs from the original discrete-time information ratchet with an infinite bit sequence in Ref. [17], and the consequences had been explored in Ref. [21].

We now introduce the notation used to describe the ratchet operation, with reference to Fig. 2 detailing the intermediate stages in a single bit scan O. Figure 2 reveals the bare essentials of the ratchet mechanism, with the finite tape (equivalently) presented in a linear fashion, and the heat and work reservoirs omitted to center the discussion on the tape-ratchet interaction. Each bit in this finite tape is indexed as $B_N \in \{0, 1\}$ with $N \in \{1, 2, \ldots, L\}$, typical of a classical bit. The random variable B is used to denote the input (leftmost) bit, and we arbitrarily take the forward direction spatially in which the linear tape is scanned as the left-to-right direction hereafter. The ratchet $R \in \{1, 2, \ldots, N_R\}$ is in either one of the N_R ratchet states. In Ref. [21], the (total of $2^L N_R$) *joint* tape-ratchet states with their *joint* probability distribution $\mathbf{p}$ was used to mathematically describe the ratchet mechanism as the consideration of these *joint* states would capture the statistical behavior of all the microstates of the (finite tape-ratchet) system. We now proceed to illuminate the different stages in a single bit scan O and the evolution of the joint distribution $\mathbf{p}$ by the corresponding operator matrix O, i.e.,

$$\mathbf{p}'(\tau) = O\,\mathbf{p}(0), \tag{1}$$

with the ratchet interacting sequentially with each bit for a fixed period τ.

At the start of each *bit* scan O (stage 1), the ratchet (in state R) is initially not attached to any bit before the incoming input bit $\mathrm{B} = \mathrm{B}_N$ attaches to it (stage 2). The interacting bit-ratchet subsystem $\mathrm{B} \otimes \mathrm{R}$ (formed by both random variables together) now undergoes a stochastic thermal transition M (driven by thermal fluctuations from the heat reservoir) to $\mathrm{B}' \otimes \mathrm{R}'$ (stage 3), with the joint (tape-ratchet) distribution $\mathbf{p}$ transformed to an *intermediate* distribution $\tilde{\mathbf{p}} = M\mathbf{p}$. The output bit $\mathrm{B}' = \mathrm{B}'_N$ subsequently detaches from the ratchet (stage 4) and shifts to the right end of the tape via switching S, completing this *bit* scan O (stage 5) with the new joint distribution $\mathbf{p}' = S\tilde{\mathbf{p}}$. The ratchet retains its state R' from this interaction and is now ready to attach to the next input bit $\mathrm{B} = \mathrm{B}_{N+1}$ [the bit index is more accurately $(N \bmod L) + 1$], repeating the cycle. The sequential interaction of each bit with the information ratchet thus proceeds in this manner.

The bit scan matrix O accounts for the thermal transition M involving the interacting subsystem comprising the ratchet and the attached leftmost bit of the finite tape (from stage 2 to 3), and switching S required to shift the output bit from the leftmost end of tape to the right end (from stage 4 to 5) as modeled by the linear tape in Fig. 2. Each *bit* scan thus effectively constitutes a two-step *composite* operation $O = SM$. For an L-bit tape scanned by an information ratchet with N_R ratchet states, the size of $O = SM$ (determined by the *joint* tape-ratchet states) is $2^L N_\mathrm{R} \times 2^L N_\mathrm{R}$, with substeps M and S necessarily having the same dimensions as O. The details of the explicit construction of these matrices, which act on the *joint* distribution $\mathbf{p}$, can be gleaned from Ref. [21], although the joint (tape-ratchet) states in $\mathbf{p}$ will be ordered differently in this paper and discussed subsequently.

The thermodynamics involved in a bit scan O between the information ratchet and the respective reservoirs in Fig. 1 is as follows. Unlike a piston, our (finite) tape-ratchet system does not perform work (in the absence of external perturbation). It basically mediates a transfer of energy between the heat reservoir and the work reservoir. The energy that is being transferred is heat as the tape-ratchet system interacts with the heat reservoir during the thermal transition substep M (from stage 2 to 3 in Fig. 2). The heat energy is then converted to gravitational potential energy capable of doing work in the work reservoir, analogous to Refs. [17,19]. It is in this context that $Q = -W$ over every bit scan O, where Q is the heat dissipated by the ratchet into the heat reservoir and W is the mechanical energy stored (which manifests as work) in the work reservoir by the ratchet, since the ratchet does not retain energy. The interaction with the work reservoir to accumulate (or expend) this mechanical energy from the work conversion occurs during the attachment of input bit B (from stage 1 to 2) to and detachment of output bit B' (from stage 3 to 4) from the ratchet, with the joint distribution unchanged. Thus, this attachment and detachment mechanism does not feature in the mathematical modeling of the bit scan operation (matrix) $O = SM$, which governs the evolution of the joint $\mathbf{p}$ in Eq. (1). Note that the switching substep S does not involve either the heat or work reservoir nor play a role in the energetics or informational change within the cycle O; see Appendix A (energetics) and Sec. V (informational change) for further details. Nonetheless, this switching S is an intrinsic attribute of this information ratchet necessary for its sequential interaction with each bit (of the finite tape).

In our earlier paper [21], we have established (and proven) the *joint* IPSL which states that the *expected* work extracted from the information ratchet is bounded above by the change in Shannon entropy of the *joint tape-ratchet* distribution $\mathbf{p}$ over a single bit scan O, i.e., $\langle W \rangle \leqslant \Delta H[\mathbf{p}]$ with $\Delta H[\mathbf{p}] = H[\mathbf{p}'] - H[\mathbf{p}]$ from Eq. (1). It is the aim of this paper to alternatively consider the dynamics within the information ratchet system from the perspective of the interacting bit-ratchet subsystem $\mathrm{B} \otimes \mathrm{R}$ with the *marginal bit-ratchet* distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$.[1] This is motivated by the sequential interaction of the ratchet with each bit (in the finite tape), implying that there will instantaneously only be *one* interacting bit B which the ratchet (in state R) can be attached to in its operation. Intuitively we expect the marginal distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ to be sufficient to capture the corresponding free energy difference, which arises solely from the $\mathrm{B} \otimes \mathrm{R}$ subsystem. Refer to Fig. 2 for a schematic illustrating the difference between this marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ (a $2N_\mathrm{R} \times 1$ vector) and the joint $\mathbf{p}$ (a $2^L N_\mathrm{R} \times 1$ vector).

Note that the leftmost bit of the finite tape in Fig. 2 is always either the *input* bit B or *output* bit B'. We will thus simply use $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ to denote the marginal *leftmost* bit-ratchet distribution, irrespective of whether this leftmost bit is an input or output bit.

Our next task now is to demonstrate the validity of the *marginal* IPSL in terms of the *marginal* distribution, stating the *expected* work is bounded above by the change in Shannon entropy of the *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ over the stochastic thermal transition substep M in a single bit scan O, i.e., $\langle W \rangle \leqslant \Delta H[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}]$ with $\Delta H[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = H[\tilde{\mathbf{p}}_{\mathrm{B}\otimes\mathrm{R}}] - H[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}]$. This is essentially a recast of our original IPSL using the *joint* distribution $\mathbf{p}$ in Ref. [21]. To this end, the *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ will be defined from its *joint* $\mathbf{p}$ [Eq. (11)].

III. IPSL WITH MARGINAL DISTRIBUTION

To establish the required relation [Eq. (18)] which the *marginal* IPSL is premised on (Sec. III C), we first need to show the work $\langle W \rangle$ [Eq. (2)], previously in terms of the *joint* (tape-ratchet) distribution $\mathbf{p}$ in Ref. [21], can be equivalently formulated using the *marginal* (bit-ratchet) distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ [Eq. (10)]. This will support the physical intuition that the interacting bit-ratchet subsystem $\mathrm{B} \otimes \mathrm{R}$ is sufficient to describe the dynamics of the tape-ratchet system with just the *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$. We will start with the simplest case for

[1]We mention that our use of the $\otimes$ symbol here is for the purpose of labeling the interacting bit-ratchet space $\mathrm{B} \otimes \mathrm{R}$ with distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$, and also for the labels of the respective interacting bit-ratchet states (see subsequent Figs. 3 and 5). It is not to be implied that there is no correlation between the interacting bit and ratchet such that $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ can be decomposed into its respective marginals. However, we note an exception in Eqs. (57)–(59) where we invoked the mathematical interpretation of the $\otimes$ symbol when discussing a specific ratchet design without correlation. Nonetheless, it will be apparent from the context on the intended meaning of the $\otimes$ symbol.

a 1-bit tape ($L = 1$) with its joint and marginal distributions coinciding, i.e., $\mathbf{p} = \mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$, before proceeding to $L > 1$ to show the equivalent formulations for $\langle W\rangle$ with either $\mathbf{p}$ or $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$. This motivates the *marginal* IPSL which considers the evolution of the *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ over one thermal transition substep M, in a similar manner to the *joint* IPSL in Ref. [21]. A proof for the marginal IPSL is furnished separately in Appendix A.

We had shown in Ref. [21] that the *expected* or *ensemble-averaged* work, in units of k_BT (i.e., setting $k_BT = 1$), reads

$$\langle W\rangle = \sum_{i,j} M_{ji} p_i \ln\left(\frac{M_{ij}}{M_{ji}}\right), \tag{2}$$

using the *joint* tape-ratchet distribution $\mathbf{p}$ and thermal transition (square) matrix M (of dimension $2^L N_{\mathrm{R}}$). The formulation for this $\langle W\rangle$ in Eq. (2) is detailed in Appendix A, together with the necessary assumptions behind this work expression.

We now proceed to show explicitly the calculation of $\langle W\rangle$ for $L = 1$ and $L > 1$ using Eq. (2) before proving that the work $\langle W\rangle$ for an arbitrary L-bit tape can also be equivalently expressed in terms of the *marginal* bit-ratchet distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$.

A. $\langle W\rangle$ for 1-bit tape

We first calculate the work $\langle W\rangle$ for $L = 1$ with the *joint* (tape-ratchet) distribution $\mathbf{p}$ with the corresponding L-bit thermal transition matrix $M^{(L)}$, i.e., $M^{(1)}$ here. The indexing notation employed in this calculation takes into account both the dimensions of $M^{(1)}$ and the corresponding $\mathbf{p}$ (for consistency) to obtain the subsequent expression in Eq. (6), with the explicit details as follows.

The thermal transition operator (matrix) for the 1-bit tape is given by

$$M_{\text{1-bit}} = \left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right), \tag{3}$$

where E, F, G, and H are submatrices of $M_{\text{1-bit}}$ with size $N_{\mathrm{R}} \times N_{\mathrm{R}}$. To ease the subsequent notation from what we had used in our previous paper [21], we will use the shorthand $M^{(1)}$ for $M_{\text{1-bit}}$ and denote the entry or element of $M^{(1)}$ as $M^{(1)}_{i,j}$. We mention at the outset that our design of $M^{(1)}$ will be *tridiagonal*, a requirement of our ratchet mechanism which we will return to address its necessity in Sec. V.

Let us now define new variables r, s, u, and v where $r, s \in \{0, 1\}$ and $u, v \in \{1, 2, \ldots, N_{\mathrm{R}}\}$. The indices i and j can then be defined as

$$i(r, u) = rN_{\mathrm{R}} + u, \tag{4a}$$

$$j(s, v) = sN_{\mathrm{R}} + v. \tag{4b}$$

The variables r and s determine which submatrix the indices are referring to, i.e.,

$$\begin{aligned} E &\to r = 0, \quad s = 0 \\ F &\to r = 0, \quad s = 1 \\ G &\to r = 1, \quad s = 0 \\ H &\to r = 1, \quad s = 1 \end{aligned} \tag{5}$$

TABLE I. Relationship among the variables r and u, index i, and the tape-ratchet configuration for $L = 1$ and $N_{\mathrm{R}} = 2$.

r	u	i	Tape-ratchet configuration
0	1	1	$0 \otimes A$
0	2	2	$0 \otimes B$
1	1	3	$1 \otimes A$
1	2	4	$1 \otimes B$

and the variables u and v refer to the row and column of the particular submatrix, respectively.

From Eq. (2), the work can now be written in terms of r, s, u, and v as

$$\begin{aligned} \langle W\rangle &= \sum_{i,j} M^{(1)}_{j,i} p_i \ln\left(\frac{M^{(1)}_{i,j}}{M^{(1)}_{j,i}}\right) \\ &= \sum_{r=0}^{1}\sum_{s=0}^{1}\sum_{u=1}^{N_{\mathrm{R}}}\sum_{v=1}^{N_{\mathrm{R}}} M^{(1)}_{(sN_{\mathrm{R}}+v,\, rN_{\mathrm{R}}+u)}\, p_{(rN_{\mathrm{R}}+u)} \\ &\quad \times \ln\left[\frac{M^{(1)}_{(rN_{\mathrm{R}}+u,\, sN_{\mathrm{R}}+v)}}{M^{(1)}_{(sN_{\mathrm{R}}+v,\, rN_{\mathrm{R}}+u)}}\right]. \end{aligned} \tag{6}$$

As for the probability distribution $p_i = p_{(rN_{\mathrm{R}}+u)}$, Table I gives the relationship among the variables r and u, index i, and the corresponding tape-ratchet configurations for $L = 1$ and $N_{\mathrm{R}} = 2$ (with A, B labeling the ratchet states).

B. $\langle W\rangle$ for L-bit tape

We now proceed to generalise the $\langle W\rangle$ expression in Eq. (6) for arbitrary L to obtain Eq. (9), by recognizing the repetition of elements of $M^{(1)}$ in $M^{(L)}$ [see Eq. (7)] and thus we can utilize a similar indexing notation for the calculation here. The definition of the *marginal* (interacting bit-ratchet) distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ in Eq. (11) then follows from Eq. (10) and we will have shown the work equivalence with either the *joint* distribution $\mathbf{p}$ or *marginal* distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$, i.e., Eq. (13).

The thermal transition operator for the L-bit tape $M^{(L)}$ is given by

$$M_{L\text{-bit}} = \mathbb{1}_{2^{L-1}} \otimes M_{\text{1-bit}} = \mathbb{1}_{2^{L-1}} \otimes \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \tag{7}$$

on the basis of the ratchet interacting with only the leftmost bit at any instant (and the values of the remaining bits are unchanged). The transition probabilities between the $\mathrm{B} \otimes \mathrm{R}$ states are thus given by the respective elements in $M^{(1)}$, which explains the relation between $M^{(1)}$ and $M^{(L)}$ above. Notice that $M^{(L)}$ will also be tridiagonal as a consequence of the tridiagonality of $M^{(1)}$ which we alluded to earlier; see subsequent Eq. (72) for an explicit expression for $M^{(L)}$ constructed from Eq. (7) here. Each of the submatrices E, F, G, and H in $M^{(L)}$ is repeated 2^{L-1} times. Outside of these submatrices, all other elements of $M^{(L)}$ are zero.

Since there are repeating elements of submatrices E, F, G, and H in $M^{(L)}$, variables r, s, u, and v introduced earlier can be used again here to refer to the elements of these submatrices.

Let us now define new variables k, α, and β, where $k \in \{0, 1, \ldots, 2^{L-1} - 1\}$ and

$$\alpha(r, u, k) = 2kN_R + rN_R + u, \tag{8a}$$

$$\beta(s, v, k) = 2kN_R + sN_R + v. \tag{8b}$$

It can be checked that $M^{(L)}_{\alpha,\beta} = M^{(1)}_{i,j}$ (and $M^{(L)}_{\beta,\alpha} = M^{(1)}_{j,i}$) for any k. The variables r and s determine which submatrix the indices are referring to, as given in Eq. (5). Variable k points to the $(k+1)$th copy of the submatrix, counting from top to bottom (or left to right). Since k can take 2^{L-1} different values, it indicates that the repetition of the elements occurs 2^{L-1} times. The variables u and v refer to the row and column of the particular submatrix, respectively.

The elements of $M^{(L)}$ outside of indices α and β defined above are all zero. Thus, the work in Eq. (2) can be calculated by summing over the indices α and β, i.e.,

$$\begin{aligned}\langle W\rangle &= \sum_{\alpha,\beta} M^{(L)}_{\beta,\alpha} p_\alpha \ln\left(\frac{M^{(L)}_{\alpha,\beta}}{M^{(L)}_{\beta,\alpha}}\right)\\ &= \sum_{r=0}^{1}\sum_{s=0}^{1}\sum_{u=1}^{N_R}\sum_{v=1}^{N_R}\sum_{k=0}^{2^{L-1}-1} M^{(L)}_{(2kN_R+sN_R+v,\,2kN_R+rN_R+u)}\\ &\quad\times p_{(2kN_R+rN_R+u)}\\ &\quad\times \ln\left[\frac{M^{(L)}_{(2kN_R+rN_R+u,\,2kN_R+sN_R+v)}}{M^{(L)}_{(2kN_R+sN_R+v,\,2kN_R+rN_R+u)}}\right].\end{aligned} \tag{9}$$

Since $M^{(L)}_{\alpha,\beta} = M^{(1)}_{i,j}$ and $M^{(L)}_{\beta,\alpha} = M^{(1)}_{j,i}$, then $M^{(L)}_{(2kN_R+rN_R+u,\,2kN_R+sN_R+v)} = M^{(1)}_{(rN_R+u,\,sN_R+v)}$ and similarly $M^{(L)}_{(2kN_R+sN_R+v,\,2kN_R+rN_R+u)} = M^{(1)}_{(sN_R+v,\,rN_R+u)}$. Subsequently,

$$\begin{aligned}\langle W\rangle &= \sum_{r=0}^{1}\sum_{s=0}^{1}\sum_{u=1}^{N_R}\sum_{v=1}^{N_R} M^{(1)}_{(sN_R+v,\,rN_R+u)}\\ &\quad\times\left[\sum_{k=0}^{2^{L-1}-1} p_{(2kN_R+rN_R+u)}\right]\\ &\quad\times\ln\left[\frac{M^{(1)}_{(rN_R+u,\,sN_R+v)}}{M^{(1)}_{(sN_R+v,\,rN_R+u)}}\right]\\ &= \sum_{i,j} M^{(1)}_{j,i}\, p^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)} \ln\left(\frac{M^{(1)}_{i,j}}{M^{(1)}_{j,i}}\right),\end{aligned} \tag{10}$$

where $p^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)}$ is the *marginal* probability defined as

$$p^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)} = \sum_{k=0}^{2^{L-1}-1} p_{(2kN_R+rN_R+u)}. \tag{11}$$

This shows that the work can be calculated from the marginals in Eq. (10) following the exact same formulation as the work for 1-bit tape in Eq. (6), where the marginal and joint distributions are the same, i.e., $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}} = \mathbf{p}$ for $L = 1$ from Eq. (11).

Table II gives the relationship among the variables r, u, k, the indices i and α, and the corresponding tape-ratchet configurations for $L = 2$ and $N_R = 2$. The list of the marginals in this example is given by

$$\begin{aligned}p^{(\mathrm{B}\otimes\mathrm{R})}_{0\otimes A} &= p_1 + p_5 = p_{00\otimes A} + p_{01\otimes A},\\ p^{(\mathrm{B}\otimes\mathrm{R})}_{0\otimes B} &= p_2 + p_6 = p_{00\otimes B} + p_{01\otimes B},\\ p^{(\mathrm{B}\otimes\mathrm{R})}_{1\otimes A} &= p_3 + p_7 = p_{10\otimes A} + p_{11\otimes A},\\ p^{(\mathrm{B}\otimes\mathrm{R})}_{1\otimes B} &= p_4 + p_8 = p_{10\otimes B} + p_{11\otimes B}.\end{aligned} \tag{12}$$

TABLE II. Relationship among the variables r, u, k, the indices i and α, and the tape-ratchet configuration for $L = 2$ and $N_R = 2$.

k	r	u	α	Tape-ratchet configuration	i
0	0	1	1	$00 \otimes A$	1
0	0	2	2	$00 \otimes B$	2
0	1	1	3	$10 \otimes A$	3
0	1	2	4	$10 \otimes B$	4
1	0	1	5	$01 \otimes A$	1
1	0	2	6	$01 \otimes B$	2
1	1	1	7	$11 \otimes A$	3
1	1	2	8	$11 \otimes B$	4

We have thus shown that for all L, the work $\langle W\rangle$ can be expressed in terms of the *marginal* bit-ratchet distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ with the 1-bit thermal transition matrix $M^{(1)}$, which is equivalent to the original formulation in Ref. [21] with the *joint* tape-ratchet distribution $\mathbf{p}$ and L-bit thermal transition matrix $M^{(L)}$, i.e.,

$$\langle W\rangle(\mathbf{p}, M^{(L)}) = \langle W\rangle(\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}, M^{(1)}) \quad \forall\, L. \tag{13}$$

Note that the matrix for $M^{(L)}$ in Eq. (7) is different from its previous construction in Eq. (4) of Ref. [21] as the joint (tape-ratchet) states in $\mathbf{p}$ here have their bit sequences ordered such that the leftmost bit is iterated first (after enumerating the ratchet states) for the same remaining bit sequence, as exemplified in Table II. Nonetheless, the physical operation of $M^{(L)}$ is the same in both cases with the sequential interaction of our information ratchet with each bit governed by the same transition probabilities.

With this reordering of joint states here from Ref. [21], we are led to Eq. (71) subsequently in this paper from Eq. (42) of Ref. [21] for consistency. Interestingly, however, the switching matrix $S^{(L)}$ remains the same as Eq. (8) in Ref. [21] even after the (joint) state reordering. We can understand this intuitively by observing that the ordering of joint states applies to both the rows and columns of $S^{(L)}$, and thus $S^{(L)}$ is independent of this (arbitrarily chosen) order adopted for the joint states.

C. Marginal distribution before and after one thermal transition

We next address how the *marginal* distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ evolves over a thermal transition substep M and seek its relation [Eq. (18)]. This will provide the premise for which the *marginal* IPSL is based on, with the proof and its mathematical details (similar to the proof for the *joint* IPSL in Ref. [21]) followed through in Appendix A.

The indices of $M^{(L)}$ outside of α and β have previously been excluded in Eq. (9) because the elements outside of these indices are all zero due to the structure of $M^{(L)}$ given in Eq. (7). However, to provide further clarity, we redefine indices α and β here to include these elements as well, i.e.,

$$\alpha(r, u, k_\alpha) = 2k_\alpha N_\mathrm{R} + rN_\mathrm{R} + u, \tag{14a}$$

$$\beta(s, v, k_\beta) = 2k_\beta N_\mathrm{R} + sN_\mathrm{R} + v, \tag{14b}$$

with $k_\alpha, k_\beta \in \{0, 1, \ldots, 2^{L-1} - 1\}$. Note that if $k_\alpha \neq k_\beta$, then the element is definitely zero. If $k_\alpha = k_\beta = k$, then the indices refer to the elements of one of the submatrices.

The relation between *joint* probabilities before and after one *thermal transition* M is given by

$$\begin{aligned}
\tilde{p}_\alpha &= \sum_\beta M^{(L)}_{\alpha,\beta}\, p_\beta \\
\tilde{p}_{(2k_\alpha N_\mathrm{R}+rN_\mathrm{R}+u)} &= \sum_{s=0}^{1}\sum_{v=1}^{N_\mathrm{R}}\sum_{k_\beta=0}^{2^{L-1}-1} M^{(L)}_{(2k_\alpha N_\mathrm{R}+rN_\mathrm{R}+u,\, 2k_\beta N_\mathrm{R}+sN_\mathrm{R}+v)}\, p_{(2k_\beta N_\mathrm{R}+sN_\mathrm{R}+v)} \\
\tilde{p}_{(2kN_\mathrm{R}+rN_\mathrm{R}+u)} &= \sum_{s=0}^{1}\sum_{v=1}^{N_\mathrm{R}} M^{(L)}_{(2kN_\mathrm{R}+rN_\mathrm{R}+u,\, 2kN_\mathrm{R}+sN_\mathrm{R}+v)}\, p_{(2kN_\mathrm{R}+sN_\mathrm{R}+v)},
\end{aligned} \tag{15}$$

where the summation is performed only for $k_\beta = k_\alpha = k$ in the last line, since the terms with $k_\beta \neq k_\alpha$ are all zero. Note that Eq. (15) is considered over substep M and not a complete bit scan O, thus we have used a tilde symbol to denote the *intermediate* probabilities (distribution) in contrast with the prime symbol in Eq. (1).

We are now ready to relate the *marginal* probabilities before and after a thermal transition. Using the definition of marginal probability in Eq. (11), the marginal probability after one thermal transition is given by

$$\tilde{p}^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)} = \sum_{k=0}^{2^{L-1}-1} \tilde{p}_{(2kN_\mathrm{R}+rN_\mathrm{R}+u)}. \tag{16}$$

Next, inserting Eq. (15), we have

$$\begin{aligned}
\tilde{p}^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)} = &\sum_{k=0}^{2^{L-1}-1}\sum_{s=0}^{1}\sum_{v=1}^{N_\mathrm{R}} M^{(L)}_{(2kN_\mathrm{R}+rN_\mathrm{R}+u,\, 2kN_\mathrm{R}+sN_\mathrm{R}+v)} \\
&\times p_{(2kN_\mathrm{R}+sN_\mathrm{R}+v)}.
\end{aligned} \tag{17}$$

$M^{(L)}_{(2kN_\mathrm{R}+rN_\mathrm{R}+u,\, 2kN_\mathrm{R}+sN_\mathrm{R}+v)} = M^{(1)}_{(rN_\mathrm{R}+u,\, sN_\mathrm{R}+v)}$ follows from $M^{(L)}_{\alpha,\beta} = M^{(1)}_{i,j}$ earlier. Continuing,

$$\begin{aligned}
\tilde{p}^{(\mathrm{B}\otimes\mathrm{R})}_{i(r,u)} &= \sum_{s=0}^{1}\sum_{v=1}^{N_\mathrm{R}} M^{(1)}_{(rN_\mathrm{R}+u,\, sN_\mathrm{R}+v)}\left[\sum_{k=0}^{2^{L-1}-1} p_{(2kN_\mathrm{R}+sN_\mathrm{R}+v)}\right] \\
&= \sum_j M^{(1)}_{i,j}\, p^{(\mathrm{B}\otimes\mathrm{R})}_{j(s,v)},
\end{aligned} \tag{18}$$

where the marginal probabilities in $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ from Eq. (11) is invoked again. This shows that the *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ is evolved by $M^{(1)}$ over a thermal transition, which is the same relation as Eq. (1) for the *joint* $\mathbf{p}$ with a 1-bit tape (over a bit scan O).

Unlike analysis through the joint $\mathbf{p}$, our analysis with the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ giving the thermal transition substep in Eq. (18) cannot similarly be followed through with a switching substep to yield a complete bit scan (composite) operation. The exception, however, applies only to a 1-bit tape $L = 1$ (independent of N_R), since the (leftmost) input bit is always the same bit so that switching $S^{(1)}$ is the identity operation and $O^{(1)} = M^{(1)}$; see Ref. [21].

With the formulation of the work and probability evolution in terms of the marginal distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ and the 1-bit thermal transition matrix $M^{(1)}$ [i.e., Eqs. (10) and (18), respectively], we can prove the IPSL (for any L) recasted in terms of *marginal* distribution in the same manner as the IPSL proof for 1-bit tape in Sec. IV A of Ref. [21]; see Appendix A for the mathematical details. Thus, the original IPSL established in Ref. [21] (with the *joint* tape-ratchet distribution $\mathbf{p}$)

$$\langle W\rangle(\mathbf{p}, M^{(L)}) \leqslant \Delta H[\mathbf{p}], \tag{19}$$

now also has a *marginal* formulation

$$\langle W\rangle(\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}, M^{(1)}) \leqslant \Delta H[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}], \tag{20}$$

where the right-hand side (RHS) is the change in Shannon entropy of the *marginal* bit-ratchet distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ instead. We will proceed to discuss the implications of Eq. (20) and continue to utilize the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ to probe the effects of correlation responsible for the different behaviors of the information ratchet (tape-ratchet) system in steady state.

IV. TIGHTER IPSL BOUND ON $\langle W\rangle$ WITH MARGINAL DISTRIBUTION

The IPSL had been formulated with both the joint distribution $\mathbf{p}$ and marginal distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ in Sec. III earlier. We saw that the *joint* IPSL in Eq. (19) upper bounds the bit scan work $\langle W\rangle_k$ by the change in Shannon entropy of the *joint* (tape-ratchet) distribution $\Delta H_k[\mathbf{p}]$ in a single bit scan operation k. We show here that we can replace the RHS of Eq. (19) with a tighter bound on $\langle W\rangle_k$ using the *marginal* distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$, thus recovering the *marginal* IPSL in Eq. (20) established in the preceding section.

First, we seek to establish a relation between the informational bounds in Eqs. (19) and (20). Here we consider the change over the thermal transition substep M_k in bit scan k where the leftmost bit remains attached to the ratchet (Fig. 2). Explicitly, the input bit $\mathrm{B} = \mathrm{B}_N$ (before M_k) and the output bit $\mathrm{B}' = \mathrm{B}'_N$ (after M_k) refers to the same bit (with index N), although its value (0 or 1) which this bit realises may have changed. In addition, we can collectively group all the remaining bits in the finite tape not attached to and thus not interacting with the ratchet (and whose values are unchanged) as the noninteracting subsystem $B_{\setminus\{N\}}$. We can now expand the *joint* entropy of the composite tape-ratchet system in terms of their marginals and mutual information (perceived alternatively as the interacting bit-ratchet subsystem $B \otimes R$ and noninteracting subsystem $B_{\setminus\{N\}}$):

$$\begin{aligned}\Delta H_{M_k}[\mathbf{p}] &= \Delta H_{M_k}(B, R, B_{\setminus\{N\}})\\ &= H(B', R', B_{\setminus\{N\}}) - H(B, R, B_{\setminus\{N\}})\\ &= H(B', R') + H(B_{\setminus\{N\}}) - I(B', R'; B_{\setminus\{N\}})\\ &\quad - H(B, R) - H(B_{\setminus\{N\}}) + I(B, R; B_{\setminus\{N\}})\\ &= \Delta H_{M_k}(B, R) + \Delta C_{M_k}\\ &= \Delta H_{M_k}[\mathbf{p}_{B\otimes R}] + \Delta C_{M_k},\end{aligned} \tag{21}$$

where we have used the identity $H(X, Y) = H(X) + H(Y) - I(X; Y)$ and denoted the mutual information terms $\Delta C_{M_k} \equiv -\Delta I_{M_k}(B, R; B_{\setminus\{N\}}) = -[I(B', R'; B_{\setminus\{N\}}) - I(B, R; B_{\setminus\{N\}})]$.

We next proceed to show that the contribution from ΔC_{M_k} is always nonnegative by noting that the thermal transition substep M_k only involves the interacting bit-ratchet subsystem $B \otimes R$, i.e., $B \otimes R \to B' \otimes R'$, independent of the noninteracting subsystem $B_{\setminus\{N\}}$ and leaving it unchanged. We thus have the physical equivalence between the mathematical operations $\tilde{\mathbf{p}} = M^{(L)}\mathbf{p}$ at the joint level and $\tilde{\mathbf{p}}_{B\otimes R} = M^{(1)}\mathbf{p}_{B\otimes R}$ at the marginal level (for any L), owing to the sequential interaction of each bit with the ratchet. This gives $I(B', R'; B_{\setminus\{N\}}|B, R) = 0$; see Appendix B for a mathematical proof of this. We will drop the $\otimes$ symbol subsequently to simplify the notation. An alternative perspective is BR statistically "shields" B′R′ from $B_{\setminus\{N\}}$ as alluded to in Refs. [17,22]. Using $I(X; Y; Z) = I(X; Y) - I(X; Y|Z)$, we can now express ΔC_{M_k} into a single conditional mutual information $I(X; Y|Z)$:

$$\begin{aligned}\Delta C_{M_k} &\equiv -\Delta I_{M_k}(B, R; B_{\setminus\{N\}})\\ &= I(B, R; B_{\setminus\{N\}}) - I(B', R'; B_{\setminus\{N\}})\\ &= I(B, R; B_{\setminus\{N\}}; B', R') + I(B, R; B_{\setminus\{N\}}|B', R')\\ &\quad - I(B, R; B_{\setminus\{N\}}; B', R') - \underbrace{I(B', R'; B_{\setminus\{N\}}|B, R)}_{=0}\\ &= I(B, R; B_{\setminus\{N\}}|B', R') \geqslant 0.\end{aligned} \tag{22}$$

The nonnegativity of ΔC_{M_k} can also be interpreted in terms of the data processing inequality; see Ref. [23]. Furthermore, this informational term is responsible for the minimum inevitable cost of *modularity dissipation* which Boyd *et al.* identified in Ref. [22],

$$\begin{aligned}\frac{\langle \Sigma_k^{\mathrm{mod}}\rangle_{\min}}{k_B \ln 2} &= I(B, R; B_{\setminus\{N\}}) - I(B', R'; B_{\setminus\{N\}})\\ &= \underbrace{-\Delta I_{M_k}(B, R; B_{\setminus\{N\}})}_{\Delta C_{M_k}} \geqslant 0,\end{aligned} \tag{23}$$

and arises from the loss of global correlations as only the interacting bit-ratchet subsystem is involved in the modular operation of the ratchet independent of the noninteracting subsystem comprising the remaining bits.

The *joint* IPSL in Eq. (19) can now be recast as

$$\begin{aligned}&\langle W\rangle_{M_k}(\mathbf{p}, M^{(L)}) \leqslant \Delta H_{M_k}[\mathbf{p}]\\ &\qquad\Updownarrow\\ &\langle W\rangle_{M_k}(\mathbf{p}_{B\otimes R}, M^{(1)}) \leqslant \Delta H_{M_k}[\mathbf{p}_{B\otimes R}] + \underbrace{\Delta C_{M_k}}_{\geqslant 0}.\end{aligned} \tag{24}$$

The equivalence of the work terms has been shown earlier in Eq. (13) and comparing with the *marginal* IPSL in Eq. (20), we have thus established that the *marginal* distribution $\mathbf{p}_{B\otimes R}$ for the IPSL in Eq. (20) yields a tighter informational bound than Eq. (19) using *joint* distribution $\mathbf{p}$. In fact, the term ΔC_{M_k} is related to the presence of correlation within our finite-tape information ratchet system. Similar to the case of Boyd *et al.*'s infinite-tape information ratchet, the consideration of correlation in the ratchet system has refined the inequality bounds of the IPSL [19].

We mention that $\langle W\rangle_{M_k}$ in Eq. (24) above actually refers to the heat transferred out of the heat reservoir in the thermal transition substep M_k. Nonetheless, this is equal to the eventual work output from the attachment and detachment mechanism (involving the interacting bit-ratchet $B \otimes R$; see Fig. 2) in the *same* bit scan k, with details of the thermodynamics discussed earlier in Sec. II. We will thus adopt $\langle W\rangle_{M_k}$ as a shorthand for this work while considering the informational change $\Delta H_{M_k}[\cdot]$ for the respective distribution over substep M_k on the RHS of the IPSL subsequently in this paper.

To further shed light on the link between ΔC_{M_k} and correlations within this finite-tape information ratchet system, we first recognize that

$$\begin{aligned}I(B, R; B_{\setminus\{N\}}) &= I(B; B_{\setminus\{N\}}) + I(R; B_{\setminus\{N\}}|B)\\ &= I(B; B_{\setminus\{N\}}) + I(B, B_{\setminus\{N\}}; R) - I(B; R),\end{aligned} \tag{25}$$

by applying $I(X, Y; Z) = I(X; Z) + I(Y; Z|X)$ twice. The ΔC_{M_k} in Eq. (21) can now be alternatively expressed as

$$\begin{aligned}\Delta C_{M_k} &\equiv -\Delta I_{M_k}(B, R; B_{\setminus\{N\}})\\ &= \Delta I_{M_k}(B; R) - \Delta I_{M_k}(B; B_{\setminus\{N\}})\\ &\quad - \Delta I_{M_k}(B, B_{\setminus\{N\}}; R),\end{aligned} \tag{26}$$

which shows that it is related to the change in correlations between the different components (as captured by the respective mutual information terms) within the composite tape-ratchet system. This insight will facilitate the discussion in our subsequent analysis of correlation effects in two separate ratchet designs, referred to as the tridiagonal Markov chain with one-state (T1S) ratchet [20,24] in Sec. VII A and the tridiagonal Markov chain with two-state (T2S) ratchet based on the ratchet design of Boyd *et al.* [17,19,20,22] in Sec. VII B.

V. STATIONARY STATE

We are now ready to perform an in-depth (mathematical) analysis of the stationary (steady-state) behavior of our finite-tape information ratchet. Previously in Ref. [21], we have uncovered the possibility for negative *asymptotic* work in the *stationary* phase. Here we show that this negativity of the asymptotic work $\langle W\rangle_\infty$ is dependent on the presence of the *correlation* term ΔC_{M_∞} defined earlier in Sec. IV. Specifically, we identify two distinct stationary states which either correspond to the absence or presence of ΔC_{M_∞}: the *equilibrium* and *nonequilibrium* stationary states, respectively. The necessary prerequisites will first be introduced to illustrate the subtleties involved between the different stationary states before culminating in the theorems detailing the differences to be presented.

In our construction of the bit scan matrix $O^{(L)}$ of dimension $2^L N_\mathrm{R}$ modeling a single bit scan operation, the design is such that it is *regular* (so eventual convergence to a stationary joint distribution $\boldsymbol{\pi}_{L\text{-bit}} = O^{(L)}\boldsymbol{\pi}_{L\text{-bit}}$ is guaranteed in the stationary or steady state) and this $\boldsymbol{\pi}_{L\text{-bit}}$ is unique from Perron-Frobenius theorem [25]. A regular $O^{(L)}$ corresponds to an *ergodic* Markov chain with all (finite) tape-ratchet states accessible (irreducible) and acyclic (no periodic distributions) necessary to invoke this theorem.

Note that throughout the ratchet operation, switching S_k merely reorders the probabilities within the *joint* distribution so $\Delta H_{S_k}[\tilde{\mathbf{p}}] = 0$ always. It is required to transform the intermediate distribution $\tilde{\mathbf{p}} = M_k\mathbf{p}$ (after the thermal transition M_k) to $\mathbf{p}'$ in each bit scan k, i.e., $\mathbf{p}' = S_k\tilde{\mathbf{p}}$; see Ref. [21]. Since $\Delta H_k[\mathbf{p}] = \Delta H_{S_k}[\tilde{\mathbf{p}}] + \Delta H_{M_k}[\mathbf{p}]$ over a complete (bit scan) cycle $O_k = S_k M_k$, this implies $\Delta H_k[\mathbf{p}] = \Delta H_{M_k}[\mathbf{p}]$ for arbitrary bit scan k. In the stationary state when the system has converged to $\boldsymbol{\pi}_{L\text{-bit}}$, the change in Shannon entropy of the joint (tape-ratchet) distribution vanishes, i.e., $\Delta H_\infty[\mathbf{p}] = 0$, since now $\mathbf{p} \to \boldsymbol{\pi}_{L\text{-bit}}$ is unchanged over O_∞, and thus $\Delta H_{M_\infty}[\mathbf{p}] = \Delta H_\infty[\mathbf{p}] = 0$.

Crucially, we observe the presence of both zero and nonzero (specifically negative) asymptotic work $\langle W\rangle_\infty \leqq 0$ in this stationary state. To explain this phenomenon, we will consider the evolution of the stationary (joint) distribution $\boldsymbol{\pi}_{L\text{-bit}}$ over the substeps M and S, respectively. We associate the *equilibrium* stationary state with

$$M^{(L)}\boldsymbol{\pi}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}, \tag{27}$$

$$S^{(L)}\boldsymbol{\pi}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}. \tag{28}$$

The justification for defining the equilibrium stationary state with the above condition is $M^{(L)}$ (and also $S^{(L)}$) preserves the stationary distribution of $O^{(L)}$, $\boldsymbol{\pi}_{L\text{-bit}}$ in Eq. (27), i.e., this $\boldsymbol{\pi}_{L\text{-bit}}$ is the stationary distribution of both $O^{(L)}$ and $M^{(L)}$. Note that our $M^{(L)}$ is *tridiagonal* [from Eq. (7)] which implies its stationary distribution, i.e., $\boldsymbol{\pi}_{L\text{-bit}}$ here satisfies *detailed balance*. This property of detailed balance arising from the tridiagonality of $M^{(L)}$ will be further discussed in Corollary 1.1 subsequently. Conversely, the *nonequilibrium* stationary state satisfies

$$M^{(L)}\boldsymbol{\pi}_{L\text{-bit}} = \tilde{\boldsymbol{\pi}}_{L\text{-bit}}, \tag{29}$$

$$S^{(L)}\tilde{\boldsymbol{\pi}}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}, \tag{30}$$

where $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} \neq \boldsymbol{\pi}_{L\text{-bit}}$, i.e., the intermediate (joint) distribution $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ after substep M is different from $\boldsymbol{\pi}_{L\text{-bit}}$. This implies $\boldsymbol{\pi}_{L\text{-bit}}$ in this case is not the stationary distribution of $M^{(L)}$ and does not satisfy detailed balance.

The physical action of the switching substep S shifts the leftmost bit of the finite tape to its right end (with the ratchet state fixed) always. It transforms $\mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\cdots\mathrm{B}_L \xrightarrow{S} \mathrm{B}_2\mathrm{B}_3\cdots\mathrm{B}_L\mathrm{B}_1$ for an L-bit tape with all its bit values $\mathrm{B}_i \in \{0,1\}$ unchanged, with index $i \in \{1,2,\ldots,L\}$. An arbitrary tape sequence in general will thus correspond to a different tape sequence after S, and similarly a different tape-ratchet configuration. In consequence, the mathematical operation $S^{(L)}$ serves to reorder the probabilities of the tape-ratchet configuration in $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$. At the stationary state, this reordering by $S^{(L)}$ necessarily restores $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ back to $\boldsymbol{\pi}_{L\text{-bit}}$, i.e., $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} \xrightarrow{S^{(L)}} \boldsymbol{\pi}_{L\text{-bit}}$ [Eq. (30) in general, with Eq. (28) a special case when $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}$]. Specifically, $S^{(L)}$ is a transformation that transfers the probability of one tape-ratchet configuration to that of another tape-ratchet configuration. Out of all the transfers, there are two unique *tape sequences* where $S^{(L)}$ transfers the probability of the corresponding tape-ratchet configuration back to the same tape-ratchet configuration. These are tape sequences of either all “0” s or all “1” s, independent of the ratchet state. As a result, we expect (a total of $2N_\mathrm{R}$ of) these tape-ratchet configurations to maintain the same probabilities before and after the switching substep S in the steady-state, i.e., $\tilde{\pi}_{\{0\}^L\otimes\mathrm{R}} = \pi_{\{0\}^L\otimes\mathrm{R}}$ and $\tilde{\pi}_{\{1\}^L\otimes\mathrm{R}} = \pi_{\{1\}^L\otimes\mathrm{R}}$ with an arbitrary ratchet state R. From the perspective of either the circular (Fig. 1) or linear (Fig. 2) finite tape, it is apparent that

$$\begin{aligned}\tilde{\pi}_{\{0\}^L\otimes\mathrm{R}} &\xrightarrow{S^{(L)}} \pi_{\{0\}^L\otimes\mathrm{R}} = \tilde{\pi}_{\{0\}^L\otimes\mathrm{R}},\\ \tilde{\pi}_{\{1\}^L\otimes\mathrm{R}} &\xrightarrow{S^{(L)}} \pi_{\{1\}^L\otimes\mathrm{R}} = \tilde{\pi}_{\{1\}^L\otimes\mathrm{R}},\end{aligned} \tag{31}$$

with these linear (circular) tape sequences $\{0\}^L$ and $\{1\}^L$ possessing translational (rotational) symmetry and thus correspond to its original tape sequence after S.

With the establishing of $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ containing reordered probabilities in $\boldsymbol{\pi}_{L\text{-bit}}$, we can now deduce that $\boldsymbol{\pi}_{L\text{-bit}} \xrightarrow{M^{(L)}} \tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ in Eqs. (27) and (29) is also a reordering on $\boldsymbol{\pi}_{L\text{-bit}}$ (only in the stationary state) such that it will be the inverse of the subsequent S. In particular,

$$\begin{aligned}\pi_{\{0\}^L\otimes\mathrm{R}} &\xrightarrow{M^{(L)}} \tilde{\pi}_{\{0\}^L\otimes\mathrm{R}} = \pi_{\{0\}^L\otimes\mathrm{R}},\\ \pi_{\{1\}^L\otimes\mathrm{R}} &\xrightarrow{M^{(L)}} \tilde{\pi}_{\{1\}^L\otimes\mathrm{R}} = \pi_{\{1\}^L\otimes\mathrm{R}}.\end{aligned} \tag{32}$$

Tape-ratchet configurations of the finite-tape information ratchet with tape sequences of either all “0” s or all “1” s, independent of the ratchet state, will thus have the same probabilities in $\boldsymbol{\pi}_{L\text{-bit}}$ and $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ (before and after the thermal transition substep M, respectively, in the steady-state behavior).

We are now well-poised to account for this steady-state behavior of our finite-tape information ratchet in terms of the *equilibrium* and *nonequilibrium* stationary states.

A. Equilibrium stationary state

Theorem 1. For a finite-tape (length L) information ratchet with its corresponding 1-bit thermal transition matrix $M^{(1)}$ ($= O^{(1)}$) that is tridiagonal (and regular), the ratchet system will converge to an *equilibrium* stationary state [Eqs. (27) and (28)] if and only if

$$\Delta C_{M_\infty} = 0. \tag{33}$$

Proof. First, we will prove stationary states in equilibrium necessarily have $\Delta C_{M_\infty} = 0$. For such a state, its stationary *joint* distribution $\boldsymbol{\pi}_{L\text{-bit}}$ is preserved over substep M from Eq. (27), which implies its *marginal* $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ is also unchanged over this M and is the stationary distribution of $M^{(1)}$, i.e., $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}} = \boldsymbol{\pi}_{1\text{-bit}}$ satisfying

$$\boldsymbol{\pi}_{1\text{-bit}} = M^{(1)}\boldsymbol{\pi}_{1\text{-bit}}, \tag{34}$$

since the marginal distribution is evolved by $M^{(1)}$ in Eq. (18). This gives $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = 0$, and $\Delta C_{M_\infty} = 0$ follows subsequently from Eq. (21) since $\Delta H_{M_\infty}[\mathbf{p}] = 0$ in the stationary state after the convergence of $\mathbf{p} \to \boldsymbol{\pi}_{L\text{-bit}}$.

For the reverse relation, we had shown earlier in Eq. (22) that $\Delta C_{M_\infty} = I(\mathrm{BR};\mathrm{B}_{\setminus\{N\}}|\mathrm{B'R'})$. The conditional mutual information reads

$$\begin{aligned} I(X;Y|Z) &= \sum_{z\in\mathcal{Z}} p_Z(z)\, I(X;Y|Z=z) \\ &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}\sum_{z\in\mathcal{Z}} p_{X,Y,Z}(x,y,z) \\ &\quad\times \log\left[\frac{p_Z(z)\, p_{X,Y,Z}(x,y,z)}{p_{X,Z}(x,z)\, p_{Y,Z}(y,z)}\right], \end{aligned} \tag{35}$$

which vanishes if and only if the argument of the logarithm in the sum is equal to one, giving the condition

$$p_{X,Y,Z}(x,y,z) = \frac{1}{p_Z(z)}\, p_{X,Z}(x,z)\, p_{Y,Z}(y,z) \quad \forall\, \{x,y,z\}. \tag{36}$$

This stems from $I(X;Y|Z=z) \geqslant 0$, which is the relative entropy between the conditional joint distribution of X and Y given $Z = z$ and product distribution of $\mathbf{p}(X|Z)$ and $\mathbf{p}(Y|Z)$.

For $I(\mathrm{BR};\mathrm{B}_{\setminus\{N\}}|\mathrm{B'R'}) = \Delta C_{M_\infty} = 0$, we have

$$p(\mathrm{BR},\mathrm{B}_{\setminus\{N\}},\mathrm{B'R'}) = \frac{1}{p(\mathrm{B'R'})}\, p(\mathrm{BR},\mathrm{B'R'})\, p(\mathrm{B}_{\setminus\{N\}},\mathrm{B'R'}) \tag{37}$$

with $X \equiv \mathrm{BR}$, $Y \equiv \mathrm{B}_{\setminus\{N\}}$, and $Z \equiv \mathrm{B'R'}$. In addition, we had earlier shown that $I(\mathrm{B'R'};\mathrm{B}_{\setminus\{N\}}|\mathrm{BR}) = 0$ in deriving Eq. (22), so the joint distribution also satisfies

$$p(\mathrm{B'R'},\mathrm{B}_{\setminus\{N\}},\mathrm{BR}) = \frac{1}{p(\mathrm{BR})}\, p(\mathrm{B'R'},\mathrm{BR})\, p(\mathrm{B}_{\setminus\{N\}},\mathrm{BR}), \tag{38}$$

with $X \equiv \mathrm{B'R'}$, $Y \equiv \mathrm{B}_{\setminus\{N\}}$, and $Z \equiv \mathrm{BR}$.

Equating the above two expressions yields

$$\begin{aligned} &\frac{1}{p(\mathrm{B'R'})}\, p(\mathrm{BR},\mathrm{B'R'})\, p(\mathrm{B}_{\setminus\{N\}},\mathrm{B'R'}) \\ &\qquad = \frac{1}{p(\mathrm{BR})}\, p(\mathrm{B'R'},\mathrm{BR})\, p(\mathrm{B}_{\setminus\{N\}},\mathrm{BR}) \\ &\frac{p(\mathrm{B'R'},\mathrm{B}_{\setminus\{N\}})}{p(\mathrm{B'R'})} = \frac{p(\mathrm{BR},\mathrm{B}_{\setminus\{N\}})}{p(\mathrm{BR})} \\ &\frac{p(\mathrm{BR})}{p(\mathrm{B'R'})} = \frac{p(\mathrm{BR},\mathrm{B}_{\setminus\{N\}})}{p(\mathrm{B'R'},\mathrm{B}_{\setminus\{N\}})}. \end{aligned} \tag{39}$$

For our finite-tape information ratchet, the respective random variables realize $\mathrm{B} \in \{0,1\}$, $\mathrm{R} \in \{1,2,\ldots,N_\mathrm{R}\}$, and $\mathrm{B}_{\setminus\{N\}} \in \{0,1\}^{L-1}$. Note that the *marginal* probabilities $p(\mathrm{BR})$ are the same for all *joint* probabilities $p(\mathrm{BR},\mathrm{B}_{\setminus\{N\}})$ with the same $\mathrm{B} = b$, $\mathrm{R} = r$ independent of the realizations of $\mathrm{B}_{\setminus\{N\}}$, and similarly for the probabilities $\tilde{p}(\mathrm{B'R'})$ and $\tilde{p}(\mathrm{B'R'},\mathrm{B}_{\setminus\{N\}})$ (denoted now with a tilde symbol) with the same $\mathrm{B'} = b'$, $\mathrm{R'} = r'$ in the respective (intermediate) distributions $\tilde{\mathbf{p}}_{\mathrm{B}\otimes\mathrm{R}}$ and $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ after the thermal transition M substep. For the purpose of our proof, we will choose $b = b'$, $r = r'$ and applying this choice to the relation in Eq. (39), we obtain

$$\begin{aligned} \frac{p(\mathrm{B}=b,\mathrm{R}=r)}{\tilde{p}(\mathrm{B'}=b,\mathrm{R'}=r)} &= \frac{p(\mathrm{B}=b,\mathrm{R}=r,\mathrm{B}_{\setminus\{N\}}=y)}{\tilde{p}(\mathrm{B'}=b,\mathrm{R'}=r,\mathrm{B}_{\setminus\{N\}}=y)} \\ &= \frac{p(\mathrm{B}=b,\mathrm{R}=r,\mathrm{B}_{\setminus\{N\}}=\bar{y})}{\tilde{p}(\mathrm{B'}=b,\mathrm{R'}=r,\mathrm{B}_{\setminus\{N\}}=\bar{y})}, \end{aligned} \tag{40}$$

with $y \neq \bar{y}$. From Eq. (32), there always exist a $\mathrm{B}_{\setminus\{N\}} = \{b\}^{L-1}$ given $\mathrm{B} = b \in \{0,1\}$ such that the joint probabilities before and after substep M are the same:

$$\frac{p(\mathrm{B}=b,\mathrm{R}=r,\mathrm{B}_{\setminus\{N\}}=\{b\}^{L-1})}{\tilde{p}(\mathrm{B'}=b,\mathrm{R'}=r,\mathrm{B}_{\setminus\{N\}}=\{b\}^{L-1})} = 1 \quad \forall\, \{b,r\}, \tag{41}$$

in the RHS of Eq. (39). Combining this with the relation in Eq. (39) to relate the ratios of marginal and joint probabilities (before and after substep M) for the other corresponding tape-ratchet states with the same $\mathrm{B} = b$, $\mathrm{R} = r$ but different $\mathrm{B}_{\setminus\{N\}} \neq \{b\}^{L-1}$, it implies $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}$ corresponding to the equilibrium stationary state obeying Eqs. (27) and (28). ■

To illuminate details of the switching S used to deduce the relation between $\boldsymbol{\pi}_{L\text{-bit}}$ and $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ and Eqs. (31) and (32), we furnish an explicit example with $L = 2$ and $N_\mathrm{R} = 2$ (similar to Table II) and determine the corresponding tape-ratchet configurations before and after S in Table III, by noting the leftmost bit is shifted to the right end of finite tape from the physical interpretation of S. This implies

$$\begin{pmatrix} \pi_{00\otimes A} \\ \pi_{00\otimes B} \\ \pi_{10\otimes A} \\ \pi_{10\otimes B} \\ \pi_{01\otimes A} \\ \pi_{01\otimes B} \\ \pi_{11\otimes A} \\ \pi_{11\otimes B} \end{pmatrix} = S^{(2)} \begin{pmatrix} \tilde{\pi}_{00\otimes A} \\ \tilde{\pi}_{00\otimes B} \\ \tilde{\pi}_{10\otimes A} \\ \tilde{\pi}_{10\otimes B} \\ \tilde{\pi}_{01\otimes A} \\ \tilde{\pi}_{01\otimes B} \\ \tilde{\pi}_{11\otimes A} \\ \tilde{\pi}_{11\otimes B} \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_{00\otimes A} \\ \tilde{\pi}_{00\otimes B} \\ \tilde{\pi}_{01\otimes A} \\ \tilde{\pi}_{01\otimes B} \\ \tilde{\pi}_{10\otimes A} \\ \tilde{\pi}_{10\otimes B} \\ \tilde{\pi}_{11\otimes A} \\ \tilde{\pi}_{11\otimes B} \end{pmatrix}, \tag{42}$$

with the first equality arising from the preservation [Eq. (28)] or restoration [Eq. (30)] of $\boldsymbol{\pi}_{L\text{-bit}}$ after each bit scan $O^{(L)}$ (in the stationary state) and the second equality from the action of S detailed in Table III, noting that the positions of $\tilde{\pi}_{10\otimes A}$ ($\tilde{\pi}_{10\otimes B}$)

TABLE III. Tape-ratchet configurations before and after switching substep S for $L = 2$ and $N_\mathrm{R} = 2$. The index α (and similarly for α' after switching S) here has been defined earlier in Eq. (8a) and used in Table II to enumerate the respective joint states.

α	Tape-ratchet configurations before and after S	α'
1	$00\otimes A \to 00\otimes A$	1
2	$00\otimes B \to 00\otimes B$	2
3	$10\otimes A \to 01\otimes A$	5
4	$10\otimes B \to 01\otimes B$	6
5	$01\otimes A \to 10\otimes A$	3
6	$01\otimes B \to 10\otimes B$	4
7	$11\otimes A \to 11\otimes A$	7
8	$11\otimes B \to 11\otimes B$	8

and $\tilde{\pi}_{01\otimes A}$ ($\tilde{\pi}_{01\otimes B}$) are swapped. It can next be deduced that

$$\begin{pmatrix}\tilde{\pi}_{00\otimes A}\\ \tilde{\pi}_{00\otimes B}\\ \tilde{\pi}_{10\otimes A}\\ \tilde{\pi}_{10\otimes B}\\ \tilde{\pi}_{01\otimes A}\\ \tilde{\pi}_{01\otimes B}\\ \tilde{\pi}_{11\otimes A}\\ \tilde{\pi}_{11\otimes B}\end{pmatrix} = M^{(2)}\begin{pmatrix}\pi_{00\otimes A}\\ \pi_{00\otimes B}\\ \pi_{10\otimes A}\\ \pi_{10\otimes B}\\ \pi_{01\otimes A}\\ \pi_{01\otimes B}\\ \pi_{11\otimes A}\\ \pi_{11\otimes B}\end{pmatrix} = \begin{pmatrix}\pi_{00\otimes A}\\ \pi_{00\otimes B}\\ \pi_{01\otimes A}\\ \pi_{01\otimes B}\\ \pi_{10\otimes A}\\ \pi_{10\otimes B}\\ \pi_{11\otimes A}\\ \pi_{11\otimes B}\end{pmatrix}, \tag{43}$$

such that the action of M is the inverse of the subsequent S. The (joint) probabilities $p(\mathrm{BR}, \mathrm{B}_{\setminus\{N\}})$ in $\boldsymbol{\pi}_{L\text{-bit}}$ are thus reordered by the M substep in the stationary state to give the intermediate $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ with probabilities $\tilde{p}(\mathrm{B'R'}, \mathrm{B}_{\setminus\{N\}})$. We see that the probabilities $\pi_{00\otimes A}$ and $\pi_{11\otimes A}$ (similarly for $\pi_{00\otimes B}$ and $\pi_{11\otimes B}$, respectively, with the same tape sequences) which are unchanged over $M^{(2)}$ (and also $S^{(2)}$) have respective tape sequences "00" and "11", which correspond precisely to the $\mathrm{B}_{\setminus\{N\}} = \{b\}^{L-1}$ given $\mathrm{B} = b$ which satisfy Eq. (41). A tape sequence with all bits realizing the same value thus correspond to its original sequence over substep M (and also S) for an arbitrary ratchet state which also remains fixed in this stationary state. This guarantees (tape-ratchet) configurations with $\mathrm{B} = b$, $\mathrm{R} = r$, $\mathrm{B}_{\setminus\{N\}} = \{b\}^{L-1}$ fulfill Eq. (41). Coupled with the relation in Eq. (39), assuming $\Delta C_{M_\infty} = 0$ in this example, to relate the ratios of marginal and joint probabilities (before and after substep M) yields

$$\begin{aligned} 1 &= \frac{\pi_{00\otimes A}}{\tilde{\pi}_{00\otimes A}} = \frac{\pi_{01\otimes A}}{\tilde{\pi}_{01\otimes A}} \quad (\mathrm{B}=0,\ \mathrm{R}=A)\\ 1 &= \frac{\pi_{00\otimes B}}{\tilde{\pi}_{00\otimes B}} = \frac{\pi_{01\otimes B}}{\tilde{\pi}_{01\otimes B}} \quad (\mathrm{B}=0,\ \mathrm{R}=B)\\ 1 &= \frac{\pi_{11\otimes A}}{\tilde{\pi}_{11\otimes A}} = \frac{\pi_{10\otimes A}}{\tilde{\pi}_{10\otimes A}} \quad (\mathrm{B}=1,\ \mathrm{R}=A)\\ 1 &= \frac{\pi_{11\otimes B}}{\tilde{\pi}_{11\otimes B}} = \frac{\pi_{10\otimes B}}{\tilde{\pi}_{10\otimes B}} \quad (\mathrm{B}=1,\ \mathrm{R}=B). \end{aligned} \tag{44}$$

implying $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}$ for the equilibrium stationary state [Eqs. (27) and (28)] as set out in Theorem 1.

Corollary 1.1. This finite-tape information ratchet system attains a zero asymptotic work:

$$\langle W\rangle_\infty = 0, \tag{45}$$

if and only if it is in the equilibrium stationary state [Eqs. (27) and (28)].

Proof. We will first show $\langle W\rangle_\infty = 0 \Rightarrow \Delta C_{M_\infty} = 0$. Earlier knowledge of the marginal distribution as a tighter IPSL bound than the joint distribution in Eqs. (19), (20), and (24) gives

$$\langle W\rangle_{M_k} \leqslant \Delta H_{M_k}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] \leqslant \Delta H_{M_k}[\mathbf{p}], \tag{46}$$

where $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ is the marginal distribution of $\mathbf{p} \to \boldsymbol{\pi}_{L\text{-bit}}$ in the stationary state with $\Delta H_{M_\infty}[\mathbf{p}] = \Delta H_\infty[\mathbf{p}] = 0$. Switching S_k also does not involve any work so $\langle W\rangle_{M_k} = \langle W\rangle_k$. Squeezed by both the vanishing lower and upper bounds, we necessarily have $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = 0$, implying $\Delta C_{M_\infty} = 0$ in Eq. (21). Theorem 1 then implies this stationary state is in equilibrium.

We will now invoke the design of our ratchet system to prove equilibrium stationary states necessarily have zero asymptotic work: $\langle W\rangle_\infty = 0$. As our Markov chains (governing stochastic thermal transitions in the interacting bit-ratchet state space $\mathrm{B}\otimes\mathrm{R}$; see subsequent Figs. 3 and 5) are random walks with partial reflective boundaries whose $M^{(1)}$ is tridiagonal, its corresponding (block diagonal) $M^{(L)}$ will also be tridiagonal from its construction in Eq. (7). One property of tridiagonal matrices is its stationary distribution obeys detailed balance. For equilibrium stationary states, its stationary (joint) distribution $\boldsymbol{\pi}_{L\text{-bit}}$ is also the stationary distribution of $M^{(L)}$ from Eq. (27). This confers the property of detailed balance and the probability flux between the interacting bit-ratchet states $\mathrm{B}\otimes\mathrm{R}$:

$$J_{ji} = J_{j\leftarrow i} \equiv M_{ji}p_i - M_{ij}p_j \tag{47}$$

for all pairs of states $\{i, j\}$ vanishes. By expressing Eq. (2) in terms of the respective fluxes J_{ij}, i.e.,

$$\langle W\rangle = \sum_{i<j} \underbrace{(M_{ji}p_i - M_{ij}p_j)}_{J_{ji}=J_{j\leftarrow i}} \ln\left(\frac{M_{ij}}{M_{ji}}\right), \tag{48}$$

we observe the asymptotic work $\langle W\rangle_\infty = 0$ under the condition of detailed balance. These are characteristics of a stationary state in *equilibrium*. ∎

This establishes the if-and-only-if relation

$$\Delta C_{M_\infty} = 0 \;\Leftrightarrow\; \langle W\rangle_\infty = 0 \tag{49}$$

for the system in its *equilibrium* stationary state.

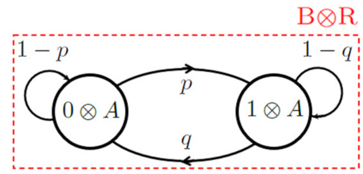

FIG. 3. Transition state diagram for the T1S ratchet, governing the transitions between the interacting bit-ratchet states $\mathrm{B}\otimes\mathrm{R}$ with the corresponding transition probabilities.

B. Nonequilibrium stationary state

Theorem 2. For the same finite-tape information ratchet as in Theorem 1, the ratchet system will converge to a *nonequilibrium* stationary state [Eqs. (29) and (30)] if and only if

$$\Delta C_{M_\infty} > 0. \tag{50}$$

Proof. First, we will prove nonequilibrium stationary states imply $\Delta C_{M_\infty} > 0$. For these states, its stationary *joint* distribution $\boldsymbol{\pi}_{L\text{-bit}}$ is transformed over substep M to an intermediate distribution $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} \neq \boldsymbol{\pi}_{L\text{-bit}}$ from Eq. (29). The consequence is Eq. (39) in Theorem 1 do not hold, voiding the premise required for Eq. (37), i.e., $I(\mathrm{BR}; \mathrm{B}_{\setminus\{N\}}|\mathrm{B'R'}) = \Delta C_{M_\infty} \neq 0$. Since $\Delta C_{M_\infty} \geqslant 0$ from (22), the only possibility is $\Delta C_{M_\infty} > 0$ for this nonequilibrium stationary state.

The reverse relation follows similarly since the stationary states of our ratchet system are either in equilibrium or nonequilibrium, characterized by Eqs. (27) and (28), and Eqs. (29) and (30), respectively, with respect to $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$. With the establishment of Theorem 1 stating the consequence of $\Delta C_{M_\infty} = 0 \Leftrightarrow \tilde{\boldsymbol{\pi}}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}$, $\Delta C_{M_\infty} > 0$ must then necessarily result in a different $\tilde{\boldsymbol{\pi}}_{L\text{-bit}} \neq \boldsymbol{\pi}_{L\text{-bit}}$, associated with a nonequilibrium stationary state [Eqs. (29) and (30)]. ■

Corollary 2.1. This finite-tape information ratchet system attains a negative asymptotic work:

$$\langle W \rangle_\infty < 0, \tag{51}$$

if and only if it is in the nonequilibrium stationary state [Eqs. (29) and (30)].

Proof. We will first show $\langle W \rangle_\infty < 0 \Rightarrow \Delta C_{M_\infty} > 0$. The inequality (46) with $\langle W \rangle_\infty < 0$ and $\Delta H_{M_\infty}[\mathbf{p}] = 0$ (stationary state) gives $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] \leqslant 0$. However, an inconsistency arises should $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = 0$ since this will also imply $\Delta C_{M_\infty} = 0$ from Eq. (21) with $\Delta H_{M_\infty}[\mathbf{p}] = 0$ in the stationary state, which is associated earlier with $\langle W \rangle_\infty = 0$ in Eq. (49). Only $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] < 0$ is thus possible, and $\Delta C_{M_\infty} > 0$ follows from Eq. (21), implying this stationary state is in nonequilibrium from Theorem 2.

For the reverse relation starting from $\Delta C_{M_\infty} > 0$, this implies $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] < 0$ from Eq. (21) in the stationary state. Plugging this into the inequality (46) with $\Delta H_{M_\infty}[\mathbf{p}] = 0$, we recover again the negative asymptotic work: $\langle W \rangle_\infty < 0$. ■

This establishes the if-and-only-if relation

$$\Delta C_{M_\infty} > 0 \;\; \Leftrightarrow \;\; \langle W \rangle_\infty < 0 \tag{52}$$

for the system in its *nonequilibrium* stationary state.

A closer inspection of $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] < 0$ (or more generally $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] \neq 0$) since $\Delta C_{M_\infty} > 0$ from Eq. (21) in the stationary state reveals its consistency with the breaking of detailed balance for such nonequilibrium stationary states. Corresponding to Eqs. (29) and (30), the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ of $\boldsymbol{\pi}_{L\text{-bit}}$ is also not preserved and evolved by the (tridiagonal) 1-bit thermal transition matrix $M^{(1)}$ to $\tilde{\mathbf{p}}_{\mathrm{B}\otimes\mathrm{R}}$ (over substep M) since $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}} \neq \boldsymbol{\pi}_{1\text{-bit}}$. In consequence, a nonequilibrium stationary state ensues even though the subsequent switching substep S in the same bit scan will transform $\tilde{\mathbf{p}}_{\mathrm{B}\otimes\mathrm{R}}$ ($\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$) back to the original $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ ($\boldsymbol{\pi}_{L\text{-bit}}$) [Eq. (30)] since the stationary joint $\boldsymbol{\pi}_{L\text{-bit}}$ is unchanged by $O^{(L)}$ for every bit scan in the steady state, i.e., $\boldsymbol{\pi}_{L\text{-bit}} = O^{(L)} \boldsymbol{\pi}_{L\text{-bit}}$.

For all our finite-tape information ratchets with $L = 1$ (independent of N_{R}), its joint and marginal distributions are the same, i.e., $\mathbf{p} = \mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$, with convergence $\mathbf{p} \to \boldsymbol{\pi}_{1\text{-bit}}$ in the stationary state necessarily in equilibrium since the thermal transition matrix $M^{(1)}$ $(= O^{(1)})$ is tridiagonal and this $\boldsymbol{\pi}_{1\text{-bit}}$ is thus the stationary distribution of both $O^{(1)}$ and $M^{(1)}$. However, this is not guaranteed for $L > 1$ with the presence of the noninteracting subsystem $\mathrm{B}_{\setminus\{N\}}$, previously absent for the 1-bit tape with only a single (interacting) bit constituting the entire tape itself and its *local* bit scan operation $O^{(1)}$ is essentially a *global* operation where the informational states of all the bits within the entire tape can be simultaneously changed. This *global* operation is thus *reversible* with zero entropy production (in the stationary state) and the change in Shannon (informational) entropy of the system exactly cancels the change in thermodynamic (physical) entropy of the heat reservoir due to heat dissipation.

On the contrary for $L > 1$, the *local* modular operation of our information ratchet sequentially interacting with each bit effectively decouples the interacting subsystem $\mathrm{B} \otimes \mathrm{R}$ from the noninteracting subsystem $\mathrm{B}_{\setminus\{N\}}$ held fixed in each bit scan. This yields a marginalized nonequilibrium (in general) free energy difference arising solely from the interacting $\mathrm{B} \otimes \mathrm{R}$ subsystem which leads to a tighter bound on the work, i.e., the inequality in Eq. (46) with the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ as compared with the joint $\mathbf{p}$ from the global nonequilibrium free energy difference of the composite ratchet system. The localized ratchet operation here is unable to leverage global correlations as a thermodynamic resource [19,20], which possibly renders it thermodynamically inefficient and results in a modularity dissipation $\langle \Sigma^{\mathrm{mod}} \rangle_{\min}$—a minimum entropy production which quantifies the irreversibility of this modular operation [22]. One of its manifestations is the two different (upper) bounds in the respective joint and marginal formulations of the IPSL in Eqs. (19) and (20).

We had earlier identified in Eq. (23) the relation between this $\langle \Sigma^{\mathrm{mod}} \rangle_{\min}$ and the ΔC_{M_k} term reflecting the different correlations within the finite-tape ratchet system. The implications for the stationary states of our information ratchet system are as follows. Equilibrium stationary states have $\Delta C_{M_\infty} = 0$ and zero asymptotic work $\langle W \rangle_\infty = 0$ (from Theorem 1), and saturate both IPSL inequalities in Eqs. (19) and (20) with both bounds the same, i.e., $\Delta H_{M_\infty}[\mathbf{p}] = \Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = 0$. This ratchet operation is thus *reversible* in its steady-state behavior with $\langle \Sigma^{\mathrm{mod}} \rangle_{\min} = 0$. Pertaining to the nonequilibrium stationary states, however, $\Delta C_{M_\infty} > 0$ (Theorem 2) implies the modular operation of the ratchet here is *irreversible* with a minimum entropy production $\langle \Sigma^{\mathrm{mod}} \rangle_{\min} > 0$ in its stationary state and work is continuously expended to transform the stationary $\boldsymbol{\pi}_{L\text{-bit}}$ (of $O^{(L)}$) to the intermediate $\tilde{\boldsymbol{\pi}}_{L\text{-bit}}$ over substep M. Both distributions are not stationary distributions of the thermal transition matrix $M^{(L)}$ from Eq. (29), and results in a difference in the joint and marginal IPSL bounds in Eqs. (19) and (20), respectively, i.e., $\Delta H_{M_\infty}[\mathbf{p}] = 0$ and $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] < 0$. This explains the negative asymptotic work $\langle W \rangle_\infty < 0$ (work expenditure) to drive and sustain this nonequilibrium steady-state behavior.

VI. TRANSIENT PHASE

The discussion thus far has revolved around the stationary (steady-state) behavior of the finite-tape information ratchet. This inevitably leads to the question of what happens in the transitional period between the start of ratchet operation and its steady-state behavior. To address this question, a consideration of the tape-ratchet dynamics before the convergence $\mathbf{p} \to \boldsymbol{\pi}_{L\text{-bit}}$ occurs, i.e., in the *transient* phase, of our information ratchet system is as follows in this section. It turns out that the finite-tape ratchet in the transient phase promises a rich array of thermodynamic functionalities which had been elucidated in Ref. [21] (and similarly in Ref. [17]), although the mathematics is in general intractable here in comparison with the stationary states. It is also in this transient phase that maximum mechanical work production is possible with specific ratchet design from the amassing of (mechanical) energy [21].

Before the finite-tape information ratchet converges to its stationary distribution $\boldsymbol{\pi}$, both its instantaneous joint (tape-ratchet) $\mathbf{p}$ and marginal (bit-ratchet) $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ distributions are continuously evolving with successive bit scans. At this stage, the ratchet is said to be in the *transient* phase. Crucially, the signs of $\Delta H_{M_k}[\mathbf{p}]$ and $\Delta H_{M_k}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}]$ are undetermined in this transient phase as (in general) $\Delta H_{M_k}[\mathbf{p}] =$

$\Delta H_k[\mathbf{p}] \neq 0$, in comparison with the previous stationary states in Theorems 1 and 2. [It is possible here for two different (joint) distributions to have the same Shannon entropy, i.e., $\Delta H_{M_k}[\mathbf{p}] = \Delta H_k[\mathbf{p}] = 0$, but this will only be instantaneous as $\mathbf{p}$ continues to evolve before its convergence to $\boldsymbol{\pi}$; similarly for the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$.] Nonetheless, $\Delta C_{M_k} \geqslant 0$ from Eq. (22) and key relations such as Eqs. (1), (18)–(21), (24), and (46) derived earlier are valid at all times and we can continue to utilize them to study the behavior of our finite-tape ratchet in its transient phase.

During the transient phase, our earlier work [21] had deduced ratchet operations in the engine, eraser or dud mode, illuminating a diverse range of ratchet behavior. It is also at this phase that useful work is performed and converted to mechanical energy, specifically gravitational potential energy, which is accumulated and stored in the work reservoir (a source for future work extraction) in Fig. 1 for practical purposes (as manifested by the raised mass in the mass-pulley system). We found ratchet possessing memory to have a greater capacity for this work-energy transfer (to mechanical energy in its engine mode) through exploiting correlation. This is in contrast with the stationary behavior of the finite-tape information ratchet which is unable to operate as an engine with $\langle W\rangle_\infty \leqslant 0$ and is therefore unlikely of much promise from a practical perspective.

VII. RESULTS ON CORRELATION EFFECT

Here we will probe the correlation term $\Delta C_{(\cdot)}$ further (over thermal transition M and now also switching S substeps separately) by delving into the respective mutual information between the different subsystems within the finite-tape information ratchet system [see Eq. (26)] in both transient and stationary phases. This will be applied to two specific ratchet designs: the tridiagonal Markov chain with one-state (T1S) ratchet in Sec. VII A and the tridiagonal Markov chain with two-state (T2S) ratchet in Sec. VII B, to explicitly observe their different stationary (steady-state) behavior as expounded upon in the preceding sections. They are the consequences of accounting for correlation in the tape-ratchet system. We complete the discussion by working out the mathematical condition for equilibrium stationary states (with $\langle W\rangle_\infty = 0$) of information ratchets that harness correlation, and applying the condition to the T2S ratchet explicitly, in Sec. VII C as our last main result in this paper. This is yet another manifestation of correlation within the ratchet system, with the identified states serving as promising candidates for the maximum work-(mechanical) energy conversion for practical purposes [21].

We now illustrate how correlation manifest in different ratchet designs (specifically the T1S and T2S ratchets in Ref. [21]) and utilize the expression for ΔC_{M_k} in Eq. (26) from our earlier analysis of the IPSL bounds. We saw that such correlations are captured by the mutual information terms in ΔC_{M_k} which read

$$\begin{aligned}\Delta C_{M_k} &\equiv -\Delta I_{M_k}(\mathrm{B},\mathrm{R};\mathrm{B}_{\backslash\{N\}})\\ &= \underbrace{\Delta I_{M_k}(\mathrm{B};\mathrm{R})}_{\text{(i)}} - \underbrace{\Delta I_{M_k}(\mathrm{B};\mathrm{B}_{\backslash\{N\}})}_{\text{(ii)}}\\ &\quad - \underbrace{\Delta I_{M_k}(\mathrm{B},\mathrm{B}_{\backslash\{N\}};\mathrm{R})}_{\text{(iii)}}. \end{aligned} \tag{53}$$

Explicitly, the respective changes over the thermal transition substep M_k are

$$\text{(i)}\ \Delta C_{M_k}^{(\mathrm{B};\mathrm{R})} \equiv I(\mathrm{B}';\mathrm{R}') - I(\mathrm{B};\mathrm{R}), \tag{54a}$$

$$\text{(ii)}\ \Delta C_{M_k}^{(\mathrm{B};\mathrm{B}_{\backslash\{N\}})} \equiv I(\mathrm{B}';\mathrm{B}_{\backslash\{N\}}) - I(\mathrm{B};\mathrm{B}_{\backslash\{N\}}), \tag{54b}$$

$$\text{(iii)}\Delta C_{M_k}^{(\mathrm{B},\mathrm{B}_{\backslash\{N\}};\mathrm{R})} \equiv I(\mathrm{B}',\mathrm{B}_{\backslash\{N\}};\mathrm{R}') - I(\mathrm{B},\mathrm{B}_{\backslash\{N\}};\mathrm{R}), \tag{54c}$$

where the leftmost bit is transformed from the input bit $\mathrm{B} = \mathrm{B}_N$ to the output bit $\mathrm{B}' = \mathrm{B}'_N$, and we will denote this leftmost bit as B or B′. Correlations thus exist between (i) the leftmost bit (B or B′) and the ratchet (R or R′), (ii) the leftmost bit (B or B′) and all the remaining bits $\mathrm{B}_{\backslash\{N\}}$, and (iii) the finite tape comprising all the bits B or B′, $\mathrm{B}_{\backslash\{N\}}$ and the ratchet (R or R′), over the thermal transition substep M_k within an arbitrary bit scan k. We have also used the shorthand $\mathrm{B}_{\backslash\{N\}}$ to refer to the bit sequence $\mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}$ excluding the leftmost bit (B_N or B'_N), from the circular tape perspective in Fig. 1 and the possibility of repeated *tape* scans with a finite tape.

For completeness, we also explore into the mutual informational changes over the subsequent switching substep S_k. The quantities we look into here are: (a) $\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\backslash\{N\}})} \equiv I(\mathrm{B};\mathrm{B}_{\backslash\{N,N+1\}},\mathrm{B}') - I(\mathrm{B}';\mathrm{B}_{\backslash\{N\}})$, and (b) $\Delta C_{S_k}^{(\mathrm{B},\mathrm{B}_{\backslash\{N\}};\mathrm{R})} \equiv I(\mathrm{B},\mathrm{B}_{\backslash\{N,N+1\}},\mathrm{B}';\mathrm{R}') - I(\mathrm{B}',\mathrm{B}_{\backslash\{N\}};\mathrm{R}')$, where in the superscript we have simply used B and R (without prime symbols) as labels to denote the leftmost bit and ratchet, respectively. Crucially, the leftmost bit (B or B′) here involves two *different* bits. Specifically, the output bit $\mathrm{B}' = \mathrm{B}'_N$ is replaced by the new input bit $\mathrm{B} = \mathrm{B}_{N+1}$, unlike over M_k where the leftmost bit (B or B′) is the same bit (with index N). Note that the notation $\mathrm{B}_{\backslash\{N,N+1\}}$ denotes the remaining bits $\mathrm{B}_{N+2}\mathrm{B}_{N+3}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}$ excluding the leftmost bit B_{N+1} and rightmost bit B'_N after S_k. For (a), we found after a straightforward evaluation in Appendix C that

$$\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\backslash\{N\}})} = I(\mathrm{B}_{N+1};\mathrm{B}_{\backslash\{N,N+1\}}) - I(\mathrm{B}'_N;\mathrm{B}_{\backslash\{N,N+1\}}), \tag{55}$$

where we have made explicit the new input bit $\mathrm{B} = \mathrm{B}_{N+1}$ and old output bit $\mathrm{B}' = \mathrm{B}'_N$ above and the tape sequence has changed from $\mathrm{B}'_N\mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}$ to $\mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}\mathrm{B}'_N$ after switching S_k. Thus, in general $\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\backslash\{N\}})} \neq 0$ in the *transient* phase for $L \neq 1, 2$, but always vanishes for $L = 2$ as there are no additional bits in $\mathrm{B}_{\backslash\{N,N+1\}}$ besides the outgoing output bit $\mathrm{B}' = \mathrm{B}'_N$ and incoming input bit $\mathrm{B} = \mathrm{B}_{N+1}$. However, a similar evaluation for (b) in Appendix C has uncovered that $\Delta C_{S_k}^{(\mathrm{B},\mathrm{B}_{\backslash\{N\}};\mathrm{R})} = 0$ for all L since the entire information ratchet (tape-ratchet system) is self-contained with the recirculation of existing bits.

We now proceed to investigate how the behavior of these correlations affect the tape-ratchet dynamics and nature of the corresponding stationary states.

A. Tridiagonal Markov chain with one-state (T1S) ratchet

The permissible thermal transitions between the interacting bit-ratchet states $\mathrm{B}\otimes\mathrm{R}$ for the T1S ratchet are schematically represented in Fig. 3, which has the smallest $\mathrm{B}\otimes\mathrm{R}$ state space of dimension 2, i.e., $\{0\otimes A, 1\otimes A\}$. We have used A to arbitrarily label the single ratchet state ($N_\mathrm{R} = 1$) which we will drop from hereon. The left stochastic matrix corresponding to this thermal transition substep M is

$$M^{\mathrm{T1S}}_{\text{1-bit}} = \begin{matrix} & \begin{matrix} 0 & \quad 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \end{matrix} = \left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right), \tag{56}$$

with transition probabilities $0 \leqslant p, q \leqslant 1$ and its partition labels match those in Eq. (3). Specifically for $L = 1$, the marginal and joint distributions are identical with $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}} = \mathbf{p}$ and $M^{\mathrm{T1S}}_{\text{1-bit}} = O^{\mathrm{T1S}}_{\text{1-bit}}$ governs their dynamics and is responsible for their convergence to the stationary distribution $\boldsymbol{\pi}^{\mathrm{T1S}}_{\text{1-bit}} = [\frac{q}{p+q}, \frac{p}{p+q}]^{\top}$.

A pertinent feature of this T1S ratchet is its single ratchet state ($N_{\mathrm{R}} = 1$) so itself lacks memory and unable to generate and introduce correlations within the tape-ratchet system. Thus, correlations involving the ratchet identified earlier in Eqs. (54a) and (54c) vanish and we now seek to study the correlation given by Eq. (54b) between the leftmost bit (B or B′) and remaining bits in the finite tape with an uncorrelated and correlated initial joint distributions separately.

1. Uncorrelated initial joint distribution

An uncorrelated initial joint distribution $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(0)$ can be broken into its marginal bit distributions (superscript denoting the individual bit), i.e.,

$$\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(0) = \bigotimes_{i=1}^{L} \mathbf{p}^{\mathrm{T1S},\,(i)}_{\text{1-bit}}(0), \tag{57}$$

with no initial correlations between the bits and $I(\mathrm{B} = \mathrm{B}_1; \mathrm{B}_{\setminus\{1\}}) = 0$ at the beginning of ratchet operation. With the T1S ratchet (without memory), the sequential processing of the bits are independent of each other:

$$O^{\mathrm{T1S}}_{L\text{-bit}}\left(\bigotimes_{i=1}^{L} \mathbf{p}^{\mathrm{T1S},\,(i)}_{\text{1-bit}}(0)\right) = \left(\bigotimes_{i=2}^{L} \mathbf{p}^{\mathrm{T1S},\,(i)}_{\text{1-bit}}(0)\right) \otimes \mathbf{p}^{\mathrm{T1S},\,(1)}_{\text{1-bit}}(\tau). \tag{58}$$

The processing is performed over an interaction period τ for a single bit scan where only the (leftmost) first bit B_1 interacts with the ratchet. Thus, the joint distribution $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}$ here is *always* a product of marginals over either substep M or S from Eq. (58) above. This implies $C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})} = 0$ at all times for the uncorrelated initial $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(0)$. This means that Eq. (54b) vanishes which implies $\Delta C_{M_k} = -\Delta C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}_{M_k} = 0$ with $\Delta H_{M_k}[\mathbf{p}] = \Delta H_{M_k}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}]$ from Eq. (21). Note that the mutual information terms involving this T1S ratchet, i.e., $I(\mathrm{B};\mathrm{R}) = I(\mathrm{B}, \mathrm{B}_{\setminus\{N\}};\mathrm{R}) = 0$ vanish, respectively, at the outset of its operation because there is no uncertainty for a single ratchet state R, thus Eqs. (54a) and (54c) are both zero. In the stationary state, $\Delta H_{M_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = \Delta H_{S_\infty}[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}] = 0$, i.e., the marginal $\mathbf{p}^{\mathrm{T1S}}_{\mathrm{B}\otimes\mathrm{R}} = \boldsymbol{\pi}^{\mathrm{T1S}}_{\text{1-bit}}$ is unchanged over either substep M or S for this steady-state behavior in equilibrium; see Eqs. (27) and (28).

Since $\Delta H_\infty[\mathbf{p}] = 0$ in the steady state, $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}$ will have converged to its stationary distribution:

$$\lim_{k\to\infty} \mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(k) = \boldsymbol{\pi}^{\mathrm{T1S}}_{L\text{-bit}} = \bigotimes_{i=1}^{L} \boldsymbol{\pi}^{\mathrm{T1S},\,(i)}_{\text{1-bit}}. \tag{59}$$

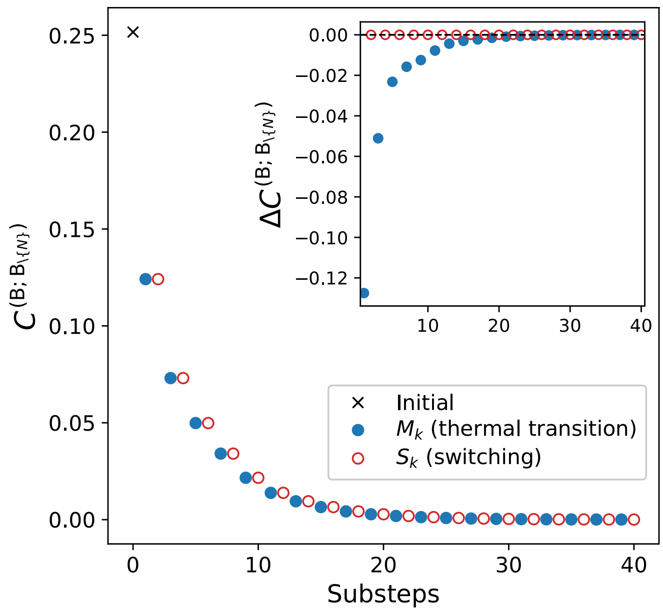

FIG. 4. Plot of $C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}$ over $k = 20$ bit scans for T1S ratchet 2-bit tape with correlated initial distribution $\mathbf{p}^{\mathrm{T1S}}_{\text{2-bit}}(0) = [p_{00}, p_{01}, p_{10}, p_{11}]^{\top} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0]^{\top}$ and transition probabilities $(p, q) = (0.872, 0.949)$ after every (thermal transition M_k and switching S_k) substep within the two-step composite bit scan operation $O_k = S_k M_k$. Inset depicts the corresponding change in $C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}$, i.e., $\Delta C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}_{M_k}$ and $\Delta C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}_{S_k}$ over the respective M_k and S_k substep.

This follows from the independent individual bit-ratchet interactions earlier in Eq. (58). Due to the lack of correlation between the bits and itself, all its stationary states are in equilibrium fulfilling detailed balance in the steady-state behavior with zero asymptotic work $\langle W\rangle_\infty = 0$ necessarily. This is consistent with the relation in Eq. (49) derived from our earlier analysis of equilibrium stationary states: $\Delta C_{M_\infty} = 0 \Leftrightarrow \langle W\rangle_\infty = 0$.

2. Correlated initial joint distribution

A correlated initial joint distribution $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(0)$ cannot be broken into its marginals as in Eq. (57) and correlations exist within the bits prior to ratchet operation, i.e., $I(\mathrm{B} = \mathrm{B}_1; \mathrm{B}_{\setminus\{1\}}) > 0$. Since the T1S ratchet is unable to generate or introduce correlations into the tape-ratchet system, any initial correlations present between the bits of the finite tape will eventually vanish as all initial joint distributions $\mathbf{p}^{\mathrm{T1S}}_{L\text{-bit}}(0)$ converge to the same stationary distribution $\boldsymbol{\pi}^{\mathrm{T1S}}_{L\text{-bit}}$ in Eq. (59) which is necessarily uncorrelated. This implies $C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})} = 0$ in the steady-state behavior and also satisfy the relation in Eq. (49), implying these stationary states are in equilibrium.

We illustrate with a correlated initial distribution for $L = 2$ (the simplest nontrivial L): $\mathbf{p}^{\mathrm{T1S}}_{\text{2-bit}}(0) = [p_{00}, p_{10}, p_{01}, p_{11}]^{\top} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0]^{\top}$ and transition probabilities $(p, q) = (0.872, 0.949)$ to observe the behavior of correlations between these two bits in the transient phase before vanishing once equilibrium has been attained in the stationary state. Quantified by $C^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}$, this quantity is plotted in Fig. 4 after every (thermal transition M_k and switching

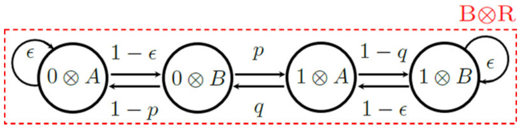

FIG. 5. Transition state diagram for the T2S ratchet, governing the transitions between the interacting bit-ratchet states B ⊗ R with the corresponding transition probabilities.

S_k) substep within a bit scan $O_k = S_k M_k$ (specifically $O^{\rm T1S}_{\text{2-bit}} = S^{\rm T1S}_{\text{2-bit}} M^{\rm T1S}_{\text{2-bit}}$) for $k = 20$ bit scans. We see that indeed $\Delta C^{(\rm B;B_{\setminus\{N\}})}_{M_k} \leqslant 0$ over any M_k and $\Delta C^{(\rm B;B_{\setminus\{N\}})}_{k} = \Delta C^{(\rm B;B_{\setminus\{N\}})}_{M_k}$, with $\Delta C^{(\rm B;B_{\setminus\{N\}})}_{S_k} = 0$ and switching does not alter the information manifested as correlations within the tape in this case when $L = 2$ which is in line with our observation based on Eq. (55). Eventually $C^{(\rm B;B_{\setminus\{N\}})} = 0$ (steady-state) for all L also implies $\Delta C_{M_\infty} = 0$ (similarly for ΔC_{S_∞}) which is consistent with (49) for an equilibrium stationary state.

B. Tridiagonal Markov chain with two-state (T2S) ratchet

In contrast with the T1S ratchet, the simplest design of our information ratchet with memory [21] is $N_{\rm R} = 2\,(>1)$, i.e., the T2S ratchet (Fig. 5) with its two ratchet states (arbitrarily labeled A, B) and corresponding tridiagonal thermal transition matrix M for $L = 1$:

$$M^{\rm T2S}_{\text{1-bit}} = \begin{array}{c} \\ 0\otimes A \\ 0\otimes B \\ 1\otimes A \\ 1\otimes B \end{array} \begin{array}{c} \begin{array}{cccc} 0\otimes A & 0\otimes B & 1\otimes A & 1\otimes B \end{array} \\ \begin{pmatrix} \epsilon & 1-p & 0 & 0 \\ 1-\epsilon & 0 & q & 0 \\ 0 & p & 0 & 1-\epsilon \\ 0 & 0 & 1-q & \epsilon \end{pmatrix} \end{array}, \tag{60}$$

which doubles the interacting bit-ratchet state space B ⊗ R as compared to the T1S ratchet with its $M^{(1)}$ in Eq. (56). Due to the T2S ratchet possessing memory and its subsequent ability to generate and introduce correlations from its modular operation of the finite tape, all the correlation terms in Eqs. (54a)–(54c) are possibly nonzero. Moreover, we will show they manifest differently for equilibrium and nonequilibrium stationary states which stem from the effect of correlation in their steady-state behavior.

Considering first the trivial case for $L = 1$, all initial joint (and here equivalently marginal) distributions $\mathbf{p}^{\rm T2S}_{\text{1-bit}}(0)$ will converge to

$$\boldsymbol{\pi}^{\rm T2S}_{\text{1-bit}} = [2(p+q-pq) - (p+q)\epsilon]^{-1} \begin{pmatrix} (1-p)q \\ (1-\epsilon)q \\ (1-\epsilon)p \\ p(1-q) \end{pmatrix}, \tag{61}$$

which is the right eigenvector of $O^{\rm T2S}_{\text{1-bit}} = M^{\rm T2S}_{\text{1-bit}}$ in Eq. (60) with eigenvalue 1, and the asymptotic work $\langle W \rangle_\infty = 0$ vanishes in the stationary state with $\Delta H_\infty[\mathbf{p}] = 0$. All stationary states of the T2S ratchet with 1-bit tape, independent of transition parameters (p, q, ϵ), are thus in equilibrium from our tridiagonal design of the thermal transition matrix $M^{(1)}$ alluded to previously and the earlier relation (49).

We will now proceed to explore this T2S ratchet with finite tape $L = 2$, the simplest analytically tractable case for

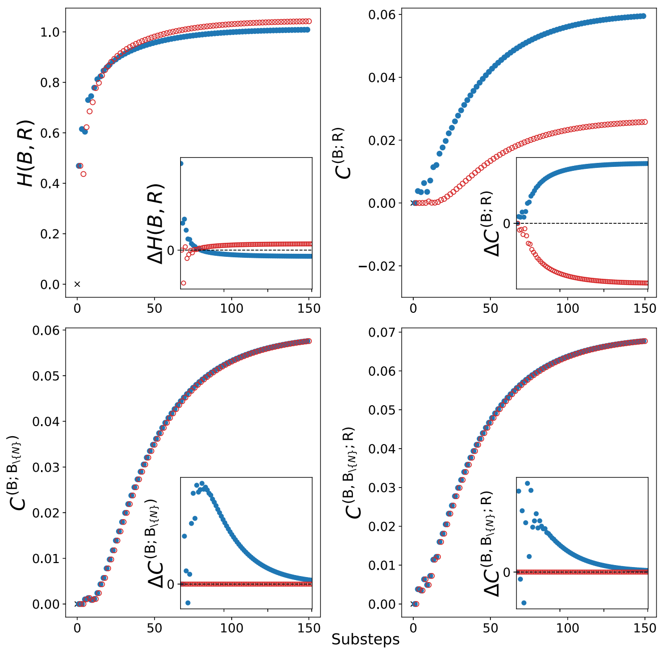

FIG. 6. Plots of the Shannon entropy of the marginal bit-ratchet distribution $H[\mathbf{p}_{\rm B\otimes R}]$ (top left) and the respective mutual informations in ΔC from Eq. (26) for a system that eventually converges to a *nonequilibrium* stationary state: (i) $C^{(\rm B;R)}$ between the leftmost bit and ratchet (top right), (ii) $C^{(\rm B;B_{\setminus\{N\}})}$ between the leftmost bit and all the remaining bits (bottom left), and (iii) $C^{(\rm B,B_{\setminus\{N\}};R)}$ between the finite tape comprising all the bits and ratchet (bottom right). These quantities are plotted prior to ratchet operation (black cross) and after every M_k (blue solid circle) and S_k (red hollow circle) substep within a bit scan $O_k = S_k M_k$ over $k = 75$ bit scans with an (uncorrelated) initial definite-state (joint) distribution $\mathbf{p}^{\rm T2S}_{\text{2-bit}}(0)$ in Eq. (62), i.e., the system is initially in the joint tape-ratchet state $00 \otimes A$ definitively, with transition probabilities $(p, q, \epsilon) = (0.276, 0.680, 0.9)$. The insets depict the *changes* in the mutual information over the corresponding M_k and S_k substep for ease of discerning their signs.

$L > 1$, to highlight the effects of correlation on the finite-tape information ratchet in ΔC_{M_k} which presents both equilibrium and nonequilibrium stationary states.

1. Nonequilibrium stationary state with $\langle W \rangle_\infty < 0$

We will first discuss the case for nonequilibrium stationary states with negative asymptotic work $\langle W \rangle_\infty < 0$ as it turns out to have an overwhelmingly higher incidence from our sampling in (p, q, ϵ) parameter space. We found $(p, q, \epsilon) = (0.276, 0.680, 0.9)$ corresponds to such a case and plotted the respective informational quantities in Fig. 6: the Shannon entropy of the marginal distribution $H[\mathbf{p}_{\rm B\otimes R}]$ (top left) and the mutual information in C from Eq. (26), i.e., (i) $C^{(\rm B;R)}$ between the leftmost (either input or output) bit and ratchet (top right), (ii) $C^{(\rm B;B_{\setminus\{N\}})}$ between the leftmost bit and all the remaining bits (bottom left), and (iii) $C^{(\rm B,B_{\setminus\{N\}};R)}$ between the finite tape comprising all the bits and ratchet (bottom right). These quantities are plotted *after* every (thermal transition M_k and switching S_k) substep within a bit scan $O_k = S_k M_k$ over $k = 75$ bit scans with an (uncorrelated) initial definite-state (joint) distribution [21], i.e.,

$$\mathbf{p}^{\text{T2S}}_{\text{2-bit}}(0) = \begin{pmatrix} p_{00\otimes A} \\ p_{00\otimes B} \\ p_{10\otimes A} \\ p_{10\otimes B} \\ p_{01\otimes A} \\ p_{01\otimes B} \\ p_{11\otimes A} \\ p_{11\otimes B} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (62)$$

First, we see that $H[\mathbf{p}_{\text{B}\otimes\text{R}}]$ and $C^{(\text{B;R})}$ (top panel) alternate between every M_∞ and S_∞ substep within each bit scan O_∞ in the steady-state behavior. The other $C^{(\text{B;B}_{\setminus\{N\}})}$ and $C^{(\text{B,B}_{\setminus\{N\}};\text{R})}$, however, saturate to some positive (nonzero) value (bottom panel). Both observations are consistent with the respective changes over a bit scan O necessarily vanishing in the stationary state after $\mathbf{p}^{\text{T2S}}_{\text{2-bit}}(0) \to \boldsymbol{\pi}^{\text{T2S}}_{\text{2-bit}}$. We will continue to consider the interval over M_k for the respective informational changes to describe these nonequilibrium (and also for equilibrium) stationary states (subsequently).

The respective informational changes given by Eqs. (54b) and (54c) are identical over either a complete bit scan O_k or corresponding substep M_k, i.e., $\Delta C_k^{(\text{B;B}_{\setminus\{N\}})} = \Delta C_{M_k}^{(\text{B;B}_{\setminus\{N\}})}$ and $\Delta C_k^{(\text{B,B}_{\setminus\{N\}};\text{R})} = \Delta C_{M_k}^{(\text{B,B}_{\setminus\{N\}};\text{R})}$, so $\Delta C_{S_k}^{(\text{B;B}_{\setminus\{N\}})} = \Delta C_{S_k}^{(\text{B,B}_{\setminus\{N\}};\text{R})} = 0$.

Although there are no initial correlations in $\mathbf{p}^{\text{T2S}}_{\text{2-bit}}(0)$, $C^{(\text{B;B}_{\setminus\{N\}})} > 0$ and $C^{(\text{B,B}_{\setminus\{N\}};\text{R})} > 0$ saturate eventually (i.e., $\Delta C^{(\text{B;B}_{\setminus\{N\}})}$ and $\Delta C^{(\text{B,B}_{\setminus\{N\}};\text{R})}$ become zero) as the T2S ratchet is able to generate and introduce correlations from its operation unlike the T1S ratchet. We also observe this behavior in the stationary state for longer L. Of particular interest is the marginal distribution $\mathbf{p}_{\text{B}\otimes\text{R}}$ and its correlation captured by the mutual information in its steady-state behavior: $\Delta H_{M_\infty}[\mathbf{p}_{\text{B}\otimes\text{R}}] < 0$ and $\Delta C_{M_\infty} = \Delta C_{M_\infty}^{(\text{B;R})} > 0$ and their magnitudes are equal which satisfy Eq. (21) over the interval M_∞. This is consistent with the earlier analysis of nonequilibrium stationary states characterized by $\Delta C_{M_\infty} > 0 \Leftrightarrow \langle W \rangle_\infty < 0$ in Eq. (52). We emphasize that these informational changes are considered over M_∞ as the corresponding change over S_∞ will be of opposite sign (with equal magnitude) since the net change over a bit scan $O_\infty = S_\infty M_\infty$ must vanish in the stationary state.

Probing deeper into the underlying mathematical structure reveals the intriguing role of switching in the finite-tape ratchet. We recognized in Sec. V that $\Delta H_k[\mathbf{p}] = \Delta H_{M_k}[\mathbf{p}]$ for arbitrary bit scan k and $\Delta H_\infty[\mathbf{p}] = \Delta H_{M_\infty}[\mathbf{p}] = 0$ in the stationary state. However, we also have $\Delta H_{M_\infty}[\mathbf{p}_{\text{B}\otimes\text{R}}] < 0$ in this stationary state according to our earlier analysis. This must imply that the mathematical effect of M_∞ now is the opposite of the subsequent (physical) switching by S_∞ which reorders the probabilities of joint states in $\boldsymbol{\pi}^{\text{T2S}}_{\text{2-bit}}$, reflecting tape sequences whose respective leftmost bits are shifted to the right end of the finite tape. This will thus be consistent with preserving $\boldsymbol{\pi}^{\text{T2S}}_{\text{2-bit}}$ after one complete bit scan O_∞ in the steady-state behavior. We had earlier exemplified this T2S ratchet ($N_{\text{R}} = 2$) with $L = 2$ using the same indexing notation in Eq. (62) for its joint states, and detailed these states with reordered probabilities in Table III in Sec. V A.

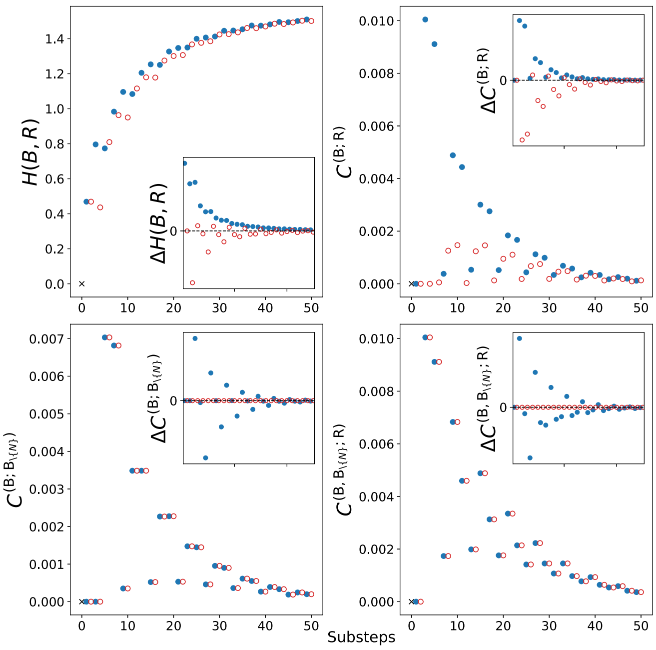

FIG. 7. Plots of informational quantities similar to Fig. 6 except for a system that eventually converges to an *equilibrium* stationary state with transition probabilities $(p, q, \epsilon) = (0.710, 0.965, 0.9)$.

2. Equilibrium stationary state with $\langle W \rangle_\infty = 0$

We are now ready to address the case for equilibrium stationary states with zero asymptotic work $\langle W \rangle_\infty = 0$ which are of practical interest as these are potential candidates for maximizing the cumulative *tape* scan work [21] since $\langle W \rangle_\infty < 0$ for the majority of the (nonequilibrium) stationary states which we saw earlier. We were thus seeking for parametric combinations (p, q, ϵ) corresponding to equilibrium stationary states with $\langle W \rangle_\infty = 0$ in that maximization and found the required $(p, q, \epsilon) = (0.710, 0.965, 0.9)$ in Ref. [21] for this T2S ratchet ($L = 2$). The same informational quantities in Fig. 6 are now plotted in Fig. 7 over $k = 25$ bit scans with the same (uncorrelated) initial definite-state (joint) distribution $\mathbf{p}^{\text{T2S}}_{\text{2-bit}}(0)$ given by (62). We point out the salient differences between Figs. 6 and 7 arising from the nature of their stationary states. The behavior of the correlations as given by Eqs. (54b) and (54c) are similar except $C^{(\text{B;B}_{\setminus\{N\}})} = C^{(\text{B,B}_{\setminus\{N\}};\text{R})} = 0$ for stationary states in equilibrium (bottom panel of Fig. 7) but $C^{(\text{B;B}_{\setminus\{N\}})} > 0$ and $C^{(\text{B,B}_{\setminus\{N\}};\text{R})} > 0$ saturate (bottom panel of Fig. 6) for those nonequilibrium states after convergence $\mathbf{p}^{\text{T2S}}_{\text{2-bit}}(0) \to \boldsymbol{\pi}^{\text{T2S}}_{\text{2-bit}}$. Nonetheless, $\Delta C^{(\text{B;B}_{\setminus\{N\}})} = \Delta C^{(\text{B,B}_{\setminus\{N\}};\text{R})} = 0$ over substep M_∞ (and also S_∞) in these equilibrium and nonequilibrium stationary states and both have no contribution to ΔC_{M_∞} (ΔC_{S_∞}). The notable difference (in their steady-state behavior) is thus $\Delta H_{M_\infty}[\mathbf{p}_{\text{B}\otimes\text{R}}] = 0$ and $\Delta C_{M_\infty} = \Delta C_{M_\infty}^{(\text{B;R})} = 0$ [considering Eq. (21) over the interval M_∞] for these equilibrium stationary states. Indeed, this is consistent with the earlier analysis: $\Delta C_{M_\infty} = 0 \Leftrightarrow \langle W \rangle_\infty = 0$ in Eq. (49). Although the net change over a (bit scan) cycle $O_\infty = S_\infty M_\infty$ must vanish in the stationary phase, all the respective informational changes vanish even over the substep interval M_∞ (and equivalently S_∞ here).

In addition, we infer from $\Delta H_{M_\infty}[\mathbf{p}_{B\otimes R}] = 0$ in the stationary behavior that the marginal distribution $\mathbf{p}_{B\otimes R}$ of $\boldsymbol{\pi}^{T2S}_{2\text{-bit}}$ is the stationary distribution of $M^{(1)}$ for T2S ratchet in Eq. (60), i.e., $\boldsymbol{\pi}^{T2S}_{1\text{-bit}}$ in Eq. (61), from Sec. V A. Moreover, since both $M^{(1)}$ and $M^{(L)}$ leave the marginal and joint distributions unchanged, respectively [see Eqs. (34) and (27)], the implication is the subsequent switching $S^{(L)}$ must also mathematically leave the joint $\mathbf{p} = \boldsymbol{\pi}^{T2S}_{L\text{-bit}}$ unchanged [see Eq. (28)], such that $\Delta H_\infty[\mathbf{p}] = 0$ in this steady state. However, $S^{(1)} = \mathbb{1}_{2N_R}$ is the identity matrix since there are no other bits to be switched in a 1-bit tape, but $S^{(L)}$ is a permutation matrix which permutes the indices $\{\alpha\} \rightarrow \{\alpha'\}$ in Table III with its (α', α)-entries (equal to unity) the only nonzero entries [21]. We thus deduced that those joint states whose probabilities were interchanged by M_∞ and S_∞ must have *equal* probabilities for such equilibrium stationary states. This condition gives the mathematical constraint on the transition probabilities (p, q, ϵ) for these states with $\langle W\rangle_\infty = 0$.

We proceed to work out this constraint in general and explicitly apply to the T2S ratchet with 2-bit tape in the immediate subsection before furnishing a proof to show that the constraint is identical for finite tapes of all length L with the same ratchet design.

C. Mathematical constraint for $\langle W\rangle_\infty = 0$

Intriguingly, the condition for equilibrium stationary states with $\langle W\rangle_\infty = 0$ for our information ratchet with tape of length L depends on the stationary distribution $\boldsymbol{\pi}_{1\text{-bit}}$ for the 1-bit tape. Thus, it is useful to begin our derivation with $L = 1$.

1. 1-bit tape

Given the thermal transition operator (matrix) $M^{(1)}$ in Eq. (3) for the 1-bit tape, $\boldsymbol{\pi}_{1\text{-bit}}$ obeys the relation

$$M^{(1)}\boldsymbol{\pi}_{1\text{-bit}} = \boldsymbol{\pi}_{1\text{-bit}}$$
$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}, \tag{63}$$

where π_0 and π_1 are vectors of size $N_R \times 1$ since E, F, G, and H are submatrices of $M^{(1)}$ with size $N_R \times N_R$. Note that $O^{(1)} = M^{(1)}$ for $L = 1$ since $S^{(1)} = \mathbb{1}_{2N_R}$ (identity) and there is effectively no contribution (both physically and mathematically) from switching on the (single) bit-ratchet dynamics here.

For the T2S ratchet, its $\boldsymbol{\pi}^{T2S}_{1\text{-bit}}$ in Eq. (61) implies

$$\begin{aligned} \pi_0^{T2S} &= [2(p+q-pq) - (p+q)\epsilon]^{-1}\begin{pmatrix} (1-p)q \\ (1-\epsilon)q \end{pmatrix} \\ \pi_1^{T2S} &= [2(p+q-pq) - (p+q)\epsilon]^{-1}\begin{pmatrix} (1-\epsilon)p \\ p(1-q) \end{pmatrix}. \end{aligned} \tag{64}$$

2. 2-bit tape

The thermal transition operator (matrix) for the 2-bit tape following Eq. (7) is

$$M^{(2)} = \begin{pmatrix} M^{(1)} & \\ & M^{(1)} \end{pmatrix} = \begin{pmatrix} E & F & & \\ G & H & & \\ & & E & F \\ & & G & H \end{pmatrix}. \tag{65}$$

As we are seeking for stationary states in equilibrium where switching leaves its stationary distribution unchanged, the corresponding $\boldsymbol{\pi}_{2\text{-bit}}$ obeys [see Eq. (27)]

$$M^{(2)}\boldsymbol{\pi}_{2\text{-bit}} = \boldsymbol{\pi}_{2\text{-bit}}$$
$$\begin{pmatrix} E & F & & \\ G & H & & \\ & & E & F \\ & & G & H \end{pmatrix}\begin{pmatrix} \pi_{00} \\ \pi_{10} \\ \pi_{01} \\ \pi_{11} \end{pmatrix} = \begin{pmatrix} \pi_{00} \\ \pi_{10} \\ \pi_{01} \\ \pi_{11} \end{pmatrix}. \tag{66}$$

Note that $\pi_{\times\times}$ are again all vectors of size $N_R \times 1$ and the subscripts denote the respective tape sequences. We can see that the linear equations involving π_{00} and π_{10} only have the terms π_{00} and π_{10}. Similarly, the linear equations involving π_{01} and π_{11} only depend on π_{01} and π_{11}. The equations can thus be decoupled into

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} \pi_{00} \\ \pi_{10} \end{pmatrix} = \begin{pmatrix} \pi_{00} \\ \pi_{10} \end{pmatrix}, \quad \begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} \pi_{01} \\ \pi_{11} \end{pmatrix} = \begin{pmatrix} \pi_{01} \\ \pi_{11} \end{pmatrix}. \tag{67}$$

Each of these set of equations is simply for the 1-bit tape in Eq. (63). Therefore, the solution for $\boldsymbol{\pi}_{2\text{-bit}}$ follows the same expression as $\boldsymbol{\pi}_{1\text{-bit}}$, up to some proportionality factor. With this linearly independent set of solutions, the general solution for $\boldsymbol{\pi}_{2\text{-bit}}$ can be expressed as

$$\begin{pmatrix} \pi_{00} \\ \pi_{10} \\ \pi_{01} \\ \pi_{11} \end{pmatrix} = k_1\begin{pmatrix} \pi_0 \\ \pi_1 \\ 0 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 0 \\ \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} k_1\pi_0 \\ k_1\pi_1 \\ k_2\pi_0 \\ k_2\pi_1 \end{pmatrix}, \tag{68}$$

where k_1 and k_2 are normalization constants obeying the conservation of probability $k_1 + k_2 = 1$.

Mathematically, switching swaps the position of π_{10} and π_{01} (see Table III), but since switching preserves the distribution $\boldsymbol{\pi}_{2\text{-bit}}$ [see Eq. (28)], we must have $\pi_{10} = k_1\pi_1 = k_2\pi_0 = \pi_{01}$. This is the condition we are looking for. Explicitly with the T2S ratchet using Eq. (64),

$$\begin{pmatrix} k_1(1-\epsilon)p \\ k_1 p(1-q) \end{pmatrix} = \begin{pmatrix} k_2(1-p)q \\ k_2(1-\epsilon)q \end{pmatrix}, \tag{69}$$

where the normalization factor in Eq. (61) cancels out.

Evaluating the ratio k_1/k_2, we have

$$\begin{aligned} \frac{k_1}{k_2} = \frac{(1-p)q}{(1-\epsilon)p} &= \frac{(1-\epsilon)q}{p(1-q)} \\ (1-p)(1-q) &= (1-\epsilon)^2 \\ pq - p - q &= \epsilon^2 - 2\epsilon. \end{aligned} \tag{70}$$

This is the constraint on (p, q, ϵ) which gives stationary states in equilibrium for the T2S ratchet with 2-bit tape. As a consistency check, we note that we obtain the same condition (70) by equating the probabilities for $10 \otimes A$ and $01 \otimes A$, and $10 \otimes B$ and $01 \otimes B$ after explicitly solving for $\boldsymbol{\pi}_{2\text{-bit}}$ (right eigenvector

TABLE IV. Relationship between the indices before (i, j) and after (i', j') switching $\pi_{i,j} \to \pi_{i',j'}$ for $L > 1$.

Case	i	j	i'	j'	Outcome
I	0	$1 \leqslant j \leqslant 2^{L-2}$	0	$2j-1$	$k_j\pi_0 = k_{j'}\pi_0$
II	0	$2^{L-2}+1 \leqslant j \leqslant 2^{L-1}$	1	$2j-2^{L-1}-1$	$k_j\pi_0 = k_{j'}\pi_1$
III	1	$1 \leqslant j \leqslant 2^{L-2}$	0	$2j$	$k_j\pi_1 = k_{j'}\pi_0$
IV	1	$2^{L-2}+1 \leqslant j \leqslant 2^{L-1}$	1	$2j-2^{L-1}$	$k_j\pi_1 = k_{j'}\pi_1$

of $O^{\text{T2S}}_{\text{2-bit}}$ with eigenvalue 1):

$$\boldsymbol{\pi}^{\text{T2S}}_{\text{2-bit}} = (\cdots) \begin{array}{c} 00\otimes A \\ 00\otimes B \\ 10\otimes A \\ 10\otimes B \\ 01\otimes A \\ 01\otimes B \\ 11\otimes A \\ 11\otimes B \end{array} \begin{pmatrix} (1-p)q^2\epsilon(2-p-\epsilon) \\ q^2\epsilon(1-\epsilon)(2-p-\epsilon) \\ pq\epsilon(1-\epsilon)(2-p-\epsilon) \\ pq(1-\epsilon)[p+q-q(p+\epsilon)] \\ pq(1-\epsilon)[p+q-p(q+\epsilon)] \\ pq\epsilon(1-\epsilon)(2-q-\epsilon) \\ p^2\epsilon(1-\epsilon)(2-q-\epsilon) \\ p^2(1-q)\epsilon(2-q-\epsilon) \end{pmatrix}, \tag{71}$$

where we have omitted the normalization factor $(\cdots)$ to avoid unnecessary clutter.

We next show that the same constraint (70) on equilibrium stationary states (with zero asymptotic work) holds for all L-bit tapes with the T2S ratchet and in general, such a specific condition on its transition probabilities (corresponding to these equilibrium states) exists given any ratchet design with $N_{\text{R}} > 1$ and independent of L.

3. L-bit tape

The stationary distribution for the L-bit tape follows [see Eq. (27)]

$$M^{(L)}\boldsymbol{\pi}_{L\text{-bit}} = \boldsymbol{\pi}_{L\text{-bit}}$$

$$\begin{pmatrix} E & F & & & & & & & & \\ G & H & & & & & & & & \\ & & E & F & & & & & & \\ & & G & H & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & E & F & & & \\ & & & & & G & H & & & \\ & & & & & & & E & F \\ & & & & & & & G & H \end{pmatrix} \begin{pmatrix} \pi_{0,1} \\ \pi_{1,1} \\ \pi_{0,2} \\ \pi_{1,2} \\ \vdots \\ \pi_{0,2^{L-1}-1} \\ \pi_{1,2^{L-1}-1} \\ \pi_{0,2^{L-1}} \\ \pi_{1,2^{L-1}} \end{pmatrix} = \begin{pmatrix} \pi_{0,1} \\ \pi_{1,1} \\ \pi_{0,2} \\ \pi_{1,2} \\ \vdots \\ \pi_{0,2^{L-1}-1} \\ \pi_{1,2^{L-1}-1} \\ \pi_{0,2^{L-1}} \\ \pi_{1,2^{L-1}} \end{pmatrix}, \tag{72}$$

where we have introduced new notation $\pi_{i,j}$ with $i \in \{0, 1\}$ denoting the value of the leftmost bit and $j \in \{1, 2, \ldots, 2^{L-1}-1, 2^{L-1}\}$ is the index enumerating the respective (binary) sequences of the remaining $L-1$ bits. As before, this equation can be decoupled to a set of 2^{L-1} equations corresponding to the 1-bit tape,

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} \pi_{0,j} \\ \pi_{1,j} \end{pmatrix} = \begin{pmatrix} \pi_{0,j} \\ \pi_{1,j} \end{pmatrix}, \tag{73}$$

as in Eq. (63). The general solution for $\boldsymbol{\pi}_{L\text{-bit}}$ can then be expressed as

$$\begin{pmatrix} \pi_{0,1} \\ \pi_{1,1} \\ \pi_{0,2} \\ \pi_{1,2} \\ \vdots \\ \pi_{0,2^{L-1}-1} \\ \pi_{1,2^{L-1}-1} \\ \pi_{0,2^{L-1}} \\ \pi_{1,2^{L-1}} \end{pmatrix} = \begin{pmatrix} k_1\pi_0 \\ k_1\pi_1 \\ k_2\pi_0 \\ k_2\pi_1 \\ \vdots \\ k_{2^{L-1}-1}\pi_0 \\ k_{2^{L-1}-1}\pi_1 \\ k_{2^{L-1}}\pi_0 \\ k_{2^{L-1}}\pi_1 \end{pmatrix}, \tag{74}$$

where k_j are the normalization constants with $\sum_j k_j = 1$, and π_0 and π_1 are the stationary distributions of $\boldsymbol{\pi}_{\text{1-bit}}$ in Eq. (63) for the 1-bit tape.

Switching will reorder the elements of the (joint tape-ratchet) distribution, which is denoted by $\pi_{i,j} \to \pi_{i',j'}$, where i, j and i', j' are the indices before and after switching, respectively. The relationship between these indices before and after switching for $L > 1$ is given in Table IV.

From Table IV, there are two possibilities for the (mathematical) action of switching. The first possibility includes cases with the reorder of $\pi_{0,j} \to \pi_{0,j'}$ (case I) and $\pi_{1,j} \to \pi_{1,j'}$ (case IV). To exemplify for $L = 3$ (using both notations $\pi_{\times\cdots\times}$ and $\pi_{i,j}$), π_{001} becomes π_{010}, and π_{110} becomes π_{101} after switching. Since switching leaves $\boldsymbol{\pi}_{L\text{-bit}}$ unchanged, then $\pi_{0,j} = \pi_{0,j'}$ $(\pi_{1,j} = \pi_{1,j'})$ or $k_j\pi_0 = k_{j'}\pi_0$ $(k_j\pi_1 = k_{j'}\pi_1)$. This only tells us that the proportionality constants should be equal, i.e., $k_j = k_{j'}$, but it tells us nothing about the condition.

The second possibility includes cases with the reorder of $\pi_{0,j} \to \pi_{1,j'}$ (case II) and $\pi_{1,j} \to \pi_{0,j'}$ (case III). To exemplify for $L = 3$ (using both notations $\pi_{\times\cdots\times}$ and $\pi_{i,j}$), π_{010} becomes π_{100}, and π_{101} becomes π_{011} after switching. Since switching leaves $\boldsymbol{\pi}_{L\text{-bit}}$ unchanged, then $\pi_{0,j} = \pi_{1,j'}$ $(\pi_{1,j} =$

$\pi_{0,j'}$) or $k_j\pi_0 = k_{j'}\pi_1$ ($k_j\pi_1 = k_{j'}\pi_0$). This is precisely the condition we are looking for, which can be obtained by evaluating the ratio between the normalization constants ($k_j/k_{j'}$ or $k_{j'}/k_j$) as in Eq. (70) for T2S ratchet. This serves as the general condition for any $L > 1$ and explains why the condition for 2-bit tape also applies to finite tapes of longer length.

From a physical perspective of this switching operation (with the leftmost bit shifted to the right end of finite tape), we can deduce the condition(s) required for an equilibrium stationary state with $\langle W \rangle_\infty = 0$: tape sequences which are cyclic permutations of one another have equal (stationary) probabilities. Summarising the outcomes of the reordering in the example for $L = 3$ (with $2^3 = 8$ different tape sequences) exemplified above, this implies $\pi_{001} = \pi_{010} = \pi_{100}$ and $\pi_{110} = \pi_{101} = \pi_{011}$. The remaining two sequences, corresponding to π_{000} and π_{111} whose probabilities are unchanged after switching [see Eq. (31)], are precisely $\{0\}^L$ and $\{1\}^L$, respectively, which satisfy Eq. (41) in Theorem 1 earlier.

This completes the proof to show that the same constraint applies to all (finite) tapes of length L for the corresponding equilibrium stationary states (and distributions) with the same ratchet design.

VIII. CONCLUSION

In conclusion, we have recast the information processing second law (IPSL) for a finite-tape information ratchet in terms of the *marginal* (leftmost) bit-ratchet distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ in Eq. (20), analogous to the original IPSL established by He *et al.* with the *joint* tape-ratchet distribution $\mathbf{p}$ in Eq. (19), by showing the equivalence of the work $\langle W \rangle$ expressions using either the marginal or joint distribution in Eq. (13). Moreover, this marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ yields a tighter informational bound than with the joint $\mathbf{p}$ for the IPSL. Crucially, we also showed that the marginal probabilities in $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ are evolved by the 1-bit thermal transition operator (matrix) $M^{(1)}$, independent of the length L of the finite tape, over the corresponding (thermal transition) substep M, in Eq. (18). This reflects the sequential interaction of each bit with the ratchet physically, and significance accorded to this (leftmost) either interacting bit B or interacted bit B′ in the ratchet operation. We subsequently utilized this marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ extensively to establish the relationship between joint and marginal distributions [Eq. (21)] and capture the correlations manifested within the information ratchet (tape-ratchet) system with the introduction of the respective mutual information terms [Eq. (26)].

We analyzed in detail the differences between equilibrium and nonequilibrium stationary states and identified this as a manifestation of the absence (presence) of such changes in correlation in Eq. (26) with zero (negative) asymptotic work $\langle W \rangle_\infty$, respectively, for these stationary states in general. The analysis is applied to two designs of the finite-tape information ratchet, the T1S ratchet (without memory) and the T2S ratchet (with memory), respectively, where correlations can be generated and introduced by the latter (T2S) but not the former (T1S), from their modular operation of the finite tape. We saw how the lack of these correlations resulted in stationary states (distributions) which are all in equilibrium for the T1S ratchet, and their presence (of correlations) can bias the T2S ratchet away from equilibrium such that its asymptotic work $\langle W \rangle_\infty < 0$ in the steady state is nonvanishing, specifically negative. We realized this case for nonequilibrium stationary states has an overwhelmingly higher incidence from our exploration of different parameters (p, q, ϵ) with the T2S ratchet, and subsequently uncovered the mathematical condition [Eq. (70)] for its stationary states to be in equilibrium which intriguingly is independent of the length L of the finite tape. This can be generalized to any ratchet design (with $N_\mathrm{R} > 1$) giving rise to such a specific constraint (independent of L) on its respective transition parameters. Such equilibrium states are of practical interest as they are potential candidates for maximizing the cumulative tape scan work [21].

Finally, we showed how correlation is responsible for the aforementioned phenomenon of negative asymptotic work $\langle W \rangle_\infty < 0$ through the mutual informational terms in ΔC_{M_∞} for those nonequilibrium stationary states. Using the marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ to probe the mathematical structure behind the ratchet mechanism (specifically the bit scan operation $O = SM$) turns out to be fruitful in surfacing the distinguishing features between equilibrium and nonequilibrium stationary states. Solely with the joint $\mathbf{p}$, the respective changes $\Delta H_\infty[\mathbf{p}] = 0$ for both states all vanish as they are in their steady-state behavior. The marginal $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ facilitates an insightful study of the underlying bit-ratchet dynamics over the respective (M_k or S_k) substep interval. The constraint for arbitrary ratchet design, and explicitly for the T2S ratchet, i.e., Eq. (70), was obtained through the understanding that switching S (and also M) preserves both joint and marginal distributions for stationary states in equilibrium. To possibly further these results, an extended investigation into the information ratchet mechanism through the framework of *stochastic thermodynamics* [26–29] seems promising.

APPENDIX A: MATHEMATICAL PROOF OF MARGINAL IPSL

A mathematical proof for the marginal IPSL in Eq. (20) asserted in the main text is given below, with an explanation behind the work expression $\langle W \rangle$ in Eq. (2) and its assumptions involved pertaining to the operation mechanism of the finite-tape information ratchet.

For this information ratchet with an arbitrary number of ratchet states N_R and a finite tape of L bits, we can obtain the corresponding *marginal* (interacting bit-ratchet) probabilities in $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ (a $2N_\mathrm{R} \times 1$ vector) from its *joint* (tape-ratchet) distribution $\mathbf{p}$ (a $2^L N_\mathrm{R} \times 1$ vector), as given in Eq. (11), at the beginning of a thermal transition substep M (and also the beginning of a two-step bit scan $O = SM$). We have also shown in the main text that the evolution of $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ over substep M is governed by the 1-bit thermal transition matrix $M^{(1)}$ in Eq. (18), i.e.,

$$\tilde{\mathbf{p}}_{\mathrm{B}\otimes\mathrm{R}} = M^{(1)}\,\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}, \tag{A1}$$

with the tilde symbol used to denote the *intermediate* probability distribution after the thermal transition substep M, in contrast with the prime symbol [in Eq. (1)] over a complete bit scan O. Note that the dimension of the (square) matrix $M^{(1)}$ is $2N_\mathrm{R}$; see Eq. (3). This $M^{(1)}$ essentially governs the transitions between the interacting bit-ratchet $\mathrm{B} \otimes \mathrm{R}$ states driven by the heat reservoir, with matrix elements the respective transition probabilities repeated in $M^{(L)}$; see Eq. (7) for the details.

We next proceed to obtain the work $\langle W\rangle$ given by Eq. (2) in the main text. Unlike a piston, our (finite) tape-ratchet system does not perform work (in the absence of external perturbation). It basically mediates a transfer of energy between the heat reservoir and the work reservoir. The energy that is being transferred is heat as the tape-ratchet system interacts with the heat reservoir during the thermal transition substep M (from stage 2 to 3 in Fig. 2). The heat energy is then converted to gravitational potential energy capable of doing work in the work reservoir, analogous to Refs. [17,19]. It is in this context that $Q = -W$ over every bit scan O, where Q is the heat dissipated by the ratchet into the heat reservoir and W is the mechanical energy stored (which manifests as work) in the work reservoir by the ratchet, since the ratchet does not retain energy. We assume that the mediation of energy transfer by the tape-ratchet system is performed through its energy levels. Because the energy flow is solely through heat during the thermalization of the tape-ratchet system with the heat reservoir (in substep M), the energy levels $\Delta E_{ji} \equiv E_j - E_i$ of the tape-ratchet system is fixed. Furthermore, thermalization implies that the dynamics of the tape-ratchet system obey *detailed balance*. Our assumption is that detailed balance follows the Markovian dynamics during *each* thermal transition substep M involving the interacting bit-ratchet $B\otimes R$ states:

$$M_{ji}^{(1)}\pi_i^{(\text{1-bit})} = M_{ij}^{(1)}\pi_j^{(\text{1-bit})} \quad \forall\,\{i,j\} \quad \text{with} \quad i,j\in\{B\otimes R\}, \tag{A2}$$

where the stochastic transition from state i to state j is quantified by the transition probability $M_{ji}^{(1)}$, which is the (j,i) entry of the left stochastic 1-bit thermal transition matrix $M^{(1)}$. In addition, the probabilities in the equilibrium (stationary) distribution

$$\pi_i^{(\text{1-bit})} = \frac{e^{-\beta E_i}}{\sum_k e^{-\beta E_k}} = \exp\left[\beta(F - E_i)\right] \tag{A3}$$

are given by the canonical ensemble in equilibrium statistical mechanics with $\beta \equiv 1/k_BT$ the inverse temperature and F being its Helmholtz free energy. Note that this $\boldsymbol{\pi}_{\text{1-bit}} = M^{(1)}\boldsymbol{\pi}_{\text{1-bit}}$ is also the stationary distribution of the *tridiagonal* $M^{(1)}$; see Eq. (34) for an earlier discussion on this. Equations (A2) and (A3) then lead to

$$\frac{M_{ij}}{M_{ji}} = \frac{\pi_i}{\pi_j} = \exp[\beta(E_j - E_i)], \tag{A4}$$

where we have withheld the superscripts (1) and (1-bit) to simplify notation. This implies

$$\Delta E_{ji} \equiv E_j - E_i = k_BT\ \ln\left(\frac{M_{ij}}{M_{ji}}\right). \tag{A5}$$

Specifically, the interaction with the work reservoir to accumulate (or expend) the mechanical energy from the work conversion (with $Q = -W$ earlier) occurs during the attachment of input bit to and detachment of output bit from the ratchet; see Fig. 2. Thus, the *expected* (or *ensemble-averaged*) work production $\langle W\rangle$ from a single bit scan O at any arbitrary time can be expressed as

$$\langle W\rangle = k_BT\sum_{i,j} M_{ji}p_i \ln\left(\frac{M_{ij}}{M_{ji}}\right), \tag{A6}$$

which is the sum of the respective work $W_{ji} \equiv k_BT\ \ln(M_{ij}/M_{ji})$ (originating from thermal transition $i\to j$) weighted by the corresponding probability $M_{ji}p_i$ for such a transition to occur. We have subsequently set $k_BT = 1$ to simplify the $\langle W\rangle$ expression and recover Eq. (2) in the main text with W_{ji} and thus $\langle W\rangle$ now in units of k_BT. We emphasize that the probabilities p_i here are the *instantaneous* probability distribution which is continuously evolving as the information ratchet operates, and not from the time-independent equilibrium stationary distribution π_i in Eq. (A3).

Next, let us evaluate the change in entropy of the $B\otimes R$ subsystem (over substep M) using the Shannon entropy [23]:

$$\begin{aligned}\Delta H &= H[\tilde{\mathbf{p}}_{B\otimes R}] - H[\mathbf{p}_{B\otimes R}] = \sum_i p_i\ln p_i - \sum_j \tilde{p}_j\ln\tilde{p}_j\\ &= \sum_i\left(\sum_j M_{ji}\right)p_i\ln p_i - \sum_j\left(\sum_i M_{ji}p_i\right)\ln\tilde{p}_j,\end{aligned} \tag{A7}$$

where $\sum_j M_{ji}^{(1)} = 1$ (from probability conservation) is inserted in the first parentheses and the second parentheses results from applying Eq. (A1), i.e., the action of $M^{(1)}$ on $\mathbf{p}_{B\otimes R}$ (matrix multiplication). We have again suppressed the superscripts (1) and $(B\otimes R)$ in the above and subsequent equations here; we explicitly refer to the matrix elements in $M^{(1)}$ and marginal probabilities in $\mathbf{p}_{B\otimes R}$. Putting the double sums and terms together, the informational term ΔH thus reads

$$\Delta H = \sum_{i,j} M_{ji}p_i\ln\left(\frac{p_i}{\tilde{p}_j}\right). \tag{A8}$$

Note that $H[\,\cdot\,]$ is the base 2 binary entropy function but we will be using the natural logarithm (with base e) here, to eliminate the prefactor ln 2 appearing in the IPSL of Ref. [17]. In addition, both the $\langle W\rangle$ and ΔH terms are expressed in the same natural ln, necessary for the subsequent step in the proof.

Now consider the difference between ΔH and $\langle W\rangle$:

$$\Delta H - \langle W\rangle = \sum_{i,j} M_{ji}p_i\ln\left(\frac{M_{ji}p_i}{M_{ij}\tilde{p}_j}\right). \tag{A9}$$

The log sum inequality is first applied to the inner sum (within the square brackets) to yield

$$\begin{aligned}\Delta H - \langle W\rangle &= \sum_i p_i\left[\sum_j M_{ji}\ln\left(\frac{M_{ji}p_i}{M_{ij}\tilde{p}_j}\right)\right]\\ &\geqslant \sum_i p_i\left[\left(\sum_j M_{ji}\right)\ln\left(\frac{\sum_{j'} M_{j'i}}{\sum_{j''}\frac{1}{p_i}M_{ij''}\tilde{p}_{j''}}\right)\right]\\ &= \sum_i p_i\ln\left(\frac{p_i}{\tilde{\tilde{p}}_i}\right),\end{aligned} \tag{A10}$$

where we have used $\sum_j M_{ji} = 1$ and $\sum_j M_{ij}\tilde{p}_j = \tilde{\tilde{p}}_i$, with $\tilde{\tilde{p}}_i$ denoting the intermediate (marginal) probabilities after one thermal transition starting from an initial distribution $\tilde{p}_i$.

Applying the log sum inequality once again to the last expression, we obtain

$$\Delta H - \langle W\rangle \geqslant \sum_i p_i \ln\left(\frac{p_i}{\tilde{\tilde{p}}_i}\right) \geqslant \left(\sum_i p_i\right)\ln\left(\frac{\sum_j p_j}{\sum_k \tilde{\tilde{p}}_k}\right) = \ln 1 = 0, \tag{A11}$$

since probability conservation holds at any discrete time. We have thus established our proposed marginal IPSL:

$$\langle W\rangle \leqslant \Delta H[\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}]. \tag{A12}$$

Note that the proof for the *marginal* IPSL here is similar to the *joint* IPSL proof for 1-bit tape in Sec. IV A of Ref. [21], since $M^{(L)} = M^{(1)}$ and $\mathbf{p}_{\text{1-bit}} = \mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ (i.e., joint and marginal probabilities are the same) for $L = 1$.

APPENDIX B: MATHEMATICAL PROOF OF $I(\mathrm{B}', \mathrm{R}'; \mathrm{B}_{\setminus\{N\}}|\mathrm{B}, \mathrm{R}) = 0$

Here we furnish a mathematical proof of $I(\mathrm{B}'\mathrm{R}'; \mathrm{B}_{\setminus\{N\}}|\mathrm{BR}) = 0$ where we have dropped the commas in the labels of joint states.

For our information ratchet with N_R ratchet states and a (finite) tape of L bits, the number of possible marginal states BR (B′R′) before (after) a thermal transition substep M is $2N_\mathrm{R}$, with the marginal (bit-ratchet) distribution $\mathbf{p}_{\mathrm{B}\otimes\mathrm{R}}$ a $2N_\mathrm{R} \times 1$ vector established earlier. We will denote the marginal probabilities by p_i^{BR} and $\tilde{p}_j^{\mathrm{B'R'}}$ with indices $i, j \in \{1, 2, \dots, 2N_\mathrm{R}\}$ labeling the marginal states. Subsequently, the number of possible tape sequences for the remaining $L-1$ bits, i.e., $\mathrm{B}_{\setminus\{N\}}$ not involved in this thermal transition M, is 2^{L-1}, and its state will be denoted with the index $k \in \{1, 2, \dots, 2^{L-1}\}$. In addition, we denote with $p_{i,k}$ ($\tilde{p}_{j,k}$) the joint probability with the marginal BR (B′R′) in state i (j) and the noninteracting bits $\mathrm{B}_{\setminus\{N\}}$ in state k.

The conditional mutual information is defined as

$$\begin{aligned} I(X;Y|Z) &= \sum_{z\in\mathcal{Z}} p_Z(z) I(X;Y|Z=z) \\ &= \sum_{z\in\mathcal{Z}} p_Z(z) \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y|Z=z}(x,y|z) \\ &\quad \times \log\left[\frac{p_{X,Y|Z=z}(x,y|z)}{p_{X|Z=z}(x|z)\, p_{Y|Z=z}(y|z)}\right]. \end{aligned} \tag{B1}$$

To evaluate $I(\mathrm{B}'\mathrm{R}'; \mathrm{B}_{\setminus\{N\}}|\mathrm{BR})$ with $X \equiv \mathrm{B}'\mathrm{R}'$, $Y \equiv \mathrm{B}_{\setminus\{N\}}$, and $Z \equiv \mathrm{BR}$, we seek to establish the respective conditional probabilities in Eq. (B1).

First, $p_{X|Z=z}(x|z) \equiv p(j|i)$ is the (conditional) probability of finding B′R′ in state j given BR is in state i. This is simply the transitional probability from state i to state j in a thermal transition given by M_{ji}, i.e., the (j,i)-entry of the thermal transition matrix $M^{(1)}$.

Next, $p_{Y|Z=z}(y|z) \equiv p(k|i)$ is the (conditional) probability of $\mathrm{B}_{\setminus\{N\}}$ in state k given BR is in state i, which is $p_{i,k}/p_i^{\mathrm{BR}}$.

Last, $p_{X,Y|Z=z}(x,y|z) \equiv p(j,k|i)$ is the joint probability of the tape-ratchet system in state j, k (after thermal transition M) given BR is in state i. Owing to the sequential interaction of the ratchet with each bit, only the leftmost bit interacts with the ratchet (constituting the interacting bit-ratchet subsystem BR), with the remaining bits $\mathrm{B}_{\setminus\{N\}}$ (noninteracting subsystem) in state k before and after this M. We can thus obtain the joint probability after M from the joint probability before M with $\mathrm{B}_{\setminus\{N\}}$ fixed in state k throughout. With BR in state i, the joint probability prior to M is $p_{i,k}/p_i^{\mathrm{BR}}$. This joint state i, k is then transformed to j, k with transition probability M_{ji} in the thermal transition. We thus have $p(j,k|i) = M_{ji}p_{i,k}/p_i^{\mathrm{BR}}$.

Collecting the above expressions,

$$p_{X|Z=z}(x|z) \equiv p(j|i) = M_{ji}, \tag{B2a}$$

$$p_{Y|Z=z}(y|z) \equiv p(k|i) = \frac{p_{i,k}}{p_i^{\mathrm{BR}}}, \tag{B2b}$$

$$p_{X,Y|Z=z}(x,y|z) \equiv p(j,k|i) = M_{ji}\frac{p_{i,k}}{p_i^{\mathrm{BR}}}, \tag{B2c}$$

and inserting these into the conditional mutual information expression in Eq. (B1) yields

$$\begin{aligned} &I(\mathrm{B}'\mathrm{R}'; \mathrm{B}_{\setminus\{N\}}|\mathrm{BR}) \\ &= \sum_{i=1}^{2N_\mathrm{R}} p_i^{\mathrm{BR}} \sum_{j=1}^{2N_\mathrm{R}}\sum_{k=1}^{2^{L-1}} p(j,k|i)\log\left[\frac{p(j,k|i)}{p(j|i)\,p(k|i)}\right] \\ &= \sum_{i=1}^{2N_\mathrm{R}} p_i^{\mathrm{BR}} \sum_{j=1}^{2N_\mathrm{R}}\sum_{k=1}^{2^{L-1}} M_{ji}\frac{p_{i,k}}{p_i^{\mathrm{BR}}}\log\left[\frac{M_{ji}\frac{p_{i,k}}{p_i^{\mathrm{BR}}}}{M_{ji}\frac{p_{i,k}}{p_i^{\mathrm{BR}}}}\right] = 0. \end{aligned} \tag{B3}$$

The argument in the logarithm is always unity as a consequence of (i) the joint probabilities $p_{i,k}$ and $\tilde{p}_{j,k}$ (before and after a thermal transition, respectively) are related by $\tilde{p}_{j,k} = \sum_i M_{ji}p_{i,k}$, and (ii) the transition probabilities for the joint states $p_{i,k} \to \tilde{p}_{j,k}$ and marginal states $p_i^{\mathrm{BR}} \to \tilde{p}_j^{\mathrm{B'R'}}$ are the same, i.e., M_{ji} [see Eq. (18)]. Crucially, (i) and (ii) stem from the thermal transition leaving $\mathrm{B}_{\setminus\{N\}}$ in state k unchanged and independent of this arbitrary $\mathrm{B}_{\setminus\{N\}}$.

We have thus shown $I(\mathrm{B}', \mathrm{R}'; \mathrm{B}_{\setminus\{N\}}|\mathrm{B}, \mathrm{R}) = 0$ in our finite-tape information ratchet system, which processes the bits of the finite tape *modularly*.

APPENDIX C: EVALUATION OF $\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})}$ AND $\Delta C_{S_k}^{(\mathrm{B},\mathrm{B}_{\setminus\{N\}};\mathrm{R})}$ IN SWITCHING S

The details of the evaluation for the following mutual informational terms involved in switching S are provided here, utilizing information-theoretic properties.

In the following, we will consider the transformation of a tape sequence before and after the switching substep S_k (in an arbitrary bit scan k):

$$\begin{gathered} \mathrm{B}'_N\mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1} \\ \downarrow \\ \mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}\mathrm{B}'_N. \end{gathered} \tag{C1}$$

Specifically, we considered (in the main text)

$$\text{(a) } \Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})} \equiv I(\mathrm{B}_{N+1};\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N) - I(\mathrm{B}'_N;\mathrm{B}_{\setminus\{N\}}), \tag{C2a}$$

$$\text{(b) } \Delta C_{S_k}^{(\mathrm{B},\mathrm{B}_{\setminus\{N\}};\mathrm{R})} \equiv I(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N;\mathrm{R}') - I(\mathrm{B}'_N,\mathrm{B}_{\setminus\{N\}};\mathrm{R}'), \tag{C2b}$$

where we have made explicit the new input bit $\mathrm{B}=\mathrm{B}_{N+1}$ and old output bit $\mathrm{B}'=\mathrm{B}'_N$. The notation $\mathrm{B}_{\setminus\{N\}}$ denotes the bit sequence $\mathrm{B}_{N+1}\mathrm{B}_{N+2}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}$ (excluding the leftmost bit with index N), and similarly $\mathrm{B}_{\setminus\{N,N+1\}}$ for $\mathrm{B}_{N+2}\mathrm{B}_{N+3}\cdots\mathrm{B}_L\mathrm{B}'_1\cdots\mathrm{B}'_{N-1}$ (excluding the leftmost bit B_{N+1} and rightmost bit B'_N) after S_k.

For (a), we have

$$\begin{aligned}\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})} &= I(\mathrm{B}_{N+1};\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N) - I(\mathrm{B}'_N;\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}})\\ &= [H(\mathrm{B}_{N+1}) + H(\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N) - H(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N)]\\ &\quad - [H(\mathrm{B}'_N) + H(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}}) - H(\mathrm{B}'_N,\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}})]\\ &= [H(\mathrm{B}_{N+1}) + H(\mathrm{B}_{\setminus\{N,N+1\}}) - H(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}})] - [H(\mathrm{B}'_N) + H(\mathrm{B}_{\setminus\{N,N+1\}}) - H(\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N)]\\ &= I(\mathrm{B}_{N+1};\mathrm{B}_{\setminus\{N,N+1\}}) - I(\mathrm{B}'_N;\mathrm{B}_{\setminus\{N,N+1\}}),\end{aligned} \tag{C3}$$

corresponding to Eq. (55). We had noted in general $\Delta C_{S_k}^{(\mathrm{B};\mathrm{B}_{\setminus\{N\}})} \neq 0$ in the *transient* phase for $L \neq 1, 2$, but always vanishes for $L = 2$ (see bottom left plots in Figs. 6 and 7) as there are no additional bits in $\mathrm{B}_{\setminus\{N,N+1\}}$ besides the outgoing output bit $\mathrm{B}'=\mathrm{B}'_N$ and incoming input bit $\mathrm{B}=\mathrm{B}_{N+1}$.

Similarly for (b),

$$\begin{aligned}\Delta C_{S_k}^{(\mathrm{B},\mathrm{B}_{\setminus\{N\}};\mathrm{R})} &= I(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N;\mathrm{R}') - I(\mathrm{B}'_N,\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}};\mathrm{R}')\\ &= [H(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N) + H(\mathrm{R}') - H(\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{B}'_N,\mathrm{R}')]\\ &\quad - [H(\mathrm{B}'_N,\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}}) + H(\mathrm{R}') - H(\mathrm{B}'_N,\mathrm{B}_{N+1},\mathrm{B}_{\setminus\{N,N+1\}},\mathrm{R}')] = 0,\end{aligned} \tag{C4}$$

for all L since the entire information ratchet (tape-ratchet system) is self-contained with the recirculation of existing bits after each *tape* scan.

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