Ergodicity and slow relaxation in the one-dimensional self-gravitating system

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We show that homogeneous and nonhomogeneous states of the one-dimensional self-gravitating sheets models have different ergodic properties. The former are nonergodic and the one-particle distribution function has a zero collision term if a proper limit is taken for the periodic boundary conditions. As a consequence, homogeneous states of the sheets model are nonergodic and do not relax to the equilibrium state, while nonhomogeneous states are ergodic in a time window of the order of the relaxation time to equilibrium, as similarly observed in other systems with a long range interaction.

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I. INTRODUCTION

Toy models retaining the main characteristics of realistic systems have always been an important tool to grasp the phenomenology of many-body systems. They have been particularly important in understanding the nonequilibrium dynamics and equilibrium properties of systems with long range interactions, which often present unusual properties not observed if the interaction is short ranged, such as nonergodicity, anomalous diffusion, non-Gaussian quasistationary states, negative microcanonical heat capacity, ensemble inequivalence, and a very long relaxation time to thermodynamic equilibrium, diverging with the particle number N [1–19]. Some one-dimensional models have been extensively studied in the literature, such as one-dimensional plasmas [20], one-dimensional self-gravitating systems such as the sheets and shell models [21], and derived models, e.g., the ring [22] and the Hamiltonian mean field (HMF) models [23]. The dynamics of systems with long range interactions can typically be divided into three stages: A violent collisionless relaxation from the initial condition into a quasistationary state (or an oscillating state close to it), occurring in a very short time [24], followed by a very slow evolution towards thermodynamic equilibrium, caused by the small cumulative effects of collisions (graininess). The final and third stage is the thermodynamic equilibrium, that may never be attained in the $N \to \infty$ limit, when the mean-field description becomes exact and the collisional contributions to the Kinetic equation vanish. In this limit, and under suitable conditions, the dynamics is exactly described by the Vlasov equation [7,25].

Let us consider a system of identical particles described by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{N} \sum_{i<j=1}^{N} V_{ij}, \tag{1}$$

with the interparticle potential $V_{ij} \equiv V(|\mathbf{r}_i - \mathbf{r}_j|)$; $\mathbf{p}_i$, $\mathbf{r}_i$ are the momentum and position for particle i, respectively, and m is the mass of the particles. The factor $1/N$ in the potential energy term in Eq. (1) is introduced such that the total energy is extensive [26] (the so-called Kac factor). In the limit $N \to \infty$ the one-particle distribution function $f(\mathbf{p}, \mathbf{r}; t)$ satisfies the Vlasov equation [25],

$$\dot{f} \equiv \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \tag{2}$$

where the mean-field force is given by

$$\mathbf{F}(\mathbf{r}; t) = -\frac{\partial}{\partial \mathbf{r}} \int V(\mathbf{r} - \mathbf{r}') f(\mathbf{r}', \mathbf{p}'; t) d\mathbf{p}' d\mathbf{r}'. \tag{3}$$

Collisional effects modify the Vlasov equation such that $\dot{f} = I[f]$, where the collisional integral $I[f]$ is a functional of f, usually obtained using some approximation such as the weak coupling limit, with the interparticle force taken to be of order $\alpha \ll 1$ and $I[f]$ computed up to order α^2, or retaining terms of order $1/N$. The resulting kinetic equations are called respectively the Landau and Balescu-Lenard equations [27]. For one-dimensional systems the collisional integral in the Balescu-Lenard, Landau, and Boltzmann equations vanish identically in a homogeneous state and one must go to the next term in the approximation, i.e., by computing $I[f]$ up to order α^3 or $1/N^2$ [8,9,28,29].

Let us consider the example of a system with a vanishing collisional integral, for both homogeneous and nonhomogeneous states, given by N identical particles in one dimension interacting only through zero-distance hard-core potential. In this case the interaction causes a swap of particle velocities, and by simply relabeling the particles at the time of the collision one obtains a statistically equivalent system of free particles, such that the one-particle distribution function only evolves due to the free flux. The corresponding kinetic

equation is then given by the one-dimensional Liouville equation with zero force:

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} = 0, \tag{4}$$

where m is the mass, x is the position, and p the momentum. For a homogeneous state the one-particle distribution function is thus strictly constant.

Yet another simple model, but with long range interacting particles and real collisions (due to the discontinuity in the force at zero distance) is the one-dimensional self-gravitating system of identical particles, with unit mass and Hamiltonian [30]

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{N}\sum_{i<j=1}^{N} |x_i - x_j|. \tag{5}$$

In Eq. (5) units were chosen such that we have a unit mass and the Kac factor in the potential energy term. The force on particle i is given by $F_i = (N_+^{(i)} - N_-^{(i)})/N$, where $N_+^{(i)}$ ($N_-^{(i)}$) is the number of particles with coordinate $x > x_i$ ($x < x_i$). In this model particles can cross each other freely. The potential in the Hamiltonian is obtained from the solution of the Poisson equation in one spatial dimension, and corresponds to a system of N infinite sheets with total finite mass.

The initial violent relaxation of this model is mainly a collisionless process, and is well described by the Vlasov equation (see [18,19] and references therein). On the other hand, after the violent relaxation, the evolution towards thermodynamic equilibrium is due to collisional corrections to the Vlasov equation, which are required to properly describe it. This relaxation process has been studied in the literature in the last few decades, with the recurrent question, does the system relax to thermodynamic equilibrium, due to the extremely slow dynamics of its macroscopic parameters [30–34]? Joyce and Worrakitpoonpon introduced an order parameter to measure the distance to equilibrium and showed that this system in a nonhomogeneous state evolves to thermodynamic equilibrium [35]. They showed this for a number of particles up to $N = 800$, and yet requiried a very large simulation time to observe the complete relaxation. This implies that the contribution of the collisional integral of the corresponding kinetic equation is very small.

The very slow relaxation towards equilibrium also manifests in the ergodic properties of the system. In order to assess if a given system is ergodic, and to discuss the time windowed required to evidence this property, we use an approach introduced in Refs. [10,15], which is based on the fact that the mean-field description is valid for long range systems with sufficiently large N (up to very small corrections) [25]. In this case, correlations among particles are negligible, and observable averages can be computed from one-particle reduced distribution functions $f_1(p, x; t)$ (see for instance Ref. [27]). Thence one can formulate the physics in μ space (one-particle phase space), such that all information required on the statistics of the system derive from the time evolution of the one-particle distribution function $f_1(p, x; t)$. The usual definition of ergodicity, i.e., that that the time average of an observable along a trajectory in the N-particle phase space is equal to the ensemble equilibrium (a large number mental copies of the N-particle system), then translates into the one-particle phase space as the equality of averages over the history of a single particle to averages over the N particles at a given time. This amounts to considering the N particles as our ensemble, which is only valid for long range systems where the mean-field description is valid for sufficiently large N. As shown in Ref. [10] for two-dimensional self-gravitating systems, this approach is equivalent to other methods used to assess ergodicity. Therefore, a system with long range interactions is ergodic if averages of observables over the history of a single particle are equal to the ensemble average, i.e., to an average computed at a fixed time for the N particles in the system. This approach was used for the HMF model [15,36] and for a two-dimensional self-gravitating system [10]. It was shown that in the limit $N \to \infty$ these systems are nonergodic, and never reach the true thermodynamic equilibrium, while for finite N they are ergodic only after a time window of the order of the relaxation time to equilibrium.

Here we show that these results are also valid for the one-dimensional self-gravitating system with Hamiltonian in Eq. (5) for a nonhomogeneous state, but not for the homogeneous case. Indeed in the latter we show that by properly considering periodic boundary conditions and then taking the limit of the size of the unit cell going to infinity, while keeping the density constant, the one-particle distribution function does not evolve in time, i.e., the collisional effects vanish. The present paper is structured as follows: In Sec. II we discuss separately the ergodic properties of homogeneous and nonhomogeneous states of the sheets model. The kinetic equation for the homogeneous state is obtained in Sec. III with identically vanishing collisional contributions. We close the paper with some concluding remarks in Sec. IV.

II. SLOW DYNAMICS AND ERGODICITY

We investigate the ergodic properties of the sheets model system using the approach in Ref. [10]. The system is ergodic if time averages, computed over a given time window of length t_e, equals the ensemble average over the N particles at this same fixed time t_e, which we call ergodicity time. The value of t_e must be seen as an estimate of the order of magnitude such that the difference between the time and ensemble averages, in the sense discussed above, is sufficiently negligible. Following the discussion in the Introduction, we define the time average of the momentum of the kth particle,

$$\overline{p}_k(t) = \frac{1}{M}\sum_{j=1}^{M} p_k(j\Delta t), \tag{6}$$

and similarly the time average of its position,

$$\overline{x}_k(t) = \frac{1}{M}\sum_{j=1}^{M} x_k(j\Delta t), \tag{7}$$

with a fixed time step Δt, $M = t/\Delta t$. We also consider the time dependent standard deviations (supposing the averages over all particles vanish $\langle x\rangle = 0$ and $\langle p\rangle = 0$):

$$\sigma_p \equiv \sqrt{\frac{1}{N}\sum_{k=1}^{N} \overline{p}_k^2(t)} \tag{8}$$

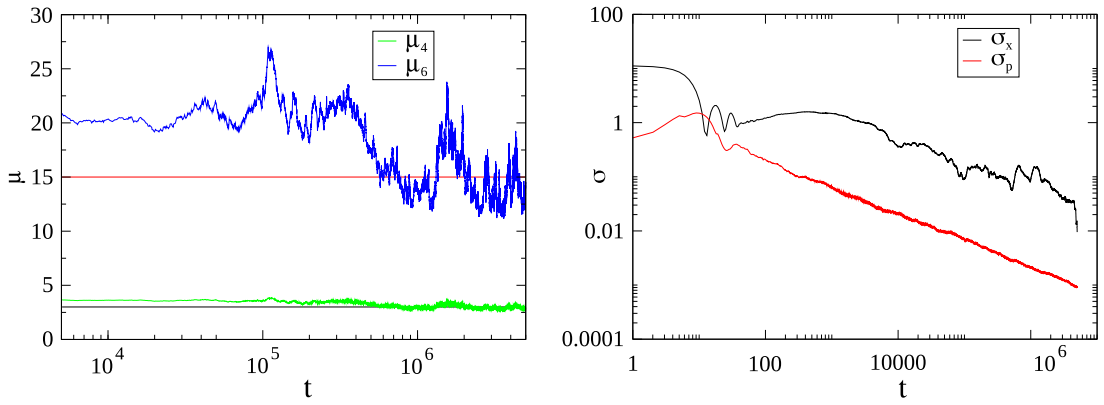

FIG. 1. Left: Reduced moments μ_4 and μ_6 as a function of time for $N = 100$ and a waterbag initial state with $x_0 = 10.0$ and $p_0 = 0.5$ for the system with the Hamiltonian in Eq. (5) and open boundary conditions. A running average was performed over a time window of $\delta t = 10\,000$. The straight lines correspond to the equilibrium values of $\mu_4 = 3$ and $\mu_6 = 15$ introduced for comparison purposes. Right: Standard deviations for $\overline{p}_k$ and $\overline{x}_k$ in Eqs. (8) and (9).

and

$$\sigma_x \equiv \sqrt{\frac{1}{N}\sum_{k=1}^{N}\overline{x}_k^2(t)}. \tag{9}$$

Ergodicity for a system with long range interaction is then equivalent to stating that $\sigma_p(t) \to 0$ and $\sigma_x(t) \to 0$ for sufficiently large t. The order of magnitude of t required such that $\sigma_p(t)$ and $\sigma_x(t)$ are significantly small is thus ergodicity time t_e. It was discussed for the HMF model and for a two-dimensional self-gravitating system that $t_e \approx t_r$, with t_r the relaxation time to thermodynamic equilibrium [10,15,36].

We now consider separately the ergodic properties of nonhomogeneous and homogeneous states of the sheets model. Nonhomogeneous states of this model can for instance be used to model the dynamics of stars moving in a direction perpendicular to highly flattened galactic planes [37], while homogeneous states were studied as simplified cosmological models (see [38] and references therein). For the latter case, homogeneous states representing a homogeneous simplified cosmological model are of particular interest. A purely attractive potential can only have a homogeneous equilibrium state (for sufficiently high temperature) if it is infinite, and are modeled using periodic boundary conditions, considering the forces due to the images of the unit cell (see below). A thorough discussion of the equilibrium properties for both cases is given in Ref. [19].

A. Nonhomogeneous state

In order to find evidence of the very large value of the ergodic time t_e, we implemented a molecular dynamics simulation of an open N-particle system (no spatial boundary conditions) with the Hamiltonian in Eq. (5) using an event-driven algorithm [39]. The dynamics between two successive particle crossings is integrable, and can be computed up to machine precision. Collisions are then implemented straightforwardly by updating the force on the particles after each crossing. Due to very high local densities at the core of the spatial distribution, a high numeric precision is required, and we used quadruple precision in order to avoid missing any collision due to round-off errors (which indeed occur for double precision). The initial state is a waterbag state defined by

$$f(x,p;0) = \begin{cases} 1/(4p_0x_0) & \text{if } -x_0 < x < x_0 \text{ and } -p_0 < p < p_0, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

with x_0 and p_0 given constants. To measure the distance to the Gaussian distribution we use the reduced moments,

$$\mu_k \equiv \frac{\langle p^k \rangle}{\langle p^2 \rangle^{k/2}}. \tag{11}$$

The reduced moment of order 4 is called the kurtosis of the distribution, and for any Gaussian distribution we have that $\mu_4 = 3$ and $\mu_6 = 15$. The left panel of Fig. 1 shows the time evolution of μ_4 and μ_6 for the system, with $x_0 = 10.0$ and $p_0 = 0.5$ for the initial condition and $N = 100$. In this case the relaxation time to equilibrium is of the order of $t_r \approx 10^6$. The right panel of Fig. 1 shows that the condition for ergodicity stated above is satisfied for a value of time of the order of magnitude of the relaxation time $t_e \approx t_r$. By increasing the number of particles in the simulation the relaxation time t_r rapidly becomes too large for a feasible simulation using the event driven algorithm. We also note that collisional effects are more important for smaller values of N.

In order to discuss the physical meaning of ergodicity for a long range interacting system, we define the one-particle momentum and position probability densities $\phi(p;t)$ and $\rho(x;t)$ for the N particles at a given time t as the probability density for the given values of p and x, respectively. We also define the density distribution for the values of p and x of a fixed particle, say the kth particle, along its history up to time t, and denoted by $g(p;t)$ and $h(x;t)$, respectively. Then, in the

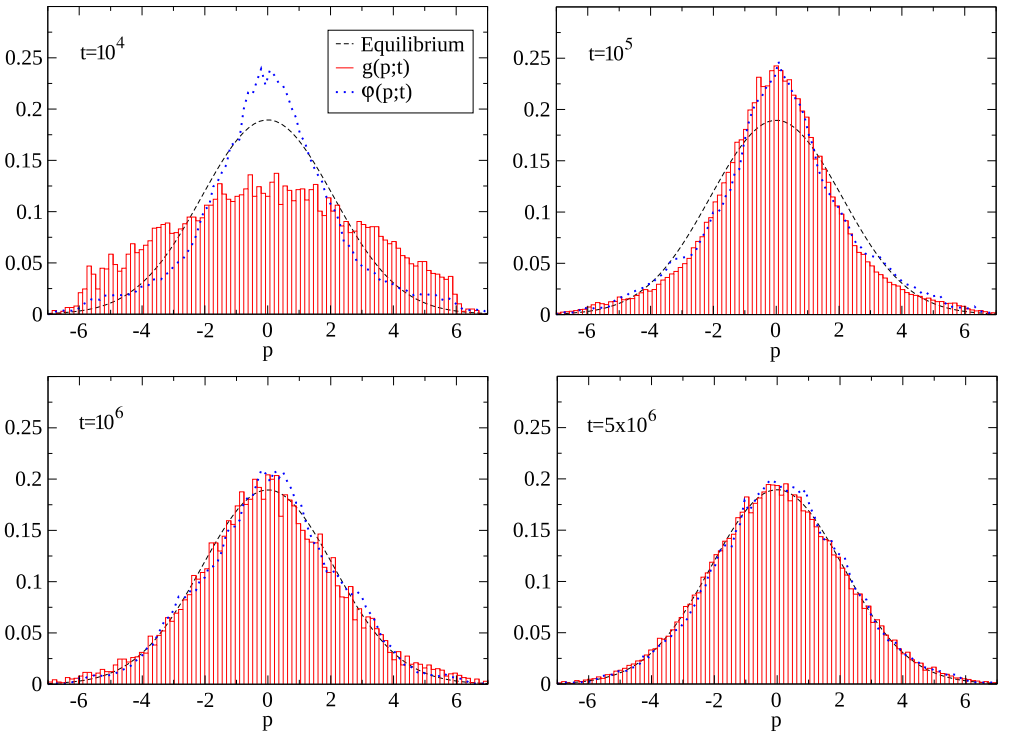

FIG. 2. Distributions $\phi(p;t)$ (dotted line) and $g(p;t)$ (histogram) for the same initial condition as in Fig. 1 with $N = 100$, and for few values of t. The dashed line is the equilibrium Gaussian with inverse temperature $\beta = 0.225$. This value of β as obtained by averaging the kinetic energy for a time window of size $\delta t = 10\,000$ at the end of the simulation. The precision for the histogram for $p_k(t)$ was increased by collecting the values of the momenta of all particle from time $t - 100$ up to t, justified by an expected negligible change in the statistical distribution for a relatively short period of time.

present case, ergodicity is equivalent to the relations

$$\phi(p;t) = g(p;t) \tag{12}$$

and

$$\rho(x;t) = h(x;t), \tag{13}$$

for $t \gtrapprox t_r \approx t_e$. Figures 2 and 3 show these distributions for a few values of time, and also the spatial distribution function at equilibrium given by $\rho(x) = C\text{sech}(x/\Lambda)$, with $\Lambda = 4e/3$ and e the mean-field energy per-particle [40], and the momentum Gaussian distribution at equilibrium. It is evident that the time and ensemble distributions become very close as t approaches t_r. So the momentum ϕ and g, and spatial ρ and h, distribution functions satisfy Eqs. (12) and (13) and are also equal to the equilibrium distribution for a time of the order of magnitude of the relaxation time to equilibrium, as was also observed for other long range interacting systems [10,15,36].

B. Homogeneous state

We now turn to the case of a homogeneous state. Periodic boundary conditions can be implemented using an Ewald sum with a unit cell $x \in [-L, L]$ such that the force on each particle, due to the particles in the unit cell and the infinite number of images, is determined by a direct sum over replicas [41]. For the one-dimensional self-gravitating system a closed analytical form was obtained by Miller and Rouet [42] as an additional potential representing all replicas, and given with our choice of units by

$$V_{\text{Ewald}} = -\frac{1}{N}\sum_{i=1}^{N}\frac{(x_{CM} - x_i)^2}{2L}, \tag{14}$$

where $x_{CM} \equiv (1/N)\sum_{i=1}^{N} x_i$ is the position of the center of the mass in the unit cell. Therefore the total force on the ith particle, for $x_i \neq x_j$ for $i, j = 1, \ldots, N$, is given by

$$\begin{aligned} F_i &= -\frac{\partial}{\partial x_i}\left[\frac{1}{N}\sum_{i,j=1}^{N}|x_i - x_j| + V_{\text{Ewald}}\right] \\ &= \frac{N_>^{(i)} - N_<^{(i)}}{N} + \frac{1}{LN}(x_i - x_{CM}), \end{aligned} \tag{15}$$

where $N_>^{i)}$ ($N_<^{(i)}$) is the number of particles at the right (left) of particle i on the straight line. The full effective Hamiltonian with periodic boundary conditions is then

$$H = \sum_{i=1}^{N}\frac{p_i^2}{2} + V(\mathbf{x}), \tag{16}$$

with $\mathbf{x} \equiv (x_1, \ldots, x_N)$ and

$$V(\mathbf{x}) = \frac{1}{N}\sum_{i<j=1}^{N}|x_i - x_j| - \frac{1}{N}\sum_{i=1}^{N}\frac{(x_{CM} - x_i)^2}{2L}. \tag{17}$$

The resulting equations of motion are then integrated using a fourth-order symplectic integrator [43,44]. Figure 4 shows the reduced moments μ_4 and μ_6 as a function of time for

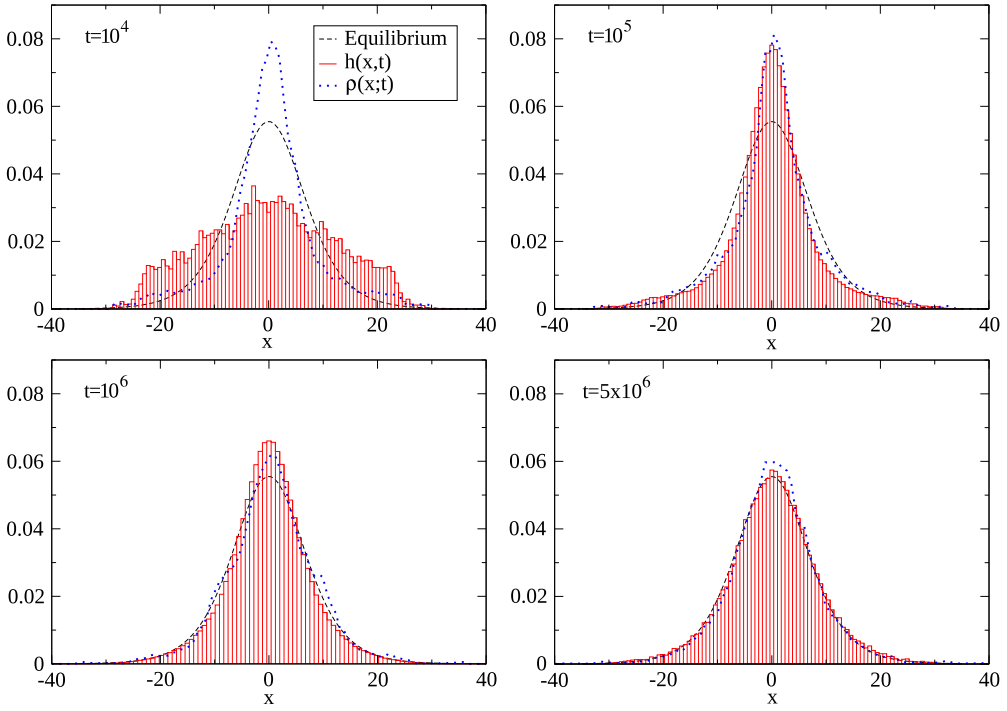

FIG. 3. Same as Fig. 2 but for $h(x;t)$ and $\rho(x;t)$. The dashed line is the spatial distribution function at equilibrium $\rho(x) = C\mathrm{sech}(x/\Lambda)$, $\Lambda = 4e/3$ and e the mean-field energy per-particle [40].

$N = 128$, up to $t = 10^5$, for an initial waterbag state with $x_0 = L = 1$ and $p_0 = 3$. The system remains in a homogeneous state for the whole simulation time. We observe that the time evolution is extremely slow if compared to the nonhomogeneous case, with only a very small variation in μ_6 visible in the graphic. Figure 5 shows the distribution functions $g(p;t)$ and $h(x;t)$ at the final time, also clearly at variance to what is observed for the nonhomogeneous cases. Although the spatial distribution $g(x;t)$ is roughly uniform, as expected since particles can cross each other and the mean-field force is very small, the distribution of the momentum $g(p;t)$ over the history of one single particle is not even symmetrical, as is the case for the nonhomogeneous systems at all time values, except for a very short initial time. The particle in Fig. 5 will eventually change the sign of its momentum after a sufficiently long time (see for instance Fig. 2 of Ref. [36]). The fact that in this example it keeps the same sign up to the referenced time results from the value of time in the figure being much less than the ergodicity time. We note here that, although the average momentum over the N particles is zero, the time average of the momentum of a single particle does not have to vanish, unless the system is ergodic. We conclude that the time for ergodicity, if finite, is certainly many orders of magnitude greater than for a nonhomogeneous state. We will shed some light and explain the physical origin of this difference, and of the peculiar dynamics of the homogeneous state, in the next section by discussing the kinetic theory for a homogeneous state.

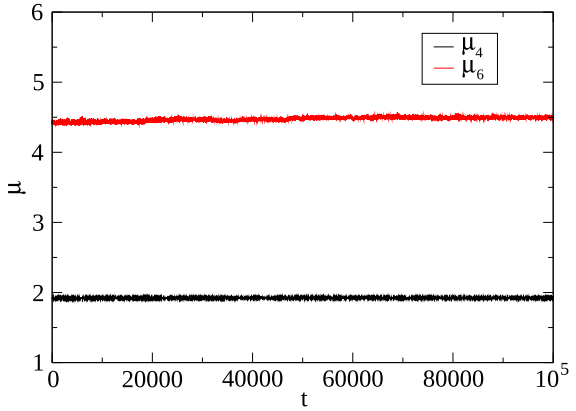

FIG. 4. Reduced moments of p for a homogeneous state with a waterbag initial condition with $N = 128$, $x_0 = 1.0$, and $p_0 = 3.0$.

III. KINETIC EQUATION FOR A HOMOGENEOUS STATE

The statistical dynamics of a system of many-particle systems can be studied by determining a kinetic equation describing the time evolution of the one-particle distribution function. We first define the N-particle distribution function $f_N(x_1, p_1, \ldots, x_n, p_n; t)$ as the probability density in the N-particle phase space, which satisfies the Liouville equation, and is assumed fully symmetric by particle permutations. The s-particle reduced distribution function is obtained by integrating out $N - s$ particles as

$$f_s(x_1, p_1, \ldots, x_s, p_s; t) \equiv \int dx_{s+1} p_{s+1} \cdots x_N p_N f_N(x_1, p_1, \ldots, x_N, p_N; t). \quad (18)$$

An usual starting point to determine a kinetic equation is the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY)

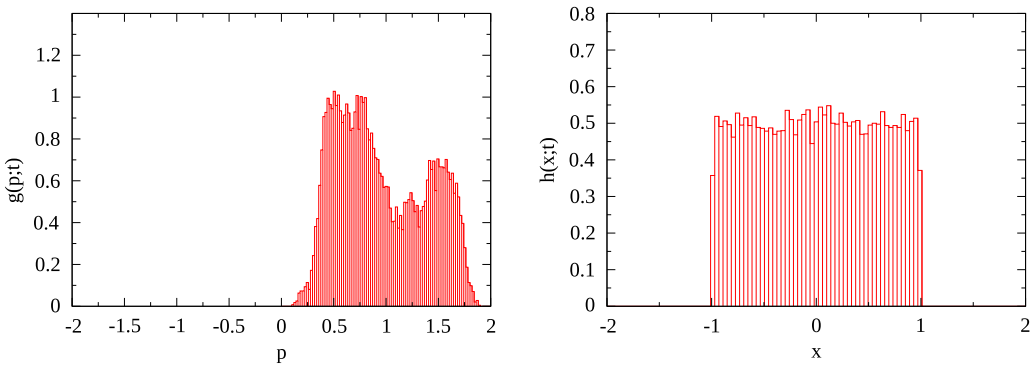

FIG. 5. Left: Momentum distribution function $g(p;t)$ over the history of a fixed single particle at $t = 10^5$ for the same simulation as in Fig. 4. Right: Position distribution function $h(x;t)$ over the history of a fixed single particle for the same simulation.

hierarchy for the reduced distribution functions [27,45]:

$$\frac{\partial}{\partial t} f_s(1,\ldots,s;t) = -\sum_{k=1}^{s} p_k \frac{\partial}{\partial x_k} f_s(1,\ldots,s;t) + \frac{1}{2}\sum_{\substack{k,l=1 \\ (k\neq l)}}^{s} v'_{kl}\,\partial_{kl} f_s(1,\ldots,s;t) + (N-s)\sum_{k=1}^{s}\int d(s+1)\, v'_{k,(s+1)} \frac{\partial}{\partial p_k} f_{s+1}(1,\ldots,s+1;t), \tag{19}$$

where $v_{jk} \equiv v(x_j - x_k)$ is the interparticle potential, $\partial_{kl} \equiv p_k\partial/\partial x_k - p_l\partial/\partial x_l$, $1 \equiv x_1, p_1$, $2 \equiv x_2, p_2$, ..., $d1 \equiv dx_1 dp_1$, $d2 \equiv dx_2 dp_2$, ..., and so on. The potential in Eq. (17) corresponds to the interparticle potential

$$v_{ij} = v(x_i - x_j) = \frac{1}{N}|x_i - x_j| - \frac{1}{2LN^2}\sum_{i,j=1}^{N}(x_i - x_j)^2, \tag{20}$$

such that $V = \sum_{i,j=1}^{N} v_{i,j}$, after some straightforward algebra and using the expression for the center of mass position. The case with $s = 1$ leads to the prototypical kinetic equation,

$$\left[\frac{\partial}{\partial t} + p_1\frac{\partial}{\partial x_1}\right] f_1(1;t) = (N-1)\frac{\partial}{\partial p_1}\int d2\, v'_{12}\, f_2(1,2;t). \tag{21}$$

In order to obtain a closed-form expression for the kinetic equation one must determine an expression for the two-particle distribution f_2 in terms of f_1. For uncorrelated particles we have $f_2(1,2;t) = f_1(1;t)f_1(2;t)$, and Eq. (21) then results in the Vlasov equation (2).

We perform the cluster expansion [27]:

$$\begin{aligned} f_2(1,2;t) &= f_1(1;t)f_1(2;t) + g_2(1,2;t), \\ f_3(1,2,3;t) &= f_1(1;t)f_1(2;t)f_1(3;t) + f_1(1;t)g_2(2,3;t) \\ &\quad + f_1(2;t)g_2(1,3;t) + f_1(3;t)g_2(1,2;t) \\ &\quad + g_3(1,2,3;t), \end{aligned} \tag{22}$$

and so on, where g_s is the s-particle correlation function. By plugging Eq. (22) into Eq. (19) for $s = 1$ we have

$$\frac{\partial}{\partial t} f_1(1;t) = N\int d2\, v'_{12}\partial_{12}[f_1(1;t)f_1(2;t) + g_2(1,2;t)]. \tag{23}$$

From Eqs. (19), (22), and (23) we obtain the following equation for the two-particle correlation function [27]:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + p_1\frac{\partial}{\partial x_1} + p_2\frac{\partial}{\partial x_2}\right) g_2(1,2,t) \\ &= v'_{12}\partial_{12} f_1(1;t)f_1(2;t) + v'_{12}\partial_{12} g_2(1,2;t) \\ &\quad + N\int d3[v'_{13}\partial_{13} f_1(1;t)g_2(2,3;t) + v'_{23}\partial_{23} f_1(2;t) \\ &\quad \times g_2(1,3;t) + (v'_{13}\partial_{13} + v'_{23}\partial_{23})\{f_1(3;t)g_2(1,2;t) \\ &\quad + g_3(1,2,3;t)\}]. \end{aligned} \tag{24}$$

By neglecting three-particle correlations in Eq. (24) we obtain a closed integrodifferential equation for g_2, and plugging its solution into Eq. (23) yields the Balescu-Lenard kinetic equation [27]. The case with non-negligible three particle correlations was considered in Refs. [8,29].

The derivative of the interparticle potential v in Eq. (20) appears in Eq. (24) and we must consequently account for the singularity of its derivative at zero interparticle distance. In order to deal with this, we consider the following relabeling of particle indices: At the moment two particles (sheets) cross each other, we interchange their labels. In this way, at each collision (at zero distance) particles simple exchange their momenta and the force is constant in time. If the particles are initially labeled such that $x_i < x_j$ if $i < j$, the ordering in position is preserved. Then the force on particle i due to particle j is now given by

$$F_{i,j} = -\partial v_{i,j}/\partial x_i + F_{\mathrm{HC}}, \tag{25}$$

and F_{HC} stands for the hard-core force that swaps particle momenta when they collide at zero distance. It is important here to note that although the resulting kinetic equation for the one-particle reduced distribution function is the same in both pictures (particles not being able to cross each other), the resulting ergodic properties are different, as the history of a fixed given particle is different in both. The contribution of the first term in the right-hand sides of Eqs. (25) to (24), then

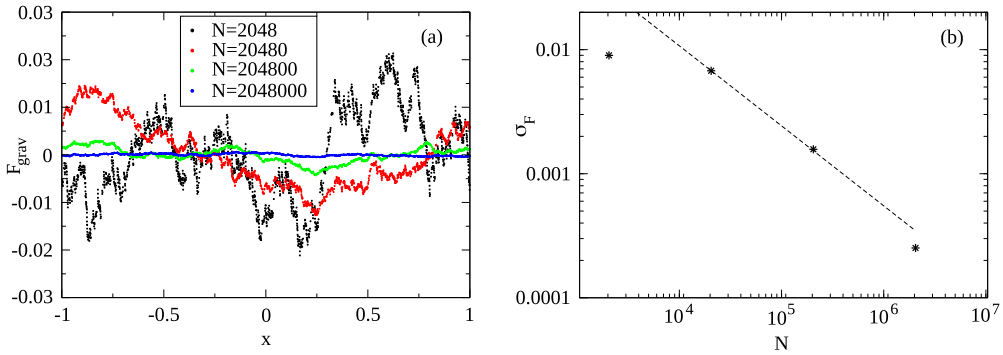

FIG. 6. (a) Force on a particle as a function of its position as given by Eq. (15) for the one-dimensional self-gravitating system with an additional potential from the Ewald sum, for different number of particles but same density $n = N/L$. The position was rescaled to the interval $[-1, 1]$ for comparison purposes. The unit cell for $N = 2048$ is given by $L = 20$, and is obtained accordingly for other values of N in order to keep the spatial density constant. (b) Standard deviations σ_F for the force in (a) for each value of N. The dashed line is the best fit of a power law for the three larger values of N with $\sigma_F \propto N^{-0.645} \propto L^{-0.645}$.

it vanishes in the limit $L \to \infty$ as the gravitational force in a homogeneous state vanishes. To illustrate this fact, Fig. 6(a) shows the force $F_{\rm grav}$ due to both the self-gravitating potential and the Ewald sum, for a few values of the number of particles N but keeping the density $n = N/L$ constant. The standard deviation σ_F of the force is shown in Fig. 6(b) with the best fit of a power law in N and L, evidencing that the force on each particle does tend to zero as N (and L) increase; i.e., that, as the size $L = N/n$ of the unit cell increases, $F_{\rm grav}$ approaches zero. We conclude that increasing N in this way is not equivalent to considering the Vlasov limit, that would correspond to taking $N \to \infty$ but keeping L constant. As a consequence, only contributions from hard-core collisions are retained in Eq. (24). This result in fact proves the validity of the "Jeans swindle" for the model considered here, i.e., that the contribution of the background interaction to the infinite homogeneous contribution vanishes, and one must consider only the effects of local fluctuations in density [46,47]. These fluctuations vanish as the size of the unit cell goes to infinity, with constant particle density.

The same reasoning can be used in an analogous way for the BBGKY hierarchy, which then takes exactly the same form as the hierarchy obtained for a system of particles with a hard-core potential at zero distance as the only interaction. For such a system in an homogeneous state, the one-particle distribution function $f_1(p;t)$ is strictly constant in time as the interaction only swaps the momenta of two particles at each collision, and three-particles processes are nonexistent (the probability that three particles collide at the same time at the same point is zero). For the same initial condition, the BBGKY hierarchy being identical for both systems, the time evolution for the reduced distribution functions must be the same, and therefore the distribution $f_1(p;t)$ for a homogeneous one-dimensional self-gravitating system is constant in time. Small deviations from this are expected to occur in numerical simulations due to spurious nonphysical effects resulting from a finite value of L, that result in small fluctuations of the value of the force around zero. We also note that for smaller values of the energy the homogeneous state is unstable, in the sense that any small deviation from homogeneity creates a runaway effect out of this state, and a corresponding violent relaxation ensues.

IV. CONCLUDING REMARKS

We showed that the sheets model describing a one-dimensional self-gravitating system has profoundly different dynamic properties depending on whether it is in a homogeneous or a nonhomogeneous state. In the former case we showed that by considering a proper limit in the periodic boundary conditions the one-particle evolution function does not evolve in time, as its kinetic equation is essentially a Boltzmann-like equation, with the exception that if the homogeneous state is unstable (for smaller energies) then an evolution of the distribution function occurs due to the amplification of small deviations from homogeneity. For the nonhomogeneous state, the system has a slow dynamics to equilibrium, with a relaxation time much greater than for other long range interacting systems if one uses the violent relaxation time for comparison. The nonhomogeneous system is ergodic but only after a time of the order of the relaxation time to equilibrium, as also observed for other long range interacting systems, but it is nonergodic in a homogeneous state, as illustrated by simulations presented here.

It is interesting to compare the physical properties of the homogeneous state of the model considered here with those for the Hamiltonian mean field (HMF) model. The HMF model does indeed relax to thermodynamic equilibrium for both the homogeneous and nonhomogeneous states. For the antiferromagnetic HMF model, quasistationary states formed after the violent relaxation are always homogeneous, and it was shown in Ref. [48] that the motions of the particles are approximately ballistic, with small corrections described in terms of independent Brownian noises. For the one-dimensional homogeneous self-gravitating system, collisions are pointlike and occur only when two particles are at the same position, and otherwise the dynamics is integrable, while for the HMF model this is not the case. Although the HMF can be derived from the one-dimensional self-gravitating system by truncating at first order the Fourier expansion of the corresponding Green function, the physical properties are profoundly different. For the Hamiltonian in Eq. (5) and after taking the limit $L \to \infty$, $N \to \infty$ and N/L constant, collisions are hard core pointlike, i.e., they occur only at zero distance due to the discontinuity in the potential. Nevertheless,

if the unit cell is kept finite (as N), then the force between collisions are not zero, resulting in graininess, although no clear relaxation to equilibrium could be observed in our simulations, possibly due for a too large timescale involved. It is worth noticing that the number of particles is infinite for both finite or infinite L, due to the periodic boundary condition and the infinite number of replicas. What occurs is that graininess is removed when a homogeneous truly (nonperiodic) infinite system is considered. For the HMF model, and for other similar models, graininess is always present if N is finite, resulting in corrections to the mean field description in the force acting on each particle at every position, with no discontinuity in the interparticle potential.

A possible way to shed some light on the slow dynamics of this system in nonhomogeneous states is to obtain a kinetic equation, which for the present model is a challenging task as it requires the determination of action-angle variables for the mean-field description of the system [49,50], and has been possible only for very special cases (see [51] and references therein). This is the subject of ongoing research.

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