Lasting effects of static magnetic field on classical Brownian motion

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The Bohr-Van Leeuwen theorem states that an external static magnetic field does not influence the state of a classical equilibrium system: There is no equilibrium classical magnetism, since the magnetic field does not do work. We revisit this famous no-go result and consider a classical charged Brownian particle interacting with an equilibrium bath. We confirm that the Bohr-Van Leeuwen theorem holds for the long-time (equilibrium) state of the particle. But the external static, homogeneous magnetic field does influence the long-time state of the thermal bath, which is described via the Caldeira-Leggett model. In particular, the magnetic field induces an average angular momentum for the (uncharged) bath, which separates into two sets rotating in opposite directions. The effect relates to the bath going slightly out of equilibrium under the influence of the Brownian particle and persists for arbitrarily long times. In this context we studied the behavior of the two other additive integrals of motion, energy, and linear momentum. The situation with linear momentum is different, because it is dissipated away by (and from) the bath modes. The average energy of the bath mode retains the magnetic field as a small correction. Thus, only the bath angular momentum really feels the magnetic field for long times.

I. INTRODUCTION

The Bohr-Van Leeuwen theorem governs the response of equilibrium systems to external magnetic fields [1–3]. It states that charged classical particles do not feel the magnetic field in equilibrium. Thus, equilibrium influences of the magnetic field are to be restricted to the quantum domain, where the theorem does not apply due to the noncommutativity of quantum observables [4]. The essence of the theorem is that the equilibrium state can be represented as a function of the energy; e.g., the Gibbs coordinate-velocity distribution or the microcanonical distribution. Since the magnetic field does not do work, it does not appear in the energy that enters equilibrium distributions [1,4]; see Sec. III for a reminder on this theorem and its origin.

The Bohr-Van Leeuwen theorem prevents the existence of equilibrium classical magnetism [1], a subject that could have potential applications in various fields including real and complex plasma [5–7], where the influence of magnetic fields is relevant for the fusion research, charged colloidal liquids [6,8], macroions [9], etc. Especially interesting are biophysical applications [10–14]: Since a stationary magnetic field is not screened by a living body, it makes an interesting diagnostic tool and also a potential abusive factor for biosystems. Moreover, there is a massive body of experimental results witnessing observable effects of weak, static magnetic fields in biological systems; see, e.g., Refs. [10,11]. A natural target for the magnetic field is metal ions (Na^+, K^+, Ca^{2+}, etc.) that are crucial in molecular biology: Nearly 1/3 of all proteins employ metal ions for their functioning [13]. Ions are important in bioenergetics, communication (e.g., nerve impulse generation), osmotic regulation, metabolism, energy

storage, etc. [14]. However, the translational motion of ions is always classical, and hence the Bohr-Van Leeuwen theorem prohibits their equilibrium magnetic response [12]. Biophysical responses to static magnetic fields might be looked for in the quantum domain; see, e.g., Refs. [15–17]. It is, however, not likely that all biophysical influences of magnetic fields can be accounted for by quantum models [12].

In all these fields one deals with classical charges moving in the thermal baths. Thus, it is necessary to understand which effects are not prohibited by the Bohr-Van Leeuwen theorem and can support long-time (i.e., lasting) influences of a static magnetic field for classical Brownian motion of a charged particle.

Consider a Brownian charged particle described via the Langevin equation. The equilibrium state feels no magnetic field according to the Bohr-Van Leeuwen theorem. This is, however, not the end of the story, because during its relaxation (whatever short) the particle perturbs the equilibrium state of the bath, which is now slightly out of equilibrium. Using the Caldeira-Leggett (CL) model [18–22] that reproduces the Langevin dynamics, we show that an external static magnetic field leads to long-time changes in the bath state. The (uncharged) bath oscillators acquire a sizable average angular momentum and separate (in the frequency space) into two groups that rotate in different directions. This bath angular momentum can (but need not) be driven by the conservation law of the total (for the bath + Brownian particle), effective angular momentum. Thus, the magnetic field can persist in the long-time limit, though its influence is to be looked for not in the state of the Brownian charge but rather in its environment.

Recall that CL is a concrete microscopic model composed of undamped harmonic oscillators. It has a wide range of

applications for baths composed of weakly interacting particles (photons, phonons) [20,21]. Its applicability to the real Brownian dynamics was clarified within molecular dynamic theories, where the oscillators refer to effective modes of a liquid bath [22–24].

Our study of angular momentum led us to a more general physical question: How do the three additive integrals of motion (angular momentum, linear momentum, and energy) behave at the interface between the system (Brownian particle) and bath? For example, is the bath capable of storing linear momentum in the same way it stores angular momentum? The answer to this question is negative: The bath angular momentum differs from the linear momentum and energy; see Secs. V and VI for details. For the energy the situation is less interesting, since the influence of the Brownian particle amounts to a small perturbation of the bath energy. The linear momentum is more interesting, because it is transferred to the bath can be a dominant effect for finite times but is eventually dissipated from its observables.

This paper is organized as follows. The next section discusses the CL model: an ion interacting with many independent harmonic oscillators representing an equilibrium thermal bath. Section III presents the famous Bohr-Van Leeuwen theorem with discussion. In Secs. IV and V we study the angular momentum and total energy of individual modes, respectively. We compare the behavior of angular momentum with the linear momentum in Sec. VI. We summarize in the last section and provide a perspective on future research. We relegated detailed derivations of our results to Appendices making our conclusions self-contained. Appendix A discusses the solution of the Langevin equation (6) and studies pertinent correlation functions of the Brownian motion. Appendix B discusses the system-bath (Caldeira-Leggett) model and the derivation of the Langevin equation. in Appendix C calculated the mean angular momentum of a single bath oscillator, while Appendix D studies its mean energy.

II. THE MODEL

Consider a classical particle with coordinates $\mathbf{R} = (X, Y, Z)$, unit charge, and unit mass that interacts with magnetic field. The particle is subject to an external, rotation symmetric, harmonic potential with frequency ω_0. Particle's Lagrangian reads

$$\mathcal{L}_S = \frac{1}{2}\dot{\mathbf{R}}^2 - \frac{\omega_0^2}{2}\mathbf{R}^2 + \mathbf{A}(\mathbf{R})\dot{\mathbf{R}}, \qquad (1)$$

where $A(\mathbf{R})$ is a vector potential that generates a static, homogeneous magnetic field $\boldsymbol{B}$ with the magnitude $b = |\boldsymbol{B}|$ along the z-axes (the normal vector $\mathbf{e}_z$):

$$\boldsymbol{B} = \operatorname{rot}\mathbf{A} = \mathbf{e}_z b, \quad \mathbf{A}(\mathbf{R}) = \tfrac{1}{2}(-bY, bX, 0). \qquad (2)$$

The particle couples with a bath made of N harmonic oscillators (modes) with coordinates $\mathbf{r}_k = (x_k, y_k, z_k)$, masses m_k, frequencies ω_k, and coupling constants c_k. The potential energy of particle-bath interaction is assumed to be non-negative and bilinear over the particle and bath coordinates (Caldeira-Leggett model). Hence the bath + interaction Lagrangian reads [18–22]

$$\mathcal{L}_B = \sum_{k=1}^{N} \left[\frac{m_k}{2}\dot{\mathbf{r}}_k^2 - \frac{m_k\omega_k^2}{2}\left(\mathbf{r}_k - \frac{c_k\mathbf{R}}{m_k\omega_k^2}\right)^2 \right], \qquad (3)$$

where the full Lagrangian is $\mathcal{L}_S + \mathcal{L}_B$. Due to (2) and linearity of equations of motion generated by $\mathcal{L}_S + \mathcal{L}_B$, the motion along z coordinates for all particles involved in (1) and (3) decouples from the motion along (x, y) coordinates.

We get from Noether's theorem, or directly from equations of motion generated by $\mathcal{L}_S + \mathcal{L}_B$, that the following quantity is conserved:

$$L = X\dot{Y} - Y\dot{X} + \sum_{k=1}^{N} m_k(x_k\dot{y}_k - y_k\dot{x}_k)$$
$$+ \frac{b}{2}(X^2 + Y^2). \qquad (4)$$

L is a sum of the particle's angular momentum $X\dot{Y} - Y\dot{X}$ along the z direction, angular momenta of all bath oscillators and a contribution $\frac{b}{2}(X^2 + Y^2)$ from the charged particle related to the magnetic field. Solving Euler-Lagrange equations of motion generated by $\mathcal{L}_S + \mathcal{L}_B$ for $\mathbf{r}_k$ we get:

$$\mathbf{r}_k(t) = \mathbf{r}_k(0)\cos(\omega_k t) + \frac{\dot{\mathbf{r}}_k(0)}{\omega_k}\sin(\omega_k t)$$
$$+ \frac{c_k}{m_k\omega_k}\int_0^t dt'\sin[\omega_k(t - t')]\mathbf{R}(t'). \qquad (5)$$

Plugging (5) into the Euler-Lagrange equations of motion for $\mathbf{R}$, we get the Langevin equation for the charged particle in magnetic field [25] (see Appendix A):

$$\ddot{\mathbf{R}} = b\mathbf{e}_z \times \dot{\mathbf{R}} - \omega_0^2\mathbf{R} - \int_0^t du\,\zeta(t - u)\dot{\mathbf{R}}(u) + \boldsymbol{\xi}, \quad (6)$$

$$\zeta(t) = \sum_{k=1}^{N} \frac{c_k^2}{m_k\omega_k^2}\cos(\omega_k t), \qquad (7)$$

$$\boldsymbol{\xi}(t) = \sum_{k=1}^{N}\left[c_k\bar{\mathbf{r}}_k(0)\cos(\omega_k t) + \frac{c_k\mathbf{v}_k(0)}{\omega_k}\sin(\omega_k t) \right], \quad (8)$$

$$\bar{\mathbf{r}}_k(0) \equiv \mathbf{r}_k(0) - \mathbf{R}(0)\frac{c_k}{m_k\omega_k^2}, \quad \mathbf{v}_k = \dot{\mathbf{r}}_k. \qquad (9)$$

The cumulative force from the bath is decomposed into friction with a kernel ζ and noise $\boldsymbol{\xi}(t)$ that emerges due to the initially random state of the bath. Let the initial state of the particle and bath is given by the density

$$\mathcal{P}(\mathbf{r}_k, \mathbf{v}_k, \mathbf{R}, \mathbf{V}) \propto e^{-H_B(\bar{\mathbf{r}}_k, \mathbf{v}_k)/T}\mathcal{P}(\mathbf{R}, \mathbf{V}), \ \mathbf{V} = \dot{\mathbf{R}}, \quad (10)$$

where T is the temperature ($k_B = 1$) and where the interaction-dressed bath energy H_B reads from (3):

$$H_B = -\mathcal{L}_B + \sum_{k=1}^{N} m_k\dot{\mathbf{r}}_k^2 \qquad (11)$$

$$= \sum_{k=1}^{N}\left[\frac{m_k}{2}\dot{\mathbf{r}}_k^2 + \frac{m_k\omega_k^2}{2}\left(\mathbf{r}_k - \frac{c_k\mathbf{R}}{m_k\omega_k^2}\right)^2 \right]. \quad (12)$$

Equation (10) shows that bath's initial density is Gibbsian. Particle's initial density $\mathcal{P}(\mathbf{R}, \mathbf{V})$ is arbitrary. The initial state (10) is not independent over the particle and bath, since $e^{-H_B(\bar{\mathbf{r}}_k, \mathbf{v}_k)/T}$ contains the shifted coordinate $\bar{\mathbf{r}}_k$; cf. (9). Equation (10) refers to the timescale separation, where the bath is

prepared in equilibrium under fixed coordinate and momentum of the particle. This is realistic for Brownian motion.

Equations (8) and (10) imply the fluctuation-dissipation relation,

$$\langle \xi_\alpha(t)\xi_\beta(t') \rangle = T\delta_{\alpha\beta}\zeta(t-t') \quad \alpha, \beta = x, y, z, \qquad (13)$$

where $\langle \dots \rangle$ is the average over initial state (10).

The thermodynamic limit $N \to \infty$ for the bath is taken together with the dense frequency limit $\delta\omega \to 0$ ($n = 1, \ldots, N$) under condition $N\delta\omega \gg 1$ (see Appendix B for details). Simultaneously, we take a weak coupling $c_n \to 0$ to each oscillator [18–22]. The specific choice

$$\omega_n = \delta\omega\, n, \quad c_n = \sqrt{2\gamma\omega_n^2 m_n \delta\omega/\pi}, \qquad (14)$$

$$\zeta(t) = 2\gamma\,\delta(t), \qquad (15)$$

reproduces in the Langevin equation [Eqs. (6), (7), and (8)] the Ohmic friction with magnitude γ and the white noise [18–22].

III. THE BOHR-VAN LEEUWEN THEOREM

Fluctuation-dissipation relation (13) ensures from (6) (together with the thermodynamic limit for the bath) that the particle's state relaxes to the Gibbsian density [18–22]:

$$\mathcal{P}(\mathbf{R}, \mathbf{V}; t \to \infty) \propto e^{-\frac{1}{2}(\mathbf{V}^2 + \omega_0^2 \mathbf{R}^2)/T}, \qquad (16)$$

that does not (and cannot) contain the magnetic field b. Obviously, the average angular momentum $\langle XV_y - YV_x \rangle$ calculated via (16) is zero. Equation (16) implies the Bohr-Van Leeuwen theorem for the Brownian motion [1–4]. The absence of the magnetic field $\boldsymbol{B}$ from the equilibrium density holds for any confining potential [1,3]. This theorem is an equilibrium result and it does not hold for nonequilibrium steady states [26–31]. In particular, the theorem may be broken by a weak white noise, which is sufficient for generating a sizable diamagnetic angular momentum for the Brownian particle [31].

Note as well that the magnetic field shows up in correlation functions of the Brownian particle and in its relaxation times to the equilibrium state [31]; see also Appendix A. This, however, is not the long-time response we seek here, since correlation functions decay within the relaxation time.

IV. ANGULAR MOMENTUM OF BATH MODES

Let us assume that the initial state of the Brownian particle in (10) holds

$$\mathcal{P}(\mathbf{R}, \mathbf{V}) = \mathcal{P}_1(X)\mathcal{P}_1(Y)\mathcal{P}_1(Z)\mathcal{P}_2(V_x)\mathcal{P}_2(V_y)\mathcal{P}_2(V_z),$$
$$\mathcal{P}_1(-a) = \mathcal{P}_1(a), \quad \mathcal{P}_2(-a) = \mathcal{P}_2(a). \qquad (17)$$

Equation (17) allows a large class of initial nonequilibrium states for the particle. Note that the initial angular momentum of the particle nullifies: $\langle XV_y - YV_x \rangle = 0$. The virtue of (17) and (10) is that all averages hold specific symmetry features related to the invariance of (17) with respect to $\pi/2$ rotations in the (x, y) plane; see Appendix B 3.

For initial second moments in (17) we denote

$$\langle X^2 \rangle = \langle Y^2 \rangle = \sigma_X\, T/\omega_0^2, \quad \langle V_x^2 \rangle = \langle V_y^2 \rangle = \sigma_V\, T, \qquad (18)$$

where $\sigma_X = \sigma_V = 1$ in (18) refers to equilibrium second moments of the initial state; cf. (16).

The long-time average angular momentum $L(\omega)$ of a bath oscillator with frequency $\omega = \omega_k$ is calculated from Eqs. (5), (6), and (17) using the Laplace transform (see Appendix C):

$$L(\omega) \equiv m_k \langle x_k v_{yk} - y_k v_{xk} \rangle_{t \to \infty} = \delta\omega\, \frac{4\gamma bT}{\pi}$$

$$\times \frac{(\omega^2 - \omega_0^2)[(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2]}{[(\omega^2 - \omega_0^2 - b\omega)^2 + \gamma^2\omega^2][(\omega^2 - \omega_0^2 + b\omega)^2 + \gamma^2\omega^2]}. \qquad (19)$$

Equation (19) shows that even though bath oscillators are not charged, they acquire a nonzero angular momentum: $L(\omega) \neq 0$ for $t \to \infty$, i.e., for times much larger than the relaxation time of the particle. This means that the long-time state of the bath feels the magnetic field b. Equations (10) and (17) show that the initial mean angular momentum nullifies both for the bath oscillator and the Brownian particle. Hence the fact of $L(\omega) \neq 0$ for $t \to \infty$ means that the final state of the bath is out of equilibrium.

Now $L(\omega_k) \propto c_k^2 \propto \delta\omega$ since $L(\omega_k)$ is driven by the coupling of the corresponding mode with the Brownian particle; cf. (5). In view of (19), the observable collective mode momentum is well defined and amounts to integral between two finite frequencies:

$$\sum_{k=k_1}^{k_2} L(\omega_k) = \sum_{\omega=\omega_1}^{\omega_2} L(\omega) = \int_{\omega_1}^{\omega_2} \frac{d\omega}{\delta\omega} L(\omega). \qquad (20)$$

Equation (19) [and (27) below] can describe a finite-bath situation [32–36]. Here the bath is large [for Langevin's equation (6) to apply for certain times] but finite.

We emphasize that $L(\omega) \neq 0$ due to initially nonequilibrium state of the particle, since $L(\omega) = 0$ whenever $\sigma_X = \sigma_V = 1$ in (17); cf. (19). Next, we confirm that (19) is consistent with the average angular momentum conservation (4) between $t = 0$ and $t = \infty$:

$$\int_0^\infty d\omega\, L(\omega) = -\frac{Tb(1 - \sigma_X)}{\omega_0^2} = b[\langle X^2 \rangle_0 - \langle X^2 \rangle_\infty], \qquad (21)$$

where $\langle X^2 \rangle_0$ and $\langle X^2 \rangle_\infty$ are the initial and final values; cf. (18). Even for $\sigma_X = 1$ we can have $L(\omega) \neq 0$ due to $\sigma_V \neq 1$ in (19). Hence, $L(\omega) \neq 0$ need not be driven by the conservation law; see Fig. 1. Now $L(\omega)$ changes its sign at $\omega = \omega_0$ and goes to zero as $\sim \omega^{-4}$ for $\omega \gg \max[\omega_0, \gamma, b]$; i.e., two sets of oscillators rotate in different directions, as Fig. 1 shows.

V. ENERGY OF BATH MODES

Let us now see that the magnetic field also shows up in interaction-induced corrections to the energy of the bath mode. Equations (10) and (11) imply that the average energy
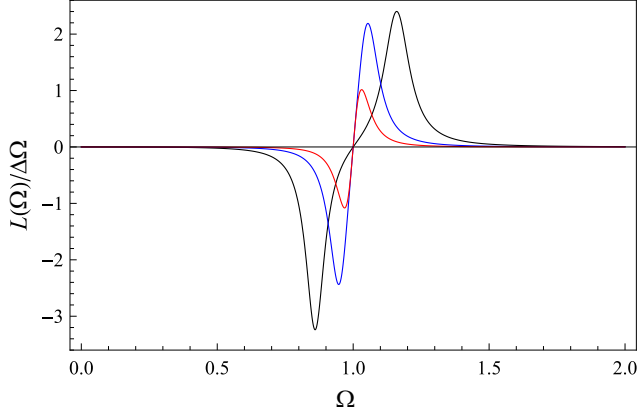
FIG. 1. $L(\omega)/\delta\omega$ versus ω for $T = 1$, $\omega_0 = 1$, $\gamma = 0.1$, $\sigma_X = 1$, $\sigma_V = 0.1$; cf. (19) and (18). Curves refer to different magnetic fields (2) (from bottom to top): $b = 0.3$ (black), $b = 0.1$ (blue), and $b = 0.03$ (red).

of the bath mode with the frequency $\omega = \omega_k$ can be defined as

$$E(\omega_k) = \frac{m_k}{2}\langle \dot{\mathbf{r}}_k^2 \rangle + \frac{m_k \omega_k^2}{2}\langle \mathbf{r}_k^2 \rangle$$
$$+ \langle \mathbf{R}^2 \rangle \frac{c_k^2}{2m_k \omega_k^2} - c_k \langle \mathbf{R} \cdot \mathbf{r}_k \rangle, \qquad (22)$$

where the average is taken over the initial state (10). The full conserved energy of the particle $+$ bath amounts to the sum of (22) over all modes, $\sum_k E(\omega_k)$, plus the mean energy $\langle \frac{1}{2}\dot{\mathbf{R}}^2 + \frac{\omega_0^2}{2}\mathbf{R}^2 \rangle$ of the particle; cf. Eqs. (1) and (11). Equation (22) in the limit $t \to \infty$ is worked out in Appendix D:

$$E(\omega) = 3T - \delta\omega[2\varepsilon(\omega) + \varepsilon(\omega)|_{b=0}], \qquad (23)$$

$$\varepsilon(\omega) = \frac{T\gamma}{2\pi} \times \frac{\omega^2(b^2 + \gamma^2) + (\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2 - b\omega)^2 + \gamma^2\omega^2}$$
$$\times \frac{(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2}{(\omega^2 - \omega_0^2 + b\omega)^2 + \gamma^2\omega^2}, \qquad (24)$$

where σ_X and σ_V are defined in (18). Now $3T$ in $E(\omega)$ is the thermal energy of the free mode; cf. the equipartition theorem from (10). The factor $\propto \delta\omega$ in (24) is due to interaction with the Brownian particle: $2\varepsilon(\omega)$ comes from the (x, y) components of (22) that feel the magnetic field b [cf. (6)], while $\varepsilon(\omega)|_{b=0}$ comes from the z component that does not feel it. Similarly to (19), the interaction-induced factor in (24) nullifies for $\sigma_X = \sigma_V = 1$; see (10) and (18). Thus, for long times the mode mean energy (24) (in contrast to particle's mean energy) depends on the magnetic field, though this dependence is weaker than for (19). The interaction-driven factor in (23) and (24) that contains the magnetic field is a small correction to the leading term $2T$. This is different from (19), where the average angular momentum without interaction is zero, and hence the magnetic field induced effect is the main one.

VI. LINEAR MOMENTUM

Given the above finding for the angular momentum, it is reasonable to ask how the linear momentum is transferred from the Brownian particle to the bath. Answering this question has intrinsic value and contrasts with the angular momentum's behavior.

Note from (3) and (1) that for a free Brownian particle (i.e., $\omega_0 = \mathbf{B} = 0$) there is a conservation of linear momentum:

$$\frac{d}{dt}\left[\sum_{k=1}^{N} \frac{c_k}{\omega_k^2} \dot{\mathbf{r}}_k(t) + \mathbf{V}(t) \right] = 0. \qquad (25)$$

Equation (25) contains the full momentum $\mathbf{V}(t)$ of the Brownian particle (recall that its mass is taken 1), which is added to $\frac{c_k}{\omega_k^2} \dot{\mathbf{r}}_k(t)$ for each mode. The latter is a part of the mode momentum $m_k \dot{\mathbf{r}}_k$. We emphasize that the full mode momentum $m_k \dot{\mathbf{r}}_k$ cannot appear in the conservation law, since the model lacks the full translation invariance. Indeed, (3) shows that each mode feels a harmonic potential. A relations similar to (25) is formally mentioned in Ref. [37].

Instead of (17) and (18) consider initial conditions $\langle \mathbf{R}(0) \rangle = 0$, $\langle \mathbf{V}(0) \rangle \neq 0$ that hold together with (10). As follows from Eqs. (6) and (15), and is generally well known, the velocity density of the particle thermalizes for long times [cf. (16)]:

$$\mathcal{P}(\mathbf{V}; t \to \infty) \propto e^{-\frac{1}{2T}\mathbf{V}^2}. \qquad (26)$$

But $\mathbf{R}$ is subject to an unbound Brownian motion and hence it is not thermalized.

For $\gamma t \gg 1$ we find from (5), (6), (15), and (10) for the average linear momentum of a bath mode with frequency $\omega = \omega_k$ (see Appendix B 4):

$$\frac{c_k}{\omega_k^2}\langle \dot{\mathbf{r}}_k \rangle = \delta\omega \times \frac{2\gamma \langle \mathbf{V}(0) \rangle}{\pi} \frac{\frac{\gamma}{\omega}\sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2}. \qquad (27)$$

Equation (27) shows that the single mode (partial) linear momentum $\frac{c_k}{\omega_k^2}\langle \dot{\mathbf{r}}_k \rangle$ is a time-dependent oscillating function. In contrast, relaxation occurs once we consider a collective quantity and take the bath's thermodynamic limit, i.e., using the integral instead of the sum:

$$\sum_{\omega_1}^{\omega_2} \frac{c_k}{\omega_k^2}\langle \dot{\mathbf{r}}_k \rangle = \frac{2\gamma \langle \mathbf{V}(0) \rangle}{\pi} \int_{\omega_1}^{\omega_2} d\omega \frac{\frac{\gamma}{\omega}\sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2}, \qquad (28)$$

where we used (27) and where naturally $\omega_2 > \omega_1$.

The behavior of the integral in (28) for $t \to \infty$ essentially depends on whether $\omega_1 > 0$ or $\omega_1 = 0$, i.e., whether the zero frequency is included in the collective quantity or not. For $\omega_1 > 0$, (28) implies

$$\sum_{\omega_1 > 0}^{\omega_2} \frac{c_k}{\omega_k^2}\langle \dot{\mathbf{r}}_k \rangle = O\left(\frac{1}{t}\right) \quad \text{for} \quad t \to \infty, \qquad (29)$$

as becomes clear after changing the variable $\omega t \to \omega$ in (28). Hence the collective linear momentum for nonzero frequencies nullifies for large times. Note that these times are larger than $1/\gamma$.

However, the total bath linear momentum is nonzero, as follows from (25). Indeed, applying (25) at times $t = 0$ and

TABLE I. Comparison between the linear momentum and angular momentum is convoluted. Hence we separated four cases depending on the following three factors. (i) Initial conditions for the Brownian particle, where $\langle \mathbf{V}(0) \rangle$ and $\langle \mathbf{L}(0) \rangle$ mean the initial velocity and angular momentum, respectively. (ii) External fields electric $\mathbf{E}$ and magnetic fields $\mathbf{B}$. Both are constant and homogeneous in space and only one of them is present for each case. (iii) The external harmonic potential with magnitude ω_0 for the particle. For each of these four cases we indicate whether and which conservation law is present, whether and to which extent the particle thermalizes for long times, and how collective bath observable behave. It is seen that the most reasonable comparison is to be done between the second and fifth columns, and this is the road we followed in the main text.

	$\omega_0 = 0$ $\mathbf{E} = 0$ $\mathbf{B} = 0$ $\langle \mathbf{V}(0) \rangle \neq 0$	$\omega_0 = 0$ $\mathbf{E} \neq 0$ $\mathbf{B} = 0$ $\langle \mathbf{V}(0) \rangle = 0$	$\omega_0 \neq 0$ $\mathbf{B} = 0$ $\mathbf{E} = 0$ $\langle \mathbf{L}(0) \rangle \neq 0$	$\omega_0 \neq 0$ $\mathbf{B} \neq 0$ $\mathbf{E} = 0$ $\langle \mathbf{L}(0) \rangle = 0$
Conservation law	Yes, effective linear momentum	No	Yes orbital momentum	Yes, effective orbital momentum
Thermalization of particle	Yes, for velocity distribution only	No do not forget $\mathbf{E}$	Yes	Yes Bohr-Van Leeuwen theorem
Collective bath observables	Forget $\langle \mathbf{V}(0) \rangle$	Do not forget $\mathbf{E}$	Forget $\langle \mathbf{L}(0) \rangle$	do not forget $\mathbf{B}$

$t \gg 1/\gamma$ together with conditions $\langle \dot{\mathbf{r}}_k \rangle = 0$, $\langle \mathbf{V}(0) \rangle \neq 0$, and $\langle \mathbf{V}(t) \rangle \cong 0$ we find for the total final linear momentum:

$$\sum_{k=1}^{N} \frac{c_k}{\omega_k^2} \dot{\mathbf{r}}_k(t) = \frac{2\gamma \langle \mathbf{V}(0) \rangle}{\pi} \int_0^\infty d\omega \, \frac{\frac{\gamma}{\omega} \sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2}$$

$$= \langle \mathbf{V}(0) \rangle. \tag{30}$$

Note that if ω_1 is extended to zero in (28), then we get back from the $t \to \infty$-limit of (28) the same total momentum $\langle \mathbf{V}(0) \rangle$. This means that the total momentum $\langle \mathbf{V}(0) \rangle$ is (for $t \to \infty$) transferred to the zero-frequency mode $\omega = 0$, which corresponds to the homogeneous shift. Thus, the linear momentum dissipates away from both the particle velocity and collective bath observables $\sum_{\omega_1 > 0}^{\omega_2} \frac{c_k}{\omega_k^2} \langle \dot{\mathbf{r}}_k \rangle$. However, the velocity $\langle \dot{\mathbf{r}}_{\omega \to 0} \rangle$ of the zero-frequency mode is not observable, because it is infinitesimaly small due to a $\lim_{\omega \to 0} \frac{c_\omega}{\omega^2} = \infty$; see (14). Hence the total linear momentum does not result into any visible motion for long times.

The comparison between the linear momentum and the angular momentum can be carried out in several different set-ups. We choose the most reasonable set-up for showing that the linear momentum is different from the angular momentum. Within this set-up the behavior of the angular momentum for the Brownian particle subject to the external potential with frequency $\omega_0 \neq 0$ and magnetic field $\mathbf{B} \neq 0$ is compared with the linear momentum under $\omega_0 = 0$ and $\mathbf{B} = 0$; see (25)–(30). In the former case the overall angular momentum is conserved [cf. (4)], while the final state of the Brownian particle thermalizes; see (16). In the latter case, the partial linear momentum is conserved, and there is a partial long-time thermalization; see (25) and (26). Other possible set-up for comparing the linear and angular momentum are discussed in Appendix B 4; see Table I.

There are two major differences between the angular momentum and linear momentum. First, the single mode angular momentum (19) converges to a well-defined, time-independent value, in contrast to the single mode linear momentum (27) that is an oscillating function. Consequently, the angular momentum is sustained in the long-time limit for finite bath observables in the thermodynamic limit, while the linear momentum dissipates away; cf. (29) with (20). Second,

(27) and (28) trivialize for zero initial momentum $\langle \mathbf{V}(0) \rangle = 0$, which again contrasts (19) that does not need an initial angular momentum for the particle.

VII. SUMMARY

We restored the influence of the static (homogeneous) magnetic field on the long-time limit of a classical Brownian motion. For long times the influence of the magnetic field is found not in the particle (as correctly claimed by Bohr-Van Leeuwen theorem), but in the orbital momentum of (uncharged) bath modes. The effect is not enforced by the conservation law of the angular momentum, i.e., there are situations where its absence is fully consistent with the conservation. This behavior is specific for the angular momentum, as compared to the two other additive integrals of motion: The linear momentum is dissipated away by (and from) bath modes, while the bath energy can feel the magnetic field, but only as a small correction.

At this point, the major open problem is whether these effects are specific for the Caldeira-Leggett model of the bath—which we emphasize does apply for describing liquids [22,24]—or it will generalize to more realistic bath models. This question is currently under investigation, but we can mention two preliminary hypotheses. The nonspontaneous aspect of the bath angular momentum is based on the conservation law (4). Hence it is likely to persist for more general bath models. The spontaneous aspect could be a specific point of linear bath models, including the Caldeira-Leggett model.

In this context, we plan to study bath models that are especially relevant for cellular ions. Here the bath is described as a fluctuating hydrodynamic system [38]. It is known that such a bath also reproduces the Langevin equation for the Brownian particle immersed into it [39]. Hence we anticipate that the effect of transferring the influence of the magnetic field to the bath will show up also here and will be reflected in the angular momentum of the fluid. The major difference of this class of models compared to the Caldeira-Leggett model is that bath modes are subject to viscosity, i.e., there is an additional source of irreversibility.

APPENDIX A: BROWNIAN STOCHASTIC MOTION AND CORRELATION FUNCTIONS

We start by presenting the solution for the following Langevin equation with Ohmic friction and white noise [cf. (6) and (15)]:

$$\ddot{X}(t) = b\dot{Y}(t) - \omega_0^2 X(t) - \gamma \dot{X}(t) + \xi_x(t), \qquad (A1)$$

$$\ddot{Y}(t) = -b\dot{X}(t) - \omega_0^2 Y(t) - \gamma \dot{Y}(t) + \xi_y(t), \qquad (A2)$$

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2\gamma\, T \delta_{\alpha\beta}\delta(t-t') \quad \alpha, \beta = x, y. \qquad (A3)$$

Note that X and Y couple only via the magnetic field b; cf. (2). The third equation in (6) is decoupled from (A1) and (A2).

We solve (A1)–(A3) via the Laplace transform:

$$\begin{pmatrix} s^2 + \omega_0^2 + \gamma s & -bs \\ bs & s^2 + \omega_0^2 + \gamma s \end{pmatrix}\begin{pmatrix} \hat{X} \\ \hat{Y} \end{pmatrix}$$

$$= \begin{bmatrix} \hat{\xi}_x + X(0)(s+\gamma) + V_x(0) - bY(0) \\ \hat{\xi}_y + Y(0)(s+\gamma) + V_y(0) + bX(0) \end{bmatrix}. \qquad (A4)$$

Inverting the matrix on the left-hand side amounts to transposing it and dividing it on its determinant. Hence, we get

$$\hat{X}(s) = X(0)(\hat{K}_0 + b\hat{H}_1) + Y(0)(\hat{K}_1 - b\hat{H}_0) \qquad (A5)$$

$$+ V_x(0)\hat{H}_0 + V_y(0)\hat{H}_1 + \hat{H}_0\hat{\xi}_x + \hat{H}_1\hat{\xi}_y, \qquad (A6)$$

$$\hat{Y}(s) = X(0)(-\hat{K}_1 + H_0 b) + Y(0)(\hat{K}_0 + b\hat{H}_1) \qquad (A7)$$

$$- V_x(0)\hat{H}_1 + V_y(0)\hat{H}_0 - \hat{H}_1\hat{\xi}_x + \hat{H}_0\hat{\xi}_y, \qquad (A8)$$

where

$$\hat{H}_0 = \frac{s^2 + \gamma s + \omega_0^2}{\Delta} \quad \hat{H}_1 = \frac{bs}{\Delta},$$

$$\hat{K}_0 = \hat{H}_0(s+\gamma) \quad \hat{K}_1 = \hat{H}_1(s+\gamma), \qquad (A9)$$

and the Δ is the determinant of the matrix in (A4):

$$\Delta = (bs)^2 + (s^2 + \gamma s + \omega_0^2)^2 = \prod_{i=1}^4 (s - p_i), \qquad (A10)$$

where p_i are roots of $\Delta = 0$:

$$p_{1,2} = \tfrac{1}{2}\left[-\gamma + ib \pm \sqrt{(\gamma - ib)^2 - 4\omega_0^2}\right] \quad p_{3,4} = p_{1,2}^*. \qquad (A11)$$

We also define residues of $\frac{1}{\Delta}$ which has four simple poles $\{p_i\}$ as

$$r_i = \frac{1}{\prod_{j \neq i}(p_i - p_j)} \quad i, j = 1, 2, 3, 4, \qquad (A12)$$

and thus the inverse Laplace transform of the above kernels are (recall Cauchy's residue theorem)

$$H_0(t) = \sum_{i=1}^4 r_i(p_i^2 + \gamma p_i + \omega_0^2)e^{p_i t} \quad H_1(t) = \sum_{i=1}^4 r_i b\, p_i\, e^{p_i t}, \qquad (A13)$$

and, similarly, we get analytical expressions for $K_0(t)$, $K_1(t)$. Using these functions together with the convolution theorem for the Laplace transform, we get

$$X(t) = X(0)[K_0(t) + bH_1(t)] + Y(0)[K_1(t) - bH_0(t)]$$

$$+ V_x(0)H_0(t) + V_y(0)H_1(t)$$

$$+ \int_0^t dt' H_0(t - t')\xi_x(t') + \int_0^t dt' H_1(t - t')\xi_y(t') \qquad (A14)$$

$$Y(t) = X(0)[-K_1(t) + bH_0(t)] + Y(0)[K_0(t) + bH_1(t)]$$

$$- V_x(0)H_1(t) + V_y(0)H_0(t)$$

$$- \int_0^t dt' H_1(t - t')\xi_x(t') + \int_0^t dt' H_0(t - t')\xi_y(t'). \qquad (A15)$$

We assume that distribution of random variables $X(0)$ and $Y(0)$ (initial values) are unbiased, symmetric, and independent from each other:

$$\mathcal{P}(X(0), Y(0)) = \mathcal{P}(X(0))\mathcal{P}(Y(0)), \quad \mathcal{P}(-x) = \mathcal{P}(x), \qquad (A16)$$

which in particular implies:

$$\langle X(0)\rangle = \langle Y(0)\rangle = 0, \qquad (A17)$$

$$\langle X(0)^2\rangle = \langle Y(0)^2\rangle \equiv \langle X_0^2\rangle, \qquad (A18)$$

$$\langle V_x(0)^2\rangle = \langle V_y(0)^2\rangle \equiv \langle V_0^2\rangle, \qquad (A19)$$

$$\langle X(0)Y(0)\rangle = \langle V_x(0)V_y(0)\rangle = 0. \qquad (A20)$$

Also note that $X(0)$, $Y(0)$, $\xi_x(t \geqslant 0)$, and $\xi_y(t \geqslant 0)$ are independent random variables due to the assumed system-bath initial state. Then the correlation function $C_{XY}(t, t') = \langle X(t)Y(t')\rangle$ will be

$$C_{XY}(t, s) = \langle V_0^2\rangle[H_1(t)H_0(s) - H_0(t)H_1(s)] \qquad (A21)$$

$$+ \langle X_0^2\rangle\{[K_0(t) + bH_1(t)][-K_1(s) + bH_0(s)] \qquad (A22)$$

$$+ [K_1(t) - bH_0(t)][K_0(s) + bH_1(s)]\} \qquad (A23)$$

$$+ \int_0^t \int_0^s dt' ds' \langle \xi(t')\xi(s')\rangle[-H_0(t - t')$$

$$\times H_1(s - s') + H_0(s - s')H_1(t - t')], \qquad (A24)$$

where the last line can be simplified as noise is white:

$$C_{XY}(t, s) = \langle V_0^2 \rangle [H_1(t)H_0(s) - H_0(t)H_1(s)] \tag{A25}$$

$$+ \langle X_0^2 \rangle \{[K_0(t) + bH_1(t)] \tag{A26}$$

$$\times [-K_1(s) + bH_0(s)] \tag{A27}$$

$$+ [K_1(t) - bH_0(t)][K_0(s) + bH_1(s)]\} \tag{A27}$$

$$+ 2\gamma T \int_0^{\min(t,s)} dy [-H_0(t-y)H_1(s-y)$$

$$+ H_0(s-y)H_1(t-y)]. \tag{A28}$$

Note that C_{XY} is antisymmetric:

$$C_{XY}(t, s) = -C_{XY}(s, t). \tag{A29}$$

Similarly, the correlation function $C_{XX}(t, s) = \langle X(t)X(s) \rangle$ will be

$$C_{XX}(t, s) = \langle V_0^2 \rangle [H_0(t)H_0(s) + H_1(t)H_1(s)] \tag{A30}$$

$$+ \langle X_0^2 \rangle \{[K_0(t) + bH_1(t)][K_0(s) + bH_1(s)] \tag{A31}$$

$$+ [K_1(t) - bH_0(t)][K_1(s) - bH_0(s)]\} \tag{A32}$$

$$+ 2\gamma T \int_0^{\min(t,s)} dy [H_0(t-y)H_0(s-y)$$

$$+ H_1(t-y)H_1(s-y)]. \tag{A33}$$

Therefore, the position-velocity autocorrelation functions can be derived by:

$$C_{XU_x}(t, s) \equiv \langle X(t)U_x(s) \rangle = \frac{d}{ds}\langle X(t)X(s) \rangle = \frac{d}{ds}C_{XX}(t, s), \tag{A34}$$

$$C_{XU_y}(t, s) \equiv \langle X(t)U_y(s) \rangle = \frac{d}{ds}\langle X(t)Y(s) \rangle = \frac{d}{ds}C_{XY}(t, s). \tag{A35}$$

APPENDIX B: EQUATIONS OF MOTION AND LANGEVIN EQUATIONS

Recall the full Lagrangian $\mathcal{L}_S + \mathcal{L}_B$ of our model [cf. (1) and (3)]:

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{R}}^2 - \frac{\omega_0^2}{2}\mathbf{R}^2 + \mathbf{AR}$$

$$+ \sum_{k=1}^N \left[\frac{m_k}{2}\dot{\mathbf{r}}_k^2 - \frac{m_k\omega_k^2}{2}\left(\mathbf{r}_k - \frac{c_k\mathbf{R}}{m_k\omega_k^2}\right)^2 \right] \tag{B1}$$

with $\mathbf{R} = (X, Y)$, $\mathbf{r}_k = (x_k, y_k)$ and the vector potential for the stationary and homogeneous magnetic field reads $\mathbf{A} = (A_x, A_y) = (-\frac{bY}{2}, \frac{bX}{2})$; cf. (1)–(3). For simplicity we assumed that the motion is two dimensional [cf. (A1) and (A2)], i.e., we did not account explicitly for the third coordinate of the bath, since for the harmonic external potential of the Brownian particle (central oscillator), the third coordinates decouple from the other two coordinates.

From Lagrange equations we get equations of motion for the Brownian particle:

$$\ddot{X} = b\dot{Y} - \omega_0^2 X + \sum_k c_k \left(x_k - X\frac{c_k}{m_k\omega_k^2} \right), \tag{B2}$$

$$\ddot{Y} = -b\dot{X} - \omega_0^2 Y + \sum_k c_k \left(y_k - Y\frac{c_k}{m_k\omega_k^2} \right), \tag{B3}$$

and for bath oscillators:

$$m_k\ddot{x}_k + m_k\omega_k^2 x_k = c_k X(t), \tag{B4}$$

$$m_k\ddot{y}_k + m_k\omega_k^2 y_k = c_k Y(t). \tag{B5}$$

We can find exact solutions for $x_k(t)$, $y_k(t)$ in (B4) and (B5) assuming that $X(t)$, $Y(t)$ are given. Employ the Laplace transform in (B4):

$$\hat{x}_k(s) = v_{xk}(0)\frac{1}{s^2 + \omega_k^2} + x_k(0)\frac{s}{s^2 + \omega_k^2} + \frac{c_k}{m_k(s^2 + \omega_k^2)}\hat{X}(s). \tag{B6}$$

Then the inverse Laplace transform produces:

$$x_k(t) = x_k(0)\cos(\omega_k t) + \frac{v_{xk}(0)}{\omega_k}\sin(\omega_k t) + \frac{c_k}{m_k\omega_k}$$

$$\times \int_0^t dt' \sin[\omega_k(t-t')]X(t'). \tag{B7}$$

A similar formula holds for $y(t)$. Now integrate by parts the final integral in (B7):

$$x_k(t) - X(t)\frac{c_k}{m_k\omega_k^2} = \left[x_k(0) - X(0)\frac{c_k}{m_k\omega_k^2} \right]\cos(\omega_k t)$$

$$+ \frac{v_{xk}(0)}{\omega_k}\sin(\omega_k t) \tag{B8}$$

$$- \frac{c_k}{m_k\omega_k^2}\int_0^t dt' \cos[\omega_k(t-t')]V_x(t'), \tag{B9}$$

and insert these results back to (B2) and (B3)

$$\ddot{X} = b\dot{Y} - \omega_0^2 X - r_x(t) + \xi_x(t), \tag{B10}$$

$$\ddot{Y} = -b\dot{X} - \omega_0^2 Y - r_y(t) + \xi_y(t). \tag{B11}$$

Equations (B10) and (B11) are Langevin equations for the Brownian particle, where we defined for friction r_s and noise ξ_s:

$$r_s(t) = \int_0^t dt' \zeta(t-t')V_s(t'), \quad s = x, y, \tag{B12}$$

$$\zeta(t) = \sum_k \frac{c_k^2}{m_k\omega_k^2}\cos(\omega_k t), \tag{B13}$$

$$\xi_x(t) = \sum_k c_k\overline{x_k}\cos(\omega_k t) + \frac{c_k}{\omega_k}\overline{v_{xk}}\sin(\omega_k t), \tag{B14}$$

$$\xi_y(t) = \sum_k c_k\overline{y_k}\cos(\omega_k t) + \frac{c_k}{\omega_k}\overline{v_{yk}}\sin(\omega_k t), \tag{B15}$$

where we define

$$\left[x_k(0) - X(0)\frac{c_k}{m_k\omega_k^2} \right] = \overline{x_k}, \quad v_{xk}(0) = \overline{v_{xk}}, \qquad \text{(B16)}$$

$$\left[y_k(0) - Y(0)\frac{c_k}{m_k\omega_k^2} \right] = \overline{y_k}, \quad v_{yk}(0) = \overline{v_{yk}}. \qquad \text{(B17)}$$

1. Equilibrium bath

If we take Gibbs distribution for initial conditions for oscillators, i.e.,

$$(x_k(0), y_k(0), v_{xk}(0), v_{yk}(0)) \sim \mathcal{N}e^{-H/T}, \qquad \text{(B18)}$$

where $\mathcal{N}$ is the normalization constant and

$$H = \sum_k \left[\frac{1}{2}m_k\omega_k^2\,\overline{y_k}^2 + \frac{1}{2}m_k\omega_k^2\,\overline{y_k}^2 + \frac{1}{2}m_k\,\overline{v_{xk}}^2 + \frac{1}{2}m_k\,\overline{v_{yk}}^2 \right], \qquad \text{(B19)}$$

we get

$$\left\langle \xi_\alpha(t)\xi_\beta(t') \right\rangle = T\delta_{\alpha\beta}\zeta(t - t') \quad \alpha, \beta = x, y. \qquad \text{(B20)}$$

We see that $\zeta(t)$ appears both in friction memory (B12) and noise correlation (B20), so this implies fluctuation-dissipation relation (13).

2. Continuous spectrum of oscillators

Now we consider some specific bath, which has uniformly spaced frequencies with Drude-Ullersma's spectrum:

$$\omega_n = \delta\omega\, n \quad n = 1, 2, 3, \ldots, \qquad \text{(B21)}$$

$$c_n = \sqrt{\frac{2\gamma m_n\omega_n^2\delta\omega}{\pi}\frac{\theta^2}{\omega_n^2 + \theta^2}}. \qquad \text{(B22)}$$

We are going to take continuum limit where the spacing of frequencies $\delta\omega \to 0$. The memory kernel (7) and (B13) (which is also the noise correlation function) will be:

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{2\gamma\,\delta\omega}{\pi}\cos(\omega_n t)\frac{\theta^2}{\omega_n^2 + \theta^2} \qquad \text{(B23)}$$

$$\to \int_0^\infty \frac{2\gamma}{\pi}\cos(\omega t)\frac{\theta^2}{\omega^2 + \theta^2}d\omega \quad \text{as} \quad \delta\omega \to 0 \quad \text{(B24)}$$

$$= \gamma\,\theta e^{-\theta t}. \qquad \text{(B25)}$$

This memory function represents memory time $1/\theta$. In the limit $\theta \to \infty$ we are going to have memoryless (Ohmic) friction and white noise:

$$\zeta(t) = 2\gamma\,\delta(t), \qquad \text{(B26)}$$

$$c(\omega) = \sqrt{\frac{2\gamma\omega^2}{\pi}}, \qquad \text{(B27)}$$

$$m \equiv 1, \qquad \text{(B28)}$$

here we took $m = 1$ for all the oscillators; cf. (15).

3. Symmetries of the model

We have seen that by the initial bath distribution (B18) and the spectrum of the oscillator couplings (B22) we recover the same Langevin equation (A1)–(A3). Note that these Langevin equations are invariant under rotation in the $X - Y$ plane. Additionally, the initial distribution of $X(0), Y(0), V_x(0), V_y(0)$ is invariant under $90°$ rotations; see (A17). Thus, the solution of these equations must also be invariant under $90°$ rotations. More specifically, for any function $f(X, V_x, Y, V_y)$ we have

$$\langle f(X(t), V_x(t), Y(s), V_y(s)) \rangle$$
$$= \langle f(Y(t), V_y(t), -X(s), -V_x(s)) \rangle, \qquad \text{(B29)}$$

which corresponds to the $90°$ rotation. Equation (A29) is a consequence of it.

Similarly, the Lagrangian (B1) and initial distribution (B18) and (A17) are rotation symmetric, and for any function $f(X, V_x, \{x_k, v_{xk}\}, Y, V_y, \{y_k, v_{yk}\})$ with $\{x_k, v_{xk}\} \equiv \{x_1, x_2, \ldots, v_{x1}, v_{x2}, \ldots\}$ we have

$$\langle f(X(t), V_x(t), \{x_k(t), v_{xk}(t)\}, Y(s), V_y(s), \{y_k(s), v_{yk}(s)\}) \rangle, \qquad \text{(B30)}$$

$$= \langle f(Y(t), V_y(t), \{y_k(t), v_{yk}(t)\}, -X(s), -V_x(s),$$
$$\{-x_k(s), -v_{xk}(s)\}) \rangle. \qquad \text{(B31)}$$

Again, this follows from $90°$ rotation. Or in other words, when we take the average over the distribution of the solution, we can replace $X \to Y$ and $Y \to -X$ and the time derivatives accordingly. For example,

$$\langle x_k(t)v_{yk}(t) - y_k(t)v_{xk}(t) \rangle = 2\langle x_k(t)\,v_{yk}(t) \rangle, \quad \text{(B32)}$$

$$\langle X(t)v_{yk}(s) \rangle = -\langle Y(t)v_{xk}(s) \rangle, \qquad \text{(B33)}$$

$$\langle X(t)\overline{y_k} \rangle = -\langle Y(t)\overline{x_k} \rangle, \qquad \text{(B34)}$$

$$\langle X^2 \rangle = \langle Y^2 \rangle, \ \langle Xx_k \rangle = \langle Yy_k \rangle, \ \langle v_{xk}^2 \rangle = \langle v_{yk}^2 \rangle, \ \text{etc.} \quad \text{(B35)}$$

4. Linear momentum of bath modes

Starting from equations of motion generated by (B1) [see also (1) and (3)] we can derive for ω_0 and $\mathbf{B} = \mathbf{A} = 0$—i.e., no external potential and no magnetic field for the Brownian particle—the following conservation law for the linear momentum:

$$\frac{d}{dt}\left[\sum_k \frac{c_k}{\omega_k^2}\,\dot{\mathbf{r}}_k + \dot{\mathbf{R}} \right] = 0, \qquad \text{(B36)}$$

where we note that only a part of the mode linear momentum $m_k\dot{\mathbf{r}}_k$ participates in the conservation law; $m_k\dot{\mathbf{r}}_k$ is obviously not conserved, since each mode feels an external potential. We now employ (B8) with initial conditions [cf. (10)]:

$$\langle \mathbf{r}_k(0) \rangle = \langle \dot{\mathbf{r}}_k(0) \rangle = \langle \mathbf{R}(0) \rangle = 0, \qquad \text{(B37)}$$

$$\langle \dot{\mathbf{R}}(0) \rangle \neq 0, \qquad \text{(B38)}$$

and also use Langevin Eq. (6) that under (B37) and (B38) and then (B26) and (B28) reads

$$\frac{d}{dt}\langle\mathbf{V}\rangle = -\gamma\langle\mathbf{V}\rangle, \quad \langle\mathbf{V}(t)\rangle = e^{-\gamma t}\langle\mathbf{V}(0)\rangle, \quad \mathbf{V}(t) = \dot{\mathbf{R}}(t). \tag{B39}$$

Recall that for a free Brownian particle the thermalization according to the Langevin equation is incomplete: The velocity distribution relaxes toward Maxwell's density [in particular, $\langle\mathbf{V}(t)\rangle$ goes to zero according to (B39)], while the coordinate does not relax, since it makes a free Brownian motion.

We find

$$\langle\dot{\mathbf{r}}_k(t)\rangle = \frac{c_k}{m_k\omega_k}\int_0^t ds\,\sin[\omega_k s]\langle\mathbf{V}(t-s)\rangle \tag{B40}$$

$$= \frac{c_k\langle\mathbf{V}(0)\rangle}{m_k\omega_k}\left[\frac{\gamma\sin[\omega_k t] - \omega_k\cos[\omega_k t]}{\gamma^2 + \omega_k^2} + \frac{\omega_k e^{-\gamma t}}{\gamma^2 + \omega_k^2}\right]. \tag{B41}$$

Hence we obtain from (B41):

$$\sum_k \frac{c_k}{\omega_k^2}\langle\dot{\mathbf{r}}_k(t)\rangle = \frac{2\gamma}{\pi}\langle\mathbf{V}(0)\rangle\int_0^\infty d\omega\left[\frac{\frac{\gamma}{\omega}\sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2}\right.$$
$$\left. + \frac{e^{-\gamma t}}{\gamma^2 + \omega^2}\right], \tag{B42}$$

where employing [$t > 0$ and $\gamma > 0$]

$$\int_{-\infty}^\infty d\omega\,\frac{\cos[\omega t]}{\gamma^2 + \omega^2} = \frac{\pi e^{-\gamma t}}{\gamma},$$
$$\int_{-\infty}^\infty d\omega\,\frac{\frac{\gamma}{\omega}\sin[\omega t]}{\gamma^2 + \omega^2} = \frac{\pi}{\gamma}(1 - e^{-\gamma t}), \tag{B43}$$

we confirm from (B42) that the conservation (B36) indeed holds.

Returning to (B41) we see that the mean linear momentum of each mode oscillates in time. This fact contrasts the mean angular momentum of each mode that (under a nonzero magnetic field) relaxes in time to a well-defined, frequency-dependent value. Another pertinent difference is that a nonzero linear momentum for bath modes is achieved only due to $\langle\mathbf{V}(0)\rangle \neq 0$, i.e., the Brownian particle should have a nonzero initial momentum for transferring it to the bath modes according to the conservation law (B36). Such a direct relation need not hold for the angular momentum, where the bath modes acquire angular momentum also without the initial angular momentum of the particle [c.f. (19)].

Note that generic finite collective observables of the bath lose the nonzero linear momentum in the long-time limit. Indeed, we note from (B41):

$$\sum_{\omega_1}^{\omega_2} \frac{c_k}{\omega_k^2}\langle\dot{\mathbf{r}}_k(t)\rangle = \frac{2\gamma}{\pi}\langle\mathbf{V}(0)\rangle\int_{\omega_1}^{\omega_2} d\omega\left[\frac{\frac{\gamma}{\omega}\sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2}\right.$$
$$\left. + \frac{\omega_k e^{-\gamma t}}{\gamma^2 + \omega^2}\right], \tag{B44}$$

where $\omega_1 < \omega_2$ are finite frequencies that define the collective observable. For $t \gg 1/\gamma$ the contribution $O(e^{-\gamma t})$ in (B44) can be neglected. The remaining two integrals converge to

zero as $O(1/t)$ for long times t and for finite and nonzero values of ω_1 and ω_2:

$$\int_{\omega_1}^{\omega_2} d\omega\,\frac{\frac{\gamma}{\omega}\sin[\omega t]}{\gamma^2 + \omega^2} \simeq \frac{1}{\gamma}\int_{\omega_1 t}^{\omega_2 t} d\omega\,\frac{\sin[\omega]}{\omega} = \frac{1}{\gamma}O\left(\frac{1}{t}\right), \tag{B45}$$

$$\int_{\omega_1}^{\omega_2} d\omega\,\frac{\cos[\omega t]}{\gamma^2 + \omega^2} \simeq \frac{\sin[\omega_2 t] - \sin[\omega_1 t]}{t\gamma^2}, \tag{B46}$$

where we note that the last relation in (B45) does not apply for $\omega_1 = 0$. At that specific (nongeneric) value of ω_1 the integral in (B45) converges to a finite value for long-times, since $\int_0^\infty d\omega\,\frac{\sin[\omega]}{\omega} = \frac{\pi}{2}$.

Altogether, we conclude that for the Brownian particle and finite observables of the bath the linear momentum is dissipated away. This discussion can be conveniently summarized as follows: If we define linear momentum density over the modes at time t:

$$\mathbf{\Pi}(\omega; t) = \langle\mathbf{V}(0)\rangle\frac{2\gamma}{\pi}\left[\frac{\frac{\gamma}{\omega}\sin[\omega t] - \cos[\omega t]}{\gamma^2 + \omega^2} + \frac{e^{-\gamma t}}{\gamma^2 + \omega^2}\right], \tag{B47}$$

then for $t \to \infty$ this quantity weakly converges to

$$\mathbf{\Pi}(\omega; t \to \infty) = 2\langle\mathbf{V}(0)\rangle\delta(\omega),$$
$$\int_0^\infty d\omega\,\mathbf{\Pi}(\omega; t \to \infty) = \langle\mathbf{V}(0)\rangle. \tag{B48}$$

For completeness, let us also consider a situation where (B37) holds, but instead of (B38) we take $\langle\mathbf{V}(0)\rangle = 0$. However, now there is a constant and homogeneous electric field $\mathbf{E}$ that is acting on the Brownian particle. Hence the Langevin equation reads instead of (B39):

$$\frac{d}{dt}\langle\mathbf{V}\rangle = -\gamma\langle\mathbf{V}\rangle + \mathbf{E}, \quad \langle\mathbf{V}(t)\rangle = \frac{\mathbf{E}}{\gamma}(1 - e^{-\gamma t}), \tag{B49}$$

where we assumed a unit charge. Note that this situation is different from that of (B39), since now there is no thermalization even for the velocity of the Brownian particle, because now this particle moves for long times with a constant velocity $\mathbf{E}/\gamma$; cf. (B49). Also, the conservation law (B36) changes to

$$\frac{d}{dt}\left[\sum_k \frac{c_k}{\omega_k^2}\dot{\mathbf{r}}_k + \dot{\mathbf{R}} - \mathbf{E}t\right] = 0, \tag{B50}$$

which means that the overall momentum $\sum_k \frac{c_k}{\omega_k^2}\dot{\mathbf{r}}_k + \dot{\mathbf{R}}$ increases monotonously in time.

Now (B40) still holds and putting there (B49) we find

$$\langle\dot{\mathbf{r}}_k(t)\rangle = \frac{c_k}{m_k\omega_k}\frac{\mathbf{E}}{\gamma}\frac{1 - \cos[\omega_k t]}{\omega_k} \tag{B51}$$

$$- \frac{\mathbf{E}}{\gamma}\frac{c_k}{m_k\omega_k}\left[\frac{\gamma\sin[\omega_k t] - \omega_k\cos[\omega_k t]}{\gamma^2 + \omega_k^2} + \frac{\omega_k e^{-\gamma t}}{\gamma^2 + \omega_k^2}\right]. \tag{B52}$$

The contribution coming from (B52) was studied by us above in (B45) and (B46), and hence we focus on (B51). Note that

$$\sum_k \frac{c_k}{\omega_k^2}\langle\dot{\mathbf{r}}_k\rangle = \frac{2\mathbf{E}}{\pi}\int_0^\infty d\omega\,\frac{1 - \cos[\omega t]}{\omega^2} = t\,\mathbf{E}. \tag{B53}$$

This relation validates the conservation law (B50). Now for $\omega_2 > \omega_1 > 0$ we find from (B51):

$$\sum_{\omega_1}^{\omega_2} \frac{c_k}{\omega_k^2} \langle \dot{\mathbf{r}}_k \rangle = \frac{2\mathbf{E}}{\pi} \int_{\omega_1}^{\omega_2} d\omega \frac{1 - \cos[\omega t]}{\omega^2}, \qquad \text{(B54)}$$

which converges for $t \to \infty$ to a finite positive value (for $\omega_2 > \omega_1 > 0$):

$$\sum_{\omega_1}^{\omega_2} \frac{c_k}{\omega_k^2} \langle \dot{\mathbf{r}}_k \rangle \to \frac{2\mathbf{E}}{\pi} \left[\frac{1}{\omega_1} - \frac{1}{\omega_2} \right]. \qquad \text{(B55)}$$

After (B46) we concluded that for the Brownian particle and finite observables of the bath the linear momentum is dissipated away. Here the situation is different: The linear momentum cannot be dissipated away, since for long times t the Brownian particle and finite bath observables move with a constant linear momentum $\propto \mathbf{E}$; cf. (B49) and (B55). The explanation of this is straightforward: At long times the particle moves with a constant linear momentum and drags along the bath modes. Instead, the acceleration induced by $\mathbf{E}$ is dissipated away from the particle and finite bath observables. A detailed comparison between the linear momentum and angular momentum is presented in Table I.

APPENDIX C: ANGULAR MOMENTUM OF A BATH OSCILLATOR

Here we consider a specific oscillator from (B7) and (B16):

$$x(t) = \bar{x} \cos(\omega t) + X(0) \frac{c}{m\omega^2} \cos(\omega t) + \frac{\overline{v_x}}{\omega} \sin(\omega t) \qquad \text{(C1)}$$

$$+ \frac{c}{m\omega} \int_0^t dt' \sin[\omega(t - t')] X(t'), \qquad \text{(C2)}$$

where we dropped subscript k's for simplicity. For the velocity we differentiate (C1) (written in terms of y),

$$v_y(t) = -\bar{y} \omega \sin(\omega t) - Y(0) \frac{c}{m\omega} \sin(\omega t) + \overline{v_y} \cos(\omega t) \qquad \text{(C3)}$$

$$+ \frac{c}{m} \int_0^t dt' \cos[\omega(t - t')] Y(t'), \qquad \text{(C4)}$$

Eventually, we want to calculate the correlation $\langle x v_y - y v_x \rangle$ at time t. But as the system is rotation symmetric, it is enough to consider $\langle x v_y \rangle$; see (B32),

$$\langle x v_y \rangle_t = -\frac{c}{m} \int_0^t \langle X(t') \bar{y} \rangle \cos(\omega t') dt' \qquad \text{(C5)}$$

$$- \frac{c}{m\omega} \int_0^t \langle X(t') \overline{v_y} \rangle \sin(\omega t') dt' \qquad \text{(C6)}$$

$$- \frac{c^2}{m^2 \omega^2} \int_0^t \langle X(t) Y(0) \rangle \cos(\omega t') dt' \qquad \text{(C7)}$$

$$+ \frac{c^2}{m^2 \omega} \int_0^t dt' \int_0^t dt'' \langle X(t') Y(t'') \rangle$$

$$\times \sin[\omega(t - t')] \cos[\omega(t - t'')], \qquad \text{(C8)}$$

where we used the relations (B33) and (B32) following from rotation symmetry and the independence of the initial conditions (B18) and (A17).

We can calculate $\langle X(t)[y(0) - \frac{Y(0)c}{m\omega^2}] \rangle$ and similar terms using solutions for X and Y (A14) and (A15). In (A14) only $\xi_y(t)$ depends on $\bar{y} \equiv [y(0) - \frac{Y(0)c}{m\omega^2}]$ and v_y; i.e., the initial state of the bath, see (B15). So we get

$$\langle X(t)\bar{y} \rangle = T \frac{c}{m\omega^2} \int_0^t dt' H_1(t - t') \cos(\omega t'), \qquad \text{(C9)}$$

$$\langle X(t)\overline{v_y} \rangle = T \frac{c}{m\omega} \int_0^t dt' H_1(t - t') \sin(\omega t'). \qquad \text{(C10)}$$

Now we can simplify first two lines of (C5) and (C6):

$$-T \frac{c^2}{m^2 \omega^2} \int_0^t dt' \int_0^{t'} dt'' H_1(t' - t'')$$

$$\times [\cos(\omega t') \cos(\omega t'') + \sin(\omega t') \sin(\omega t'')], \qquad \text{(C11)}$$

$$= -T \frac{c^2}{2m^2 \omega^2} \int_0^t \int_0^t dt' dt'' H_1[|t' - t''|) \cos(\omega(t' - t''))], \qquad \text{(C12)}$$

$$= -T \frac{c^2}{m^2 \omega^2} \int_0^t dt' H_1(t')(t - t') \cos(\omega t'). \qquad \text{(C13)}$$

Equation (C8) can also be simplified via (A29):

$$\frac{c^2}{m^2 \omega} \int_0^t dt' \int_0^t dt'' \langle X(t') Y(t'') \rangle \frac{1}{2} \{ \sin[2\omega t - \omega(t' + t'')]$$

$$+ \sin[\omega(t'' - t')] \} \qquad \text{(C14)}$$

$$= \frac{c^2}{m^2 \omega} \int_0^t dt' \int_0^{t'} dt'' \langle X(t') Y(t'') \rangle \sin[\omega(t'' - t')]. \qquad \text{(C15)}$$

Putting it all together we get a simplified result for $\langle x v_y \rangle$:

$$\langle x v_y \rangle_t = -T \frac{c^2}{m^2 \omega^2} \int_0^t dt' H_1(t')(t - t') \cos(\omega t') \qquad \text{(C16)}$$

$$- \frac{c^2}{m^2 \omega^2} \int_0^t C_{XY}(t', 0) \cos(\omega t') dt' \qquad \text{(C17)}$$

$$- \frac{c^2}{m^2 \omega} \int_0^t dt' \int_0^{t'} dt'' C_{XY}(t', t'') \sin[\omega(t' - t'')] \qquad \text{(C18)}$$

and the expressions for $C_{XY}(t', t'')$ and $H_1(t')$ are given in (A13) and (A25). Taking the limit $t \to \infty$ we get[1]

$$L = 2\langle x v_y \rangle_{t \to \infty} = \delta\omega \frac{4\gamma bT}{\pi} \frac{(\omega - \omega_0)(\omega + \omega_0)\big[(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2\big]}{b^4\omega^4 + \big[\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2\big]^2 + 2b^2\big[\gamma^2\omega^4 - (\omega^3 - \omega\omega_0^2)^2\big]} \tag{C19}$$

$$= \delta\omega \frac{4\gamma bT}{\pi} \frac{(\omega^2 - \omega_0^2)\big[(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2\big]}{\big[(\omega^2 - \omega_0^2 + i\gamma\omega)^2 - b^2\omega^2\big]\big[(\omega^2 - \omega_0^2 - i\gamma\omega)^2 - b^2\omega^2\big]} \tag{C20}$$

$$= \delta\omega \frac{4\gamma bT}{\pi} \frac{(\omega^2 - \omega_0^2)\big[(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2\big]}{\big[(\omega^2 - \omega_0^2 - b\omega)^2 + \gamma^2\omega^2\big]\big[(\omega^2 - \omega_0^2 + b\omega)^2 + \gamma^2\omega^2\big]}, \tag{C21}$$

where

$$\langle X(0)^2 \rangle = \langle Y(0)^2 \rangle = \sigma_X \frac{T}{\omega_0^2}$$

$$\langle V_x(0)^2 \rangle = \langle V_y(0)^2 \rangle = \sigma_V T. \tag{C22}$$

APPENDIX D: ENERGY OF A BATH OSCILLATOR

From the Lagrangian (B1) we can see that conserved energy is

$$E_{\text{tot}} = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\omega_0^2(X^2 + Y^2) \tag{D1}$$

$$- \sum_k c_k(X x_k + Y y_k) \tag{D2}$$

$$+ \sum_k \frac{1}{2}m_k(\dot{x}_k^2 + \dot{y}_k^2) + \frac{1}{2}m_k\omega_k^2(x_k^2 + y_k^2) \tag{D3}$$

$$+ (X^2 + Y^2)\sum_k \frac{1}{2}\frac{c_k^2}{m_k\omega_k^2}. \tag{D4}$$

Our concern is the average energy of the oscillator (again we drop subscript k):

$$E = \frac{1}{2}m\langle \dot{x}^2 + \dot{y}^2 \rangle + \frac{1}{2}m\omega^2\langle x^2 + y^2 \rangle \tag{D5}$$

$$+ \langle X^2 + Y^2 \rangle \frac{1}{2}\frac{c^2}{m\omega^2} - c\langle X x + Y y \rangle. \tag{D6}$$

The system is rotation invariant, see (B35), so the above expression is equivalent to

$$\frac{1}{2}E = \frac{1}{2}m\langle v_x^2 \rangle + \frac{1}{2}m\omega^2\langle x^2 \rangle \tag{D7}$$

$$+ \langle X^2 \rangle \frac{1}{2}\frac{c^2}{m\omega^2} - c\langle X x \rangle. \tag{D8}$$

The first line can be found by integration of the equation of motion (B4):

$$\frac{1}{2}m\langle v_x^2 \rangle + \frac{1}{2}m\omega^2\langle x^2 \rangle \Big|_0^t = c\int_0^t X(t')v_x(t')dt'. \tag{D9}$$

Using (C3) (in terms of v_x) we write the first line

$$(D7) = T + \sigma_X \frac{Tc^2}{2m\omega^2\omega_0^2} \tag{D10}$$

$$+ \frac{c^2}{m}\int_0^t dt' \int_0^{t'} dt'' \, C_{XX}(t', t'')\cos[\omega(t' - t'')] \tag{D11}$$

$$- \frac{Tc^2}{m\omega}\int_0^t dt' \int_0^{t'} dt'' \, H_0(t' - t'')\sin[\omega(t' - t'')] \tag{D12}$$

$$- \frac{c^2}{m\omega}\int_0^t dt' C_{XX}(t', 0)\sin(\omega t'), \tag{D13}$$

where the line (D10) comes from averages at $t = 0$. The second line (D8) can be worked out using (C1)

$$(D8) = \frac{Tc^2}{2m\omega^2\omega_0^2} - C_{XX}(t, 0)\frac{c^2}{m\omega^2}\cos(\omega t) \tag{D14}$$

$$- \frac{c^2}{m\omega}\int_0^t dt' \, C_{XX}(t, t')\sin[\omega(t - t')] \tag{D15}$$

$$- \frac{Tc^2}{m\omega^2}\int_0^t dt' \, H_0(t')\cos(\omega t'). \tag{D16}$$

At this stage, all expressions in (D7) and (D8) can be integrated analytically by *symbolic computation software* (cf. footnote 1). Then we take the limit $t \to \infty$. So we get

$$E = 2T - \delta\omega \frac{T\gamma}{\pi}\left\{\frac{(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2}{[\gamma^2 + (\omega - b)^2]\omega^2 - 2(\omega - b)\omega\omega_0^2 + \omega_0^4} + \frac{(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2}{[\gamma^2 + (\omega + b)^2]\omega^2 - 2(b + \omega)\omega\omega_0^2 + \omega_0^4}\right\} \tag{D17}$$

$$= 2T - \delta\omega \frac{T\gamma}{\pi} \frac{\big[\omega^2(b^2 + \gamma^2) + (\omega^2 - \omega_0^2)^2\big]\big[(1 - \sigma_V)\omega^2 + (1 - \sigma_X)\omega_0^2\big]}{\big[(\omega^2 - \omega_0^2 - b\omega)^2 + \gamma^2\omega^2\big]\big[(\omega^2 - \omega_0^2 + b\omega)^2 + \gamma^2\omega^2\big]} \tag{D18}$$

where σ_X and σ_V are defined in (C22).

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[1] The expression that we are interested in is expressed only by the Laplace kernels (A13) and its integrals w.r.t. time. These kernels are exponential functions of time and the integrals can be easily evaluated. However, the expression contains many exponential terms, the exact result is found using *Mathematica*.

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