Circularly distributed multipliers with deterministic moduli assessing the stability of quasiperiodic response

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The stability and bifurcation of a periodic solution of a dynamical system can be handled well by using the Floquet multipliers of the perturbed system with periodic coefficients. However, for a quasiperiodic (QP) response as a natural extension of a periodic one, it is much more difficult to do it quantitatively. Therefore, proposed here is an approach for defining and obtaining effective multipliers for QP stability. The proposed approach is based on a series of auxiliary variables via which the perturbed system with QP coefficients is transformed approximately into a constant one, whereupon the multipliers are obtained efficiently by performing eigenvalue analysis on the constant coefficient matrix. The major finding involves circularly distributed multipliers with deterministic moduli, with the QP response being stable if all the moduli are less than or equal to unity; otherwise it is unstable. When the QP response degenerates to periodic due to the reducibility of fundamental frequencies, the proposed approach exactly provides the Floquet multipliers for the periodic solution. From this respect, the obtained multipliers can be considered to some extent as being a generalization for QP response of the Floquet multipliers for a periodic solution.

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I. INTRODUCTION

Quasiperiodic (QP) responses in nonlinear dynamical systems have been investigated extensively across the sciences and engineering [1–5]. It has been known for decades that QP solutions are closely associated with the classical Kolmogorov-Arnold-Moser theory for such systems [6]. In many cases, QP responses arise from periodic ones via Neimark-Sacker [7] or Ruelle-Takens bifurcations [8], among others, and quasiperiodicity has been reported extensively as being responsible for certain routes to chaos [9–11]. From these perspectives, QP responses play a pivotal role in the dynamic evolution of nonlinear systems.

For continuous-time systems, a QP solution is characterized mathematically by two or more incommensurate fundamental frequencies, and hence is considered to be a natural extension of a periodic solution with a single fundamental frequency. Although it is much more difficult to solve for QP solutions than periodic ones, many powerful techniques for doing so have been established during the past decade. Actually, recent years have witnessed an increasing amount of research on calculating QP responses arising in either continuous systems [12–15] or dissipative mappings [16].

One fundamental issue arises inevitably regarding the stability and further bifurcation of a QP response. According to Lyapunov stability theory, the stability of the solution can be characterized by a perturbed system obtained by introducing a small perturbation to the solution itself [17]. Because the perturbed system is linear and governed by QP coefficients, under certain circumstances it can be reduced to an equivalent one with a constant matrix by the Lyapunov-Perron transformation [18,19], and the eigenvalues of this constant matrix offer necessary information about the stability of the considered QP solution. Much research has been devoted to finding this transformation quantitatively [20,21]; it is governed inherently by a square matrix with each component being a QP function, so for a system of dimension K, one must solve for $K \times K$ QP functions.

The Floquet theory is used widely for analyzing the stability of periodic motion [17,22]. In this theory, the so-called monodromy matrix is computed by integrating the perturbed system over one period T. However, a major problem is that a QP solution has no closed period, so this technique cannot be implemented straightforwardly. For this issue, researchers have suggested modifications to enable the Floquet theory to be applicable to QP responses. Guskov and Thouverez [23] defined an alternative matrix similar to monodromy in multidimensional representation and provided an approximation for this matrix via interpolation, while Sharma and Sinha [24] replaced the original system by a periodic one with an appropriate and sufficiently large principal period.

Generally, the stability of a QP solution can be assessed intuitively according to the evolution of the perturbation acting on the considered solution. Liao *et al.* [25] constructed a generalized eigenvalue problem and used it to determine the stability of a QP response. Suarez *et al.* [26] expanded the perturbation as a generalized harmonic series and solved the eigenvalue problem governing the harmonic coefficients. Guennoun *et al.* [27] simplified the perturbed system using the multiple-scales method, and predicted the stability of a QP response according to the fixed-point stability of the simplified system.

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Herein, we propose a simple yet efficient approach based on a series of auxiliary variables, through which the perturbed system with QP coefficients is converted approximately to a constant one. The multipliers are then defined according to the eigenvalues of the constant coefficient matrix, and test examples show that the multipliers are distributed circularly with deterministic moduli assessing the stability of the QP response under consideration.

II. AUXILIARY CONSTANT SYSTEM

Consider the following general nonlinear dynamical system that possibly exhibits a QP response

$$\dot{X} = f(X, t), \tag{1}$$

where X is a K-dimensional state variable whose superscript dash denotes differentiation with respect to time t. To describe a QP solution either analytically or semianalytically, we use the following truncated generalized Fourier expansion with two incommensurate frequencies (ω_1 and ω_2):

$$X_q = \sum_{m=-M}^{M} \sum_{n=-N}^{N} C_{m,n} \exp(i(m\omega_1 + n\omega_2)t), \tag{2}$$

where i is the imaginary unit and $C_{m,n}$ is a column vector of dimension K, representing the coefficients of the combined harmonics of the kth variable in X_q. Introducing a perturbation Y into the QP solution X_q and substituting $X_q + Y$ into Eq. (1), we obtain the perturbed system by neglecting higher powers Y^n ($n \geqslant 2$)

$$\dot{Y} = A(t)Y, \tag{3}$$

where the time-varying coefficient $A(t) = \frac{\partial f(X_q,t)}{\partial X}$ is a matrix of dimension $K \times K$ with each component a QP function in t.

The local stability of the QP response can be characterized via that of the perturbed system, being predicted from the evolution of the perturbation. Recall that for a periodic response, i.e., when X_q is periodic rather than QP, the time-independent monodromy matrix can be constructed by integrating the perturbed system numerically during one principal period, with the Floquet multipliers obtained as the eigenvalues of the monodromy matrix. However, for a QP response, there is no complete period, and so this strategy cannot be implemented directly. Therefore, herein we propose an effective approach to finding indicators for the stability of the QP response.

Given the QP solution X_q, the coefficient matrix can be rewritten as the generalized Fourier series

$$A(t) = \sum_{m=-M_1}^{M_1} \sum_{n=-N_1}^{N_1} A_{m,n} \exp(i(m\omega_1 + n\omega_2)t), \tag{4}$$

in which each $A_{m,n}$ is a constant matrix dependent upon the attained solution X_q. The numbers (M_1 and N_1) of combined harmonics in $A(t)$ depend on both those of X_q (M and N) and the nonlinear term in $f(X, t)$; for example, if the nonlinear term is a cubic polynomial in X, then we have $M_1 = 3M$ and $N_1 = 3N$.

The key procedure is introducing the auxiliary variables

$$Y_{p,q} = \exp(i(p\omega_1 + q\omega_2)t)Y, \tag{5}$$

where $Y_{0,0}$ is the exact perturbation Y. Differentiating $Y_{p,q}$ with respect to t gives

$$\begin{aligned}\dot{Y}_{p,q} &= [i(p\omega_1 + q\omega_2)Y + \dot{Y}] \exp(i(p\omega_1 + q\omega_2)t) \\ &= i(p\omega_1 + q\omega_2)Y_{p,q} + \exp(i(p\omega_1 + q\omega_2)t)A(t)Y \\ &= i(p\omega_1 + q\omega_2)Y_{p,q} + \sum_{m=-M_1}^{M_1} \sum_{n=-N_1}^{N_1} A_{m,n} Y_{m+p,n+q}\end{aligned} \tag{6}$$

according to which the perturbed system of Eq. (3) with time-varying coefficients can be transformed into an infinite-dimensional system. However, given the difficulty in dealing with an infinite-dimensional system, we truncate is to a finite-dimensional system with lower order combined harmonics as

$$\dot{Z} = BZ, \tag{7}$$

where the auxiliary state is $Z=[Y_{-P,-Q}, \ldots, Y_{-P,Q}, Y_{-P+1,-Q}, \ldots, Y_{-P+1,Q}, \ldots, Y_{P,-Q}, \ldots, Y_{P,Q}]^{\mathrm{T}}$ and the superscript T denotes the transpose. The integers P and Q signify that the truncated harmonics range from $\exp(i(-P\omega_1 - Q\omega_2)t)$ to $\exp(i(P\omega_1 + Q\omega_2)t)$. In Sec. III B, the auxiliary system will be shown to be effective in approximating the perturbed system accurately. Note that the vector of auxiliary variables has dimension $K(2P + 1)(2Q + 1)$, where K is the dimension of Y. Importantly, the auxiliary system is governed totally by the constant matrix B, which can be determined from each $A_{m,n}$ in $A(t)$ of the perturbed system.

III. CIRCULARLY DISTRIBUTED MULTIPLIERS

A. Definition of multipliers

Given the initial values $Y(0)$ and $Z(0)$, the time histories of the perturbation and auxiliary states, respectively, can be obtained via time-marching integration such as the Runge-Kutta (RK) scheme. Then, based on these time histories, the stability can be determined in a qualitative and intuitive manner. Because the auxiliary system is governed by the constant coefficient matrix B, its stability can be determined straightforwardly and exactly from the eigenvalues of B. For this, the closed-form solution of the auxiliary system is given by a linear combination of basis functions as

$$Z(t) = \sum_{n} \alpha_n V_n \exp(\eta_n t), \tag{8}$$

where V_n is the eigenvector of B corresponding to the complex eigenvalue η_n and each α_n is a coefficient vector calculated by matching the initial condition $Z(0)$. In this way, the perturbation state $Y(t) = Y_{0,0}$ can be extracted from $Z(t)$.

According to Eq. (8), the evolution (divergence or convergence) of the perturbation state $Z(t)$ depends on $\exp(\eta_n t)$ but neither α_n nor V_n, because the latter are fixed vectors if the initial condition is given. Consequently, the moduli $|\exp(\eta_n t)|$ play the dominant role regarding the perturbation state. Denoting each pair of conjugate complex eigenvalues as $\eta_n = \eta_n^r \pm i\eta_n^i$, we obtain $|\exp(\eta_n t)| = \exp(\eta_n^r t)$ as the rate of growth (or decay) during any time range $[0, t]$. Therefore, $|\exp(\eta_n t)|$ can be taken as stability indicators, i.e., a modulus greater (resp. less) than unity implies that the corresponding basis function causes the perturbation response to grow (resp. decay). Note that the assessment result is independent of the chosen time range because given any positive t, it holds uniformly that $\exp(\eta_n^r t) > 1$ if $\eta_n^r > 0$; $\exp(\eta_n^r t) = 1$ if $\eta_n^r = 0$; and $\exp(\eta_n^r t) < 1$ if $\eta_n^r < 0$.

In Floquet theory, the Floquet multipliers for periodic solutions are introduced by depicting the rate of growth or decay of the perturbation over one fundamental period. However, for QP responses, a complete period is unavailable, so for convenience we introduce $\lambda_n = \exp(\eta_n 2\pi/\omega_1)$ to obtain an average rate of change over a fixed time range of duration $2\pi/\omega_1$ (or $2\pi/\omega_2$ without loss of generality). Moreover, in the final example, to check the applicability of the proposed method to periodic responses, the time range is fixed to be of duration $2\pi/\omega_{12}$, where ω_{12} is the greatest common divisor of ω_1 and ω_2; note that the QP response degenerates to a periodic one if ω_1 and ω_2 are reducible; i.e., they have a common divisor.

Based on the above discussion, we now suggest the simple criterion that the QP response is stable as long as the moduli do not exceed unity, or unstable otherwise. This criterion is similar to the counterpart for periodic responses in the Floquet theory [17], and so for convenience we continue to refer to λ_n as multipliers, but now for QP responses.

B. Validation of auxiliary system

Before computing the multipliers, it is necessary to verify the effectiveness of the auxiliary system. To do so, we consider the following nonlinear system for a resonant circuit with saturable inductors [12]

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 x_2 - \frac{1}{8}(x_1^2 + 3x_3^2)x_1 + 0.22\cos\omega_1 t \\ \dot{x}_3 = -\frac{0.05}{8}(3x_1^2 + x_3^2)x_3 + 0.03 \end{cases}. \tag{9}$$

The evolution of its QP response has been investigated intensively, showing that a series of bifurcations may appear as k_1 is decreased from 0.12 to approximately 0.5. At each bifurcation, the difference halves between the two fundamental frequencies, so it is known as a torus-doubling bifurcation [12]. It is similar to the common period-doubling bifurcation for periodic responses, where the single fundamental frequency halves and the principal period doubles.

The QP solutions are obtained by the incremental harmonic balance (IHB) method [14] with $M = N = 7$. To achieve a relatively high precision with a reasonable computational burden, we select $P = Q = 3M = 3N$ to deduce the truncated auxiliary system, and the initial conditions are $Y(0) = [0 \quad 0 \quad 0]^{\mathrm{T}}$. Figure 1(a) shows the numerical results for the perturbed system as obtained by the Runge-Kutta method, and these are compared with the solution of the auxiliary system provided by Eq. (8). There is very good agreement in the early stage for $t \leqslant 100$, and then the solutions deviate gradually, with the first observable difference appearing at $t \approx 2000$.

Figure 2(a) shows all the multipliers distributed circularly in a ring structure without any deterministic moduli. For the high-dimensional auxiliary state Z, only the perturbation state Y of dimension K is directly related to the stability of the considered QP response. The coefficient α_n in Eq. (8) represents the contribution to Y of the corresponding basis function $V_n \exp(i\eta_n t)$, and Fig. 2(b) shows that the moduli of most of the coefficients are very much smaller than those of a few others, e.g., those with $|\alpha_n| \geqslant 10^{-6}$ or somewhat lower.

Therefore, as a test example, we truncate the basis functions to those with $|\alpha_n| \geqslant 10^{-6}$ to compute the perturbation response according to Eq. (8). As Fig. 1(b) shows, the solutions now remain together for much longer, indicating that the retained basis functions do indeed provide the majority of the solution. Moreover, it is interesting and surprising that the truncated basis functions give more accurate results than the linear combination of all basis functions. The likely

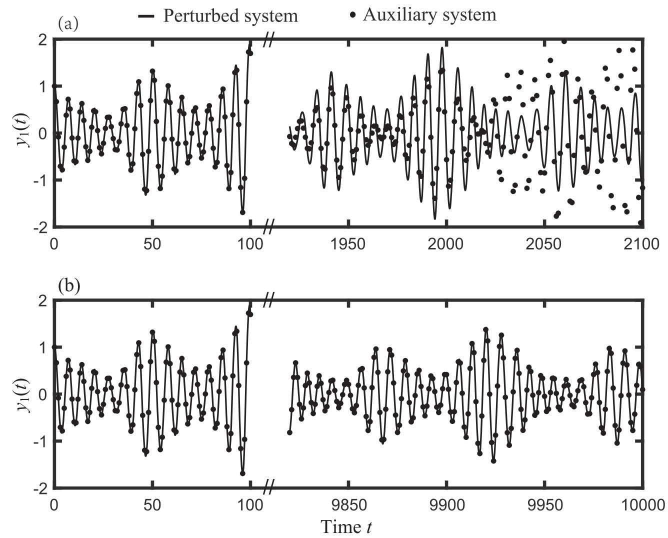

FIG. 1. Time histories of perturbation state Y provided by perturbed and auxiliary systems with $k_1 = 0.08$ and using (a) all basis functions or (b) only those with $|\alpha_n| \geqslant 10^{-6}$.

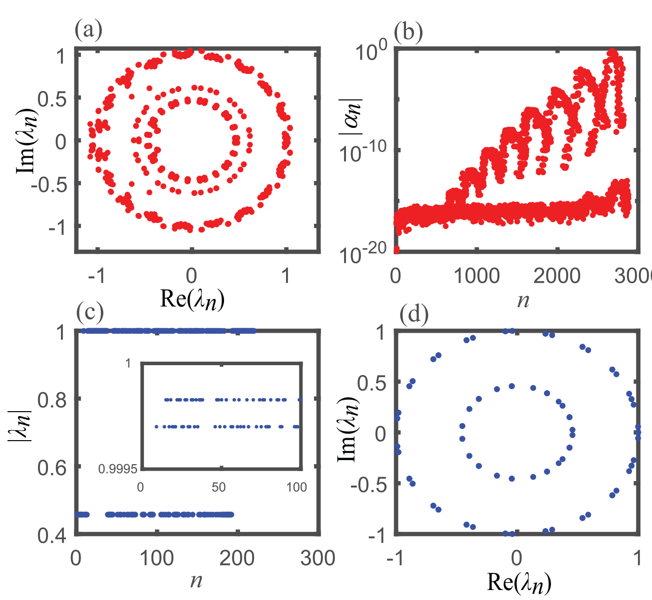

FIG. 2. (a) All multipliers (λ_n) provided by auxiliary system (7) for QP response of system (9) with $k_1 = 0.080$; (b) moduli of coefficients (α_n) vs n; (c) moduli of multipliers truncated with $|\alpha_n| \geqslant 10^{-6}$ vs n; (d) retained multipliers in complex plane.

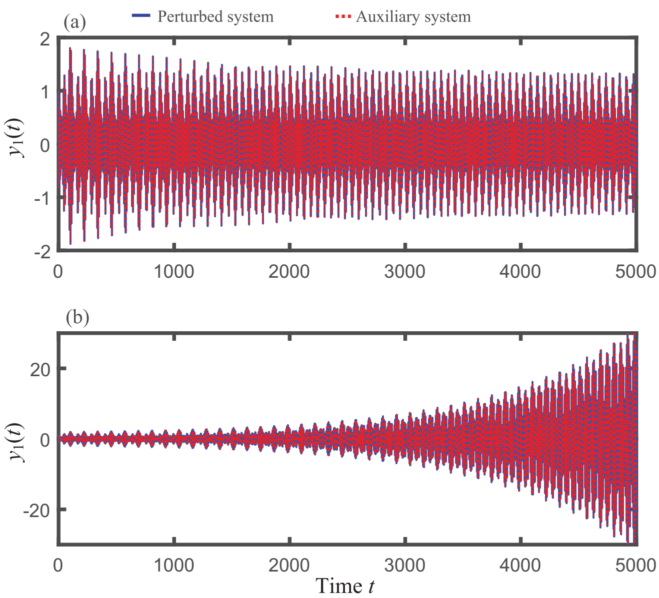

FIG. 3. Time histories of perturbation state Y provided by perturbed and auxiliary systems with (a) $k_1 = 0.081$ and (b) $k_1 = 0.079$. The basis functions with $|\alpha_n| \geqslant 10^{-6}$ are retained.

reason for this is that truncating the auxiliary system results in additional basis functions containing redundant erroneous information about the perturbation state. From this perspective, it is reasonable to truncate the basis functions to those with relatively large coefficients.

The retained multipliers with $|\alpha_n| \geqslant 10^{-6}$ are distributed circularly in rings with deterministic moduli, shown in Fig. 2(d). To examine this feature further, the multipliers are presented individually in Fig. 2(c) according to their moduli. Three deterministic moduli are clearly distinguished, although two are very close to unity. Therefore, with such high sensitivity, the deterministic moduli can be used to estimate the stability of the perturbation state and hence of the QP response itself.

Another two numerical cases are provided in Fig. 3 as examples of (a) decaying and (b) growing time responses. Now the results agree well for much longer, demonstrating the effectiveness of truncating the multipliers to those with relatively large coefficients. Intuitively, the QP solution is predicted to be stable for $k_1 = 0.081$ and unstable for $k_1 = 0.079$. Section IV will present quantitative analysis based on the circularly distributed multipliers with deterministic moduli.

C. Influence of truncation of auxiliary system

As mentioned above, two different truncation procedures were made to the auxiliary system and the basis functions, respectively. The truncation of basis functions warrants that the auxiliary system can approximate the perturbed system accurately for much longer. In this subsection, we discuss how truncating the auxiliary system possibly affects the multipliers, more specifically the deterministic moduli of circularly distributed multipliers.

Note that the basis functions are truncated microscopically based somewhat on prior knowledge about an appropriate and effective threshold, e.g., $|\alpha_n| \geqslant 10^{-6}$ as above. Furthermore,

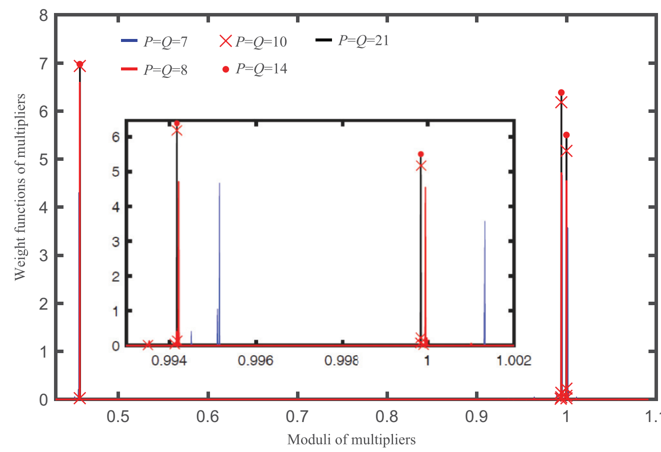

FIG. 4. Discrete weight function $W(|\lambda|)$ with $\triangle|\lambda| = 10^{-5}$ vs moduli of multipliers provided by auxiliary system (7) for system (9) with $k_1 = 0.080$.

we examine the distribution of all the multipliers at a macroscopic level without neglecting any basis functions. The key point is to define a discrete weight function of the multipliers with respect to the moduli. To do this, first the interval $[|\lambda|_{\min}, |\lambda|_{\max}]$ from the minimum to maximum modulus is discretized with an equidistance $\triangle|\lambda|$ such as $[|\lambda|_{\min}, |\lambda|_{\min} + \triangle|\lambda|, \ldots, |\lambda|_{\max} - \triangle|\lambda|, |\lambda|_{\max}]$. The discrete weight function can then be defined as

$$W(|\lambda|) = \sum_n |\alpha_n|, \quad \text{if} \quad ||\lambda_n| - |\lambda|| < \triangle|\lambda|/2, \tag{10}$$

where $|\lambda|$ takes all the discrete values over $[|\lambda|_{\min}, |\lambda|_{\max}]$ ergodically. In this manner, the non-negative function $W(|\lambda|)$ quantitatively represents the contribution to the perturbation response $Z(t)$ of the basis functions with moduli at the neighboring of λ as $(\lambda - \triangle|\lambda|/2,\ \lambda + \triangle|\lambda|/2)$.

The discrete weight function is technically threefold, involving (i) separating the modulus interval into a series of intervals, (ii) classifying each multiplier into one interval, and finally (iii) accumulating the contributions of the basis functions in each interval to the perturbation response. With the same truncation, i.e., $P = Q = 3M = 21$, Fig. 4 shows that the weight function has three peak values, with all the others being much smaller without observable magnitudes, and these peak values are reached exactly at the deterministic moduli shown in Fig. 2(c).

The weight function not only shows the distribution of the moduli macroscopically, but also makes it convenient to discuss how the system truncation influences the deterministic moduli. With $P = Q = M = 7$, there are small differences in the locations of the deterministic moduli, and there is a redundant modulus having an observable magnitude. When $P = Q = 8$, the differences in the locations of the deterministic moduli decrease significantly, meanwhile the observable redundant modulus disappears. If P and Q are increased further to 10, then the deterministic moduli can be located very accurately, although small differences still remain in the values of the weight function. With $P = Q = 2M = 14$, the differences in the values of the weight function are eliminated at the deterministic moduli. With $P = Q = 2M$ and $P = Q = 3M$,

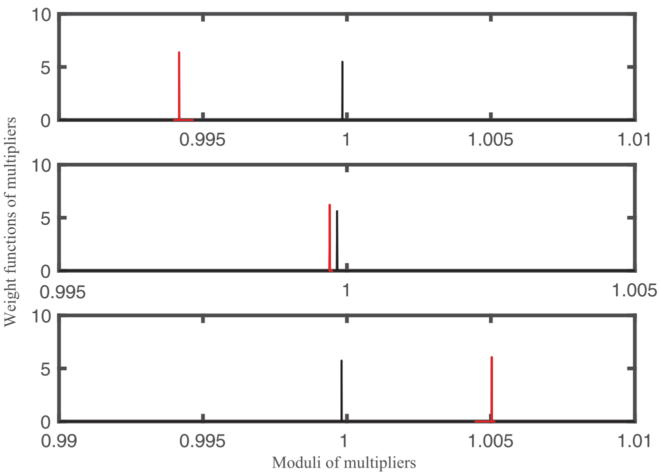

FIG. 5. Discrete weight function vs moduli close to unity for system (9) with (a) $k_1 = 0.081$, (b) $k_1 = 0.080$, and (c) $k_1 = 0.079$.

there are no observable differences regarding the distribution of the weight function between the two cases.

Based on the above results, the proposed method provides the multipliers accurately with deterministic moduli if the auxiliary system is truncated with relatively few combined harmonics, e.g., about two or three times those of the QP solution itself. To this end, P and Q are selected to be $3M$ throughout this study unless specified otherwise.

IV. STABILITY ANALYSIS BASED ON MULTIPLIERS

To discuss the implication of the multipliers for QP stability, we consider other cases close to each other with neighboring values of k_1. There are three deterministic moduli in each case, and the two close to unity are presented in Fig. 5 via the weight function. Interestingly, as k_1 is decreased very slightly, one modulus (black) remains close to unity while another (red) increases beyond unity. This regular modulus at unity for the multipliers of a QP response is similar to the universal Floquet multiplier at unity for a periodic solution of a self-excited system [28]. As is known, for a self-excited system, the frequency of a periodic solution is undetermined *a priori*. Consequently, for a QP response, it is reasonable to conjecture that there is a series of multipliers with unit modulus, with at least one of the fundamental frequencies undetermined *a priori*. Another example without this feature is discussed below.

An even more important issue is how to depict the stability of the QP response based on the obtained multipliers. According to the Floquet theory for periodic solutions, we recall that the solution is stable if and only if there is no multiplier with modulus greater than unity. As shown in Fig. 5, as k_1 is varied slightly from 0.081 to 0.079, one of the multipliers (red) has modulus greater than unity, so we predict that the QP response loses stability as k_1 crosses such a narrow range. More specifically, as Table I shows the second modulus remains the same and the third varies through unity as k_1 increases to 0.0799.

The phase planes in Fig. 6 show no observable difference between the cases with $k_1 = 0.081$ and 0.0799. For $k_1 = 0.081$, the Poincare section is one simple closed orbit with a single lap, whereas for $k_1 = 0.0799$ it is a more complex closed orbit with two laps. For $k_1 = 0.081$, the spectrum obtained by fast Fourier transform (FFT) has two independent peak frequencies, i.e., $\omega_1 = 1$ from the external force and $\omega_2 = 0.8942$ as a self-excited frequency, while for $k_1 = 0.0799$ there is the additional frequency of $0.9467 \approx (\omega_1 + \omega_2)/2$. Based on the observable differences in the Poincare sections and FFT spectra, we reason that the QP response undergoes a bifurcation in this narrow parametric range. In this bifurcation, the existing stable QP solution loses stability and another stable one appears, and it is similar to the period doubling of a periodic solution in that the frequency gap suddenly halves. The multipliers distributed on cycles provide deterministic moduli, according to which the stability of QP responses can indeed be assessed accurately and efficiently.

TABLE I. Deterministic moduli vs k_1 for QP responses of system (9).

k_1	1st modulus	2nd modulus	3rd modulus
0.0810	0.4569	0.9998	0.9942
0.0805	0.4571	0.9998	0.9970
0.0800	0.4573	0.9998	0.9997
0.0799	0.4574	0.9998	1.0002
0.0795	0.4576	0.9998	1.0024
0.0790	0.4579	0.9998	1.0050

To examine the proposed approach further, we consider the Duffing oscillator subjected to dual harmonic excitations with incommensurate frequencies, i.e.,

$$\ddot{x} + 0.1\dot{x} + x + x^3 = 5\cos\omega_1 t + 0.5\cos\omega_2 t \qquad (11)$$

with $\omega_2 = \sqrt{5}/2$ and ω_1 as the control parameter. Figure 7(a) shows the QP solutions obtained semianalytically using the IHB method plus arc-length continuation. As ω_1 is increased,

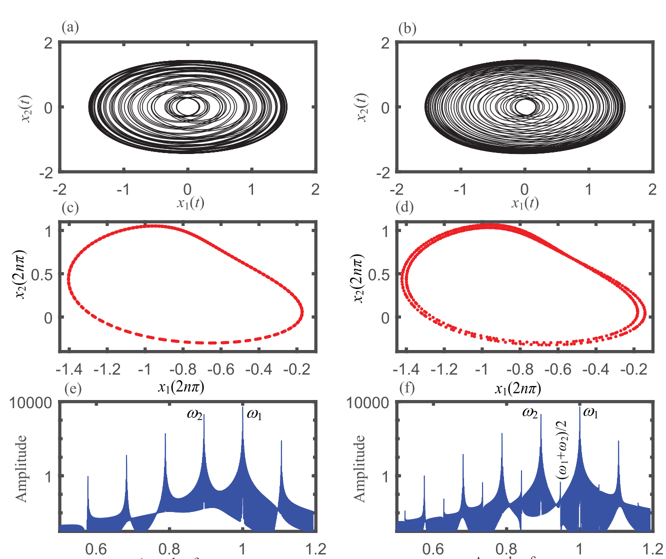

FIG. 6. The phase plane of the QP responses obtained by the RK method for system (1) with $k_1 = 0.081$ (a), the Poincare sections (c), and the FFT spectrum (e); and their counterparts for $k_1 = 0.0799$ are provided in panels (b), (d), and (f), respectively.

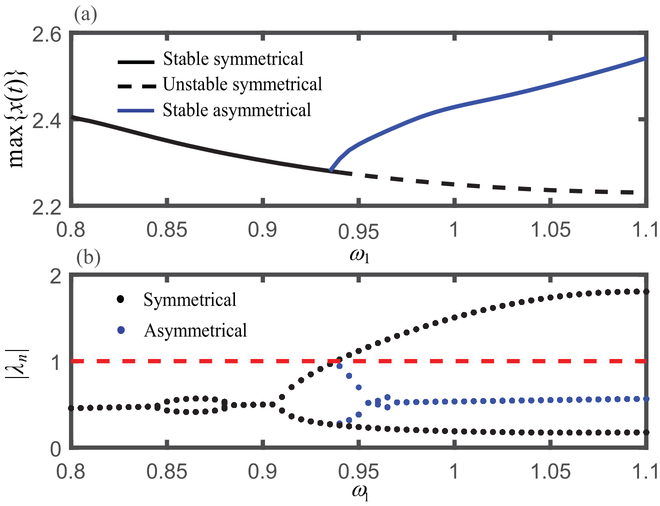

FIG. 7. (a) Amplitudes of QP responses of Duffing system vs ω_1; (b) deterministic moduli provided by weight function.

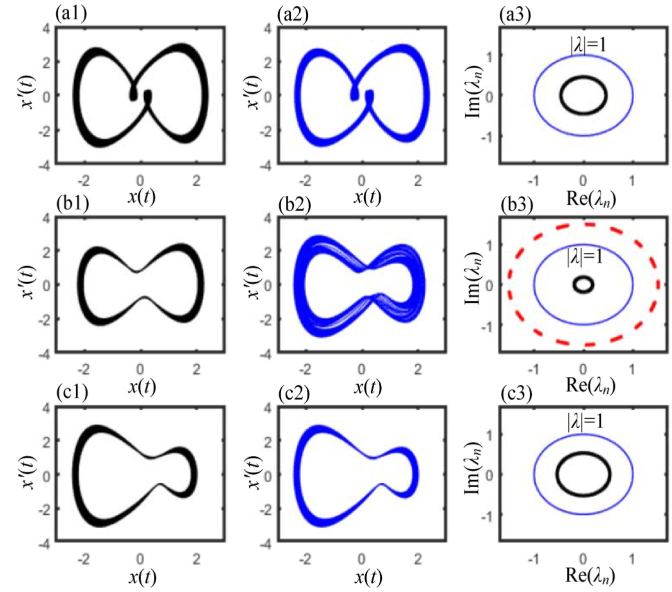

FIG. 8. (a) QP responses of Duffing system for $\omega_1 = 0.8$ (top row) and $\omega_1 = 1.0$ (middle and bottom rows) obtained by [(a1)–(c1)] IHB method and [(a2)–(c2)] RK method. [(a3)–(c3)] Moduli of corresponding multipliers provided by weight function in comparison with the unit cycle (i.e., $|\lambda| = 1$).

a branch of asymmetric QP solutions arises from the symmetric ones near $\omega_1 = 0.935$, at which a symmetry-breaking bifurcation occurs. Meanwhile, Fig. 7(b) shows the moduli of the corresponding multipliers, showing that the leading modulus exceeds unity at the bifurcation point. At the same time, the asymmetric QP solution has multipliers for which the leading modulus decreases from unity. According to the criterion for assessing stability, the symmetric QP response loses stability because of the symmetry breaking, whereupon the stable asymmetric solution arises.

Recall that the QP response of system (9) has multipliers with a universal unit modulus regardless of the control parameter, whereas that is not the case for the considered Duffing system, in which both fundamental frequencies comes from the external forces. The latter case is similar to there being no universal unit multiplier for the forced periodic vibration of a nonlinear dynamical system [28].

The assessment result is verified by the phase planes in Fig. 8 of the QP responses obtained using the IHB and RK methods. The RK method was implemented by selecting the IHB solution at time $t = 0$ as the initial condition. In Figs. 8(a1) and 8(a2) for $\omega_1 = 0.8$ and Figs. 8(c1) and 8(c2) for $\omega_1 = 1$, the modulus of each multiplier is less than unity, and the IHB results are fully consistent with the RK ones, which implies that the QP responses can be considered intuitively as being stable. By contrast, in Figs. 8(b2) and 8(b2) for $\omega_1 = 1$, one of the moduli exceeds unity, and the RK result switches from the original symmetric orbit to a new asymmetric one. The symmetric solution loses stability due to the symmetry-breaking bifurcation, accompanied with the onset of the stable asymmetric one.

Finally, we discuss the relationship between the multipliers provided by the proposed method and the Floquet multipliers of periodic responses by the monodromy matrix. To do this simply, we continue to use the Duffing system (11), but we modify its frequencies slightly to be reducible so that QP responses degenerate in essence to periodic ones. While the frequencies are still irreducible, Fig. 9(a) shows that the multipliers are distributed continuously on a circle with deterministic moduli. By contrast, when we change the frequencies slightly to $\omega_1 = 0.8$ and $\omega_2 = 1$, Fig. 9(b) shows that the multipliers gather at several discrete points.

In the case of reducible frequencies, the perturbed system has a periodic coefficient matrix, and the obtained multipliers are associated with the Floquet multipliers for the corresponding periodic response. In such cases, the frequency of the

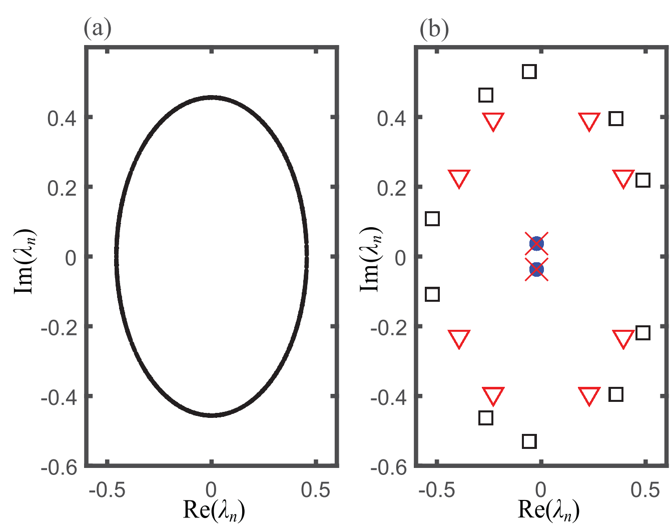

FIG. 9. Multipliers with deterministic moduli provided by weight function for Duffing system with (a) irreducible ($\omega_1 = 0.8$, $\omega_2 = \sqrt{5}/2$) or (b) reducible ($\omega_1 = 0.8$, $\omega_2 = 1$) frequencies. Red triangles, $\lambda_n = \exp(\eta_n 2\pi/\omega_1)$; black squares, $\lambda_n = \exp(\eta_n 2\pi/\omega_2)$; blue cycles, $\lambda_n = \exp(\eta_n 2\pi/\omega_{12})$ with $\omega_{12} = 0.2$ as common divisor of $\omega_1 = 0.8$ and $\omega_2 = 1$. Red crosses, Floquet multipliers of corresponding periodic response.

periodic response is the common divisor ω_{12} of ω_1 and ω_2. The number of discrete points is ω_1/ω_{12} times 2 (that is the number of Floquet multipliers) when the multipliers are defined as $\lambda_n = \exp(\eta_n 2\pi/\omega_1)$ and ω_2/ω_{12} times 2 when they are defined as $\lambda_n = \exp(\eta_n 2\pi/\omega_2)$. Interestingly, when the multipliers are given as $\lambda_n = \exp(\eta_n 2\pi/\omega_{12})$, the obtained ones are exactly the same as those given by the Floquet theory. From this, the proposed method can provide multipliers for both periodic and QP solutions.

As discussed above, for periodic responses the number of independent multipliers attained by the proposed approach is related with the definition of multiplier, whereas neither for that of the deterministic moduli. For QP responses, the number of multipliers is dependent upon not only the definition of multiplier but also the truncated number (i.e., P and Q) of variables in the auxiliary system. For any frequency ω we have that $|\lambda_n| = |\exp(\eta_n 2\pi/\omega)| = \exp(\eta_n^r 2\pi/\omega)$. Though the modulus are in relation with ω, the number of deterministic moduli is regardless of the definition. Moreover, the definition does not violate the stability criterion because whether the modulus is larger (equal or smaller) than unity is dependent upon the sign of η_n^r.

V. CONCLUSION

Presented herein was a simple yet efficient approach to defining and calculating multipliers for analyzing the stability of QP responses of nonlinear dynamical systems. In general, this approach is based on the perturbed system that depicts the stability of the QP response. Key to this is a series of auxiliary variables, according to which the perturbed system with QP coefficients can be transformed into an auxiliary one governed by a constant matrix. The multipliers can be obtained efficiently for the QP responses, with the definition based on the eigenvalues of the constant matrix.

The main finding was that the predominant multipliers are distributed circularly on cycles with deterministic moduli, which more importantly indicate the stability of the QP response. The suggested stability criterion was validated by numerical examples, i.e., that the QP response is stable if and only if no moduli exceed unity for the circularly distributed multipliers. Furthermore, the circularly distributed multipliers degenerate to Floquet multipliers when the QP response is tuned to be periodic. For this reason, the obtained multipliers for QP responses can be considered to some extent as being a logical extension of Floquet multipliers.

Numerical examples showed that the multipliers given by the proposed approach are indeed effective quantitative indicators for analyzing the stability of QP responses. However, some open problems still require further investigation, such as (i) how to depict bifurcations that are usually accompanied by stability reversals and (ii) general rules for truncating to different numbers of auxiliary variables. Also interesting would be a possible relationship between the circularly distributed multipliers and the Lyapunov-Perron transformation, which would shed light on the reducibility of QP systems.

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