Speed of wave packets and the nonlinear Schrödinger equation

Gregory Kozyreff ●[*]

Optique Nonlinéaire Théorique, Université libre de Bruxelles (U.L.B.), CP 231, Belgium

The universal theory of weakly nonlinear wave packets given by the nonlinear Schrödinger equation is revisited. In the limit where the group and phase velocities are very close together, a multiple-scale analysis carried out beyond all orders reveals that a single soliton, bright or dark, can travel at a different speed than the group velocity. In an exponentially small but finite range of parameters, the envelope of the soliton is locked to the rapid oscillations of the carrier wave. Eventually, the dynamics is governed by an equation analogous to that of a pendulum, in which the center of mass of the soliton is subjected to a periodic potential. Consequently, the soliton speed is not constant and generally contains a periodic component. Furthermore, the interaction between two distant solitons can in principle be profoundly altered by the aforementioned effective periodic potential and we conjecture the existence of new bound states. These results are derived on a wide class of wave models and in such a general way that they are believed to be of universal validity.

I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) has been with us since the end of the 1960's, when it emerged almost simultaneously in optics, hydrodynamics, and plasma physics [1–4]. It is one of the central equations of nonlinear science because it provides a unifying description of wave packets that travel in uniform media in the presence of dispersion and a weak nonlinearity. In this general frame, one considers waves of the form

$$\psi(x, t) \exp i[\beta(\omega_0)x - \omega_0 t] + \text{c.c.}, \quad (1)$$

i.e., an envelope ψ that multiplies a sinusoidal carrier wave at a given frequency ω_0. Above, c.c. means complex conjugate and the function $\beta(\omega)$ contains all the relevant linear properties of the field in question. Provided that the envelope varies slowly compared to the carrier, it generally satisfies

$$i\left(\frac{\partial \psi}{\partial x} + \beta'(\omega_0)\frac{\partial \psi}{\partial t}\right) - \frac{\beta''(\omega_0)}{2}\frac{\partial^2 \psi}{\partial t^2} + \gamma|\psi|^2\psi = 0, \quad (2)$$

which is the NLSE, where γ is a nonlinear coefficient that depends on context. What makes the success of Eq. (2), besides its universality and robustness, is its relative simplicity and the fact that the envelope is decoupled from the underlying oscillations of the carrier wave. This decoupling appears quite reasonable owing to the separation of time scales between the two, but the purpose of this paper is to challenge this view. It will be shown that the envelope can be pinned to the carrier wave in certain circumstances.

As is well known, and as directly transpires from Eq. (2) the envelope travels at the group velocity $v_g = 1/\beta'(\omega_0)$. On the other hand, the oscillations of the carrier wave move at the phase velocity $v_p = \omega_0/\beta(\omega_0)$ in the limit of a vanishing nonlinearity ($\gamma = 0$). From the point of view of the

envelope, therefore, the carrier oscillations move at relative speed $v_p - v_g$ and whatever effect the latter could have on the former, it rapidly averages to zero over time. The possibility of a nontrivial interaction therefore rests on the assumption that

$$|v_p - v_g| \ll v_p. \quad (3)$$

We thus specifically focus on wave systems whose linear response allows that

$$\beta'(\omega_0) = \beta(\omega_0)/\omega_0 \quad (4)$$

for some ω_0 and work in the vicinity of that angular frequency. A mathematically equivalent condition to (4) is that the phase velocity passes by an extremum

$$v_p'(\omega_0) = 0. \quad (5)$$

Such a situation can happen, for instance, with gravity-capillary waves, along elastic beams resting on a Winkler foundation or in cold plasmas [4]. In optics and in acoustics, the constitutive properties of the medium rarely, if ever, allows (5) to happen in free space, but wave propagation in confined geometries leads to a greater variety of dispersion relations and makes that condition achievable.

Balancing the last two terms of Eq. (2), we see that ψ evolves on a time scale $|\gamma/\beta''(\omega_0)|^{1/2}t$, to be compared with $\omega_0 t$ for the carrier wave. Hence, the separation of time scales that underlies the validity of the NLSE rests on the smallness of the following parameter:

$$\varepsilon = |\gamma/\beta''(\omega_0)|^{1/2}/\omega_0. \quad (6)$$

Equivalently, ε measures the ratio of the wavelength of the sinusoidal oscillations to the width of the envelope. It is therefore essentially a geometrical parameter that can be identified independently of the physical context. Note that the smallness of ε does not imply the smallness of γ in Eq. (2), as it involves a ratio with ω_0, which does not appear in that equation. Rather,

———————
[*]gregory.kozyreff@ulb.be

$\varepsilon \ll 1$ is implicit in the derivation of Eq. (2) from a given wave equation.

The NLSE only admits soliton solutions that travel at a constant speed. Within Eq. (2), that speed can differ from $v_g(\omega_0)$, but in that case ψ is modulated by a complex exponential, effectively shifting the frequency ω_0 to a neighboring value ω'. As a result, the soliton speed predicted by Eq. (2) remains in agreement with $v_g(\omega')$ [5].

What we will show is that, in a narrow but finite range of frequencies near ω_0 defined by (5), envelope solitons described to leading order by Eq. (2) do not move at a constant speed. Instead, their location is given by

$$x = v_p t + x_0(t), \tag{7}$$

where

$$\ddot{x}_0(t) + \eta \frac{v_l^2}{4} \kappa \beta(\omega_0) \sin[\kappa \beta(\omega_0) x_0] = 0, \tag{8}$$

$$\kappa = \begin{cases} 2, & \text{in the presence of inversion symmetry,} \\ 1, & \text{without inversion symmetry,} \end{cases} \tag{9}$$

$\eta = \pm 1$, and

$$v_l = K \varepsilon^{-1/2} e^{-\kappa \pi / 4\varepsilon} v_p. \tag{10}$$

Above, both η and K are model-dependent constants that can only be obtained numerically.

Equation (8) is the central result of this paper. It holds both for bright and dark solitons. It can only be derived from the complete wave model, of which the NLSE is the leading-order asymptotic reduction. Hence, it partially invalidates the NLSE in the vicinity of an operating point given by Eq. (5).

The number κ appearing in Eqs. (8) and (10) is connected to the set of harmonics generated by the nonlinearity. If the system is unchanged by a reversal of sign of the field, then the nonlinearity must be odd in that field. The lowest such nonlinearity is cubic and couples harmonics separated by $2\omega_0$ in the spectrum of the solution. Otherwise, the lowest possible nonlinearity is quadratic in the field. In the presence of the latter, successive peaks of the spectrum are separated by ω_0. Hence the spacing between peaks is $\kappa \omega_0$ with $\kappa = 1$ or 2, depending on the symmetry.

According to Eq. (8), the motion of the wave packet in a frame that moves at the phase velocity is analogous to that of a pendulum. It has stable stationary points given by $\kappa \beta(\omega_0) x_0 = [2n + (\eta - 1)/2]\pi$, where n is an integer. It also has unstable stationary points, $\kappa \beta(\omega_0) x_0 = [2n + (\eta + 1)/2]\pi$. Those unstable points are connected by separatrices in the phase plane $(x_0, \dot{x}_0)$, see Fig. 1. In the classical mechanical language, the closed phase plane trajectories inside the separatrices describe motion of libration; outside, the pendulum makes complete rotations about its point of attachment. The speed at $x_0 = 0$ that corresponds to the transition is v_l. In the phase portrait of Fig. 1, we thus identify closed trajectories inside the separatrices as pertaining to the locking range, where the average speed $\langle v \rangle$ of the soliton is equal to v_p.

The pendulum equation (8) becomes compatible with the classical theory of soliton motion, Eq. (2), when the kinetic energy of the pendular motion is very large. Indeed, if $|\dot{x}_0| \gg v_l$, then the phase portrait in Fig. 1 indicates that $\dot{x}_0$
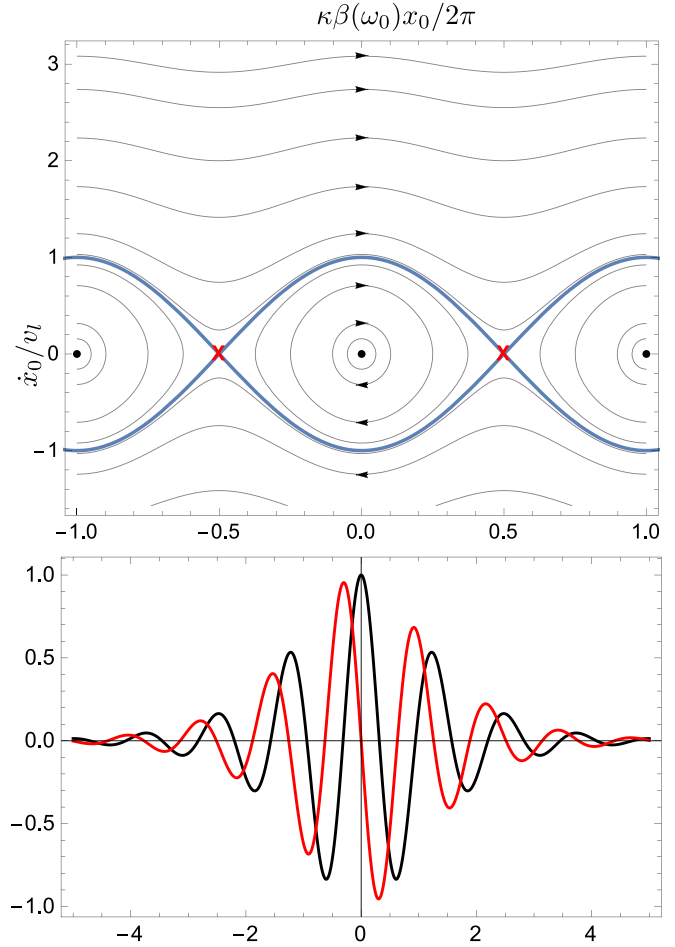

FIG. 1. Top: Phase portrait of Eq. (8) in the case $\eta = 1$. The blue lines are the separatrices that joins the unstable stationary points (red crosses). Black dots are stable stationary points. Note that well above the separatrices, $\dot{x}_0$ becomes nearly constant, so that the pinning force exerted by the carrier wave on the envelope becomes negligible. Bottom: wave packets traveling exactly at the phase velocity and corresponding to stable (black) and unstable (red) configurations in an inversion-symmetric system ($\kappa = 2$). In this illustration, $\varepsilon = 0.2$.

becomes nearly constant. Hence, one recovers in that limit the continuous family of constant-speed solitons of the NLSE [5].

However, as the energy of the effective pendulum decreases, the phase portrait in Fig. 1 clearly indicates that the soliton velocity ceases to be constant and that the motion is unsteady. The integration of Eq. (8) is a classical problem of mechanics. Outside the locking range, such that the pendulum makes complete rotations, the average speed $\langle v \rangle$ of the soliton can be found as

$$\langle v \rangle = v_p(\omega) + \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} \frac{ds}{\sqrt{(v_g(\omega) - v_p(\omega))^2 - v_l^2 \cos s}} \right]^{-1}. \tag{11}$$

This expression is illustrated in Fig. 2. Within the locking range, the soliton oscillates about a coordinate that moves at the phase velocity. Quite remarkably, this oscillation is not due to noise, inhomogeneity of the supporting medium,
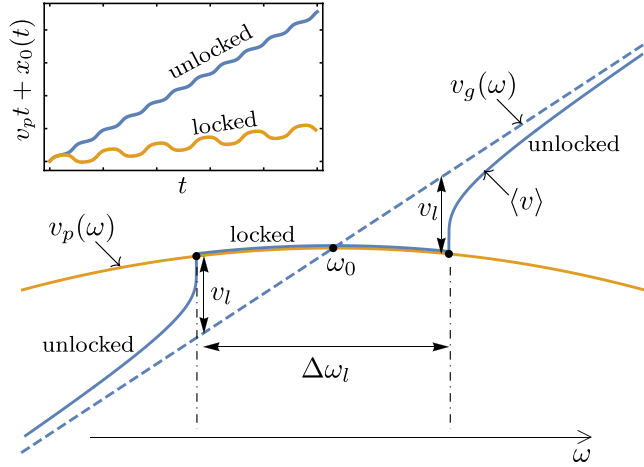
FIG. 2. Average soliton speed $\langle v \rangle$ in the vicinity of ω_0. Inside the locking range $[\omega_0 - \Delta\omega_l/2, \omega_0 + \Delta\omega_l/2]$, the soliton envelope is locked to the phase velocity $v_p(\omega)$. Moving away from the locking range, $\langle v \rangle$ gradually tends to the group velocity $v_g(\omega)$. Inset: examples of soliton trajectories just inside (orange) and just outside to the right of the locking range (blue).

or the presence of another soliton. It results solely from the oscillations of the carrier wave, which make an effective shallow periodic potential for the center of mass of envelope. Figure 2 allows us to ascertain the locking range in frequency as $2v_l/|v'_g(\omega_0)|$, that is,

$$\Delta\omega_l = \frac{2v_l}{v_g^2(\omega_0)|\beta''(\omega_0)|}. \tag{12}$$

The right-hand side depends on the unknown constant K. Its dependence on ε is plotted in Fig. 3 and shows that the effect is generally more pronounced for systems lacking inversion symmetry, due to the parameter κ.

Also, v_l makes more precise condition (3) on the closeness of the group and phase velocities. What is required in the present study is that

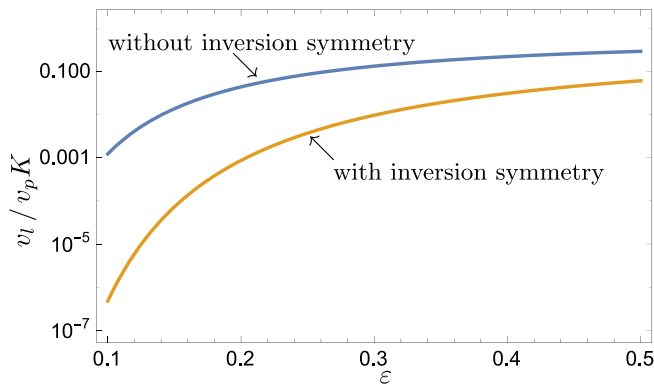$$|v_p - v_g| = O(v_l) = O(\varepsilon^{-1/2} e^{-\kappa\pi/4\varepsilon} v_p). \tag{13}$$

FIG. 3. Locking range, $\varepsilon^{-1/2} \exp{-\kappa\pi/4\varepsilon}$, as a function of the scale separation ε and of the system inversion symmetry ($\kappa = 1, 2$.)

The quantity v_l appearing in the pendulum equation is exponentially small, i.e., smaller than any finite power of ε. As a result, the physics presented here is not merely a high-order correction to the NLSE. In other words, it is of a different kind than the correction sometimes brought to the NLSE for very short pulses by adding third-order, fourth-order, or higher-order dispersion terms to the equation [6]. Rather, v_l emerges because the multiple-scale expansion that underlies the NLSE actually generates a diverging series. As far back as 1857, George Gabriel Stokes studied diverging asymptotic series that approximate the Airy function [7]. He showed that the divergence is associated to the birth of exponentially small corrections in precise regions of the complex plane, corrections that grow exponentially as one moves away from their place of birth to the point of completely invalidating the initial approximation. This process, known as Stokes phenomenon, also takes place in the present situation. We will show that a term of order v_l^2 will be switched on in the complex plane and will grow exponentially away from the center of the wave packet, thereby threatening to invalidate the soliton approximation. To compensate for this correction, a second exponentially growing term is born that is proportional to $\ddot{x}_0$ so as to cancel the effect of the former. This summarises in a few words the technique of beyond-all-orders asymptotics, which has been applied to multiple-scale problems in only a few instances [8–21]. Among these works, the one by Yang and Akylas [8] stands as particularly relevant. These authors studied the possibility of asymmetric gravity-capillary solitary waves in a simplified hydrodynamic model, namely the fifth-order KdV equation. They were looking for constant wave profiles in a frame moving at v_p and concluded that only symmetric profiles exist in that moving frame, i.e., that the wave either has a crest or a trough at its center. This is consistent with the results quoted here: the nonlinearity in that example is quadratic ($\kappa = 1$), so stationary solutions of the pendulum Eq. (8) require either $x_0 = 0$ or $\beta(\omega_0)x_0 = \pi$. In the followup research coauthored by Calvo [12], it was established that only one of them is stable, the one with a depression in the middle. This is again fully consistent with the present theory with $\eta = -1$. Further, one can immediately read off from the pendulum equation that the rate of instability of the unstable stationary state is $0.5v_l\beta(\omega_0)$ and the scaling again agrees with Calvo *et al.* The present theory thus extends the pioneering work by these authors by placing it into a universal frame, completing the picture with dynamical asymmetrical profiles and including dark solitons. Note, finally, that a single computation can in principle allow one to determine the numerical constant v_l: from what has just been said, it suffices to determine the rate of instability of the unstable stationary profile.

Finally, it follows from what precedes that the carrier wave exerts a small periodic force on the center of mass of the envelope. This can of course alter the interaction between two solitons. At large distance, two solitons are classically known to interact through a Toda potential, which may be repelling or attracting [22]. Such an interaction potential can now contain ripples associated to the pendulum restoring force. As a result, new stable bound states can appear in the vicinity of the locking range, such that the two solitons, while interacting with each other, are at the same time tied to the carrier wave.

II. A GENERAL WAVE MODEL

We will demonstrate our result on the following general class of wave equations:

$$\frac{\partial^2 E}{\partial x^2} + \int_{-\infty}^{\infty} \beta^2(\omega)\hat{E}(x,\omega)e^{-i\omega t}d\omega = N[E,\epsilon], \qquad (14)$$

where

$$\hat{E}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(x,t)e^{i\omega t}dt \qquad (15)$$

is the Fourier transform of the scalar field $E(x,t)$ of interest and $N[E,\epsilon]$ is the nonlinearity, which tends to zero as $\epsilon \to 0$. We confine our study to a weak nonlinearity, i.e., to $\epsilon \ll 1$. In Appendix A, we show how the above equation naturally arises in electromagnetic, elastic, and plasma wave propagation. The fifth-order KdV equation, allowing waves only in one direction, is in fact not included in (14). The agreement with the calculations of Calvo, Yang, and Akylas on that equation support the claim that the results presented here are general and not tied to a particular wave model. Notice that in the linear limit $\epsilon \to 0$, Eq. (14) is merely a superposition of Helmholtz equations.

The interest of formulating the wave equation in this slightly unusual way is that it directly makes apparent the all-important dispersion function $\beta(\omega)$. If we momentarily neglect the right-hand side, it is immediate to see that the general wave solution moving in the positive x direction is

$$\epsilon = 0: \quad E(x,t) = \int_{-\infty}^{\infty} A(\omega)e^{i[\beta(\omega)x - \omega t]}d\omega. \qquad (16)$$

Hence $\beta(\omega)$ is indeed the wave number, or propagation constant, as a function of angular frequency. In what follows, we will adopt time and space units such that

$$\omega_0 = 1, \quad \beta(\omega_0) = 1. \qquad (17)$$

In these units, and in the absence of nonlinearity, the phase velocity is also unity at ω_0.

When Eq. (14) models electromagnetic waves subjected to a Kerr nonlinearity, Maxwell's equations lead to

$$N[E,\epsilon] = \frac{2}{3}\epsilon^2 \frac{\partial^2}{\partial t^2}(E^3), \qquad (18)$$

where ϵ^2 is a small parameter that is proportional to the intensity of the wave and the factor $2/3$ is introduced for later convenience. However, it is common practice to simplify the differential operator $\partial^2/\partial t^2$ above by $-\omega_0^2$. In order to simplify the algebra in this paper, we will therefore present the calculation with the simpler cubic nonlinearity

$$N[E,\epsilon] = -\frac{2}{3}\epsilon^2 E^3. \qquad (19)$$

(Recall that $\omega_0 = 1$ in our choice of units.) We wish to stress, however, that we have also done the calculation with (18) and obtained, as expected, the same result as with (19). The algebra in the leading orders of the analysis is lightest with (19) but at later orders, the calculation becomes universal. We will therefore do the calculation explicitly with that nonlinearity and comment, whenever necessary in the course of our analysis, what happens when a quadratic, rather than a cubic nonlinearity is assumed.

Before embarking into a multiple-scale analysis, we note, as in Ref. [23], that for a wave packet with central frequency Ω, we may expand $\beta(\omega)$ in Taylor series around Ω in the integral of Eq. (14), giving

$$\int_{-\infty}^{\infty} \beta^2(\omega)\hat{E}(x,\omega)e^{-i\omega t}d\omega$$

$$= e^{-i\Omega t}\sum_n \frac{1}{n!}\frac{d^n(\beta^2)}{d\omega^n}\bigg|_{\Omega}$$

$$\times \int_{-\infty}^{\infty} (\omega - \Omega)^n \hat{E}(x,\omega)e^{-i(\omega-\Omega)t}d\omega + \text{c.c.}$$

$$= e^{-i\Omega t}\sum_n \frac{1}{n!}\frac{\partial^n(\beta^2)}{\partial\omega^n}\bigg|_{\Omega}\left(i\frac{\partial}{\partial t}\right)^n(Ee^{i\Omega t}) + \text{c.c.}$$

$$= e^{-i\Omega t}\beta^2(\Omega + i\partial/\partial t)(Ee^{i\Omega t}) + \text{c.c.}. \qquad (20)$$

Remark. The attentive reader will have noticed that we slightly changed the notation of the small parameter: ϵ vs ε. The two differ by a numerical factor that we will specify shortly. Using ϵ will be more useful from a notational point of view in the asymptotic treatment to follow, but ε is universally defined by (6), independently of context.

III. MULTIPLE SCALES

Let us construct a solution with the multiple-scale ansatz

$$E \sim \sum_{l \geqslant 0} \epsilon^l E_l = \sum_{l \geqslant 0} \sum_{m=-2l-1,}^{2l+1} \epsilon^l A_{l,m}(\xi, X)e^{im(x-t)}, \qquad (21)$$

with the constraint that

$$A_{l,-m} = \bar{A}_{l,m} \qquad (22)$$

on the real line for E to be real (we use an overbar to denote complex conjugation). Above, the function $A_{l,m}$ is the $O(\epsilon^l)$ contribution to the slow amplitude that modulates the mth harmonic of the fundamental carrier wave. Its evolution is assumed to take place on the following spatiotemporal scales

$$\xi = \epsilon[t - (x - x_0)(1 - \epsilon^2 v)], \quad X = \epsilon^2 x, \qquad (23)$$

where $\epsilon^2 v$ is a nonlinear correction to the phase velocity. Above, ξ is a variable that is attached to a frame moving at the speed $v_p \approx v_g$ and which serves to describe the envelope profile. On the other hand, X is a slow evolution variable, akin to time, and which allows us to monitor slow changes of the envelope in the course of propagation.

Given (21), we have, using (20),

$$\int_{-\infty}^{\infty} \beta^2(\omega)\hat{E}(x,\omega)e^{-i\omega t}d\omega$$

$$= \sum_{l,m} \epsilon^l e^{im(x-t)}[\beta(m + i\epsilon\partial/\partial\xi)]^2 A_{l,m}(\xi, X)$$

$$= \sum_{l,m,n} \epsilon^{l+n} e^{im(x-t)}\Lambda_n^m i^n \frac{\partial^n A_{l,m}}{\partial\xi^n}, \qquad (24)$$

where we have introduced

$$\Lambda_n^m = \frac{1}{n!}\frac{d^n\beta^2(\omega)}{d\omega^n}\bigg|_m. \qquad (25)$$

(Above, evaluation at $\omega = m$ means evaluation at $\omega = m\omega_0$ in a general set of units.) Furthermore, with our multiple-scale ansatz, differentiation with respect to x becomes

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} - \epsilon[1 - \epsilon^2 \nu]\frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial X}. \quad (26)$$

As a result, $\partial^2/\partial x^2$ becomes

$$\frac{\partial^2}{\partial x^2} \rightarrow \frac{\partial^2}{\partial x^2} - 2\epsilon\frac{\partial^2}{\partial x \partial \xi} + \epsilon^2\left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial x \partial X}\right)$$
$$+ 2\epsilon^3\left(\nu\frac{\partial^2}{\partial x \partial \xi} - \frac{\partial^2}{\partial \xi \partial X}\right) + \epsilon^4\left(\frac{\partial^2}{\partial X^2} - 2\nu\frac{\partial^2}{\partial \xi^2}\right)$$
$$+ 2\epsilon^5\nu\frac{\partial^2}{\partial \xi \partial X} + \epsilon^6\nu^2\frac{\partial^2}{\partial \xi^2}. \quad (27)$$

Above the $O(\epsilon^5)$ terms are unimportant for the leading orders of the asymptotic analysis and are also negligible in first approximation when it comes to late terms of the expansion. However, we give them for the sake of completeness. We thus have

$$\frac{\partial^2 E}{\partial x^2} \sim \sum_{lm} \epsilon^l e^{im(x-t)}\left(-m^2 A_{l,m} - 2im\frac{\partial A_{l-1,m}}{\partial \xi}\right.$$
$$+ \frac{\partial^2 A_{l-2,m}}{\partial \xi^2} + 2im\frac{\partial A_{l-2,m}}{\partial X} + 2im\nu\frac{\partial A_{l-3,m}}{\partial \xi}$$
$$- 2\frac{\partial^2 A_{l-3,m}}{\partial \xi \partial X} + \frac{\partial^2 A_{l-4,m}}{\partial X^2} - 2\nu\frac{\partial^2 A_{l-4,m}}{\partial \xi^2}$$
$$\left.+ 2\nu\frac{\partial^2 A_{l-5,m}}{\partial \xi \partial X} + \nu^2\frac{\partial^2 A_{l-6,m}}{\partial \xi^2}\right). \quad (28)$$

Finally,

$$E^3 = \sum_{l'l''mm'm''} \epsilon^l A_{l',m'}A_{l'',m''}A_{l-l'-l'',m-m'-m''}e^{im(x-t)}. \quad (29)$$

Putting everything together, we have to solve, for each order $l \geqslant 0$ and each harmonic m,

$$- m^2 A_{l,m} - 2im\frac{\partial A_{l-1,m}}{\partial \xi} + \frac{\partial^2 A_{l-2,m}}{\partial \xi^2} + 2im\frac{\partial A_{l-2,m}}{\partial X}$$
$$+ 2im\nu\frac{\partial A_{l-3,m}}{\partial \xi} - 2\frac{\partial^2 A_{l-3,m}}{\partial \xi \partial X} + \frac{\partial^2 A_{l-4,m}}{\partial X^2} - 2\nu\frac{\partial^2 A_{l-4,m}}{\partial \xi^2}$$
$$+ 2\nu\frac{\partial^2 A_{l-5,m}}{\partial \xi \partial X} + \nu^2\frac{\partial^2 A_{l-6,m}}{\partial \xi^2} + \sum_n \Lambda_n^m i^n\frac{\partial^n A_{l-n,m}}{\partial \xi^n}$$
$$= -\frac{2}{3}\sum_{l'l''m'm''} A_{l',m'}A_{l''m''}A_{l-2-l'-l'',m-m'-m''}. \quad (30)$$

This is the general multiple-scale translation of Eq. (14) subject to the nonlinearity (19). In what follows, we will first solve the above equations up to $l = 2$ to derive the NLSE for the amplitude $A_{0,1}$. We will next investigate the recurrence for $l \gg 1$ and show that A_{lm} grows factorially with l in that limit, making the asymptotic series (21) diverge. Finding the precise way in which this divergence occurs, we will be able

to truncate (21) optimally as

$$E = \sum_{l=0}^{L-1} \epsilon^l E_l + R \quad (31)$$

for some large L and derive an equation for the remainder R. The law governing the locking of the envelope to the carrier wave is contained in R.

A. Leading orders

At $l = 0$, $m = \pm 1$, we obtain

$$(\beta(1)^2 - 1)A_{0,1} = 0, \quad (32)$$

which is automatically satisfied, since $\beta(1) = 1$. At $l = 1$, $m = 1$, we have

$$i(\Lambda_1^1 - 2)\frac{\partial A_{0,1}}{\partial \xi} = 2i\frac{v_p - v_g}{v_g}\frac{\partial A_{0,1}}{\partial \xi} = 0. \quad (33)$$

By assumption, the difference $v_p - v_g$ is exponentially small so that the above equation is automatically satisfied at $O(\epsilon)$ of our calculation.

Next, the equation for $l = 1$, $m = 3$, yields

$$(\beta(3)^2 - 9)A_{1,3} = 0, \quad \rightarrow A_{1,3} = 0, \quad (34)$$

$\beta(3)^2$ being generally different from 9, due to dispersion. Finally, the $l = 2$, $m = 1$ equation is

$$\frac{\partial^2 A_{0,1}}{\partial \xi^2} + 2i\frac{\partial A_{0,1}}{\partial X} - \Lambda_2^1\frac{\partial^2 A_{0,1}}{\partial \xi^2} = -2A_{0,1}^2 A_{0,-1}, \quad (35)$$

or, equivalently

$$i\frac{\partial A_{0,1}}{\partial X} + \frac{1 - \Lambda_2^1}{2}\frac{\partial^2 A_{0,1}}{\partial \xi^2} + |A_{0,1}|^2 A_{0,1} = 0. \quad (36)$$

Evaluating Λ_2^1, we get

$$\Lambda_2^1 = \beta'(1)^2 + \beta(1)\beta''(1) = 1 + \beta_2, \quad (37)$$

where $\beta_2 = \beta''(1)$. Eventually, we find the classical NLSE:

$$i\frac{\partial A_{0,1}}{\partial X} - \frac{\beta_2}{2}\frac{\partial^2 A_{0,1}}{\partial \xi^2} + |A_{0,1}|^2 A_{0,1} = 0. \quad (38)$$

One may check that the above equation is indeed equivalent to Eq. (2) by substituting $\psi = A_{0,1}(\epsilon(t - \beta'(\omega_0)x), \epsilon^2 x)$ in the latter and setting $\gamma = \epsilon^2$.

B. Soliton solution

Given the sign of the nonlinear term in (38), the bright soliton solution is obtained when $\beta_2 < 0$ and is given by

$$A_{0,1} = e^{iX/2}\text{sech}\left(\frac{\xi}{|\beta_2|^{1/2}}\right), \quad (39)$$

while the leading order amplitude of the $m = -1$ harmonic is given by

$$A_{0,-1} = e^{-iX/2}\text{sech}\left(\frac{\xi}{|\beta_2|^{1/2}}\right). \quad (40)$$

On the other hand, if $\beta_2 > 0$, the NLSE admits the dark soliton solution

$$A_{0,\pm 1} = e^{\pm iX} \tanh\left(\frac{\xi}{|\beta_2|^{1/2}}\right). \tag{41}$$

Hence, the leading-order soliton solution of the complete model is

$$E_0 = \begin{cases} 2\text{sech}\left(\frac{\xi}{|\beta_2|^{1/2}}\right)\cos(x - t + X/2) & \text{if} \quad \beta_2 < 0, \\ 2\tanh\left(\frac{\xi}{|\beta_2|^{1/2}}\right)\cos(x - t + X) & \text{if} \quad \beta_2 > 0. \end{cases} \tag{42}$$

Importantly for what follows, both solutions above have complex singularities at $\xi = i(n + 1/2)\pi|\beta_2|^{1/2}$. In particular, for the bright soliton solution

$$A_{0,\pm 1} \sim \frac{i|\beta_2|^{1/2}e^{\pm iX/2}}{\xi_0 - \xi} \tag{43}$$

as $\xi \to \xi_0 = i\pi|\beta_2|^{1/2}/2$ while, in the same limit, the dark soliton solution diverges as

$$A_{0,\pm 1} \sim \frac{-\beta_2^{1/2}e^{\pm iX}}{\xi_0 - \xi}. \tag{44}$$

We may summarize the two possible behaviors in the vicinity of ξ_0 as

$$A_{0,\pm 1} \sim \frac{i^*|\beta_2|^{1/2}e^{\pm iX/2^*}}{\xi_0 - \xi}. \tag{45}$$

where we introduce the notation

$$i^* = i, \quad 2^* = 2, \quad \text{if} \quad \beta_2 < 0, \tag{46}$$

$$i^* = -1, \quad 2^* = 1, \quad \text{if} \quad \beta_2 > 0. \tag{47}$$

Note that the star does not mean complex conjugation, we use an overbar for that purpose. Next, recalling that $X = \epsilon^2 x$ in (42), the nonlinear correction to the phase velocity is

$$\epsilon^2 \nu = \begin{cases} -\epsilon^2/2 & \text{if} \quad \beta_2 < 0, \\ -\epsilon^2 & \text{if} \quad \beta_2 > 0, \end{cases} \tag{48}$$

which can be summarized by writing $\nu = -1/2^*$ in the starred notation. Finally, comparing the scales of the envelope and the carrier wave in (42), we see that the small parameter ε of the Introduction is related to ϵ as

$$\varepsilon = \epsilon/|\beta_2|^{1/2}. \tag{49}$$

C. Galilean invariance

The above formulas for $A_{0,1}$, whether they describe bright or dark solitons, belong to a family of solutions of the NLSE obtained by the transformation rule

$$\xi \to \xi + cX, \quad iX/2^* \to i\left(X/2^* + \frac{c^2X}{2\beta_2} + \frac{c\xi}{\beta_2}\right), \tag{50}$$

where c is a constant parameter that produces a change ϵc of the envelope velocity. The second transformation rule above ensures that this change of group velocity is accompanied by a change of frequency that is compatible with the dispersion relation $\beta(\omega)$.

D. Slowly accelerating soliton

If we now allow c to vary slowly in the course of propagation, then Galilean invariance is broken. Letting $c'(X) \ll 1$, the bright soliton solution is changed, to first order, as

$$A_{0,1} \sim \left[\text{sech}(\zeta) - \frac{dc}{dX}(R_a + iX\zeta)\right]e^{i[X/2 + (c^2X/2 + c\xi)/\beta_2]}, \tag{51}$$

where

$$\zeta = \frac{\xi + cX}{|\beta_2|^{1/2}}, \tag{52}$$

and R_a is the correction due to the acceleration:

$$R_a = \frac{1 + e^{4\zeta} - 2\zeta + 2\zeta^2 - 2e^{2\zeta}(5 - 5\zeta + \zeta^2)}{2(e^{3\zeta} + 2e^\zeta + e^{-\zeta})}. \tag{53}$$

Note that the far-field behavior of R_a is

$$R_a \sim \frac{e^\zeta}{2}, \quad \zeta \to \infty, \tag{54}$$

which, in principle, renders (51) physically unacceptable. However, another exponentially growing term is hidden in the remainder R of (31), which can compensate R_a and allow the existence of accelerating solitons.

For dark solitons ($\beta_2 > 0$) the same thing happens:

$$A_{0,1} \sim \left[\tanh(\zeta) + \frac{dc}{dX}(R_a + iX\zeta)\right]e^{i[X + (c^2X/2 + c\xi)/\beta_2]}, \tag{55}$$

this time with

$$R_a = \frac{\text{sech}^2\zeta}{32}[-7 + 12\zeta - 8\zeta^2 + 8\cosh 2\zeta \\ + 8(1 - \zeta)\sinh 2\zeta + \cosh 4\zeta + \sinh 4\zeta] \tag{56}$$

and

$$R_a \sim \frac{e^{2\zeta}}{8}, \quad \zeta \to \infty. \tag{57}$$

IV. LATE-TERM EXPANSION

We now investigate the large-l behavior of the amplitudes A_{lm}. We first note that the nonlinearity (19) only couples odd harmonics of the fundamental one and that two new harmonics appear at each order of the calculation. As a result nonzero amplitudes A_{lm} only exist for

$$m \text{ odd}, \quad |m| \leqslant 2\lfloor l/2 \rfloor + 1. \tag{58}$$

(With a quadratic nonlinearity, even harmonics m exist too.) As l becomes large, we anticipate from similar calculations [15–18,20] that these amplitudes grow in size as $\Gamma(l + \alpha_m)$ for some α_m. In order to determine this number, it is sufficient to study the system (30) in the vicinity of the singularity ξ_0. This is what we do first.

A. Inner expansion near $\xi = \xi_0$

In the vicinity of ξ_0, we assume that

$$A_{l,m} \sim \frac{\Gamma(l + \alpha_m)B_{l,m}e^{imX/2^*}}{(\xi_0 - \xi)^{l+1}}, \tag{59}$$

where α_m is a constant to be determined. Substituting in Eq. (30) and factoring out the factorial function, we obtain, as $\xi \to \xi_0$,

$$
\begin{aligned}
&- m^2 B_{l,m} - 2im \frac{l\Gamma(l + \alpha_m - 1)}{\Gamma(l + \alpha_m)} B_{l-1,m} \\
&+ \frac{l(l-1)\Gamma(l + \alpha_m - 2)}{\Gamma(l + \alpha_m)} B_{l-2,m} m \\
&+ \sum_n \Lambda_n^m i^n \frac{l!\Gamma(l + \alpha_m - n)}{(l-n)!\Gamma(l + \alpha_m)} B_{l-n,m} \\
&= -\frac{2}{3} \sum_{l'l''m'm''} \frac{\Gamma(l' + \alpha_{m'})\Gamma(l'' + \alpha_{m''})}{\Gamma(l + \alpha_m)} B_{l',m'} B_{l'',m''} \\
&\times \Gamma(l - 2 - l' - l'' + \alpha_{m-m'-m''}) B_{l-2-l'-l'',m-m'-m''}.
\end{aligned}
$$
(60)

Compared to (30), Eq. (60) is a set of algebraic equations rather than differential ones, and are therefore more tractable. They should in principle be solved by recurrence, starting with

$$
B_{0,\pm 1} = i^* |\beta_2|^{1/2} / \Gamma(\alpha_1).
$$
(61)

In the large-l limit, the equations simplifies to

$$
-m^2 B_{l,m} - 2im B_{l-1,m} + B_{l-2,m} + \sum_n \Lambda_n^m i^n B_{l-n,m} = O(1/l).
$$
(62)

We may then look for a solution of the form

$$
B_{l,m} \sim (i/\kappa)^l b_m^{(0)} + O(1/l),
$$
(63)

where κ is to be determined. This yields

$$
\left(\sum_n \Lambda_n^m \kappa^n - m^2 - 2m\kappa - \kappa^2 \right) b_m^{(0)} = 0.
$$
(64)

Recalling the definition of the coefficients Λ_n^m, the above equation can be written more simply as

$$
[\beta^2(m + \kappa) - (m + \kappa)^2] b_m^{(0)} = 0.
$$
(65)

Given that $\beta(\pm 1)^2 = 1$, we thus have

$$
m + \kappa = \pm 1,
$$
(66)

which can be solved for κ as a function of the harmonic m. Since m is odd, the possible values of κ are

$$
\kappa = \pm 2, \pm 4, \pm 6, \ldots.
$$
(67)

In particular, $\kappa = 2$ is obtained for $m = -1, -3$, while $\kappa = -2$ corresponds to $m = 1, 3$. (With a quadratic nonlinearity, even values are allowed for m, so we have instead $\kappa = \pm 1, \pm 2, \pm 3, \ldots$. In particular, $\kappa = 1$ is obtained for $m = -2, 0$, while $\kappa = -1$ corresponds to $m = 0, 2$.) Given the κ^{-l} dependance in Eq. (63), we may expect that the Fourier modes corresponding to smallest absolute value of κ are those that will matter at very large orders and that the other Fourier components will be subdominant as $l \to \infty$.

To make further progress, let us generalize Eq. (63) as

$$
B_{l,m} \sim (i/\kappa)^l \left(b_m^{(0)} + l^{-1} b_m^{(1)} + l^{-2} b_m^{(2)} + \cdots \right).
$$
(68)

Then we find (see Appendix B) that

$$
\frac{i^n l!\Gamma(l + \alpha_m - n)}{(l-n)!\Gamma(l + \alpha_m)} B_{l-n,m} \sim (i/\kappa)^l \sum_{q=0}^n \frac{n!\kappa^n}{(n-q)!} g_q(l)
$$
(69)

with

$$
g_0(l) \sim b_m^{(0)} + l^{-1} b_m^{(1)} + l^{-2} b_m^{(2)} + \cdots,
$$
(70)

$$
g_1(l) \sim \frac{(1 - \alpha_m) b_m^{(0)}}{l} + \frac{(2 - \alpha_m) b_m^{(1)} + (1 - \alpha_m)^2 b_m^{(0)}}{l^2},
$$
(71)

$$
g_2(l) \sim \frac{(1 - \alpha_m)(2 - \alpha_m) b_m^{(0)}}{2l^2},
$$
(72)

$$
\vdots
$$

$$
g_q(l) = O(1/l^q).
$$
(73)

Now notice that

$$
\sum_n \Lambda_n^m \kappa^n = \beta^2(m + \kappa),
$$
(74)

$$
\sum_n n \Lambda_n^m \kappa^n = \kappa \frac{d}{d\kappa}(\beta^2(m + \kappa)),
$$
(75)

$$
\sum_n n(n-1) \Lambda_n^m \kappa^n = \kappa^2 \frac{d^2}{d\kappa^2}(\beta^2(m + \kappa)), \ldots
$$
(76)

Hence, for any q,

$$
\sum_n \Lambda_n^m \frac{n!\kappa^n}{(n-q)!} = \kappa^q \frac{d^q}{d\kappa^q}(\beta^2(m + \kappa)),
$$
(77)

Combining this last expression with Eq. (69), we find that the various terms in the left-hand side of Eq. (60), can be written as

$$
\frac{il\Gamma(l + \alpha_m - 1)}{\Gamma(l + \alpha_m)} B_{l-1,m} \sim (i/\kappa)^l [g_0(l) + g_1(l)]\kappa,
$$
(78)

$$
\frac{i^2 l(l-1)\Gamma(l + \alpha_m - 2)}{\Gamma(l + \alpha_m)} B_{l-2,m} \sim (i/\kappa)^l [g_0(l) + 2g_1(l) + 2g_2(l)]\kappa^2.
$$
(79)

and

$$
\begin{aligned}
&\sum_n \Lambda_n^m i^n \frac{l!\Gamma(l + \alpha_m - n)}{(l-n)!\Gamma(l + \alpha_m)} B_{l-n,m} \\
&= (i/\kappa)^l \left[g_0(l)\beta^2(m + \kappa) + g_1(l)\kappa \frac{d}{d\kappa}(\beta^2(m + \kappa)) \right. \\
&\left. + g_2(l)\kappa^2 \frac{d^2}{d\kappa^2}(\beta^2(m + \kappa)) + \cdots \right].
\end{aligned}
$$
(80)

Putting everything together, the left-hand side of (60) is

$$
\begin{aligned}
(i/\kappa)^l \Bigg\{ &g_0(l)[\beta^2(m + \kappa) - (m + \kappa)^2] \\
&+ g_1(l)\kappa \left[\frac{d}{d\kappa} \beta^2(m + \kappa) - 2(\kappa + m) \right] \\
&+ g_2(l)\kappa^2 \left[\frac{d^2}{d\kappa^2} \beta^2(m + \kappa) - 2 \right] + O(1/l^3) \Bigg\}.
\end{aligned}
$$
(81)

Above, we compute

$$\frac{d\beta^2(\omega)}{d\omega} = 2\beta(\omega)\beta'(\omega), \tag{82}$$

$$\frac{d^2\beta^2(\omega)}{d\omega^2} = 2\beta'(\omega)^2 + 2\beta(\omega)\beta''(\omega) \tag{83}$$

If $m + \kappa = 1$, we find that $(d/d\kappa)\beta^2(m + \kappa) = 2$ and $(d^2/d\kappa^2)\beta^2(m + \kappa) = 2 + 2\beta_2$. On the other hand, from the fact that $\beta^2(-\omega) = \beta^2(\omega)$, we deduce that if $m + \kappa = -1$, then $(d/d\kappa)\beta^2(m + \kappa) = -2$ and $(d^2/d\kappa^2)\beta^2(m + \kappa) = 2 + 2\beta_2$. In both cases, the first two terms in the curly bracket of (81) vanish. Hence, the left-hand side of Eq. (60) is found to be asymptotic to $-2\beta_2(i/\kappa)^{l-2}g_2(l)$, i.e., to

$$-\beta_2(i/\kappa)^{l-2}(1 - \alpha_m)(2 - \alpha_m)b_m^{(0)}/l^2 \tag{84}$$

Therefore, the right-hand side of (60) must be $O(l^{-2})$. The right-hand side mixes and couples different harmonics. It is natural to consider a couple of harmonics related by the same κ, as in Eq. (66). Let them be m and $m + 2$ and let us assume that $\alpha_m = \alpha_{m+2} = \alpha$. It is easy to see that in the large-l limit the leading terms in the right-hand side of (60) will be those for which two of the three indices $l', l'', l - 2 - l' - l''$ are zero. Thus, using (61), (66), and the definition of i^*, the right-hand side of (60) is asymptotic to

$$\frac{-2(i/\kappa)^{l-2}}{l^2} \sum_{m',m''} \Gamma(\alpha_{m'})\Gamma(\alpha_{m''})B_{0,m'}B_{0,m''}b_{m-m'-m''}^{(0)}$$

$$= \frac{-2(i/\kappa)^{l-2}}{l^2}(i^*)^2|\beta_2| \sum_{m',m''} b_{m-m'-m''}^{(0)}$$

$$= \frac{-2\beta_2(i/\kappa)^{l-2}}{l^2} \sum_{m',m''} b_{m-m'-m''}^{(0)}$$

$$= \frac{-2\beta_2(i/\kappa)^{l-2}}{l^2}\left(b_{m-2}^{(0)} + 2b_m^{(0)} + b_{m+2}^{(0)}\right) \tag{85}$$

Eventually Eq. (60) evaluated for the two values, m and $m + 2$, corresponding to a given κ, yield

$$[(1 - \alpha)(2 - \alpha) - 4]b_m^{(0)} - 2b_{m+2}^{(0)} = 0, \tag{86}$$

$$[(1 - \alpha)(2 - \alpha) - 4]b_{m+2}^{(0)} - 2b_m^{(0)} = 0. \tag{87}$$

Then we find that either $b_{m+2}^{(0)} = b_m^{(0)}$ and

$$(1 - \alpha)(2 - \alpha) - 6 = 0, \quad \rightarrow \alpha \in \{-1, 4\}. \tag{88}$$

or $b_{m+2}^{(0)} = -b_m^{(0)}$, in which case

$$(1 - \alpha)(2 - \alpha) - 2 = 0, \quad \rightarrow \alpha \in \{0, 3\}. \tag{89}$$

Out of all the above possibilities, the case where $\alpha = 4$ will yield the dominant factorial growth as $l \gg 1$. We therefore only keep that value into consideration in what follows. We

have thus shown that

$$E_l \sim i^* \sum_\kappa \sum_{m=-\kappa\pm1} \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l + 4)}{(\xi_0 - \xi)^{l+1}} e^{im(x-t+X/2^*)} \tag{90}$$

at order l in the expansion of E. Above, λ_κ is a constant that can only be computed numerically or by actually solving the recurrence equations (60) all the way from $l = 0$ to $l \gg 1$. Note that there is a similar expression near $\xi = -\xi_0$ and indeed in the vicinity of all complex singularities of the leading-order soliton. Although the constant λ_κ is not universal, we note that, after making the change $B_{l,m} = (i/\kappa)^l b_{l,m}$, the resulting equations for $b_{l,m}$ have only real coefficients. Hence, given the factor i^* in (90), we may deduce that λ_κ is real.

B. Outer expansion away from $\xi = \xi_0$

The result of our investigation in the vicinity of ξ_0, Eq. (90), suggests to look for contributions in the late-term expansion of the form

$$E_l \sim \sum_\kappa \sum_{m=-\kappa\pm1} \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l + 4)}{(\xi_0 - \xi)^{l+4}} f_m(\xi, X)e^{im(x-t)}. \tag{91}$$

Note that the exponent in the denominator was slightly changed with respect to that in (90). This small variation, which is allowed by the undetermined factor $f_m(\xi, X)$, will significantly simplify the ensuing algebra. Thus, the A_{lm} are of the form

$$A_{l,m} = \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l + 4)}{(\xi_0 - \xi)^{l+4}} f_m(\xi, X). \tag{92}$$

Considering (30), we see that we will need to evaluate expressions of the form

$$\frac{\partial^n A_{l-n,m}}{\partial\xi^n} = \lambda_\kappa (i/\kappa)^{(l-n)}\Gamma(l + 4 - n)$$

$$\times \frac{\partial^n}{\partial\xi^n}\frac{f_m(\xi, X)}{(\xi_0 - \xi)^{l+4-n}} = \lambda_\kappa (i/\kappa)^{(l-n)}\Gamma(l + 4 - n)$$

$$\times \sum_{o=0}^n \binom{n}{o}\frac{\Gamma(l + 4 - o)}{\Gamma(l + 4 - n)}\frac{1}{(\xi_0 - \xi)^{l+4-o}}\frac{\partial^o f_m(\xi, X)}{\partial\xi^o}$$

$$= \lambda_\kappa (i/\kappa)^{(l-n)} \sum_{o=0}^n \binom{n}{o}\frac{\Gamma(l + 4 - o)}{(\xi_0 - \xi)^{l+4-o}}\frac{\partial^o f_m(\xi, X)}{\partial\xi^o}, \tag{93}$$

where we have used the binomial formula for the derivative of a product. Developing this expression further:

$$\frac{\partial^n A_{l-n,m}}{\partial\xi^n} = \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l + 4)}{(\xi_0 - \xi)^{l+4}}$$

$$\times (\kappa/i)^n \sum_{o=0}^n \binom{n}{o}\frac{\Gamma(l + 4 - o)}{\Gamma(l + 4)}(\xi_0 - \xi)^o\frac{\partial^o f_m}{\partial\xi^o}. \tag{94}$$

Expanding the sum above up to order $1/l^2$:

$$\sum_{o=0}^{n}\binom{n}{o}\frac{\Gamma(l+4-o)}{\Gamma(l+4)}(\xi_0-\xi)^o\frac{\partial^o f_m}{\partial\xi^o}$$

$$= f_m + \frac{n}{l+3}(\xi_0-\xi)\frac{\partial f_m}{\partial\xi}$$

$$+ \frac{n(n-1)}{2(l+3)(l+2)}(\xi_0-\xi)^2\frac{\partial^2 f_m}{\partial\xi^2} + \dots$$

$$\sim f_m + n\left(\frac{1}{l}-\frac{3}{l^2}\right)(\xi_0-\xi)\frac{\partial f_m}{\partial\xi}$$

$$+ \frac{n(n-1)}{2l^2}(\xi_0-\xi)^2\frac{\partial^2 f_m}{\partial\xi^2} + O(l^{-3}). \quad (95)$$

Hence, following a similar path as in the previous section, the dispersive term in the equation for A_{lm} is

$$\sum_n \Lambda_n^m i^n \frac{\partial^n A_{l-n,m}}{\partial\xi^n}$$

$$\sim \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l+4)}{(\xi_0-\xi)^{l+4}}\left(f_m\beta^2(m+\kappa)\right.$$

$$+ \left(\frac{1}{l}-\frac{3}{l^2}\right)(\xi_0-\xi)\frac{\partial f_m}{\partial\xi}\kappa\frac{d}{d\kappa}(\beta^2(m+\kappa))$$

$$+ \left.\frac{1}{2l^2}(\xi_0-\xi)^2\frac{\partial^2 f_m}{\partial\xi^2}\kappa^2\frac{d^2}{d\kappa^2}(\beta^2(m+\kappa)) + O(l^{-3})\right)$$

$$= \lambda_\kappa \frac{(i/\kappa)^l\Gamma(l+4)}{(\xi_0-\xi)^{l+4}}\left(f_m + 2\kappa(m+\kappa)\right.$$

$$\times \left(\frac{1}{l}-\frac{3}{l^2}\right)(\xi_0-\xi)\frac{\partial f_m}{\partial\xi}$$

$$+ \left.\frac{\kappa^2(1+\beta_2)}{l^2}(\xi_0-\xi)^2\frac{\partial^2 f_m}{\partial\xi^2} + O(l^{-3})\right), \quad (96)$$

where we recall that $m+\kappa = \pm 1$. Eventually we find that the left-hand side of (30) is

$$\lambda_\kappa \frac{(i/\kappa)^l\Gamma(l+4)}{(\xi_0-\xi)^{l+4}}\frac{\kappa^2(\xi_0-\xi)^2}{l^2}$$

$$\times \left[2(m+\kappa)\left(-\nu\kappa f_m - i\frac{\partial f_m}{\partial X}\right) + \beta_2\frac{\partial^2 f_m}{\partial\xi^2} + O(l^{-3})\right]. \quad (97)$$

As for the nonlinear terms of Eq. (30), the leading-order terms in the large-l limit are

$$-2\sum_{m'm''}A_{0,m'}A_{0,m''}2A_{l-2,m-m'-m''}$$

$$\sim 2\lambda_\kappa \frac{(i/\kappa)^l\Gamma(l+4)}{(\xi_0-\xi)^{l+4}}\frac{\kappa^2(\xi_0-\xi)^2}{l^2}$$

$$\times \left(A_{0,-1}^2 f_{m+2} + 2A_{0,-1}A_{0,1}f_m + A_{0,1}^2 f_{m-2}\right). \quad (98)$$

Eventually, we obtain

$$(m+\kappa)\left(\nu\kappa\beta_1 f_m + i\frac{\partial f_m}{\partial X}\right) - \frac{\beta_2}{2}\frac{\partial^2 f_m}{\partial\xi^2}$$

$$+ A_{0,1}^2 f_{m-2} + 2A_{0,-1}A_{0,1}f_m + A_{0,-1}^2 f_{m+2} = 0. \quad (99)$$

Let now f_m be given by

$$f_m = F(\xi)e^{imX/2^*}, \quad m = -\kappa \pm 1 \quad (100)$$

and be zero otherwise. Then,

$$(m+\kappa)\left(\nu\kappa - \frac{m}{2^*}\right)F - \frac{\beta_2}{2}\frac{d^2F}{d\xi^2} + 3|A_{0,1}|^2 F = 0. \quad (101)$$

The equation for F is identical for the two possible values of m because $\nu = -1/2^*$, so that

$$-\frac{F}{2^*} - \frac{\beta_2}{2}\frac{d^2F}{d\xi^2} + 3|A_{0,1}|^2 F = 0. \quad (102)$$

This is the linearized equation for the modulus of $|A_{01}|$. Therefore, we may immediately spot one solution:

$$F_1(\xi) \propto \frac{d|A_{0,1}|}{d\xi}, \quad (103)$$

and, using that particular solution to reduce the order of (102), we find a second solution:

$$F_2(\xi) \propto F_1(\xi)\int_{\xi_0}^{\xi}\frac{1}{F_1(s)^2}ds. \quad (104)$$

Another way to solve (99) is to let

$$f_m = \pm G(\xi)e^{imX/2^*}, \quad m = -\kappa \pm 1, \quad (105)$$

then we obtain

$$-\frac{G}{2^*} - \frac{\beta_2}{2}\frac{d^2G}{d\xi^2} + |A_{0,1}|^2 G = 0. \quad (106)$$

One solution is, simply,

$$G_1 \propto |A_{0,1}|. \quad (107)$$

Again, reducing the order of (106), we find a second one:

$$G_2 \propto G_1(\xi)\int_{\xi_0}^{\xi}\frac{1}{G_1(s)^2}ds. \quad (108)$$

In the case $\beta_2 < 0, 2^* = 2$, we obtain

$$F_1(\xi) = -\frac{1}{|\beta_2|}\text{sech}\left(\frac{\xi}{|\beta_2|^{1/2}}\right)\tanh\left(\frac{\xi}{|\beta_2|^{1/2}}\right), \quad (109)$$

$$F_2(\xi) = \frac{5|\beta_2|^{3/2}}{4}F_1(\xi)\left[6\frac{\xi-\xi_0}{|\beta_2|^{1/2}} + \sinh\left(\frac{2\xi}{|\beta_2|^{1/2}}\right)\right.$$

$$\left. - 4\coth\left(\frac{\xi}{|\beta_2|^{1/2}}\right)\right], \quad (110)$$

$$G_1(\xi) = \frac{-1}{|\beta_2|^{1/2}}\text{sech}\left(\frac{\xi}{|\beta_2|^{1/2}}\right), \quad (111)$$

$$G_2(\xi) = \frac{-3|\beta_2|^{3/2}}{2}G_1(\xi)\left[\frac{\xi-\xi_0}{|\beta_2|^{1/2}} + \frac{1}{2}\sinh\left(\frac{2\xi}{|\beta_2|^{1/2}}\right)\right]. \quad (112)$$

On the other hand if $\beta_2 > 0$, $2^* = 1$, we obtain

$$F_1(\xi) = \frac{1}{|\beta_2|} \text{sech}^2\left(\frac{\xi}{|\beta_2|^{1/2}}\right), \tag{113}$$

$$F_2(\xi) = \frac{5|\beta_2|^{3/2}}{4} F_1(\xi) \left[\frac{3}{2}\frac{\xi - \xi_0}{|\beta_2|^{1/2}} + \sinh\left(\frac{2\xi}{|\beta_2|^{1/2}}\right)\right.$$
$$\left. + \frac{1}{8}\sinh\left(\frac{4\xi}{|\beta_2|^{1/2}}\right)\right], \tag{114}$$

$$G_1(\xi) = \frac{-1}{|\beta_2|^{1/2}}\tanh\left(\frac{\xi}{|\beta_2|^{1/2}}\right), \tag{115}$$

$$G_2(\xi) = -3|\beta_2|\left[\frac{\xi - \xi_0}{|\beta_2|^{1/2}}\tanh\left(\frac{\xi}{|\beta_2|^{1/2}}\right) - 1\right]. \tag{116}$$

With that particular choice of integration constants, the above solutions have the following asymptotic behaviors in the vicinity of ξ_0:

$$F_1 \sim \frac{i^*}{(\xi - \xi_0)^2}, \quad F_2 \sim i^*(\xi - \xi_0)^3, \tag{117}$$

$$G_1 \sim \frac{i^*}{\xi - \xi_0}, \quad G_2 \sim i^*(\xi - \xi_0)^2. \tag{118}$$

Out of these asymptotic behaviours, only that of F_2 is compatible with (90). The other solutions connect with late terms in the vicinity of ξ_0 that correspond to smaller values of α and which are therefore subdominant in the large-l limit. Matching with (90), we thus find, away from $\xi = \xi_0$ that

$$E_l \sim \frac{\Gamma(l+4)F_2(\xi)}{(\xi_0 - \xi)^{l+4}} \sum_\kappa \sum_{m=-\kappa\pm 1} \lambda_\kappa (i/\kappa)^l e^{im(x-t+X/2^*)}. \tag{119}$$

In this sum, the contributions associated to $\kappa = \pm 2$ dominate as $l \to \infty$ and are therefore the only ones that matter to our discussion. So far, we have only been treating the singularity $\xi = \xi_0$. Additional terms arise from the singularity at $\xi = -\xi_0$. They can be deduced from the former ones by the fact that the solution must be real where x, t, and ξ are real. Eventually, we obtain

$$E_l \sim \frac{\Gamma(l+4)F_2(\xi)}{(\xi_0 - \xi)^{l+4}} \sum_\kappa \sum_{m=-\kappa\pm 1} \lambda_\kappa (i/\kappa)^l e^{im(x-t+X/2^*)}$$
$$+ \frac{\Gamma(l+4)\bar{F}_2(\xi)}{(-\xi_0 - \xi)^{l+4}} \sum_\kappa \sum_{m=-\kappa\pm 1} \lambda_\kappa (-i/\kappa)^l e^{im(x-t+X/2^*)}. \tag{120}$$

Note that the far field behavior of F_2 as $\xi \to \infty$ is

$$F_2(\xi) \sim \begin{cases} -(5/4)|\beta_2|^{1/2}e^{\xi/|\beta_2|^{1/2}} & \text{if} \quad \beta_2 < 0, \\ (5/16)|\beta_2|^{1/2}e^{2\xi/|\beta_2|^{1/2}} & \text{if} \quad \beta_2 > 0, \end{cases} \tag{121}$$

C. Location of the Stokes line

Following Dingle [24], a Stokes line occurs in the region of the complex-ξ plane where successive terms have the same phase. Here, there is a Stokes line (or ray) emanating from the singularity ξ_0. In the first line of (120) on passing from

one order of the calculation to the next, a factor $i/\kappa(\xi_0 - \xi)$ is applied. This factor must be a positive real number, say $1/\mu$. Thus the Stokes ray is given by

$$\xi = \xi_0 - i\mu/\kappa, \quad \mu \in \mathbb{R}_0^+. \tag{122}$$

The Stokes ray points in the downwards direction if $\kappa > 0$ and upwards if $\kappa < 0$. Hence, from the singularity ξ_0, only the ray with $\kappa = 2$ will cross the real axis and generate a new contribution to the solution there. Conversely, with the singularity $-\xi_0$, located below the real axis, one should only count the late terms of the series associated to $\kappa = -2$. (With a quadratic nonlinearity, the same discussion holds with the substitution $\kappa = \pm 2 \to \pm 1$.)

V. TRUNCATING THE SYSTEM

From what precedes, each new terms of the series expansion (21) gets smaller by a factor ϵ but larger by a factor $(l+4)/\kappa|\xi_0 - \xi|$. At a given distance $r = |\xi_0 - \xi|$ from the top singularity, optimal truncation thus happens at order $L - 1$, where $L = \lfloor |\kappa r|/\epsilon \rfloor$. We define

$$E^{(L-1)} = \sum_{l=0}^{L-1} \epsilon^l E_l, \tag{123}$$

so that

$$E = E^{(L-1)} + R, \tag{124}$$

and we must now derive an equation for the remainder R. An estimation of the size of R is given by $\epsilon^L E_L$, which is exponentially small in ϵ. Hence, we may safely neglect R^2 compared to R and anticipate that R satisfies a linearized version of (14) plus forcing terms:

$$\frac{\partial}{\partial x^2}R + \int_{-\infty}^{\infty} \beta^2(\omega)\hat{R}(x,\omega)e^{-i\omega t}d\omega + 2\epsilon^2 E_0^2 R$$
$$\sim \text{rhs}^+ + \text{rhs}^-, \tag{125}$$

where rhs$^\pm$ stem from the truncation of the asymptotic series (21) at order $L - 1$. More specifically, there are two contributions: rhs$^+$, associated to the singularity ξ_0 of the leading-order soliton approximation, and rhs$^-$, associated to the singularity at $-\xi_0$ in that same approximation. All of the intricate foregoing calculations precisely aimed at deriving rhs$^+$ and rhs$^-$, which is what we are about to do. But first, let us note that, by the linearity of Eq. (125), we may write R as the sum of particular solutions

$$R = R^+ + R^- \tag{126}$$

associated to rhs$^+$ and rhs$^-$, respectively. Once R^+ has been calculated, R^- can be deduced by the fact that $R^+ + R^-$ is real on the real axis. We may therefore restrict our attention to rhs$^+$, and hence, on (119) to derive it.

The various terms appearing in $E^{(L-1)}$ are such that all the $O(\epsilon^l)$ terms up to $l = L - 1$ vanish when substituting (124) in Eq. (14). Let us therefore focus on the terms of order ϵ^L and

higher. We have

$$
\frac{\partial}{\partial x^2} E^{(L-1)} \sim \ldots - 2\epsilon \frac{\partial^2}{\partial x \partial \xi} \epsilon^{L-1} E_{L-1}
$$
$$
+ \epsilon^2 \frac{\partial^2}{\partial \xi^2} (\epsilon^{L-1} E_{L-1} + \epsilon^{L-2} E_{L-2})
$$
$$
= \ldots + \sum_m e^{im(x-t)} \left[-2im\epsilon \frac{\partial}{\partial \xi} \epsilon^{L-1} A_{L-1,m} \right.
$$
$$
\left. + \epsilon^2 \frac{\partial^2}{\partial \xi^2} (\epsilon^{L-1} A_{L-1,m} + \epsilon^{L-2} A_{L-2,m}) \right], \quad (127)
$$

where we omitted to write all terms up to order L and kept only the dominant remaining ones. Similarly,

$$
\int_{-\infty}^{\infty} \beta^2(\omega) \hat{E}^{(L-1)}(x, \omega) e^{-i\omega t} d\omega
$$
$$
\sim \ldots + \sum_m e^{im(x-t)} \sum_{q \geqslant 1} \sum_{n \geqslant q} \Lambda_n^m (i\epsilon \partial/\partial \xi)^n \epsilon^{L-q} A_{L-q,m}.
$$
$$
(128)
$$

Next, the nonlinear terms are

$$
\frac{2\epsilon^2}{3} (E^{(L-1)})^3 \sim \ldots + 2\epsilon^2 E_0^2 \sum_{q=1}^{2} \epsilon^{L-q} E_{L-q}. \quad (129)
$$

Now, near optimal truncation, all terms $\epsilon^{L-q} E_{L-q}$ with $q = O(1)$ approximately have the same magnitude. Moreover, the operator $\epsilon \partial/\partial \xi$, when applied to a function of the form (92) yields a contribution that is proportional to $\epsilon(l+4)$, which is $O(1)$ when $l = O(L)$. As a result, the nonlinear terms just derived are a factor ϵ^2 smaller than those in (127) and in (128) and can be neglected in comparison. We thus have

$$
\text{rhs}^+ + \text{rhs}^- = \sum_m e^{im(x-t)} \left[2im\epsilon \frac{\partial}{\partial \xi} \epsilon^{L-1} A_{L-1,m} \right.
$$
$$
- \epsilon^2 \frac{\partial^2}{\partial \xi^2} (\epsilon^{L-1} A_{L-1,m} + \epsilon^{L-2} A_{L-2,m})
$$
$$
\left. - \sum_{q \geqslant 1} \sum_{n \geqslant q} \Lambda_n^m (i\epsilon \partial/\partial \xi)^n \epsilon^{L-q} A_{L-q,m} \right], \quad (130)
$$

Focusing on rhs^+, that is on (119), the terms with $\kappa = 2$, i.e., the harmonics $m = -1, -3$, dominate upon crossing the Stokes line, so we ignore the others. (With a quadratic nonlinearity, $\kappa = 1$, $m = -1, 0$.) With $\kappa = 2$, each successive term is identical in phase and amplitude, up to an $O(1/L)$ difference on the Stokes line: $\epsilon^{L-q} A_{L-q,m} \sim \epsilon^{L-1} A_{L-1,m}$ for all integers q of order one. rhs^+ thus simplifies as

$$
\text{rhs}^+ \sim \sum_{m=-\kappa \pm 1} e^{im(x-t)} \left[2i\epsilon \frac{\partial}{\partial \xi} \left(m + i\epsilon \frac{\partial}{\partial \xi} \right) \right.
$$
$$
\left. - \sum_{q \geqslant 1} \sum_{n \geqslant q} \Lambda_n^m (i\epsilon \partial/\partial \xi)^n \right] \epsilon^{L-1} A_{L-1,m} \quad (131)
$$

with $\kappa = 2$. Regarding the double sum above, we note the following identity:

$$
\sum_{q \geqslant 1} \sum_{n \geqslant q} a_n = \sum_{n \geqslant 1} n a_n, \quad (132)
$$

provided the right-hand side exists. Therefore,

$$
\text{rhs}^+ \sim \sum_{m=-1,-3} e^{im(x-t)} \left[2i\epsilon \frac{\partial}{\partial \xi} \left(m + i\epsilon \frac{\partial}{\partial \xi} \right) \right.
$$
$$
\left. - \sum_{n \geqslant 1} n \Lambda_n^m (i\epsilon \partial/\partial \xi)^n \right] \epsilon^{L-1} A_{L-1,m}. \quad (133)
$$

Further, and again with $O(1/L)$ accuracy, $\partial/\partial \xi \sim L/(\xi_0 - \xi)$:

$$
\text{rhs}^+ \sim \frac{i\epsilon L}{\xi_0 - \xi} \sum_{m=-1,-3} e^{im(x-t)} \left[2 \left(m + \frac{i\epsilon L}{\xi_0 - \xi} \right) \right.
$$
$$
\left. - \sum_{n \geqslant 1} n \Lambda_n^m \left(\frac{i\epsilon L}{\xi_0 - \xi} \right)^{n-1} \right] \epsilon^{L-1} A_{L-1,m}. \quad (134)
$$

A. Local behavior near the Stokes line

Berry showed on some examples that exponentially small terms are switched on not discontinuously but in an $O(\epsilon^{1/2})$-thin region comprising the Stokes line [25]. This observation has been confirmed in many instances [15,16,25–28]. Based on this knowledge, let us write

$$
\xi = \xi_0 - ir + \epsilon^{1/2} s. \quad (135)
$$

In the following, it will sometimes be convenient to write the asymptotically equivalent expression

$$
\xi_0 - \xi = ir - \epsilon^{1/2} s \sim ir e^{i\epsilon^{1/2} s/r + \frac{1}{2} \epsilon (s/r)^2}. \quad (136)
$$

In (134), we have, using (135),

$$
2 \left(m + \frac{i\epsilon L}{\xi_0 - \xi} \right) - \sum_{n \geqslant 1} n \Lambda_n^m \left(\frac{i\epsilon L}{\xi_0 - \xi} \right)^{n-1}
$$
$$
\sim 2 \left(m + \kappa - i\epsilon^{1/2} \frac{\kappa s}{r} \right) - \sum_{n \geqslant 1} n \Lambda_n^m \left(\kappa - i\epsilon^{1/2} \frac{\kappa s}{r} \right)^{n-1}
$$
$$
\sim 2 \left(m + \kappa - i\epsilon^{1/2} \frac{\kappa s}{r} \right)
$$
$$
- \sum_{n \geqslant 1} \Lambda_n^m \left(n\kappa^{n-1} - n(n-1)\kappa^{n-2} i\epsilon^{1/2} \frac{\kappa s}{r} \right)
$$
$$
= 2 \left(m + \kappa - i\epsilon^{1/2} \frac{\kappa s}{r} \right) - \frac{d}{d\kappa} \beta^2 (m + \kappa)
$$
$$
+ i\epsilon^{1/2} \frac{\kappa s}{r} \frac{d^2}{d\kappa^2} \beta^2 (m + \kappa)
$$
$$
= 2i\beta_2 \epsilon^{1/2} \frac{\kappa s}{r}. \quad (137)
$$

Next, we must evaluate $\epsilon^{L-1} A_{L-1,m}$ in the vicinity of the Stokes line. With (136) and using $\Gamma(z) \sim \sqrt{2\pi/z}(z/e)^z$, we

have, in (119), with $l = L - 1 \sim \kappa r/\epsilon$,

$$\frac{\epsilon^{L-1}\Gamma(L+3)}{(\xi_0 - \xi)^{L+3}}(i/\kappa)^{L-1}$$

$$\sim (\kappa/\epsilon)^4 \sqrt{\frac{2\pi}{L+3}} \left(\frac{i\epsilon(L+3)}{e\kappa(\xi_0 - \xi)} \right)^{L+3}$$

$$\sim (\kappa/\epsilon)^4 \sqrt{\frac{2\epsilon\pi}{\kappa r}} e^{-\kappa/\epsilon(r + i\epsilon^{1/2}s) - \frac{1}{2}\kappa s^2/r}$$

$$= (\kappa/\epsilon)^4 \sqrt{\frac{2\epsilon\pi}{\kappa r}} e^{i\kappa/\epsilon(\xi_0 - \xi) - \frac{1}{2}\kappa s^2/r}$$

$$= (\kappa/\epsilon)^4 \sqrt{\frac{2\epsilon\pi}{\kappa r}} e^{i\kappa(x - t - x_0 + X/2^*)} e^{-|\kappa\Im(\xi_0)|/\epsilon} e^{-\frac{1}{2}\kappa s^2/r}. \quad (138)$$

Hence, we obtain

$$\frac{\partial}{\partial x^2}R^+ + \int_{-\infty}^{\infty}\beta^2(\omega)\hat{R}^+(x,\omega)e^{-i\omega t}d\omega + 2\epsilon^2 E_0^2 R^+$$

$$\sim 2i\epsilon\beta_2\lambda_\kappa F_2(\xi)(\kappa/\epsilon)^4 e^{-|\kappa\Im(\xi_0)|/\epsilon}\sqrt{\frac{2\kappa\pi}{r}}e^{-i\kappa x_0}$$

$$\times \sum_m e^{i(m+\kappa)(x-t+X/2^*)}\frac{\kappa s}{r}e^{-\frac{1}{2}\kappa s^2/r}. \quad (139)$$

We thus see that the right-hand side of (139) varies very rapidly, on the s scale, becoming negligible as soon as s^2 exceeds a few times r/κ. On the ξ scale, this is an $O(\epsilon^{1/2})$-thin region. In the vicinity of the Stokes line, we let

$$R^+ \sim 4i\pi\lambda_\kappa F_2(\xi)(\kappa/\epsilon)^4 e^{-|\kappa\Im(\xi_0)|/\epsilon}e^{-i\kappa x_0}$$

$$\times \sum_m e^{i(m+\kappa)(x-t+X/2^*)}S_m(r,s) \quad (140)$$

with r fixed and s related to x and t through (136) and (23). Substitution in (139) yields

$$-(m+\kappa)^2 S_m - 2i(m+\kappa)\epsilon^{1/2}\frac{\partial S_m}{\partial s} + \epsilon\frac{\partial^2 S_m}{\partial s^2}$$

$$+ \beta^2(m+\kappa + i\epsilon^{1/2}\partial/\partial s)S_m$$

$$\sim -\epsilon\beta_2\frac{\partial^2 S_m}{\partial s^2} \sim \epsilon\beta_2\sqrt{\frac{\kappa}{2\pi r}}\frac{\kappa s}{r}e^{-\frac{1}{2}\kappa s^2/r}. \quad (141)$$

The solution of this last equation is

$$S_m(r,s) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\kappa/(2r)}s}e^{-u^2}du. \quad (142)$$

With this solution, $R^+ \to 0$ as $s \to -\infty$, and

$$R^+ \to 8i\pi\lambda_\kappa F_2(\xi)\left(\frac{\kappa}{\epsilon}\right)^4 e^{-|\kappa\Im(\xi_0)|/\epsilon}$$

$$\times e^{-i\kappa x_0}\cos(x - t + X/2^*) \quad (143)$$

as $s \to \infty$. Hence, upon crossing the Stokes line that joins ξ_0 and $-\xi_0$, one obtains the contribution

$$R^+ + R^- \sim 16\pi\kappa^4\lambda_\kappa\epsilon^{-4}e^{-\kappa\Im(\xi_0)/\epsilon}$$

$$\times \Re(iF_2(\xi)e^{-i\kappa x_0})\cos(x - t + X/2^*) \quad (144)$$

to the remainder R.

VI. SOLITON EQUATION OF MOTION

Let us first consider the bright soliton case. Adding the leading-order expression of the slowly accelerating soliton, see Eq. (51), and the remainder, Eq. (144), we finally obtain, as $\xi \to \infty$ and with c exponentially small

$$E \sim \cos(x - t + X/2)e^{\xi/|\beta_2|^{1/2}}$$

$$\times \left(-\frac{dc}{dX} - 20\pi\kappa^4|\beta_2|^{1/2}\lambda_\kappa\epsilon^{-4}e^{-\kappa\Im(\xi_0)/\epsilon}\sin\kappa x_0 \right), \quad (145)$$

where we have used the large-ξ asymptotic expressions (54) for R_a and (121) for F_2, and where $\zeta \sim \xi/|\beta_2|^{1/2}$ on account of the exponential smallness of c. In order to prevent unacceptable exponential divergence, we must set the factor between parentheses to zero. This is the sought-after result, since $\dot{x}_0 = \epsilon v_p c$ and $\dot{c} = \epsilon^2\beta_0 v_p\frac{dc}{dX}$. Eventually, in the original variables, and reverting to ε as the small parameter, we obtain

$$\ddot{x}_0 + \eta\frac{v_l^2}{4}\kappa\beta_0\sin(\kappa\beta_0 x_0) = 0 \quad (146)$$

with

$$v_l^2 = 80\pi\kappa^3|\lambda_\kappa|\varepsilon^{-1}e^{-\kappa\pi/2\varepsilon}v_p^2, \qquad \eta = \text{sign}(\lambda_\kappa). \quad (147)$$

Second, we consider dark solitons ($\beta_2 > 0$). Here, as $\xi \to \infty$,

$$E \sim 0.25\cos(x - t + X)e^{2\xi/|\beta_2|^{1/2}}$$

$$\times \left(\frac{dc}{dX} + 20\pi\kappa^4|\beta_2|^{1/2}\lambda_\kappa\epsilon^{-4}e^{-\kappa\Im(\xi_0)/\epsilon}\sin\kappa x_0 \right), \quad (148)$$

where we used the far-field expressions (57) and (121) for R_a and F_2, respectively. We thus obtain exactly the same equation of motion as with the bright soliton.

VII. CONCLUSIONS AND PERSPECTIVES

The law of propagation of wave packets at the group velocity is one of the most fundamental in physics, given its simplicity and scope of application. This paper brings an exception to the rule, in the case where phase and group velocities are very close. In an exponentially small range of parameter, the nonlinear Schrödinger equation becomes invalid as far as soliton dynamics is concerned. There, the motion of bright and dark solitons is locally equivalent to that of a pendulum. To obtain this rather simple looking result, it was necessary to carry out a calculation beyond all orders of the classical multiple scales expansion that underlies the NLSE. We have treated a general class of wave equations that describes many physical situations, which gives confidence in the generality of our findings. What is required is a weak nonlinearity and the existence of an extremum of the phase velocity. The weak nonlinearity, which exists in almost any classical physical system, leads to the existence

of harmonics of the fundamental wave. These harmonics interact very weakly with the fundamental one to produce the pinning force. The higher the separation between harmonics, the weaker the interaction. This explains why quadratic nonlinearities lead to stronger pinning forces than cubic ones, for the latter only lead to odd harmonics of the fundamental signal.

The ultimate result of the present beyond-all-orders calculation, Eqs. (145) and (148), correspond to the smallest possible absolute value of κ, i.e., to the closest nonlinear harmonics of the fundamental ones. The other harmonics, associated to larger absolute values of κ, contributes in principle to the equation of motion too, even though they are exponentially smaller than first term. This suggests that the set of all these contributions would form a trans series.

It would be desirable to numerically demonstrate the dynamics described here but this is a challenging task. One should indeed simulate the complete wave model Eq. (14), including both the fast oscillations of the carrier wave and their slow envelope at the same time, since the dynamics is precisely governed by the interplay between these widely separated spatiotemporal scales. Moreover, one should let the field evolve over distances or durations that are exponentially larger than the elementary oscillations, since the characteristic speed of the soliton center of mass relative to the carrier wave is given by the exponentially small v_l in the region of interest. This requires an efficient and stable numerical algorithm. An attempt was made in the course of this research but it was unsuccessful: our numerical code with explicit time stepping was prone to numerical instabilities as soon as the nonlinearity was present. As a result, soliton pulse evolution could be monitored for only about 40 time units and ε was constrained to be no larger than 0.1. This was not enough to confirm the phase portrait of Fig. 1, because the acceleration and deceleration of the soliton takes place over an $O(1/v_l)$ characteristic time. This is a similar situation to the one encountered with localized patterns pushed by distant boundaries [17,29], where the slow dynamics requires numerical integration over tens of thousands or even 10^6 of unit time. In the absence of a proper numerical investigation, let us stress that a partial numerical confirmation of Eq. (8) already exists: its stationary solutions and their stability are consistent with the numerical results obtained in the particular case of Refs. [8,12].

Beside challenging the conventional view of wave propagation at the fundamental level, the results of this paper may find their way to application. One example is soliton-based optical frequency combs [30,31]. In this frame, the phase of wave packets (light pulses) must be tightly controlled, and it is precisely that phase that is governed by Eq. (8). In the same vein, nonlinear optical pulses can travel over very long distances compared to their width in optical fibers with low attenuation. Over such distances, exponentially small effects, if present, may have time to qualitatively affect the propagation of wave packets.

APPENDIX A: DERIVATION OF THE WAVE EQ. (14)

1. Electromagnetic waves

In a nonmagnetic medium, the electric and magnetic fields, respectively, $\mathcal{E}$ and $\mathcal{H}$ are governed by [32],

$$\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \frac{\partial \mathcal{P}_{nl}}{\partial t}, \nabla \times \mathcal{E} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t}, \quad (A1)$$

where the displacement field is separated into a contribution $\mathcal{D}$ that is a linear functional of $\mathcal{E}$, and a nonlinear polarization term $\mathcal{P}_{nl}$. By cross differentiation and using the fact that the fields are divergence free, we may rewrite the above two equations as

$$\nabla^2 \mathcal{E} - \mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathcal{P}_{nl}}{\partial t^2}. \quad (A2)$$

Let us now particularize the equation to an electric field with a single component in the y direction, and an input amplitude E_0: $\mathcal{E} = E_0 E(x,t)\hat{\mathbf{y}}$. Then, with

$$-\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2} = \int_{-\infty}^{\infty} \beta^2(\omega)\hat{\mathcal{E}}(x,\omega)e^{-i\omega t}d\omega \quad (A3)$$

and $\mathcal{P}_{nl} = \varepsilon_0 \chi^{(3)}|\mathcal{E}|^2\mathcal{E}$ (see, for instance, Refs. [23,33]), we get

$$\frac{\partial^2 E}{\partial x^2} + \int_{-\infty}^{\infty} \beta^2(\omega)\hat{E}(x,\omega)e^{-i\omega t}d\omega = \frac{\chi^{(3)}|E_0|^2}{c^2}\frac{\partial^2}{\partial t^2}(E^3). \quad (A4)$$

Hence, we recover (14) and the nonlinearity (19) with

$$\epsilon^2 = \frac{3\chi^{(3)}|E_0|^2\omega_0^2}{2\beta(\omega_0)^2c^2} \ll 1. \quad (A5)$$

If, instead of free space, one considers wave propagation in a waveguide, then we may approximately write $\mathcal{E} = E(x,t)\mathbf{\Phi}(\mathbf{r}_\perp)$, where $\mathbf{\Phi}(\mathbf{r}_\perp)$ is the vector distribution of a waveguide mode, which depends on the transverse coordinates $\mathbf{r}_\perp$ (see, e.g., Ref. [34]). Then Eq. (A3) can again be used, provided that $\beta(\omega)$ is the dispersion function of the waveguide and not that in free space. In general $\mathbf{\Phi}(\mathbf{r}_\perp)$ also depends on frequency but for a wave packet, this dependence can safely be neglected.

2. Elastic waves

Let us next consider waves propagating along an elastic beam. There, the vertical displacement $w(x,t)$ satisfies [35]

$$\rho \frac{\partial^2 w}{\partial t^2} - T\frac{\partial^2 w}{\partial x^2} + B\frac{\partial^4 w}{\partial x^4} = f, \quad (A6)$$

where T is the tension in the beam, B its bending stiffness, and f is a distributed force, which may depend nonlinearly on w. Let us assume a nonlinear restoring force $f = -f_1 w - f_2 w^2$. Then by way of using the spatial Fourier transform of w,

$$\hat{w}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(x,t)e^{-ikx}dx, \quad (A7)$$

the beam equation can be rewritten as

$$\frac{\partial^2 w}{\partial t^2} + \int_{-\infty}^{\infty} \omega^2(k)\hat{w}(k,t)e^{ikx}dk + f_2 w^2/\rho = 0, \quad (A8)$$

with $\omega^2(k) = (Tk^2 + Bk^4 + f_1)/\rho$. This is of the same form as (14), except for the exchange of coordinate x and t and the replacement $\beta(\omega) \to \omega(k)$. A minimum of v_p is obtained at the wave number $k_0 = (f_1/B)^{1/4}$. Note in particular that if the constants B, T, f_1, f_2, and ρ are chosen such that the equation reads $\partial^2 w/\partial t^2 + \frac{1}{2}(w + \partial^4 w/\partial x^4) + \sqrt{9/38}\,\epsilon w^2 = 0$, then the critical wave and angular frequency are both unity and the NLSE reads $i\partial A/\partial T + \partial^2 A/\partial \xi^2 + |A|^2 A = 0$, where $T = \epsilon^2 t$.

3. Plasma waves

Finally, let us consider the model for cold plasmas used by Taniuti *et al.* [4]

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \tag{A9}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + n^{-1}\frac{\partial}{\partial x}\left(\frac{1}{2}|\mathcal{B}|^2\right) = 0, \tag{A10}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\mathcal{V} + iR_e^{-1}n^{-1}\frac{\partial \mathcal{B}}{\partial x}\right) - n^{-1}\frac{\partial \mathcal{B}}{\partial x} = 0, \tag{A11}$$

$$\frac{\partial}{\partial t}\left(\mathcal{B} - iR_i^{-1}\frac{\partial \mathcal{V}}{\partial x}\right) - \frac{\partial \mathcal{V}}{\partial x} = -\frac{\partial}{\partial x}\left[u\left(\mathcal{B} - iR_i^{-1}\frac{\partial \mathcal{V}}{\partial x}\right)\right], \tag{A12}$$

$$\mathcal{V} = v - iw, \quad \mathcal{B} = B_y - iB_z, \tag{A13}$$

where n is the density, (u, v, w) is the cartesian velocity field, B_y and B_z are the transverse components of the magnetic field, with a constant applied magnetic field in the x direction. Finally, R_e and R_i are normalized cyclotron frequencies for the electrons and ions, respectively. Assuming $n = 1 - \epsilon \hat{n}$ and $u = \epsilon \hat{u}$, Eqs. (A11) and (A12) read

$$\frac{\partial}{\partial t}\left(\mathcal{V} + iR_e^{-1}\frac{\partial \mathcal{B}}{\partial x}\right) - \frac{\partial \mathcal{B}}{\partial x} = \epsilon f_1(\hat{u}, \hat{n}, \mathcal{B}, \mathcal{V}), \tag{A14}$$

$$\frac{\partial}{\partial t}\left(\mathcal{B} - iR_i^{-1}\frac{\partial \mathcal{V}}{\partial x}\right) - \frac{\partial \mathcal{V}}{\partial x} = \epsilon f_2(\hat{u}, \hat{n}, \mathcal{B}, \mathcal{V}). \tag{A15}$$

Defining $\mathcal{F} = \mathcal{V} + iR_e^{-1}\frac{\partial \mathcal{B}}{\partial x}$ and $\mathcal{G} = \mathcal{B} - iR_i^{-1}\frac{\partial \mathcal{V}}{\partial x}$, we have, using (A14) and (A15), $\mathcal{V} \sim \mathcal{F} - iR_e\frac{\partial \mathcal{F}}{\partial t}$ and $\mathcal{B} \sim \mathcal{G} + iR_i^{-1}\frac{\partial \mathcal{G}}{\partial t}$ in the small-ϵ limit. Furthermore, let us introduce a new function $\mathcal{H}$ as

$$\mathcal{H} = \left(1 + iR_i^{-1}\frac{\partial}{\partial t}\right)\left(1 - iR_e^{-1}\frac{\partial}{\partial t}\right)\mathcal{F}. \tag{A16}$$

Then Eqs. (A14) and (A15) can be combined to yield

$$\frac{\partial^2 \mathcal{H}}{\partial x^2} + \int_{-\infty}^{\infty} \frac{\omega^2 \hat{\mathcal{H}}(x, \omega)e^{-i\omega t}}{\left(1 - iR_i^{-1}\omega\right)\left(1 + iR_e^{-1}\omega\right)}d\omega = \epsilon N, \tag{A17}$$

where N is a nonlinear functional of $\mathcal{H}$.

APPENDIX B: DERIVATION OF THE FUNCTIONS $g_q(l)$

In this section, we derive the expressions of the functions $g_q(l)$ appearing in (69). Omitting the label m, we have

$$\sum_{q=0}^{n} \frac{n!g_q(l)}{(n-q)!} = \frac{l!\Gamma(l + \alpha - n)(i/\kappa)^{n-l}B_{l-n}}{(l-n)!\Gamma(l+\alpha)} \tag{B1}$$

where

$$(i/\kappa)^{n-l}B_{l-n} \sim b^{(0)} + (l-n)^{-1}b^{(1)} + (l-n)^{-2}b^{(2)} + \cdots \tag{B2}$$

Evaluating (B1) with $n = 0$, we directly obtain

$$g_0(l) = (i/\kappa)^{-l}B_l \sim b^{(0)} + l^{-1}b^{(1)} + l^{-2}b^{(2)} + \cdots \tag{B3}$$

For $n \geqslant 1$, the $g_n(l)$ are obtained by recurrence:

$$g_n(l) = \frac{l!\Gamma(l + \alpha - n)(i/\kappa)^{n-l}B_{l-n}}{n!(l-n)!\Gamma(l+\alpha)} - \sum_{q=0}^{n-1}\frac{g_q(l)}{(n-q)!}, \tag{B4}$$

which, upon expansion in the large-l limit yields (70)–(72).

[1] D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys. **46**, 133 (1967).

[2] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Self-focusing and diffraction of light in a nonlinear medium, Sov. Phys. Uspekhi **10**, 609 (1968).

[3] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).

[4] T. Taniuti and H. Washimi, Self-Trapping and Instability of Hydromagnetic Waves Along the Magnetic Field in a Cold Plasma, Phys. Rev. Lett. **21**, 209 (1968).

[5] M. J. Ablowitz, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons* (Cambridge University Press, Cambridge, 2011).

[6] J. M. Dudley, G. Genty, and S. Coen, Supercontinuum generation in photonic crystal fiber, Rev. Mod. Phys. **78**, 1135 (2006).

[7] G. G. Stokes, On the discontinuity of arbitrary constants which appear in divergent developments, Trans. Cambridge Philos. Soc. **10**, 105 (1864).

[8] T. S. Yang and T. R. Akylas, On asymmetric gravity-capillary solitary waves, J. Fluid Mech. **330**, 215 (1997).

[9] T. S. Yang, On traveling-wave solutions of the kuramoto-sivashinski equation, Physica D **110**, 25 (1997).

[10] M. K. Wadee and A. P. Bassom, Effects of exponentially small terms in the perturbation approach to localized buckling, Proc. R. Soc. Lond. A **455**, 2351 (1999).

[11] M. Wadee and A. P. Bassom, Characterization of limiting homoclinic behaviour in a one-dimensional elastic buckling model, J. Mech. Phys. Solids **48**, 2297 (2000).

[12] D. C. Calvo, T.-S. Yang, and T. R. Akylas, On the stability of solitary waves with decaying oscillatory tails, Proc. R. Soc. Lond. A **456**, 469 (2000).

[13] M. K. Wadee, C. D. Coman, and A. P. Bassom, Solitary wave interaction phenomena in a strut buckling model incorporating restabilisation, Physica D **163**, 26 (2002).

[14] K. L. Adams, J. R. King, and R. H. Tew, Beyond-all-order effects in multiple-scales asymptotics: Travelling-wave solutions

to the Kuramoto-Sivashinsky equation, J. Eng. Math. **45**, 197 (2003).

[15] G. Kozyreff and S. J. Chapman, Asymptotics of Large Bound States of Localized Structures, Phys. Rev. Lett. **97**, 044502 (2006).

[16] S. J. Chapman and G. Kozyreff, Exponential asymptotics of localised patterns and snaking bifurcation diagrams, Physica D **238**, 319 (2009).

[17] G. Kozyreff, P. Assemat, and S. J. Chapman, Influence of Boundaries on Localized Patterns, Phys. Rev. Lett. **103**, 164501 (2009).

[18] A. D. Dean, P. C. Matthews, S. M. Cox, and J. R. King, Exponential asymptotics of homoclinic snaking, Nonlinearity **24**, 3323 (2011).

[19] M. A. Wadee and L. Gardner, Cellular buckling from mode interaction in i-beams under uniform bending, Proc. R. Soc. Lond. A. **468**, 245 (2012).

[20] G. Kozyreff and S. J. Chapman, Analytical Results for Front Pinning between an Hexagonal Pattern and a Uniform State in Pattern-Formation Systems, Phys. Rev. Lett. **111**, 054501 (2013).

[21] H. de Witt, Beyond all order asymptotics for homoclinic snaking in a Schnakenberg system, Nonlinearity **32**, 2667 (2019).

[22] J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).

[23] A. C. Newell and J. V. Moloney, *Nonlinear Optics* (Addison-Wesley, Boston, 1992).

[24] R. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic Press Inc., New York, 1973).

[25] M. V. Berry, Uniform asymptotic smoothing of stokes's discontinuities, Proc. R. Soc. Lond. A **422**, 7 (1988).

[26] S. J. Chapman, J. R. King, and K. L. Adams, Exponential asymptotics and stokes lines in nonlinear ordinary differential equations, Proc. R. Soc. Lond. A **454**, 2733 (1998).

[27] S. J. Chapman and J.-M. Vanden-Broeck, Exponential asymptotics and gravity waves, J. Fluid Mech. **567**, 299 (2006).

[28] N. Joshi, C. J. Lustri, and S. Luu, Stokes phenomena in discrete Painlevé II, Proc. Roy. Soc. Lond. A **473**, 20160539 (2017).

[29] G. Kozyreff and L. Gelens, Cavity solitons and localized patterns in a finite-size optical cavity, Phys. Rev. A **84**, 023819 (2011).

[30] T. Hansson, D. Modotto, and S. Wabnitz, Dynamics of the modulational instability in microresonator frequency combs, Phys. Rev. A **88**, 023819 (2013).

[31] Y. K. Chembo and C. R. Menyuk, Spatiotemporal Lugiato-Lefever formalism for Kerr-comb generation in whispering-gallery-mode resonators, Phys. Rev. A **87**, 053852 (2013).

[32] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999), p. 429.

[33] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, New York, 1989).

[34] A. W. Snyder and J. D. Love, *Optical Waveguide Theory* (Springer, New York, 1984).

[35] P. Howell, G. Kozyreff, and J. Ockendon, *Applied Solid Mechanics* (Cambridge University Press, Cambridge, 2009).