Entropic form emergent from superstatistics

Maike A. F. dos Santos,[1,2,*] Fernando D. Nobre,[2,3,†] and Evaldo M. F. Curado[2,3,‡]
[1]*Department of Physics, PUC-Rio, Rua Marquês de São Vicente, 225, 22451-900, Rio de Janeiro, Brazil*
[2]*Centro Brasileiro de Pesquisas Físicas Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, RJ Brazil*
[3]*National Institute of Science and Technology for Complex Systems Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, RJ Brazil*

(Received 19 September 2022; accepted 23 December 2022; published 25 January 2023)

The Beck-Cohen superstatistics became an important theory in the scenario of complex systems because it generates distributions representing regions of a nonequilibrium system, characterized by different temperatures $T \equiv \beta^{-1}$, leading to a probability distribution $f(\beta)$. In superstatistics, some classes have been most frequently considered for $f(\beta)$, like χ^2, χ^2 inverse, and log-normal ones. Herein we investigate the superstatistics resulting from a χ^2_η distribution through a modification of the usual χ^2 by introducing a real index η $(0 < \eta \leqslant 1)$. In this way, one covers two common and relevant distributions as particular cases, proportional to the q-exponential $(e_q^{-\beta x} = [1-(1-q)\beta x]^{\frac{1}{1-q}})$ and the stretched exponential $(e^{-(\beta x)^\eta})$. Furthermore, an associated generalized entropic form is found. Since these two particular-case distributions have been frequently found in the literature, we expect that the present results should be applicable to a wide range of classes of complex systems.

DOI: 10.1103/PhysRevE.107.014132

I. INTRODUCTION

Although Boltzmann-Gibbs (BG) statistical mechanics [1,2] represents one of the most successful theories in physics, its applicability should be restricted to ergodic systems, which are usually characterized by weak and/or short-range interactions, as well as short-time memories, leading to additivity of extensive thermodynamic quantities. Lately, complex systems [3–6] have emerged as an interesting field of research and many of them violate some of these restrictions, being adequately described through proposals that differ from BG statistical mechanics. In this scenario, many tools of BG statistical physics have been adapted, or generalized, and one should mention those emerging from generalized entropies and their associated probability distributions, which produced powerful techniques, covering a wide range of applications, particularly those of complex systems. As examples, there are classes of out-of-equilibrium systems (e.g., complex fluids in a living cell) that are composed of microenvironments in their own equilibrium condition, each of them presenting its own temperature, $T \equiv \beta^{-1}$, leading to a probability distribution $f(\beta)$; one theory that addresses such systems is Beck-Cohen superstatistics [4].

Superstatistics appeared as a proposal for dealing with complex environments that consist of parts characterized by different BG statistics so the $f(\beta)$ distribution carries information associated with variations on the intensive β parameter. For this reason, in their superstatistics proposal, Beck *et al.* referred to a "statistics of a statistics" [7], and defined an effective Boltzmann factor

$$\mathcal{B}(E) = \int_0^\infty d\beta f(\beta) e^{-\beta E}, \tag{1}$$

where E corresponds to the energy associated with a given microstate. One notices that for an equilibrium situation corresponding to an intensive β parameter that does not fluctuate, given by a single value β_0, i.e., $f(\beta) = \delta(\beta - \beta_0)$, one recovers Boltzmann weight. In more complex situations, the $f(\beta)$ distribution may admit continuous distributions that depend on the system under analysis.

Beck-Cohen superstatistics considers different universality classes associated with the $f(\beta)$ distribution, from which one should mention the χ^2, χ^2 inverse, and log-normal classes. These distributions have been relevant for describing several complex systems, like air pollution statistics [8], wind velocity fluctuations [9], frequency fluctuations in power grids [10], and anomalous non-Gaussian diffusion processes [11,12]. Interestingly, the χ^2 superstatistics yields the weight characteristic of Tsallis statistics, connected with S_q entropy [13], whereas the χ^2 inverse and log-normal distributions do not present, up to now, explicit entropic functionals. In this framework, several proposals appeared for finding entropic functionals associated with given energy probability distributions, considering the standard internal-energy definition [5,6,14,15] or generalized definitions [16], leading to the possibility of determining entropic forms associated with different classes of superstatistics.

Out of the scope of Beck-Cohen superstatistics, many entropic forms have been proposed, motivated by phenomenological approaches, and some of them present a high potential for applicability in complex systems [17–21]. Among those, one should mention (in chronological order) the proposals by Rény (1961) [22], Sharma and Mittal (1975) [23], Tsallis (1988) [13], Abe (1997) [24], Landsberg-Vedral (1998) [25],

*Corresponding author: santosmaikeaf@gmail.com
†fdnobre@cbpf.br
‡evaldo@cbpf.br

Curado (1999, 2004) [26,27], Anteneodo and Plastino (1999) [28], Kaniadakis (2001) [29], Hanel and Thurner (2011) [17], and Tsallis and Cirto (2013) [30]. In general, these entropic functionals are not additive (except for Rény entropy) but can be asymptotically additive, depending on the system considered, and present equilibrium distributions characterized by long tails, which have been important characteristics for approaching complex systems. These entropic forms have been applied in a wide range of fields of knowledge, like complex networks [31,32], economics [33–35], ecology [36,37], anomalous diffusion [38–40], and cognitive sciences [41,42].

Hence, generalized entropies (through their corresponding equilibrium distributions), as well as Beck-Cohen superstatistics, represent possible approaches for dealing with complex systems. Apart from the case of χ^2 superstatistics, which was shown to be related to Tsallis statistics, in many cases it may become a hard task to identify the entropic form associated with a given class of superstatistics, e.g., those associated with the χ^2 inverse and log-normal distributions, for which one does not know, up to now, explicit forms of their entropic functionals. In the present paper, we propose a class to be called hereafter χ^2_η superstatistics, corresponding to a modification of the usual χ^2 distribution, by introducing a real index η ($0 < \eta \leqslant 1$). We succeed in identifying an associated entropic form, which recovers well-known cases in two limits, namely, $\eta \to 1$, where we obtain the χ^2 particular case (Tsallis entropy), as well as another relevant limit characterized by a stretched exponential weight, related with Anteneodo-Plastino entropic form [28].

In the next section, we review briefly some basic results of Beck-Cohen superstatistics; In Sec. III, we introduce the Mittag-Leffler function, from which the χ^2_η distribution is obtained. Therefore, we construct a χ^2_η superstatistics that implies on a q-stretched-exponential factor, i.e., $\mathcal{B}(E) = \exp_q[-(\beta_0 E)^\eta]$ (to be defined later on), whereas an associated two-index entropic form is found in Sec. IV. Finally, in Sec. V we present our conclusions.

II. BRIEF REVIEW OF SUPERSTATISTICS

In this section, we introduce some basic concepts of superstatistics, which is an appropriate tool for approaching systems that present significant fluctuations in a given intensive parameter, e.g., inverse temperature, chemical potential, and diffusivity. A nonhomogeneous environment, such as within a living cell, is an example of a system in which diffusivity fluctuates [11,43,44] and may be considered a complex environment containing a collection of diffusivities, each one associated with a given part of the system. Another example is an out-of-equilibrium system at a given microstate with a total energy E, composed of environments, each of them at their own equilibrium condition, presenting a temperature, $T \equiv \beta^{-1}$, leading to a BG statistical weight $e^{-\beta E}$. By introducing a probability distribution $f(\beta)$, superstatistics defines a generalized weight $\mathcal{B}(E)$ following Eq. (1). In this framework, a probability distribution may be defined,

$$p(E) = \frac{\mathcal{B}(E)}{\mathcal{Z}}, \tag{2}$$

where

$$\mathcal{Z} = \int dE\, \mathcal{B}(E) = \int dE \int_0^\infty d\beta f(\beta) e^{-\beta E} \tag{3}$$

represents the partition function [see also Ref. [45] for a different, but qualitatively equivalent, definition of the distribution $p(E)$].

Usually, one assigns to each different $f(\beta)$ a distinct class of superstatistics, so when building a given superstatistics class [4], both distributions $f(\beta)$ and $p(E)$ must be normalized, as follows directly from the equations above. Moreover, in a general case, one may have a density of states $g(E)$ associated to a state with energy E, so assuming an energy spectrum defined by positive energies ($0 \leqslant E < \infty$), Eqs. (2) and (3) become, respectively,

$$p(E) = \frac{\mathcal{B}(E) g(E)}{\mathcal{Z}}, \tag{4}$$

restricted to $\int_0^\infty dE\, \mathcal{B}(E) g(E)$ being finite, so

$$\mathcal{Z} = \int_0^\infty dE\, \mathcal{B}(E) g(E) = \int_0^\infty dE\, g(E) \int_0^\infty d\beta f(\beta) e^{-\beta E}. \tag{5}$$

By inverting the order of integrations in the equation above, one has that

$$\mathcal{Z} = \int_0^\infty d\beta f(\beta) \int_0^\infty dE\, g(E) e^{-\beta E} = \int_0^\infty d\beta f(\beta) Z(\beta), \tag{6}$$

where

$$Z(\beta) = \int_0^\infty dE\, g(E) e^{-\beta E} \tag{7}$$

represents the usual partition function for a fixed value of β. A direct interpretation of Eq. (6) yields that the partition function $\mathcal{Z}$ of superstatistics represents an average of the standard partition function $Z(\beta)$ over the distribution of temperatures $f(\beta)$.

Furthermore, if one requires that the BG factor should be recovered, the distribution $f(\beta)$ should approach a delta function in some special limit. As typical examples, one should mention three different distributions $f(\beta)$ that are commonly used in superstatistics; by considering the simple case $g(E) = 1$, the corresponding partition function of Eq. (3) can be calculated in each case, as described below. The first one is

$$f(\beta) = \frac{1}{b\Gamma[\gamma]} \left(\frac{\beta}{b}\right)^{\gamma - 1} e^{-\frac{\beta}{b}} \quad (\chi^2 \text{ distribution}), \tag{8}$$

$$\mathcal{Z} = \frac{1}{b(\gamma - 1)}, \tag{9}$$

where $b > 0$ and $\gamma > 1$. This distribution represents the χ^2 superstatistics and has been applied in high-energy physics [46], wind power persistence [9], time series of leverage returns [47], and protein diffusion dynamics in bacteria [48].

The second distribution is known as χ^2 inverse,

$$f(\beta) = \frac{1}{b\gamma\Gamma[\gamma]}\left(\frac{\beta}{b}\right)^{-\gamma-2} e^{-\frac{b}{\beta}} \quad (\chi^2 \text{ inverse distribution}), \tag{10}$$

$$\mathcal{Z} = \frac{1+\gamma}{b}, \tag{11}$$

in which $b > 0$ and $\gamma > -2$. This distribution has been considered in investigations of air pollution by nitrogen oxides [8], diffusion controlled by size fluctuations of single molecules [49], as well as metastasis and cancer survival [50]. Finally, the third distribution is the log-normal, written as

$$f(\beta) = \frac{1}{\gamma\sqrt{2\pi}\beta}\exp\left[-\frac{\left(\log\frac{\beta}{\mu}\right)^2}{2\gamma^2}\right] \quad \text{(log-normal distribution)}, \tag{12}$$

$$\mathcal{Z} = \frac{\exp(\gamma^2/2)}{\mu}, \tag{13}$$

where γ and μ are positive parameters. The log-normal superstatistics has been especially relevant for the understanding of financial time series [51], Brownian particles in a complex environment [52], and Lagrangian acceleration statistics in turbulent flows [53].

Although all three classes of superstatistics defined above have been successful in the description of various complex systems, only the χ^2 superstatistics has been connected to an entropic form so far, namely, Tsallis entropy [4]. In fact, substituting the distribution $f(\beta)$ of Eq. (8) in Eqs. (1) and (2), one obtains

$$p(E) = \frac{(1+bE)^{-\gamma}}{\mathcal{Z}}, \tag{14}$$

which is identified with Tsallis weight for $1 < q < 2$,

$$p_q(E) = \frac{\exp_q(-\beta_0 E)}{\mathcal{Z}} \quad \left(\exp_q[x] = [1+(1-q)x]_+^{\frac{1}{1-q}} \quad (q \in \mathbb{R})\right), \tag{15}$$

where $[u]_+ = u$, for $u > 0$, zero otherwise, by considering $b\gamma = \beta_0$ and $\gamma^{-1} = q - 1$. Therefore, a relevant question arises, concerning the association of other superstatistics classes with entropic forms known in the literature. In the following sections, we focus our attention on a unique superstatistical class and its associated entropic form.

III. A UNIQUE CLASS OF SUPERSTATISTICS

We begin by defining a special function known as the three-parameter Mittag-Leffler function, which generalizes the exponential in the context of fractional calculus [54,55],

$$E^{\gamma}_{\eta,\sigma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma[\eta k+\sigma]}\frac{z^k}{k!}, \tag{16}$$

where $(\gamma)_k = \Gamma[\gamma+k]/\Gamma[\gamma]$ represents the Pochhammer symbol and $\sigma, \eta, \gamma, z \in \mathbb{C}$, with $\mathcal{R}\{\sigma\} > 0$, $\mathcal{R}\{\eta\} > 0$, $\mathcal{R}\{\gamma\} > 0$. The function above recovers the two-parameter Mittag-Leffler function [56] $E^1_{\eta,\sigma}(z)$, for $\gamma = 1$, and is reduced to one free parameter for $\sigma = 1$ and $\gamma = 1$. Moreover, it leads to the exponential form when $\eta = \sigma = \gamma = 1$, i.e., $E^1_{1,1}(z) = e^z$. Recently, applications of the Mittag-Leffler function appeared in several physical phenomena, like Lévy flights [57,58], relaxation properties of solid systems [59], nonsingular kernels in random-walk modelings [60,61], and other superstatistics [62,63].

Inspired by the Mittag-Leffler function of Eq. (16), herein we propose the following $f(\beta)$ (hereafter to be called χ^2_η distribution):

$$f(\beta) = \frac{1}{b}\left(\frac{\beta}{b}\right)^{\eta\gamma-1} E^{\gamma}_{\eta,\eta\gamma}\left[-\frac{\beta^\eta}{b^\eta}\right], \tag{17}$$

where γ and b are positive real parameters, whereas $0 < \eta \leqslant 1$. One should mention that this function presents the precise mathematical structure of the Havriliak-Negami model for describing dielectric relaxation [64]. To calculate the generalized factor of Eq. (1), one needs the Laplace transform of Eq. (17); its use in superstatistics becomes feasible through the following Laplace transform [54,55]:

$$\mathcal{L}\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(-\nu t^\eta)\} = \frac{s^{\eta\gamma-\sigma}}{(s^\eta+\nu)^\gamma} \quad (\mathcal{R}\{s\} > 0 \text{ and } |s| > |\nu|^{\frac{1}{\eta}}). \tag{18}$$

Let us now consider the limit to $\eta \to 1$ in Eq. (17). Using the definition of Eq. (16), one has

$$\begin{aligned}\lim_{\eta\to1} f(\beta) &= \sum_{k=0}^{\infty}\lim_{\eta\to1}\frac{1}{b}\left(\frac{\beta}{b}\right)^{\eta\gamma-1}\frac{(\gamma)_k}{\Gamma[\eta k+\eta\gamma]}\frac{\left(-\frac{\beta^\eta}{b^\eta}\right)^k}{k!}\\ &= \frac{1}{b\Gamma[\gamma]}\left(\frac{\beta}{b}\right)^{\gamma-1}\sum_{k=0}^{\infty}\frac{\left(-\frac{\beta}{b}\right)^k}{k!}\\ &= \frac{1}{b\Gamma[\gamma]}\left(\frac{\beta}{b}\right)^{\gamma-1}e^{-\frac{\beta}{b}},\end{aligned} \tag{19}$$

which is precisely the χ^2 distribution of Eq. (8). For this reason, Eq. (17) defines a unique superstatistics class, χ^2_η superstatistics, that includes χ^2 as a special case. Moreover, the generalized factor of Eq. (1) may be found by calculating the Laplace transform of Eq. (17) [i.e., using Eq. (18)]:

$$\mathcal{B}[E] = (1+(bE)^\eta)^{-\gamma}. \tag{20}$$

Similarly to what was done in the previous section, the corresponding partition function of Eq. (3) can also be calculated [by considering the simple case $g(E) = 1$]:

$$\mathcal{Z} = \int_0^\infty dE\,\mathcal{B}(E) = \frac{\Gamma[\gamma-1/\eta]\,\Gamma[1+1/\eta]}{b\Gamma[\gamma]} \quad (\gamma\eta > 1), \tag{21}$$

whereas it diverges for $\gamma\eta < 1$.

To recover some previous results, we consider the notation [65]

$$\gamma^{-1} = (q-1); \quad b = \beta_0(q-1)^{\frac{1}{\eta}} \quad (q > 1), \tag{22}$$

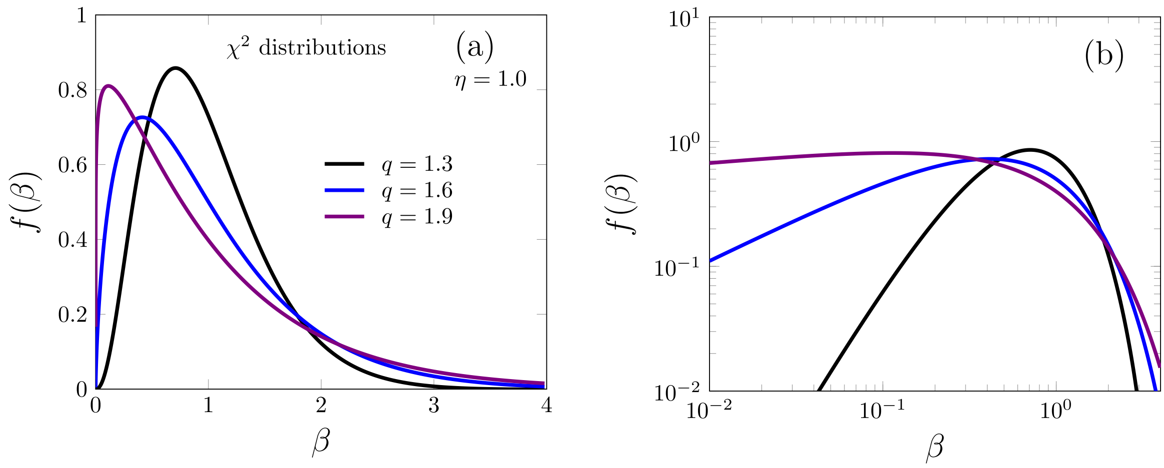

FIG. 1. The $f(\beta)$ distributions that emerge from Eq. (23) in the limit $\eta \to 1$ (i.e., χ^2 distributions) are plotted versus β in linear-linear (a) and log-log (b) representations, for $\beta_0 = 1$ and typical values of $q > 1$.

implying

$$f(\beta) = \frac{1}{\beta_0 (q-1)^{\frac{1}{q-1}}} \left(\frac{\beta}{\beta_0}\right)^{\frac{\eta}{q-1}-1} E^{\frac{1}{q-1}}_{\eta,\frac{\eta}{q-1}}\left[-\frac{\beta^\eta}{\beta_0^\eta (q-1)}\right], \tag{23}$$

so Eq. (20) becomes

$$\mathcal{B}(E) = \exp_q[-(\beta_0 E)^\eta]. \tag{24}$$

The generalized factor above is expressed as a stretched q-exponential function [see Eq. (15)], presenting two important limits as particular cases: (i) $\eta \to 1$, leading to the q-exponential function associated with Tsallis entropy and (ii) $q \to 1$, yielding a stretched exponential related to Anteneodo-Plastino entropy; in both limits, one has nonadditive entropic forms. It should be mentioned that the functional form of Eq. (24) has been observed in a wide range of properties of natural systems, like (a) velocity measurements in a turbulent Couette-Taylor flow [66]; (b) relaxation curves of RKKY spin glasses, like CuMn and AuFe [67]; (c) cumulative distribution for the magnitude of earthquakes, leading to a modification of the Gutenberg-Richter law [68]; and (d) thermal conductivity of systems of nearest-neighbor interacting XY rotators, yielding a microscopic verification of Fourier's law [69].

Moreover, the limit to $q \to 1$ in Eq. (23) leads to the convergent series proposed by Pollard (see Ref. [70] for more details),

$$\lim_{q\to 1} f(\beta) = \lim_{q\to 1} \frac{1}{\beta_0 (q-1)^{\frac{1}{q-1}}} \left(\frac{\beta}{\beta_0}\right)^{\frac{\eta}{q-1}-1} E^{\frac{1}{q-1}}_{\eta,\frac{\eta}{q-1}}\left[-\frac{\beta^\eta}{\beta_0^\eta (q-1)}\right]$$
$$= \frac{1}{\beta_0 \pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma[\eta k + 1] \sin[\pi \eta k]}{k!(\beta \beta_0^{-1})^{\eta k+1}} \quad (L_\eta \text{ distribution}), \tag{25}$$

known also as one-sided Lévy distribution.

This distribution defines a class (hereafter to be called L_η), being a particular case of the χ^2_η superstatistics in the limit to $q \to 1$, recovering the analysis of stretched exponentials from the superstatistics carried in Ref. [71].

Next we present plots of the $f(\beta)$ distributions discussed above, for typical values of their parameters; in all cases, we consider $\beta_0 = 1$. In Fig. 1, we present the χ^2 distributions obtained from Eq. (23) in the limit $\eta \to 1$, plotted versus β in both linear-linear [Fig. 1(a)] and log-log [Fig. 1(b)] representations, considering typical values of $q > 1$. One notices that by increasing q, one gets longer-tail contributions, as expected for generating q-exponential distributions through a Laplace transform. The L_η distributions resulting from Eq. (25) are exhibited versus β in Fig. 2, in both linear-linear [Fig. 2(a)] and log-log [Fig. 2(b)] representations for typical values of η in the interval $0 < \eta \leqslant 1$. As β gets larger, the corresponding L_η distributions decay faster for smaller η. In Fig. 3, we show the χ^2_η distributions [from Eq. (23)] versus β in linear-linear [Fig. 3(a)] and log-log [Fig. 3(b)] representations for $\eta = 0.5$ and typical values of $q > 1$. As β increases, the corresponding χ^2_η distributions decay faster for larger values of q. One should notice that all distributions shown in Figs. 1–3 exhibit qualitatively similar behaviors, typical of one-sided distributions, namely, presenting larger contributions for small values of β and decaying for larger values of β; such a decay varies according to the parameters q and η, leading to distinct behavior in the corresponding tails.

Hence, the present investigation deals with a unique class of superstatistics and their corresponding most relevant particular cases, given by the following $f(\beta)$ distributions:

(1) A unique class, χ^2_η superstatistics ($q \geqslant 1, 0 < \eta \leqslant 1$).

(2) χ^2 superstatistics, associated with Tsallis entropy [13], recovered in the limit $\eta \to 1$.

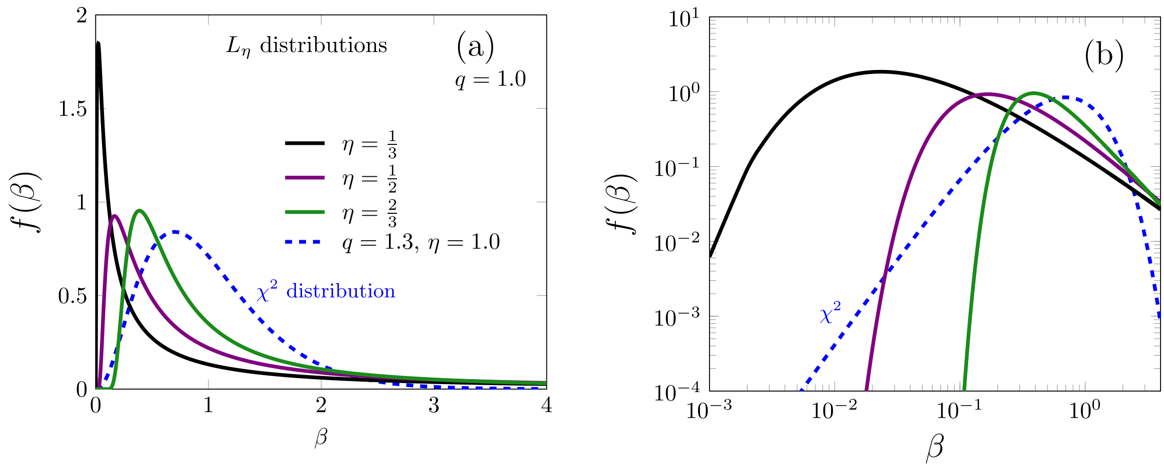

FIG. 2. The L_η distributions [cf. Eq. (25)] are plotted versus β in linear-linear (a) and log-log (b) representations for $\beta_0 = 1$ and typical values of η ($0 < \eta \leqslant 1$). For comparison, we also present a χ^2 distribution (dashed blue line) for $q = 1.3$ and $\eta = 1$

(3) L_η superstatistics, associated with Anteneodo-Plastino entropy [28], obtained in the limit $q \to 1$.

The general entropic form associated with the χ^2_η superstatistics will be found in the next section.

IV. CONSTRUCTING THE ASSOCIATED ENTROPIC FORM

The maximum entropy principle allows us to find equilibrium probability distributions for given constraints, e.g., normalization of the probability and internal-energy definition [1,3]. Less well-established, however, is the inverse procedure, namely, determining an entropic form associated with a given probability distribution; lately, several proposals appeared in the literature for such a purpose [5,6,14–16]. Next, we describe briefly how to determine the entropic form related to a continuous monotonically decreasing energy probability distribution $p(E)$ or, equivalently, with a discrete set of W probabilities, $\{p_i(E_i)\}$, associated with an energy spectrum $\{E_i\}$ (see, e.g., Ref. [15]):

$$\sum_{i=1}^{W} p_i(E_i) = 1. \tag{26}$$

The aim consists of finding an entropic form,

$$S(\{p_i\}) = k \sum_{i=1}^{W} g(p_i) \quad (g(0) = g(1) = 0), \tag{27}$$

where k represents a constant with dimensions of entropy, whereas $g(p_i)$ is a concave function of the probabilities $\{p_i\}$; moreover, from these probabilities one may define the internal energy:

$$U = \sum_{i=1}^{W} p_i E_i. \tag{28}$$

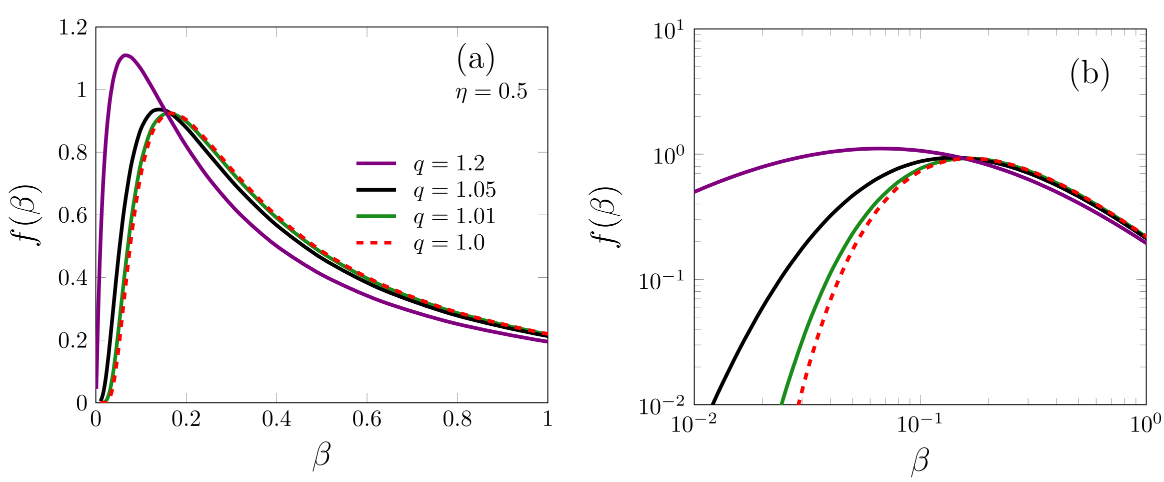

FIG. 3. The $f(\beta)$ distributions from Eq. (23) (χ^2_η distributions) are plotted versus β in linear-linear (a) and log-log (b) representations for $\beta_0 = 1$, $\eta = 0.5$, and typical values of $q > 1$. For comparison, we also present the L_η distribution (dashed red line) for $q = 1.0$ and $\eta = 0.5$.

One fundamental condition concerns the possibility of the inversion operation,

$$p_i(E_i) \quad \Rightarrow \quad E_i(p_i), \tag{29}$$

where, in the present case, the inverse function of the stretched q exponential appearing in Eq. (24), $f = e_q^{-\zeta^\eta}$, is given by

$$\zeta[f] = (-\ln_q f)^{\frac{1}{\eta}} \quad \left(\ln_q(x) = \frac{x^{1-q}-1}{1-q}\right). \tag{30}$$

The entropic functional of Eq. (27) may be expressed in terms of the above inverse function [5,15],

$$g(p_i) = \int_0^{p_i} \zeta(x)dx - a_1\, p_i, \tag{31}$$

where a_1 is a constant. Using the conditions of Eq. (27), together with the definition of Eq. (30) in Eq. (31), one obtains the following entropic form:

$$S_{q,\eta}(\{p_i\}) = k\sum_i \left\{\int_0^{p_i} [-\ln_q(x)]^{\frac{1}{\eta}} dx - p_i \int_0^1 [-\ln_q(x)]^{\frac{1}{\eta}} dx\right\}, \tag{32}$$

in which we have set $a_1 = \int_0^1 [-\ln_q(x)]^{\frac{1}{\eta}} dx$. In fact, for $q < 1+\eta$, both integrals above may be calculated analytically; the first one becomes expressed in terms of a hypergeometric function,

$$\int_0^{p_i} [-\ln_q(x)]^{\frac{1}{\eta}} dx = (q-1)^{-b_1} \frac{\Gamma[b_2]}{\Gamma[b_3]} (-\ln_q p_i)^{-b_2}\, {}_2F_1\left[b_1, b_2, b_3, \frac{b_1}{q\ln_q p_i}\right], \tag{33}$$

$$b_1 = \frac{q}{q-1}\ ; \quad b_2 = \frac{1}{q-1} - \frac{1}{\eta}\ ; \quad b_3 = \frac{q}{q-1} - \frac{1}{\eta}, \tag{34}$$

whereas the second one is given by

$$\int_0^1 [-\ln_q(x)]^{\frac{1}{\eta}} dx = \frac{\Gamma\left[1+\frac{1}{\eta}\right]\Gamma[b_2]}{(q-1)^{1/\eta}\,\Gamma\left[\frac{1}{q-1}\right]}. \tag{35}$$

The limit $\eta \to 1$ in Eq. (32) implies

$$\begin{aligned}\lim_{\eta\to 1} S_{q,\eta}(\{p_i\}) &= -k\sum_i \left\{\int_0^{p_i} \ln_q(x)dx - p_i\int_0^1 \ln_q(x)dx\right\} \\ &= -\frac{k}{2-q}\sum_i p_i \ln_q p_i,\end{aligned} \tag{36}$$

which is essentially Tsallis entropy. One should note that considering the symmetry $q \leftrightarrow 2-q$, commonly used in nonextensive statistical mechanics [3], the result above becomes

$$\lim_{\eta\to 1} S_{q,\eta}(\{p_i\}) = \frac{k}{q}\sum_i p_i \ln_q\left(\frac{1}{p_i}\right) = \frac{1}{q} S_q(\{p_i\}), \tag{37}$$

where $S_q(\{p_i\})$ represents Tsallis entropy.

Another important limit of Eq. (32) corresponds to $q \to 1$,

$$\begin{aligned}\lim_{q\to 1} S_{q,\eta}(\{p_i\}) &= k\sum_i \left\{\int_0^{p_i} [-\ln x]^{\frac{1}{\eta}} dx - p_i\int_0^1 [-\ln x]^{\frac{1}{\eta}} dx\right\} \\ &= k\sum_i \left\{\Gamma\left[\frac{1+\eta}{\eta}, -\ln p_i\right] - p_i\Gamma\left[\frac{1+\eta}{\eta}\right]\right\},\end{aligned} \tag{38}$$

where $\Gamma[a,b] = \int_b^\infty x^{a-1}e^{-x}dx$ is the incomplete Gamma function; above, one identifies precisely the Anteneodo-Plastino entropic form [28]. One should note that considering the limits $q \to 1$ in Eq. (37), or $\eta \to 1$ in Eq. (38), one recovers BG entropy.

In an alternative way, the general entropic form of Eq. (32) can also be written as

$$S_{q,\eta}(\{p_i\}) = k\sum_i \left\{\Gamma_q\left[\frac{1+\eta}{\eta}, -\ln_q p_i\right] - p_i\Gamma_q\left[\frac{1+\eta}{\eta}\right]\right\}, \tag{39}$$

where we defined a q-incomplete gamma function $\Gamma_q[a,b]$,

$$\Gamma_q[a,b] = \int_b^\infty x^{a-1}[e_q(-x)]^q dx \quad \left(\lim_{q\to 1}\Gamma_q[a,b] = \Gamma[a,b]\right), \tag{40}$$

by introducing the q-exponential function of Eq. (15).

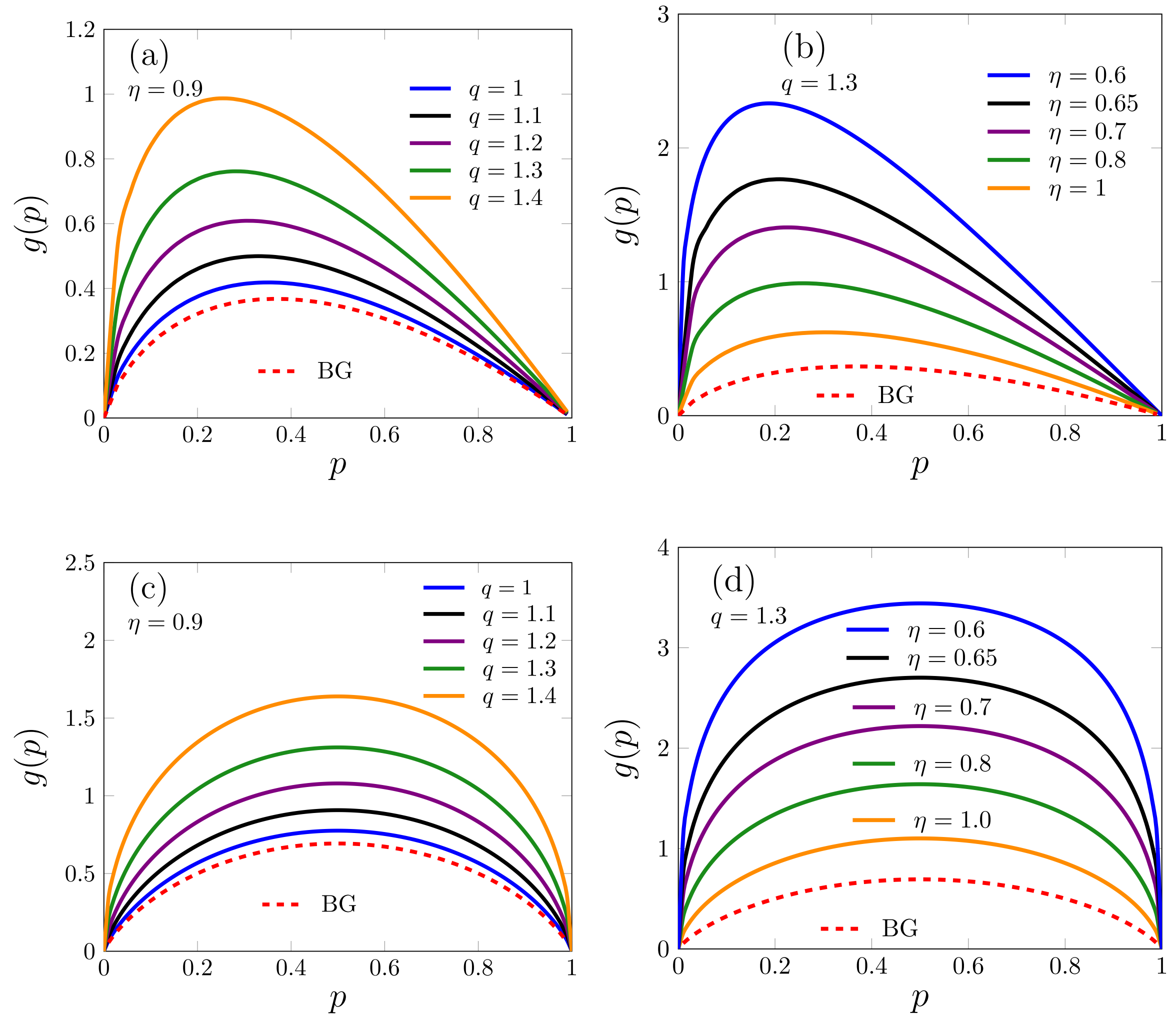

FIG. 4. The functional $g(p)$ [cf. Eqs. (41)–(43)] is represented versus p in several situations: (a) fixed η and increasing values of q; (b) fixed q and increasing values of η; (c) the same as in (a) for a two-state system, i.e., probabilities p and $1-p$; (d) the same as in (b) for a two-state system, i.e., probabilities p and $1-p$. In all cases, the functional $g(p)$ exhibits the expected concavity with respect to p.

Therefore, the entropic functional associated with the weight in Eq. (24) was determined above, being expressed in two equivalent forms:

$$S_{q,\eta}(\{p_i\}) = k\sum_{i=1}^{W} g(p_i), \tag{41}$$

$$g(p_i) = \int_0^{p_i} [-\ln_q(x)]^{\frac{1}{\eta}}dx - p_i\int_0^1 [-\ln_q(x)]^{\frac{1}{\eta}}dx \tag{42}$$

$$= \Gamma_q\left[\frac{1+\eta}{\eta}, -\ln_q p_i\right] - p_i\Gamma_q\left[\frac{1+\eta}{\eta}\right]. \tag{43}$$

Plots of the functional $g(p)$ above are exhibited in Fig. 4 for typical values of their parameters. In the upper panels, $g(p)$ is represented versus p for a fixed η and increasing values of q [Fig. 4(a)], as well as for a fixed q and increasing values of η [Fig. 4(b)]: In both cases, one notices that deviations from the BG curve ($q = \eta = 1$) get larger as the corresponding varying parameter departs from unit. Similar plots are presented in Figs. 4(c) and 4(d), where $g(p)$ is plotted versus p for $W = 2$, i.e., probabilities p and $1-p$, showing the expected maximum at equiprobability ($p = 1/2$) as well as the symmetry $p \leftrightarrow (1-p)$.

Let us now comment on the behavior of the entropic form $S_{q,\eta}(\{p_i\})$ [Eqs. (41)–(43)], taking into account Khinchin axioms [72] as described below.

(i) It depends only on the set of probabilities $\{p_i\}$.

(ii) Although the maximum at equiprobability was shown numerically in several typical cases for $W = 2$ [cf. Figs. 4(c) and 4(d)], one expects that such a behavior is preserved for general W ($W > 2$), with $p_i = 1/W$ ($i = 1, 2, \cdots W$).

(iii) It satisfies the property of expansibility by remaining unchanged through the addition of zero-probability events.

(iv) It violates the fourth Khinchin axiom, concerning the behavior of the entropy of a composite system with respect to the entropies of its subsystems, i.e., the additivity property is not satisfied.

Hence, axioms (i) and (iii) are fulfilled and axiom (ii) was shown to be satisfied in the simple case $W = 2$, whereas axiom (iv) is violated. One should mention that violation of the fourth Khinchin axiom is very common in the framework of generalized entropic forms [3,20,21].

V. CONCLUSIONS

We have introduced a potentially relevant superstatistics class, characterized by a $f(\beta)$ distribution, expressed as a modification of the usual χ^2 through the introduction of a real index η $(0 < \eta \leqslant 1)$, leading to a χ_η^2 distribution. We have shown that the resulting energy probability distribution, $p(E) = \mathcal{B}(E)/\mathcal{Z}$, is given by a stretched q-exponential function, where $\mathcal{B}(E) = \exp_q[-(\beta_0 E)^\eta]$. In this way, one covers, as particular cases, two very common forms in the realm of complex systems: (i) the q-exponential function associated with Tsallis entropy, recovered in the limit $\eta \to 1$ and (ii) the stretched exponential related to Anteneodo-Plastino entropy, obtained for $q \to 1$. Moreover, considering both limits $(q, \eta) \to (1, 1)$ one recovers the BG weight.

Futhermore, we applied methods introduced previously in the literature to find the entropic form associated with the distribution $p(E)$, given the internal energy $U = \sum_i p_i E_i$. The resulting entropic form $S_{q,\eta}(\{p_i\})$ was shown to satisfy essential Khinchin axioms.

Since the two limiting distributions, $\eta \to 1$ (Tsallis) and $q \to 1$ (stretched-exponential), are ubiquitous in complex systems, one expects that the generalized form $\mathcal{B}(E) = \exp_q[-(\beta_0 E)^\eta]$ should be relevant in the study of complex systems. As examples, one could mention the velocity measurements in a turbulent Couette-Taylor flow [66], relaxation curves of RKKY spin glasses, like CuMn and AuFe [67], cumulative distribution for the magnitude of earthquakes [68], and thermal-conductivity curves for systems of nearest-neighbor interacting XY rotators [69].

ACKNOWLEDGMENTS

We thank Constantino Tsallis for fruitful conversations. Partial financial support from CNPq, CAPES, and FAPERJ (Brazilian funding agencies) is acknowledged.

[1] R. Balian, *From Microphysics to Macrophysics* (Springer, Berlin, 1991), Vols. I and II.
[2] R. K. Pathria and P. D. Beale, *Statistical Mechanics*, 3rd ed. (Elsevier, Amsterdam, 2011).
[3] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World* (Springer, New York, 2009).
[4] C. Beck and E. G. D. Cohen, Superstatistics, Physica A **322**, 267 (2003).
[5] R. Hanel and S. Thurner, Generalized Boltzmann factors and the maximum entropy principle: Entropies for complex systems, Physica A **380**, 109 (2007).
[6] R. Hanel, S. Thurner and M. Gell-Mann, Generalized entropies and the transformation group of superstatistics, Proc. Natl. Acad. Sci. USA **108**, 6390 (2011).
[7] C. Beck, E. G. Cohen, and H. L. Swinney, From time series to superstatistics, Phys. Rev. E **72**, 056133 (2005).
[8] G. Williams, B. Schäfer, and C. Beck, Superstatistical approach to air pollution statistics, Phys. Rev. Res. **2**, 013019 (2020).
[9] J. Weber, M. Reyers, C. Beck, M. Timme, J. G. Pinto, D. Witthaut, and B. Schäfer, Wind power persistence characterized by superstatistics, Sci. Rep. **9**, 1 (2019).
[10] B. Schäfer, C. Beck, K. Aihara, D. Witthaut, and M. Timme, Non-Gaussian power grid frequency fluctuations characterized by Lévy-stable laws and superstatistics, Nat. Energy. **3**, 119 (2018).
[11] R. Metzler, Superstatistics and non-Gaussian diffusion, Eur. Phys. J.: Spec. Top. **229**, 711 (2020).
[12] M. A. F. dos Santos, E. H. Colombo, and C. Anteneodo, Random diffusivity scenarios behind anomalous non-Gaussian diffusion, Chaos Solit. Fractals **152**, 111422 (2021).
[13] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).
[14] R. Hanel, S. Thurner, and M. Gell-Mann, How multiplicity determines entropy and the derivation of the maximum entropy principle for complex systems, Proc. Natl. Acad. Sci. USA **111**, 6905 (2014).
[15] A. Plastino, E. M. F. Curado, and F. D. Nobre, Deriving partition functions and entropic functionals from thermodynamics, Physica A **403**, 13 (2014).
[16] C. Tsallis and A. M. C. Souza, Constructing a statistical mechanics for Beck-Cohen superstatistics, Phys. Rev. E **67**, 026106 (2003).
[17] R. Hanel and S. Thurner, A comprehensive classification of complex statistical systems and an axiomatic derivation of their entropy and distribution functions, Europhys. Lett. **93**, 20006 (2011).
[18] E. M. F. Curado, F. D. Nobre, and A. Plastino, Associating an entropy with power-law frequency of events, Entropy **20**, 940 (2018).
[19] V. Schwämmle, E. M. F. Curado, and F. D. Nobre, A general nonlinear Fokker-Planck equation and its associated entropy, Eur. Phys. J. B **58**, 159 (2007).
[20] M. Gell-Mann and C. Tsallis, *Nonextensive Entropy: Interdisciplinary Applications* (Oxford University Press, Oxford, 2004).
[21] C. Tsallis, Beyond Boltzmann-Gibbs-Shannon in physics and elsewhere, Entropy **21**, 696 (2019).
[22] A. Rényi, On measures of information and entropy, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press: Berkeley, LA, 1961), Vol. 1, pp. 547–561.
[23] B. D. Sharma and D. P. Mittal, New non-additive measures of entropy for discrete probability distributions, J. Math. Sci. **10**, 28 (1975).
[24] S. Abe, A note on the q-deformation-theoretic aspect of the generalized entropies in nonextensive physics, Phys. Lett. A **224**, 326 (1997).
[25] P. T. Landsberg and V. Vedral, Distributions and channel capacities in generalized statistical mechanics, Phys. Lett. A **247**, 211 (1998).
[26] E. M. F. Curado, General aspects of the thermodynamical formalism, Braz. J. Phys. **29**, 36 (1999).
[27] E. M. F. Curado and F. D. Nobre, On the stability of analytic entropic forms, Physica A **335**, 94 (2004).
[28] C. Anteneodo and A. R. Plastino, Maximum entropy approach to stretched exponential probability distributions, J. Phys. A: Math. Gen. **32**, 1089 (1999).
[29] G. Kaniadakis, Non-linear kinetics underlying generalized statistics, Physica A **296**, 405 (2001).

[30] C. Tsallis and L. J. L. Cirto, Black hole thermodynamical entropy, Eur. Phys. J. C **73**, 1 (2013).
[31] T. C. Nunes, S. Brito, L. R. da Silva, and C. Tsallis, Role of dimensionality in preferential attachment growth in the Bianconi-Barabási model, J. Stat. Mech. (2017) 093402.
[32] S. Brito, T. C. Nunes, L. R. da Silva, and C. Tsallis, Scaling properties of d-dimensional complex networks, Phys. Rev. E **99**, 012305 (2019).
[33] G. Ruiz and A. F. de Marcos, Evidence for criticality in financial data, Eur. Phys. J. B **91**, 1 (2018).
[34] L. Borland, Financial market models, in *Complexity and Synergetics* (Springer, London, 2017), p. 257.
[35] D. Xu and C. Beck, Symbolic dynamics techniques for complex systems: Application to share price dynamics, Europhys. Lett. **118**, 30001 (2017).
[36] H. B. A. Evangelista, S. M. Thomaz, R. S. Mendes, and L. R. Evangelista, Generalized entropy indices to measure α-and β-diversities of macrophytes, Braz. J. Phys. **39**, 369 (2009).
[37] Y. G. Puzachenko, *Rank Distribution in Ecology and Nonextensive Statistical Mechanics*, Archives of Zoological Museum of Lomonosov Moscow State University (Moscow, Russia, 2016), Vol. 54, p. 42.
[38] C. Tsallis and D. J. Bukman, Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis, Phys. Rev. E **54**, R2197 (1996).
[39] M. A. F. dos Santos, Analytic approaches of the anomalous diffusion: A review, Chaos Solit. Fractals **124**, 86 (2019).
[40] E. Lutz, Anomalous diffusion and Tsallis statistics in an optical lattice, Phys. Rev. A **67**, 051402(R) (2003).
[41] T. Hadzibeganovic and S. A. Cannas, A Tsallis' statistics based neural network model for novel word learning, Physica A **388**, 732 (2009).
[42] M. Siddiqui, R. S. Wedemann, and H. J. Jensen, Avalanches and generalized memory associativity in a network model for conscious and unconscious mental functioning, Physica A **490**, 127 (2018).
[43] A. V. Chechkin, F. Seno, R. Metzler, and I. M. Sokolov, Brownian Yet Non-Gaussian Diffusion: From Superstatistics to Subordination of Diffusing Diffusivities, Phys. Rev. X **7**, 021002 (2017).
[44] M. A. F. dos Santos and L. M. Junior, Random diffusivity models for scaled Brownian motion, Chaos Solit. Fractals **144**, 110634 (2021).
[45] C. Beck, Superstatistics: Recent developments and applications, in *Complexity, Metastability and Nonextensivity* (Worldscientific, 2005), pp. 33–42.
[46] C. Beck, Superstatistics in high-energy physics, Eur. Phys. J. A **40**, 267 (2009).
[47] Y. A. Katz and L. Tian, Superstatistical fluctuations in time series of leverage returns, Physica A **405**, 326 (2014).
[48] Y. Itto and C. Beck, Superstatistical modelling of protein diffusion dynamics in bacteria, J. R. Soc. Interface **18**, 20200927 (2021).
[49] M. Hidalgo-Soria and E. Barkai, Hitchhiker model for Laplace diffusion processes, Phys. Rev. E **102**, 012109 (2020).
[50] L. L. Chen and C. Beck, A superstatistical model of metastasis and cancer survival, Physica A **387**, 3162 (2008).
[51] D. Xu and C. Beck, Transition from lognormal to χ^2-superstatistics for financial time series, Physica A **453**, 173 (2016).
[52] M. A. F. dos Santos and L. M. Junior, Log-normal superstatistics for Brownian particles in a heterogeneous environment, Physics **2**, 571 (2020).
[53] C. Beck, Lagrangian acceleration statistics in turbulent flows, Europhys. Lett. **64**, 151 (2003).
[54] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffer in the kernel, Yokohama Math. J. **19**, 7 (1971).
[55] R. Garra and R. Garrappa, The Prabhakar or three parameter Mittag-Leffler function: Theory and application, Commun. Nonlinear Sci. Numer. Simul. **56**, 314 (2018).
[56] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications* (Academic Press, San Diego, CA, 1998), Vol. 198.
[57] M. A. F. dos Santos, Mittag-Leffler memory kernel in Lévy flights, Mathematics **7**, 766 (2019).
[58] M. A. F. dos Santos, Fractional Prabhakar derivative in diffusion equation with non-static stochastic resetting, Physics **1**, 40 (2019).
[59] F. Mainardi and R. Garrappa, On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics, J. Comput. Phys. **293**, 70 (2015).
[60] J. Hristov, Derivatives with non-singular kernels from the Caputo-Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models, in *Frontiers in Fractional Calculus* (Bentham Science Publishers, 2017), pp. 235–295.
[61] N. Sene, Analytical solutions of Hristov diffusion equations with non-singular fractional derivatives, Chaos **29**, 023112 (2019).
[62] M. A. F. dos Santos, Mittag-Leffler functions in superstatistics, Chaos Solit. Fractals **131**, 109484 (2020).
[63] H. Agahi and M. Khalili, Truncated Mittag-Leffler distribution and superstatistics, Physica A **555**, 124620 (2020).
[64] R. Garrappa, F. Mainardi, and M. Guido, Models of dielectric relaxation based on completely monotone functions, Fract. Calc. Appl. Anal. **19**, 1105 (2016).
[65] C. Beck, Generalized statistical mechanics for superstatistical systems, Philos. Trans. R. Soc. A **369**, 453 (2011).
[66] C. Beck, G. S. Lewis, and H. L. Swinney, Measuring nonextensitivity parameters in a turbulent Couette-Taylor flow, Phys. Rev. E **63**, 035303(R) (2001).
[67] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Generalized Spin-Glass Relaxation, Phys. Rev. Lett. **102**, 097202 (2009).
[68] Amir H. Darooneh and Ali Mehri, A nonextensive modification of the Gutenberg-Richter law: q-stretched exponential form, Physica A **389**, 509 (2010).
[69] C. Tsallis, H. S. Lima, U. Tirnakli, and D. Eroglu, Thermal transport in $d = 1, 2, 3$: A conspiracy enthroning Fourier law, arXiv:2203.00102.
[70] M. Dishon, J. T. Bendler, and G. H. Weiss, Tables of the inverse Laplace transform of the function e^{-s^β}, J. Res. Natl. Inst. Stand. Technol. **95**, 433 (1990).
[71] C. Beck, Stretched exponentials from superstatistics, Physica A **365**, 96 (2006).
[72] A. I. Khinchin, *Mathematical Foundations of Information Theory* (Dover, New York, 1957).