Probability inequalities for direct and inverse dynamical outputs in driven fluctuating systems

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When a fluctuating system is subjected to a time-dependent drive or nonconservative forces, the direct-inverse symmetry of the dynamics can be broken so inducing an average bias. Here we start from the fluctuation theorem, a cornerstone of stochastic thermodynamics, for inspecting the unbalancing between direct and inverse dynamical outputs, here called "events," in a bidirectional forward-backward setup. The occurrence of an event might correspond to the realization of a quantitative output, or to the realization of a sequence of acts that compose a complex "narrative." The focus is on mutual bounds between the probabilities of occurrence of direct and inverse events in the forward and backward mode. The inspection is made for systems in contact with a thermal bath, and by assuming Markov dynamics on the uncontrolled degrees of freedom. The approach comprises both the case of systems under a time-dependent drive and time-independent external forces. The general formulation is then used to derive (or re-derive) specialized results valid for finite-time processes, and for systems taken into steady conditions (either periodic steady states or steady states) starting from equilibrium. Among the results, we find already known forms of "generalized" thermodynamic uncertainty relations, and derive useful constraints concerning the work distribution function for systems in steady conditions.

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I. INTRODUCTION

The stochastic formulation of nonequilibrium thermodynamics is undoubtedly one of the most striking advancements in the physical statistics of the last decades [1]. The ambit is that of systems taken out of equilibrium and for which, on average, a directed dynamical behavior is allowed thanks to the induced breakdown of the direct-inverse symmetry. Something useful, or at least most interesting, comes out in contrast to the otherwise unbiased equilibrium scenario. The strength of the achievements lies in the fact that most of them are universal relations, equalities and inequalities, that are valid under general conditions and do not involve detailed features of the specific system.

Milestones in stochastic thermodynamics are the Jarzynski equality [2,3] and the Crooks fluctuation theorem (FT) [4,5], both of which came at the time when the manipulation of single biomolecules was beginning to be feasible, paired with the computer simulations counterpart. Although the initial practical impact was mainly on the charting of free energy landscapes from finite-time nonequilibrium transformations starting from thermal equilibrium, those early theorems own a much wider manifold of potentialities [6].

The FT, in particular, connects the probability of observing direct trajectories with the probability of the inverse ones under an external action, and sets a relationship between dissipation and breakdown of the reversibility. The main assumption consists in the Markov nature of the fluctuations. The FT concerns "bidirectional" processes, where the term "bidirectionality" takes on different meanings in the various forms of the theorem [7]. In the early Crooks version, the FT was obtained for the most natural kind of bidirectionality: The system is subjected to forward and backward transformations in which a parameter is changed in a certain time-dependent way (forward protocol) or in the opposite way (backward protocol). Later, the FT was reformulated [7–9] also for other conceptualizations of bidirectionality and, with a proper identification of the amounts of work and heat exchanged by system and exterior [9,10], also including external nonconservative forces that break the microreversibility.

From the FT, a wealth of corollaries and logical consequences can be derived. Among other achievements, it is worth mentioning the thermodynamic quantification of the length of time's arrow [11] and the assessment of its direction [6,12], England's viewpoint on self-replication [13] and driven self-assembly [14], and the recent derivation of so-called generalized thermodynamic uncertainty relations (TURs) concerning the interplay between relative precision on a dynamical output and required average energy cost [15–22].

Here we exploit the FT for inspecting the mutual bounds between the probability of occurrence of an event $\mathcal{E}$ in what we identify as the forward process (F), and the probability of occurrence of the inverse event $\tilde{\mathcal{E}}$ in the conjugate backward process (B). As will be explained in greater detail in Sec. III A, an event is a dynamical output that corresponds, in general terms, either to the realization of a quantitative output or to a "narrative" made of one or more clauses to be fulfilled. Such a narrative may be so complicated as to prevent its translation into an intelligible mathematical expression; rather, it could resemble more of a logical structure (like a piece of computer program with if-then-else conditions) to be checked in the course of the "execution" of the specific trajectory. By supposing to be able to observe the system in detail during its evolution, we would indeed be able to detect whether the event, in a given observation time window, actually took

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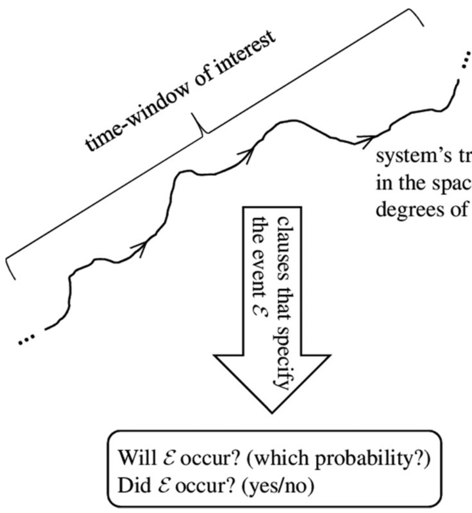

FIG. 1. Abstract representation of an event's "extraction" from a snippet of trajectory in the space of the system's degrees of freedom. The realization of event $\mathcal{E}$ corresponds to the fulfillment of a set of clauses. The fulfillment can be checked on the fly and so, ultimately, one can answer the question "Did $\mathcal{E}$ occur?" In terms of expectation, one focuses on the *probability* of occurrence of the event.

place or not. Before making the observation, it is therefore legitimate to express the probability that such an event *will* be observed. The idea is depicted in Fig. 1. While the FT is an *equality* involving the probabilities for direct and inverse trajectories, *inequalities* (i.e., the bounds we are interested in) come out from the "bundling" of all the trajectories that pertain to the realization of events $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

The system is described here at the level of continuous degrees of freedom (the reduction to the discrete state space can be done straightforwardly). We will consider both nonautonomous systems subjected to a time-dependent energy modulation, and autonomous systems subjected to constant nonconservative forces. Within these two scenarios, we treat the cases of finite-time processes starting from equilibrium, of systems kept in periodic steady states under the action of a periodic external drive, and of systems in steady states under external constant forces. In each case, the time window of interest is identified with the duration t_f of the finite-time process, or with the period τ of the external driving, or with a certain observation time t_{obs} under steady-state conditions.

It is to be noted that in our setup there are two kinds of directionality. One pertains to the direction of the process (F or B), which is determined by what we call "forward" and "backward" direction. The other directionality concerns the development of the sequence of elemental acts that compose the event of interest: "direct" (for $\mathcal{E}$) and "inverse" (for $\tilde{\mathcal{E}}$) are nothing but subjective labels that we use to identify and distinguish the two opposite ways. For instance, if the event were simply the realization of a certain trajectory in the space of the system's degrees of freedom, the inverse event would be the same trajectory traveled in the opposite way.

Starting from the FT, we derive a family of inequalities in which the probabilities of occurrence of the direct and inverse events, or even of a general number of events under the clause of their mutually exclusive occurrence, are constrained one with the other by the average amount of energy dissipation. The inequality in Eq. (12) will be the central relation: The left-hand side is a quantity built with the event's probabilities, while the right-hand side quantifies the average amount of dissipation. In each case, this inequality becomes a constraint for delimiting the range of variation of the events' probabilities given the average dissipation. It is important to highlight that this viewpoint is different from the one typically adopted, namely, exploiting the observation of the system to estimate or bound the dissipation; here the perspective is reversed. Then, using Eq. (12) as a "stepping stone," the proper choice of the dynamical output of interest allows us to derive specialized results such as those presented in Sec. V. In passing, one of the generalized TURs mentioned above will be re-derived here as a corollary of the more general results. In a specific application, we will also reverse the viewpoint by considering the likelihood that a process was F or B if it is known that a certain event did take place. Upper and lower bounds on such a likelihood will be derived.

The paper is organized as follows. In Sec. II we present the physical setup: The nature of the fluctuations, types of external drive, definition of the forward and backward processes, a brief summary of the fluctuation theorem, and the description of the cases of interest in this work. Insights are provided in Appendixes A and B. In Sec. III we introduce the concept of event, set the terminology, and define the main quantities that are built on the basis of the probabilities of the occurrence of the events. The main achievements are presented in Sec. IV (mathematical proofs are given in Appendix C and in technical footnotes). In that section, the general inequality in Eq. (12) is derived and then specified for the cases of interest. Section V contains some applications that enable us to explore different aspects of the statistical unbalancing induced by an external drive. Illustrative examples are provided in Sec. VI. Section VII is devoted to final remarks and perspectives. Throughout the text, $\beta = (k_B T)^{-1}$ with k_B the Boltzmann constant and T the absolute temperature.

II. PHYSICAL SETUP

A. Fluctuations and external drive

Let us consider a driven fluctuating system where the fluctuation theorem applies. From now on, it is best to refer to the typical situation of a nanoscopic system in a fluid phase acting as a thermal bath at a fixed temperature. The dynamics is intended to be a continuous-time Markov process on continuous degrees of freedom $\mathbf{x}$ [23]. The dynamics can be overdamped (diffusive dynamics on pure conformational degrees of freedom) or underdamped (i.e., also the momenta are included among the stochastic variables).

Two types of processes will be considered. In one situation, the system is subjected to a time-dependent drive that affects the energetics; hence the dynamics is a nonautonomous process. In the other scenario, the system is subjected to an external nonconservative force that directly acts on the degrees of freedom, $\mathbf{x}$, and induces a drift without altering the bare energetics of the system. We will only consider the case of time-independent forces. Other scenarios, like the simultaneous presence of time-dependent energy modulation and external forces, or the case of time-modulated external forces, are peculiar situations not considered here for the sake of simplicity. Let us now introduce the F and B processes in general terms; the specific cases of interest will be presented in Sec. II C.

In the situation of time-dependent energy modulation, the external driving agent could be a device which allows the controlled change of a structural parameter of the system (as in the mechanical manipulations of biomolecules), or it could be an external time-dependent field interacting with some property of the system (like an oscillating electric field interacting with an electric dipole of the molecular system of interest). Let $\lambda(t)$ be the controlled parameter. In the forward process, the initial system's microstate at time zero, $\mathbf{x}(0)$, is meant to be picked from *a* distribution $\rho_0(\mathbf{x})$ supposed to be "prepared" by previous actions on the system, and the controlled parameter is changed from λ_0 to λ_1 according to a chosen protocol $\lambda_F(t)$ with $0 \leqslant t \leqslant t_f$. In the backward process, we assume that the system's microstate is initially picked from *a* distribution $\rho_1(\mathbf{x})$, like before, supposed to be prepared by previous actions, and the controlled parameter is changed from λ_1 to λ_0 according to the inverted protocol $\lambda_B(t) = \lambda_F(t_f - t)$.

If the system is instead subjected to an external nonconservative time-independent force, the forward and backward processes are differentiated one from each other only if the initial distributions $\rho_0(\mathbf{x})$ (for F) and $\rho_1(\mathbf{x})$ (for B) are different.

In the course of a process, the system is in a nonequilibrium state, meaning that the actual distribution differs from the "underlying" thermal-equilibrium distribution. The actual distributions will be indicated with $\rho_F(\mathbf{x}, t)$ and $\rho_B(\mathbf{x}, t)$ for the forward and backward processes, respectively. Also, energy in the forms of work and heat is exchanged between the system, the external driving agent (in the form of work), and the environment (in the form of heat) [10]. Throughout, we adopt the system's viewpoint, meaning that the work done by the system and the heat released by the system both have a negative sign. With reference to the F and B processes, the amounts of work are indicated with $w_F(\gamma)$ and $w_B(\gamma)$, while the amounts of heat are $q_F(\gamma)$ and $q_B(\gamma)$. As detailed in Appendix A, work and heat are expressed in different ways depending on the specific type of process being considered. The averages obtained from the ensemble of trajectories will be indicated with $\langle w_F\rangle$, $\langle w_B\rangle$, $\langle q_F\rangle$, and $\langle q_B\rangle$.

B. Fluctuation theorem

Let γ be a trajectory in the space of the uncontrolled degrees of freedom, $\mathbf{x}$, and let $\tilde{\gamma}$ be the trajectory conjugate of γ, that is, the γ traveled backward. There is a one-to-one mapping between γ and $\tilde{\gamma}$: If a state $\mathbf{x}$ belongs to γ, the conjugate state of $\tilde{\gamma}$ is $\tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}}$ stands for the microstate $\mathbf{x}$ with all the momenta inverted (if these are included among the relevant degrees of freedom). With these positions, the well-known fluctuation theorem relates the probability $P_F(\gamma)$ of observing γ in the forward process with the probability $P_B(\tilde{\gamma})$ of observing $\tilde{\gamma}$ in the backward process [24]:

$$P_F(\gamma) e^{-\Phi_F(\gamma)} = P_B(\tilde{\gamma}), \tag{1}$$

where $\Phi_F(\gamma)$ is the trajectory-dependent function

$$\Phi_F(\gamma) = -\beta q_F(\gamma) - \ln \frac{\rho_1(\tilde{\mathbf{x}}(t_f))}{\rho_0(\mathbf{x}(0))} \tag{2}$$

with $\mathbf{x}(0)$ and $\mathbf{x}(t_f)$ the initial and final microstates of γ. [Of course, one can swap F with B and γ with $\tilde{\gamma}$. By assuming that ρ_0 and ρ_1 are invariant under the replacement of $\mathbf{x}$ with $\tilde{\mathbf{x}}$, the analog of Eq. (2) for the backward process is $\Phi_B(\tilde{\gamma}) = -\Phi_F(\gamma)$; hence Eq. (1) is obtained again.] Equation (1) holds for both time-dependent energy modulation and autonomous evolution under an external force, on condition that the heat $q_F(\gamma)$ is evaluated accordingly [see Eqs. (A2) and (A4)]. The function $\Phi_F(\gamma)$ quantifies the breakdown of reversibility at the level of the pair of conjugate trajectories. Such unbalancing results both from the specific initial distributions ρ_0 and ρ_1, and from the energy dissipation along the trajectories.

In our elaboration, a key role will be played by the average value $\langle\Phi_F\rangle$, where the average is obtained from the ensemble of trajectories γ. For the cases of interest described in Sec. II C, ρ_0 and ρ_1 are well defined and $\langle\Phi_F\rangle$ takes simple expressions used later on in Sec. IV. On physical grounds, in each case $\langle\Phi_F\rangle$ corresponds to the average energy dissipation (in k_BT units).

C. Cases of interest

Within the two types of processes illustrated in Sec. II A, let us focus on three main cases that are of interest here (see Fig. 2). For these cases, the corresponding expression of $\langle\Phi_F\rangle$ will be given; derivations and insights are provided in Appendix B.

1. *Finite-time processes starting from equilibrium*

One case [Fig. 2(a)] corresponds to finite-time processes of duration t_f starting from equilibrium. The external driving may either be a time-dependent energy modulation *or* one due to a time-independent external force. In the latter case, the labels F and B are superfluous because both ρ_0 and ρ_1 correspond to the equilibrium distribution determined by the bare energy of the system.

In the case of energy modulation, $\langle\Phi_F\rangle$ turns out to be given by $\beta[\langle w_F\rangle - \Delta A]$, where ΔA is the variation of the Helmholtz free energy of the system [25] when passing from thermal equilibrium with λ_0 to thermal equilibrium with λ_1. The difference $\langle w_F\rangle - \Delta A$ corresponds to the average energy dissipated in the active part of the driven process. Equivalently, $\langle\Phi_F\rangle$ can be expressed as $\Delta S_{\mathrm{tot}}/k_B$ where ΔS_{tot} is the average variation of the entropy of system plus environment also including the free relaxation phase after the active part of the driven process. Under a time-independent force, $\langle\Phi_F\rangle$ is equal to $\beta\langle w\rangle$, where $\langle w\rangle$ is the average work performed during the process. Equivalently, it can be expressed as $\Delta S_{\mathrm{ext}}/k_B$ where ΔS_{ext} is the average variation of the environment's entropy.

2. *Periodic steady state under cyclical energy modulation*

The second relevant case [Fig. 2(b)] is that of a system permanently subjected to a periodic energy modulation of period τ. Starting from equilibrium conditions, the system reaches a

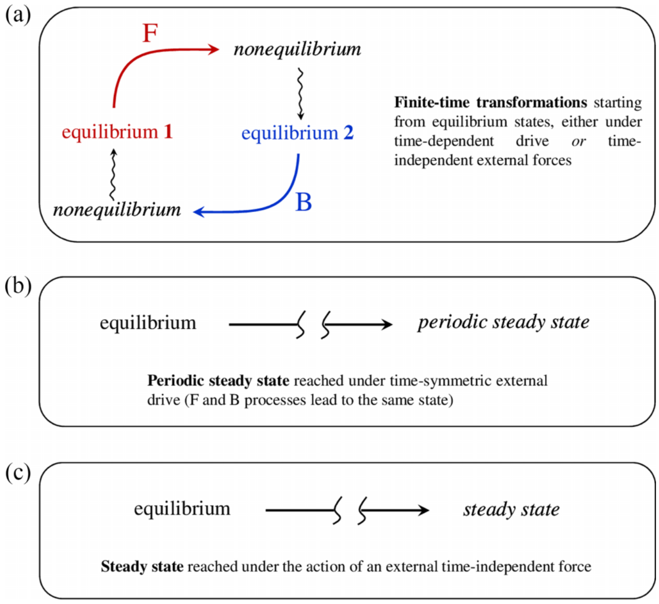

FIG. 2. The three main cases of interest here. (a) Finite-time forward (F) and backward (B) processes of duration t_f starting from equilibrium. The system can be subjected either to a time-dependent energy modulation or to a time-independent external force. In the latter case, the two equilibrium states are identical and labels F and B are superfluous. (b) The system, initially at equilibrium, is subjected to a time-symmetric periodic drive which eventually leads to a periodic steady state, which is the same in both the F and B directions. The observation time window is taken to be of duration equal to the period τ. (c) The system, initially at equilibrium, is subjected to an external time-independent force which eventually leads to a steady state. The observation time window is taken to be of some duration $t_{\rm obs}$.

periodic steady state [26,27] in which the distribution of the microstates becomes invariant under the temporal shift of τ. The observation time window will be taken to be of duration τ (see comments below) [28].

Note that, in general, two different periodic steady states are reached in the F and B directions. The two states are instead equivalent if, as intuitively expected, the external drive has a time-symmetric profile (possibly following a proper temporal shift), i.e., if

$$\lambda_F(t^* - t) \equiv \lambda_F(t) \tag{3}$$

for a certain t^* [29]. If the forward evolution owns such a kind of symmetry, the same holds for the backward evolution. In what follows, the wording “periodic steady state” will implicitly stand for this kind of situation: The same periodic steady state that is reached in both directions F and B, starting from equilibrium, under the action of a time-symmetric cyclical drive. In such case, labels F and B are immaterial.

The average $\langle \Phi_F \rangle$ turns out to be equal to $\beta \langle w_\tau \rangle$, where $\langle w_\tau \rangle$ is the average work done within a time window of duration equal to the period τ (regardless of the collocation of this time window on the timeline). Equivalently, $\langle \Phi_F \rangle$ can be expressed as $\Delta S_{\rm ext,cycle}/k_B$, where $\Delta S_{\rm ext,cycle}$ is the average entropy variation of the environment in a cycle.

3. *Steady state under time-independent forces*

The third case [Fig. 2(c)] concerns the steady-state conditions eventually reached, starting once again from equilibrium, under the persistent action of an external force. The observation time window can have a generic duration $t_{\rm obs}$. Also in this case, labels F and B are immaterial.

In this situation, $\langle \Phi_F \rangle$ is equal to $\beta \langle w_{t_{\rm obs}} \rangle$ where $\langle w_{t_{\rm obs}} \rangle$ is the average work done in the time window of duration $t_{\rm obs}$. In terms of average entropy production rate at the steady state, σ^{ss}, we can also express $\langle \Phi_F \rangle$ as $\sigma^{ss} t_{\rm obs}/k_B$.

III. EVENTS (DYNAMICAL OUTPUTS)

A. Definition of events

An event corresponds to a dynamical output in the broad sense of the term. This kind of output could be quantitative and measurable, or could correspond to a “narrative” consisting of an ensemble of clauses. At this point it proves useful to provide some examples.

An example of quantitative output is the amount of work done during the monitoring; the realization of the event could correspond to performing exactly that amount of work. More generally, a quantitative output of such a kind can be related to the net variation of an incremental property of the system that gradually changes along the trajectory by accumulation of infinitesimal variations. Another kind of quantitative output could be the realization of a preset value for a given microstate-dependent property of the system (e.g., the end-to-end distance of a fluctuating polymer molecule); if such value is attained, the event has actually occurred. Concerning the “narratives,” suppose that the dynamics of the fluctuating system takes place in a multidimensional configurational space that features four different energy wells, all interconnected with each other and enumerated as 1, 2, 3, and 4. Given an observation time window, the event $\mathcal{E}$ could be as follows: “In the first half of the observation time window, the system passes from well 2 to well 3, and well 1 is never visited.” This is an example of narrative description which also includes information about the timing (clauses about the timing, however, are not mandatory). In the underdamped regime, the event might also contain clauses involving the momenta; for instance, in passing from well 2 to well 3, we might require that each of the momenta does not surpass a given threshold in modulus.

If we imagine looking at a single trajectory we would be able to establish, unambiguously, if in the given time window the event occurs or not. Concerning the “narratives”, it might be the case that the event can take place several times. It is implicitly meant that the event occurs at least once, unless the clause “only once” is explicitly included in the specification of the event.

Alongside $\mathcal{E}$, we define the inverse event $\tilde{\mathcal{E}}$. For the “narratives”, $\tilde{\mathcal{E}}$ corresponds to the event that takes place with the elemental acts in reverse order. In the above example, $\tilde{\mathcal{E}}$ would be specified by the following: “In the second half of the observation time window, the system passes from well 3 to well 2, and well 1 is never visited.” Concerning the quantitative outputs, $\tilde{\mathcal{E}}$ has to be identified case by case. For instance, if $\mathcal{E}$ corresponds to obtaining a net cumulative response (like the amount of performed work), then in $\tilde{\mathcal{E}}$ the sign of the response is inverted. The general rule is the following: If $\mathcal{E}$ is realized for a trajectory γ, then $\tilde{\mathcal{E}}$ is realized for the conjugate trajectory $\tilde{\gamma}$, and this must hold for any pair of conjugate trajectories. This implicitly defines the type of events we deal with.

In some cases, $\mathcal{E}$ and $\tilde{\mathcal{E}}$ may coincide. For instance, suppose that the event is just the jump from one energy well to another without specifying the direction. In this case, the event and its inverse are identical. Let us use the term "symmetric" (and the subscript "s") for such a kind of event: $\tilde{\mathcal{E}}_s \equiv \mathcal{E}_s$.

In what follows, events $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots$ will be called "mutually exclusive" if only one of them can take place in the given observation time window.

Finally, we introduce our definition of "complementarity" between direct and inverse events. We say that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are complementary if they are mutually exclusive and, in addition, if one of the two events will be for sure observed in the observation time window.

B. Probabilistic expectations

Given a certain event of interest, and given the observation time window, one can express in advance the probabilistic expectation about the occurrence of such an event. For instance, in a finite-time process we would deal with the probabilities $P_F(\mathcal{E})$ and $P_F(\tilde{\mathcal{E}})$ of observing, respectively, the events $\mathcal{E}$ and $\tilde{\mathcal{E}}$ when performing the forward process, and with $P_B(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ the probabilities of observing, respectively, the events $\mathcal{E}$ and $\tilde{\mathcal{E}}$ when performing the backward process [30]. Note that $P_F(\mathcal{E}) + P_F(\tilde{\mathcal{E}}) \leqslant 1$ and $P_B(\mathcal{E}) + P_B(\tilde{\mathcal{E}}) \leqslant 1$, unless $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are complementary events. Let us recall that, for systems initially at equilibrium and subjected to an external time-independent force, labels F and B are superfluous.

In the case of monitoring at steady state or periodic steady state, the probabilities will be indicated with $P^{\mathrm{ss}}(\mathcal{E})$ and $P^{\mathrm{ss}}(\tilde{\mathcal{E}})$.

It is necessary to make a few important remarks about the periodic steady-state conditions. It is crucial to stress that if the event $\mathcal{E}$ (and hence also $\tilde{\mathcal{E}}$) has an average finite duration, then $P^{\mathrm{ss}}(\mathcal{E})$ and $P^{\mathrm{ss}}(\tilde{\mathcal{E}})$ generally depend on where the time window of duration τ is collocated on the timeline. This is because, to state that an event has actually occurred, the event must initiate and terminate within that time window. Keeping the duration τ of monitoring fixed, the occurrence of an event has its own synchronization with respect to the temporal evolution of the nonequilibrium distribution on the $\mathbf{x}$ variables; hence the event's statistics might be different within different time windows. Such a dependence on the collocation of the time window is instead absent for events $\mathcal{E}$ and $\tilde{\mathcal{E}}$ that occur instantaneously, like the crossing of some separatrix surface in the $\mathbf{x}$ space. In fact, the statistics of instantaneous events must be identical when an entire period is spawned, regardless of the collocation of the starting time of monitoring. The same also holds for any time-averaged $\mathbf{x}$-dependent property, or for an integrated average quantity (like $\langle w_\tau \rangle$) which derives from the continuous accumulation of infinitesimal contributions.

C. The key quantities

Here we introduce the key quantities that will be upper bounded in Sec. IV. These quantities, which are built on the basis of the probabilities of occurrence of the events, are expressed below by $\mathcal{C}^{\mathrm{FB}}$ for the finite-time forward-backward processes, and by $\mathcal{C}^{\mathrm{ss}}$ for systems under steady conditions (either in a periodic steady state as specified in Sec. II C, or in a steady state under a time-independent force). The subscripts a, b, c, and d will be added to address specific instances.

1. Finite-time forward-backward processes starting from equilibrium

Let us consider processes starting from thermal-equilibrium conditions. The key quantities given below are expressed with reference to forward-backward processes with time-dependent energy modulation. In this case, labels F and B are meaningful. However, the same quantities are meant to be valid also for systems subjected only to time-independent forces, just removing labels F and B.

If the focus is on a single event $\mathcal{E}$ and on its inverse $\tilde{\mathcal{E}}$, the key quantity is

$$\mathcal{C}_a^{\mathrm{FB}} = P_F(\mathcal{E}) \ln \frac{P_F(\mathcal{E})}{P_B(\tilde{\mathcal{E}})} + (1 - P_F(\mathcal{E})) \ln \left(\frac{1 - P_F(\mathcal{E})}{1 - P_B(\tilde{\mathcal{E}})} \right). \quad (4)$$

We can also consider a set of events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$ and the corresponding set $\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2, \ldots, \tilde{\mathcal{E}}_N$. Generally, different events could occur in the same time window. Here we restrict ourselves to mutually exclusive occurrences. In this case, the key quantity is

$$\mathcal{C}_b^{\mathrm{FB}} = \sum_{i=1}^{N} P_F(\text{only } \mathcal{E}_i) \ln \frac{P_F(\text{only } \mathcal{E}_i)}{P_B(\text{only } \tilde{\mathcal{E}}_i)} + \left(1 - \sum_i P_F(\text{only } \mathcal{E}_i)\right) \ln \left(\frac{1 - \sum_i P_F(\text{only } \mathcal{E}_i)}{1 - \sum_i P_B(\text{only } \tilde{\mathcal{E}}_i)} \right), \quad (5)$$

where the clauses "only" enforce the mutual exclusion.

A special case is that of a complete set of events, meaning that, for each event, the set also contains the inverse event. Let us enumerate the events with the integer n from $-N$ to N, so that $\mathcal{E}_n$ and $\mathcal{E}_{-n} \equiv \tilde{\mathcal{E}}_n$ are pairs of conjugate events. The case $n = 0$ is associated with the following: "Either none of the events takes place or two or more different events occur in the given time window." In this situation, the key quantity is

$$\mathcal{C}_c^{\mathrm{FB}} = \sum_{n=-N}^{N} P_F(\text{only } \mathcal{E}_n) \ln \frac{P_F(\text{only } \mathcal{E}_n)}{P_B(\text{only } \mathcal{E}_{-n})}, \quad (6)$$

where $P_F(\text{only } \mathcal{E}_0) = 1 - \sum_{n \neq 0} P_F(\text{only } \mathcal{E}_n)$, and $P_B(\text{only } \mathcal{E}_0) = 1 - \sum_{n \neq 0} P_B(\text{only } \mathcal{E}_n)$. In particular, for only two events $\mathcal{E}$ and $\tilde{\mathcal{E}}$ (case $N = 1$), Eq. (6) becomes

$$\begin{aligned}\mathcal{C}_d^{\mathrm{FB}} = {} & P_F(\text{only } \mathcal{E}) \ln \frac{P_F(\text{only } \mathcal{E})}{P_B(\text{only } \tilde{\mathcal{E}})} \\ & + P_F(\text{only } \tilde{\mathcal{E}}) \ln \frac{P_F(\text{only } \tilde{\mathcal{E}})}{P_B(\text{only } \mathcal{E})} \\ & + (1 - P_F(\text{only } \mathcal{E}) - P_F(\text{only } \tilde{\mathcal{E}})) \\ & \times \ln \left(\frac{1 - P_F(\text{only } \mathcal{E}) - P_F(\text{only } \tilde{\mathcal{E}})}{1 - P_B(\text{only } \mathcal{E}) - P_B(\text{only } \tilde{\mathcal{E}})} \right). \end{aligned} \quad (7)$$

2. Steady conditions starting from equilibrium

Let us consider the case of the periodic steady states defined in Sec. II C, or of steady states reached starting from equilibrium under time-independent forces. Both situations are collectively referred to as "steady conditions."

The key quantities are now expressed in terms of the probabilities of observing the events in a time window of duration τ for the case of a periodic drive, or of generic duration t_{obs} in the case of external forces. Here, labels F and B are dropped since they are immaterial. Explicitly, Eqs. (4) and (5) are replaced, respectively, by

$$\mathcal{C}_a^{\mathrm{ss}} = P^{\mathrm{ss}}(\mathcal{E})\ln\frac{P^{\mathrm{ss}}(\mathcal{E})}{P^{\mathrm{ss}}(\tilde{\mathcal{E}})} + (1-P^{\mathrm{ss}}(\mathcal{E}))\ln\left(\frac{1-P^{\mathrm{ss}}(\mathcal{E})}{1-P^{\mathrm{ss}}(\tilde{\mathcal{E}})}\right) \tag{8}$$

and

$$\mathcal{C}_b^{\mathrm{ss}} = \sum_{i=1}^{N} P^{\mathrm{ss}}(\text{only }\mathcal{E}_i)\ln\frac{P^{\mathrm{ss}}(\text{only }\mathcal{E}_i)}{P^{\mathrm{ss}}(\text{only }\tilde{\mathcal{E}}_i)} + \left(1-\sum_i P^{\mathrm{ss}}(\text{only }\mathcal{E}_i)\right)\ln\left(\frac{1-\sum_i P^{\mathrm{ss}}(\text{only }\mathcal{E}_i)}{1-\sum_i P^{\mathrm{ss}}(\text{only }\tilde{\mathcal{E}}_i)}\right), \tag{9}$$

while Eqs. (6) and (7) become

$$\mathcal{C}_c^{\mathrm{ss}} = \sum_{n=-N}^{N} P^{\mathrm{ss}}(\text{only }\mathcal{E}_n)\ln\frac{P^{\mathrm{ss}}(\text{only }\mathcal{E}_n)}{P^{\mathrm{ss}}(\text{only }\mathcal{E}_{-n})} \tag{10}$$

and

$$\mathcal{C}_d^{\mathrm{ss}} = (P^{\mathrm{ss}}(\text{only }\mathcal{E}) - P^{\mathrm{ss}}(\text{only }\tilde{\mathcal{E}}))\ln\frac{P^{\mathrm{ss}}(\text{only }\mathcal{E})}{P^{\mathrm{ss}}(\text{only }\tilde{\mathcal{E}})}. \tag{11}$$

An important remark concerns the probabilities that enter Eqs. (8)–(11) when we specifically refer to a periodic steady state. Let us bear in mind that the collocation of the time window of duration τ on the timeline might be relevant for events having an average finite duration. Thus, one should think of a *specific* pair of time windows under periodic steady-state conditions, one for the forward and one for the backward direction, within which the evolution of the nonequilibrium distributions $\rho_F(\mathbf{x},t)$ and $\rho_B(\mathbf{x},t)$ looks the same. All probabilities in Eqs. (8)–(11) as well as in all the subsequent equations, in principle, refer to these *specific* time windows. On the other hand, we are free to get rid of such a complication by intending all probabilities as *a priori* expectations, i.e., by imagining that we monitor the system over "a time-window of duration τ" without any further specification. If the event is instead related to a cumulative response deriving from the addition of infinitesimal contributions, the probabilities of occurrence are exactly the same irrespective of the collocation of the time window. With this in place, in what follows we simply term the various P^{ss} as probabilities of an event's occurrence in a time-window of duration τ at the periodic steady state.

IV. MAIN RESULTS

A. General result

Based on the above premises, the following inequality is derived in Appendix C:

$$\mathcal{C} \leqslant \langle\Phi_F\rangle, \tag{12}$$

where the quantity $\mathcal{C}$ on the left-hand side may be any of the key quantities $\mathcal{C}^{\mathrm{FB}}$ and $\mathcal{C}^{\mathrm{ss}}$ defined in Sec. III C, and where $\langle\Phi_F\rangle$ on the right-hand side takes on the corresponding expression. In each case, the specific forms of Eq. (12) will be given later in Secs. IV B–IV D.

For the finite-time forward-backward processes under a time-dependent drive, by swapping labels F and B and/or the attributes "direct" and "inverse" related to the events, one generates further inequalities which possibly bring independent constraints (this has to be inspected case by case). To see this, let us consider the forward-backward setup of Sec. III C 1. The simultaneous swap of F with B and of $\mathcal{E}$ with $\tilde{\mathcal{E}}$ generates the companion inequality of Eq. (12) in which the $\mathcal{C}$ on the left-hand side corresponds to the $\mathcal{C}_a^{\mathrm{BF}}$ formulated with $P_B(\tilde{\mathcal{E}})$ in place of $P_F(\mathcal{E})$, and vice versa; on the right-hand side, $\langle\Phi_B\rangle$, to be reformulated by adapting Eq. (2) for the backward direction, appears in place of $\langle\Phi_F\rangle$. With this double swap we get, ultimately, a pair of inequalities, namely, Eq. (12) and the companion one. These inequalities set mutual bounds between $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$. This will be resumed later in Sec. V A. By swapping only F with B, or only $\mathcal{E}$ with $\tilde{\mathcal{E}}$, one generates an analogous and separate pair of inequalities that only involve the probabilities $P_F(\tilde{\mathcal{E}})$ and $P_B(\mathcal{E})$. Totally, all permutations produce two similar and separate pairs of inequalities whose solution (mutual bounds) is of the same kind, just giving the right names to the quantities involved. Finally, if the event is symmetric as defined in Sec. III A, only two inequalities are generated, namely, Eq. (12) and the companion one with F and B swapped and where only $\mathcal{E}_s$ enters.

Let us focus now on the conditions for the saturation of Eq. (12). By following the derivation in Appendix C, it can be seen that the equality holds only if $\Phi_F(\gamma)$ [see Eq. (2)] has the same value along any trajectory for which the event of interest takes place, and also the same value along any trajectory for which the event does not take place. Only in this ideal situation, in fact, is the equality preserved in the steps where Jensen's inequality is applied. In this limit situation, one would get the tightest relationship between the probabilities of the direct and inverse events. Finding the physical conditions under which the spread of the $\Phi_F(\gamma)$ values is minimized under both clauses (the event occurs and the event does not occur) is, however, a formidable task that would have to be faced case by case for the specific system and the specific dynamical output being considered in each instance.

B. Finite-time processes starting from equilibrium under time-dependent energy modulation

Let us consider Eq. (B4), which is valid for finite-time processes starting from equilibrium. The general Eq. (12) takes the form

$$\mathcal{C}^{\mathrm{FB}} \leqslant \beta\langle w_{\mathrm{diss},F}\rangle, \tag{13}$$

where we have introduced the average dissipated work in the forward direction,

$$\langle w_{\mathrm{diss},F}\rangle = \langle w_F\rangle - \Delta A, \tag{14}$$

and where $\mathcal{C}^{\mathrm{FB}}$ can be any of $\mathcal{C}_a^{\mathrm{FB}}$, $\mathcal{C}_b^{\mathrm{FB}}$, $\mathcal{C}_c^{\mathrm{FB}}$, and $\mathcal{C}_d^{\mathrm{FB}}$. In the case of cyclic processes, $\Delta A = 0$ and so $\langle w_{\mathrm{diss},F}\rangle \equiv \langle w_F\rangle$.

Further inequalities are generated by swapping labels F and B in both members of Eq. (13). In this case, on the right-hand side we would have $\langle w_{\mathrm{diss},B}\rangle$, which corresponds to $\langle w_B\rangle + \Delta A$ if the variation ΔA still refers to the forward process.

C. Finite-time processes starting from equilibrium under time-independent forces

In this case, labels F and B are immaterial. In addition, $\Delta A = 0$ because the system's energetics remains unaltered. Thus, Eq. (13) is replaced by

$$\mathcal{C} \leqslant \beta\langle w\rangle, \tag{15}$$

where $\mathcal{C}$ can be any of the $\mathcal{C}_a^{\mathrm{FB}}$, $\mathcal{C}_b^{\mathrm{FB}}$, $\mathcal{C}_c^{\mathrm{FB}}$, and $\mathcal{C}_d^{\mathrm{FB}}$ given in Sec. III C 1, but written without labels F and B.

D. Steady conditions

Let us use the asterisk ($*$) as a subscript for indicating that the observation takes place under steady conditions starting from equilibrium, comprising both periodic steady states with time-symmetric drive and steady states under time-independent forces. Depending on the specific context, the duration t_* of the time window is identified with τ or t_{obs}. Similarly, w_* can be w_τ or $w_{t_{\mathrm{obs}}}$.

With this in place, from Eqs. (B6) and (B7) it follows that

$$\mathcal{C}^{\mathrm{ss}} \leqslant \beta\langle w_*\rangle, \tag{16}$$

where $\mathcal{C}^{\mathrm{ss}}$ can be any of the forms $\mathcal{C}_a^{\mathrm{ss}}$, $\mathcal{C}_b^{\mathrm{ss}}$, $\mathcal{C}_c^{\mathrm{ss}}$, and $\mathcal{C}_d^{\mathrm{ss}}$.

V. SPECIFIC APPLICATIONS

The general results presented in Sec. IV are here elaborated for some specific applications. For the sake of simplicity, in what follows, all relations concerning the finite-time forward-backward processes starting from equilibrium are presented and discussed only for the case of systems subjected to a time-dependent drive, which is, by the way, the most articulated situation. It is implicit that such relations can be specified for the case of time-independent forces, just (i) removing labels F and B, (ii) ignoring the considerations that concern the swap of F with B, and (iii) considering that both $\langle w_{\mathrm{diss},F}\rangle$ and $\langle w_{\mathrm{diss},B}\rangle$ have to be replaced by the average work $\langle w\rangle$.

A. Forward-backward probability bounds

Equation (13), together with Eq. (4), sets mutual bounds between the probabilities of occurrence of the events in the forward and backward processes. By making a double swap of F with B and of $\mathcal{E}$ with $\tilde{\mathcal{E}}$, a second inequality is obtained (see discussion in Sec. IV A). Explicitly, the two inequalities

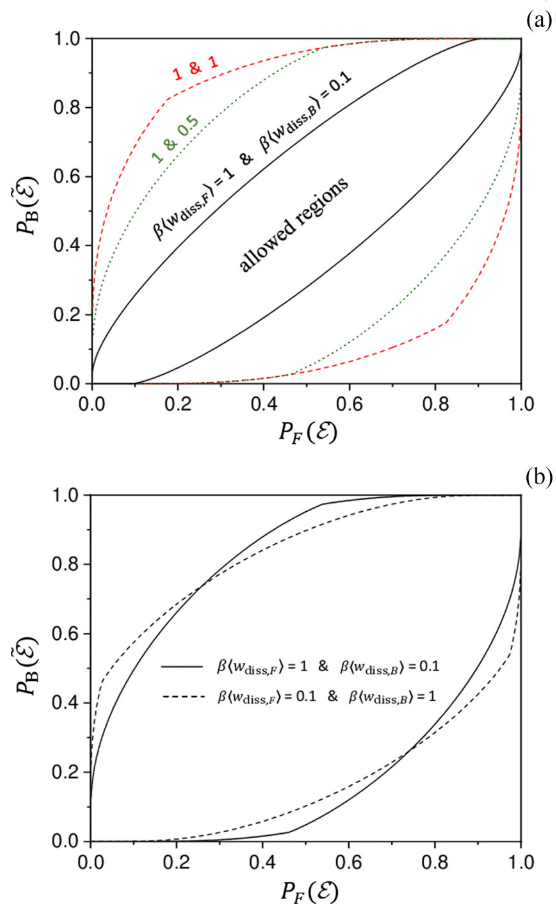

FIG. 3. Mutual bound between $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ for forward-backward processes starting from equilibrium under a time-dependent energy modulation. (a) Mutual bounds for some values of the average dissipated works (in k_BT units) $\beta\langle w_{\mathrm{diss},F}\rangle$ and $\beta\langle w_{\mathrm{diss},B}\rangle$. For each instance, the allowed region corresponds to the area between the delimiting upper and lower curves. (b) An example of how the delimiting curves change upon exchanging the values of $\beta\langle w_{\mathrm{diss},F}\rangle$ and $\beta\langle w_{\mathrm{diss},B}\rangle$ with each other.

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$$\begin{aligned}
&P_F(\mathcal{E})\ln\frac{P_F(\mathcal{E})}{P_B(\tilde{\mathcal{E}})} + (1-P_F(\mathcal{E}))\ln\left(\frac{1-P_F(\mathcal{E})}{1-P_B(\tilde{\mathcal{E}})}\right)\\
&\quad\leqslant \beta\langle w_{\mathrm{diss},F}\rangle,\\
&\qquad P_B(\tilde{\mathcal{E}})\ln\frac{P_B(\tilde{\mathcal{E}})}{P_F(\mathcal{E})}\\
&\qquad + \left(1-P_B(\tilde{\mathcal{E}})\right)\ln\left(\frac{1-P_B(\tilde{\mathcal{E}})}{1-P_F(\mathcal{E})}\right) \leqslant \beta\langle w_{\mathrm{diss},B}\rangle.
\end{aligned}\tag{17}$$

A similar and separate pair of inequalities is produced by the swap of F with B *or* by the swap of $\mathcal{E}$ with $\tilde{\mathcal{E}}$. What changes is that, on the left-hand sides, $P_F(\mathcal{E})$ is replaced by $P_F(\tilde{\mathcal{E}})$, and $P_B(\tilde{\mathcal{E}})$ is replaced by $P_B(\mathcal{E})$. Let us proceed by considering only Eq. (17), since the outcomes for the second pair of inequalities are exactly the same, just renaming the quantities.

The inequalities in Eq. (17) define an allowed region within which the values of $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ are admissible: Given

$P_F(\mathcal{E})$, the range of possible values of $P_B(\tilde{\mathcal{E}})$ is delimited and vice versa. The confines of the allowed region depend on $\langle w_{\text{diss},F}\rangle$ and $\langle w_{\text{diss},B}\rangle$, as shown in Fig. 3(a). Figure 3(b) shows how the delimiting curves change when the values of the average works are exchanged. As the average works gradually tend to zero, the allowed region collapses into the straight line of slope 1. In this case, which corresponds to observing the system at thermal equilibrium, the two events are in fact equiprobable. As the average works increase, the allowed region becomes wider and tends to cover the whole plane. Correspondingly, the mutual constraint between $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ becomes weaker and weaker. In light of this, the forbidden region is more important than the allowed region, and Eq. (17) can be seen as *no-go* conditions: Given $\langle w_{\text{diss},F}\rangle$ and $\langle w_{\text{diss},B}\rangle$, a pair of values $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ that falls in the forbidden region is *definitely* not admissible.

An explicit mutual bound between $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ can be obtained from Eq. (17) by resorting to a known inequality, due to Bretagnolle and Huber [31], between the "total variation" of two distributions and their Kullback-Leibler divergence [32]. The application of such inequality yields

$$\left|P_F(\mathcal{E}) - P_B(\tilde{\mathcal{E}})\right| \leqslant \sqrt{1 - e^{-\beta \min\{\langle w_{\text{diss},F}\rangle, \langle w_{\text{diss},B}\rangle\}}}. \tag{18}$$

If only one between $\langle w_{\text{diss},F}\rangle$ and $\langle w_{\text{diss},B}\rangle$ is known, by using that value in Eq. (18), in place of the minimum between the two, a less stringent but still valid inequality is obtained.

In the case of systems subjected to time-independent forces, the above outcomes can be easily reformulated by following the prescriptions given in the opening paragraph of this section. In particular, Eq. (18) becomes $|P(\mathcal{E}) - P(\tilde{\mathcal{E}})| \leqslant \sqrt{1 - e^{-\beta\langle w\rangle}}$ with $\langle w\rangle$ being the average work done in the given time window starting from equilibrium.

B. Probability bounds under steady conditions

Under steady conditions, Eqs. (17) are reduced to the single constraint

$$P^{\text{ss}}(\mathcal{E}) \ln \frac{P^{\text{ss}}(\mathcal{E})}{P^{\text{ss}}(\tilde{\mathcal{E}})} + (1 - P^{\text{ss}}(\mathcal{E})) \ln \left(\frac{1 - P^{\text{ss}}(\mathcal{E})}{1 - P^{\text{ss}}(\tilde{\mathcal{E}})}\right) \leqslant \beta\langle w_*\rangle \tag{19}$$

and the analog of Eq. (18) is

$$|P^{\text{ss}}(\mathcal{E}) - P^{\text{ss}}(\tilde{\mathcal{E}})| \leqslant \sqrt{1 - e^{-\beta\langle w_*\rangle}}, \tag{20}$$

where the average work and the probabilities refer to a time window of duration τ for periodic steady states, or t_{obs} for steady states.

Figure 4 displays the inequality in Eq. (19) for some values of $\beta\langle w_*\rangle$. The admissible values of $P^{\text{ss}}(\mathcal{E})$ and $P^{\text{ss}}(\tilde{\mathcal{E}})$ fall within an allowed region which becomes wider as $\beta\langle w_*\rangle$ increases. As is expected, for $\beta\langle w_*\rangle = 0$ such region shrinks into the straight line of slope 1 since the two events must have the same probability of occurrence. On the contrary, as $\beta\langle w_*\rangle$ increases, the allowed region tends to cover the whole plane. This means that, even knowing the probability of occurrence of one event, the probability for the inverse event is practically unconstrained. We see it suffices for $\langle w_*\rangle$ to be of the order of a few k_BT units to be in this situation.

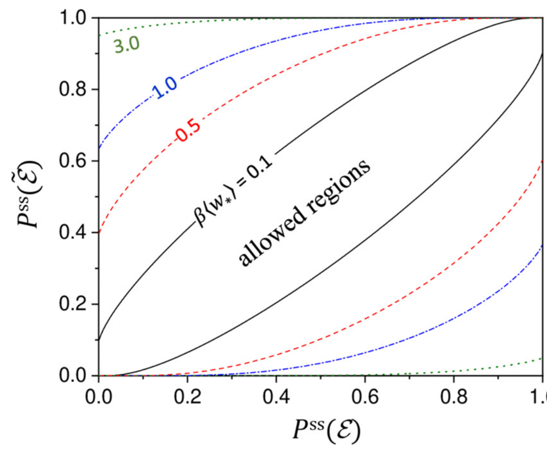

FIG. 4. Mutual bound between $P^{\text{ss}}(\mathcal{E})$ and $P^{\text{ss}}(\tilde{\mathcal{E}})$ under steady conditions starting from equilibrium. It can be either the case of a periodic steady state reached under a time-symmetric drive of period τ, or of a steady state under a time-independent force. The allowed regions are shown for some values of the average work (in k_BT units) $\beta\langle w_*\rangle$ in the observation time window of duration τ or t_{obs}.

C. Complementary events under steady conditions

Suppose that the event $\mathcal{E}$ concerns the *sign* of a certain quantitative output that derives from the continuous accumulation of small contributions in the observation time window (we come back to this kind of output in Sec. V E). This cumulative output could be a scalar quantity that can take on positive or negative values, a net response of the gain-loss kind, some developed plus-minus polarity, the net displacement along a certain coordinate, the net amount of performed work, and so on. In any given time window, a certain output of this kind is produced; hence its sign can *certainly* be determined. Thus, events $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are complementary and $P^{\text{ss}}(\mathcal{E}) + P^{\text{ss}}(\tilde{\mathcal{E}}) = 1$. To stress the fact that the events are related to the sign of the output, let us write $\mathcal{E}_+$ in place of $\mathcal{E}$, and $\mathcal{E}_-$ in place of $\tilde{\mathcal{E}}$. Without resorting to any extra information about the preference for $\mathcal{E}_-$ or $\mathcal{E}_+$, the two events have to be *a priori* treated equally. In what follows, $\mathcal{E}_\pm$ stands either for $\mathcal{E}_-$ or $\mathcal{E}_+$.

With these positions, Eq. (19) takes an interesting form:

$$f(P^{\text{ss}}(\mathcal{E}_\pm)) \leqslant \beta\langle w_*\rangle, \tag{21}$$

where we have introduced the function

$$f(x) = (2x - 1) \ln \left(\frac{x}{1 - x}\right) \tag{22}$$

of argument $x \in (0, 1)$. This function has non-negative values, is symmetric with respect to $x = 1/2$, vanishes at such point, and has limits $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = +\infty$. Given $y > 0$, the equation $f(x) = y$ has two solutions which are symmetrically placed with respect to $1/2$ and collapse on $1/2$ for $y = 0$. Let $g_\downarrow(y) \leqslant 1/2$ and $g_\uparrow(y) \geqslant 1/2$ be these solutions, which can be obtained numerically for any value of y. From Eq. (21) we get

$$g_\downarrow(\beta\langle w_*\rangle) \leqslant P^{\text{ss}}(\mathcal{E}_\pm) \leqslant g_\uparrow(\beta\langle w_*\rangle). \tag{23}$$

The bounds in Eq. (23) have to be taken as general *a priori* bounds in the absence of further information.

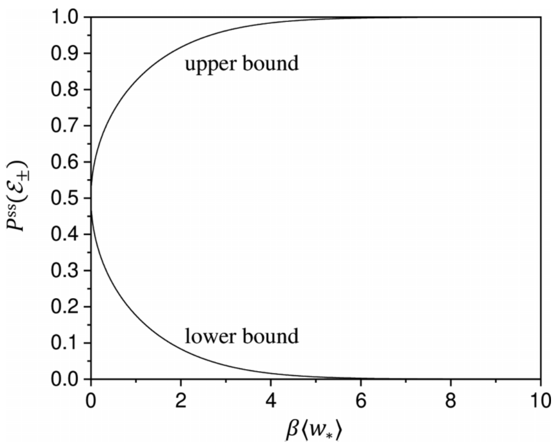

FIG. 5. Bounds on the probabilities of complementary events $\mathcal{E}_+$ and $\mathcal{E}_-$ under steady conditions (periodic steady state or steady state) starting from equilibrium. The two branches, which are symmetric with respect to the value 0.5, are the upper and lower bounds versus the average work (in k_BT units) $\beta\langle w_*\rangle$ done in the time window of duration τ for periodic steady states or $t_{\rm obs}$ for steady states.

Figure 5 gives the graphical representation of the bounds expressed by Eq. (23). It is worth noting that Eq. (23) turns out to be more stringent than the inequality obtainable from Eq. (20), namely, $[1-\sqrt{1-e^{-\beta\langle w_*\rangle}}]/2 \leqslant P^{\rm ss}(\mathcal{E}_\pm) \leqslant [1+\sqrt{1-e^{-\beta\langle w_*\rangle}}]/2$.

D. Counting statistics on net number of events

Let us consider an observation time window, and let n be the net number of events defined as

$$n := \text{number of events } \mathcal{E} - \text{number of events } \tilde{\mathcal{E}} \tag{24}$$

with $n = 0, \pm 1, \pm 2, \ldots$. The "occurrence of exactly n net events" represents, in itself, a well-defined event. Thus, the general results can be applied to this case too. In particular, the ensemble of all integers n from $-\infty$ to ∞ comprising the zero corresponds to a set of these events whose totality covers all possibilities.

With these positions, for finite-time processes starting from equilibrium, Eq. (13) with Eq. (6) yields

$$\sum_{n=-\infty}^{\infty} P_F(n) \ln \frac{P_F(n)}{P_B(-n)} \leqslant \beta\langle w_{{\rm diss},F}\rangle. \tag{25}$$

A companion inequality is obtained by swapping labels F and B.

Under steady conditions starting from equilibrium—periodic steady states or steady states—Eq. (25) readily reduces to

$$\sum_{n=-\infty}^{\infty} P^{\rm ss}(n) \ln \frac{P^{\rm ss}(n)}{P^{\rm ss}(-n)} \leqslant \beta\langle w_*\rangle. \tag{26}$$

E. Statistics on additive properties

We can go ahead a step further by identifying the occurrence of event $\mathcal{E}$ with the increment of some continuous scalar variable χ by an amount $\delta\chi$, and the occurrence of the inverse event $\tilde{\mathcal{E}}$ with the decrease by an amount $-\delta\chi$. If χ is initially set to zero and n net events have occurred, then $\chi = n\delta\chi$. Now let us take $\delta\chi$ smaller and smaller. By making a discretization of the χ domain from $-\infty$ to $+\infty$ into intervals of width $\delta\chi$ and central points located at $\chi_n = n\delta\chi$ with $n = 0, \pm1, \pm2, \ldots$, we can write $\rho(\chi_n)\delta\chi \equiv P(n)$, where $\rho(\chi)$ is the distribution on χ. From Eq. (25) it follows that

$$\mathcal{D}(\rho_F(\chi)||\rho_B(-\chi)) \leqslant \beta\langle w_{{\rm diss},F}\rangle, \tag{27}$$

where $\mathcal{D}(\rho_F(\chi)||\rho_B(-\chi))$ is the Kullback-Leibler divergence $\int_{-\infty}^{+\infty} d\chi\, \rho_F(\chi)\ln[\rho_F(\chi)/\rho_B(-\chi)]$. Equation (27) is valid for *any* additive property that increases or decreases along the trajectory followed by the system in the course of the process. Note that Eq. (27) could have been derived directly from Eq. (1) [33]. A companion inequality is obtained from Eq. (27) by swapping labels F and B.

Under steady conditions, the analog of Eq. (27) is

$$\mathcal{D}(\rho^{\rm ss}(\chi_*)||\rho^{\rm ss}(-\chi_*)) \leqslant \beta\langle w_*\rangle, \tag{28}$$

where χ_* is the net variation of the property in the observation time window.

F. TUR-like inequalities

The relations Eqs. (27) and (28) can be used as starting points for deriving generalized TURs. This can be realized by employing some recently discovered bounds [34,35] for the class of f-divergences, to which the Kullback-Leibler divergence belongs.

Let us first consider the general case of finite-time forward and backward processes starting from equilibrium. Let us introduce the quantities r_+ and r_- defined as

$$r_\pm = \frac{1}{2} + \frac{a_+ \pm 1}{2\sqrt{a_+^2 + 2a_- + 1}} \tag{29}$$

with

$$a_\pm = \frac{\sigma_{\chi,{\rm B}}^2 \mp \sigma_{\chi,{\rm F}}^2}{(\langle\chi_{\rm F}\rangle + \langle\chi_{\rm B}\rangle)^2}, \tag{30}$$

where $\langle\chi_{\rm F}\rangle$ and $\langle\chi_{\rm B}\rangle$ are the average values of χ in the forward and backward processes, and $\sigma_{\chi,{\rm F}}^2$ and $\sigma_{\chi,{\rm B}}^2$ are the variances of the distributions. Theorem 2 of Ref. [34] sets a lower bound on the Kullback-Leibler divergence $\mathcal{D}(\rho_F(\chi)||\rho_B(-\chi))$ in terms of the above quantities. By considering Eq. (27), this leads to

$$r_+ \ln\frac{r_+}{r_-} + (1-r_+)\ln\left(\frac{1-r_+}{1-r_-}\right) \leqslant \beta\langle w_{{\rm diss},F}\rangle. \tag{31}$$

Note that this relation can be rewritten as $\Delta A \leqslant \langle w_F\rangle - (\cdots)$, where the quantity $(\cdots)$ is non-negative. This improves the bound $\Delta A \leqslant \langle w_F\rangle$ (Clausius inequality in the stochastic context), but requires the additional knowledge of $\langle\chi_{\rm F}\rangle$, $\langle\chi_{\rm B}\rangle$, $\sigma_{\chi,{\rm F}}^2$, and $\sigma_{\chi,{\rm B}}^2$ for the inspected property χ. By swapping labels F and B, we get an independent companion inequality with different quantities on the left-hand side and with $\langle w_{{\rm diss},B}\rangle$ on the right-hand side:

$$r_- \ln\frac{r_-}{r_+} + (1-r_-)\ln\left(\frac{1-r_-}{1-r_+}\right) \leqslant \beta\langle w_{{\rm diss},B}\rangle. \tag{32}$$

Under steady conditions, a simplified result is obtained by setting $a_+ = 0$ and $a_- = \sigma^2_{\chi_*}/(2\langle\chi_*\rangle^2)$, where the average is meant to be taken over any time window of duration equal to the period τ (for periodic steady states) or $t_{\rm obs}$ (for steady states). In this case, $r_\pm = 2^{-1} \pm (2\sqrt{\sigma^2_{\chi_*}/\langle\chi_*\rangle^2 + 1})^{-1}$ and, taking into account that now $r_+ + r_- = 1$, from Eq. (28) it follows that $(1 - 2r_-)\ln[(1 - r_-)/r_-] \leqslant \beta\langle w_*\rangle$. Explicitly, this yields

$$\frac{1}{\sqrt{\sigma^2_{\chi_*}/\langle\chi_*\rangle^2 + 1}} \ln\left(\frac{\sqrt{\sigma^2_{\chi_*}/\langle\chi_*\rangle^2 + 1} + 1}{\sqrt{\sigma^2_{\chi_*}/\langle\chi_*\rangle^2 + 1} - 1}\right) \leqslant \beta\langle w_*\rangle. \tag{33}$$

Relations (31)–(33) correspond to already known TUR-like inequalities (generalized TURs) [16,18] which have been obtained from different angles without resorting to large-deviation-theory arguments. For instance, they do emerge as special results in the broader context of isometric uncertainty relations [20], and derive from general properties of the Euclidean geometry of the observables space [21]. The present derivation is essentially equivalent to the one of Hasegawa and co-workers [16], who extended a previous result [15] by taking inspiration from an early work on the statistical distribution of the entropy production in systems obeying the Evans-Searles fluctuation theorem [36].

Note that χ can be any additive property. In particular, Eqs. (31) and (32) can be written for χ corresponding to the total works done in the forward and backward processes, and Eq. (33) for the work in a time window under steady conditions. Let us also note that the same type of inequalities can be derived directly from Eqs. (25) and (26); hence they also hold for the ratio of variance over squared average of the discrete variable n which, let us bear in mind, is the net number of events.

It is now worth commenting on the comparison between generalized TURs and "genuine" TURs. The latter, known simply as TURs, were at first formulated under steady-state conditions [37–40] and then extended to include the external drive [41–43]. The TURs have a big impact on the analysis of the performance of molecular motors and machines, especially in biochemical contexts [44–48]. The generalized TURs are known to be less stringent than the TURs. While the generalized TURs can be derived from the basic FT alone, the derivation of the TURs is collocated within the framework of the large-deviations theory [49] (unless one restricts to the linear response regime [50]) and exploits specific features of the Markov dynamics. It seems that the enforcement of the FT alone does not allow one to reach the efficacy of the genuine TURs. In addition, the inevitable loss of information due to "contraction steps" (as is the case, in our derivation, with the reduction to the statistics of selected events) weakens the bounds. On the other hand, the generalized TURs are applicable under broader conditions where the TURs can be violated. In addition, one can easily cope with the time-dependent drive since it is included in the derivation from the beginning.

G. Inequalities on work probabilities under steady conditions

As a cumulative property, let us consider the total work performed in a time window of duration τ at the periodic steady state, or $t_{\rm obs}$ at the steady state. Equation (28) becomes

$$\int_{-\infty}^{\infty} dw_*\, \rho^{\rm ss}(w_*) \ln\frac{\rho^{\rm ss}(w_*)}{\rho^{\rm ss}(-w_*)} \leqslant \langle w_*\rangle. \tag{34}$$

This inequality sets a constraint on the work distribution function $\rho^{\rm ss}(w_*)$. In particular, from Eq. (33), by setting $\chi_* \equiv w_*$, it follows that $\eta\ln[(1+\eta)/(1-\eta)] \leqslant \beta\langle w_*\rangle$ with $\eta = (\langle w_*\rangle/\sigma_{w_*})/\sqrt{1 + (\langle w_*\rangle/\sigma_{w_*})^2}$. This inequality sets a lower bound on the standard deviation σ_{w_*} at a given average value $\langle w_*\rangle$. The profile of this lower bound features an initial growth which can be approximated by $\sigma_{w_*} \sim \sqrt{2\langle w_*\rangle/\beta}$, then reaches the maximum $\beta\sigma_{w_*} = 1.3$ at $\langle w_*\rangle = 2$, and finally decreases monotonically. Remarkably, the same kind of result was obtained in Ref. [36] for the variance of the entropy production versus the average value in the different context of the Evans-Searles fluctuation theorem.

An interesting inequality can be obtained from Eq. (23) if event $\mathcal{E}_-$ corresponds to doing negative work in the given time window, which means gaining energy. In this case, we know in advance that $P^{\rm ss}(\mathcal{E}_-) \equiv P^{\rm ss}(w_* < 0) \leqslant 1/2$; thus what matters is only the lower bound:

$$P^{\rm ss}(w_* < 0) \geqslant g_\downarrow(\beta\,\langle w_*\rangle), \tag{35}$$

which corresponds to the bottom branch in Fig. 4. This bound is more stringent than the one obtainable from Eq. (20), namely, $P^{\rm ss}(w_* < 0) \geqslant [1 - \sqrt{1 - e^{-\beta\langle w_*\rangle}}]/2$.

H. Dissipation and lag

Let us consider forward-backward processes starting from equilibrium. Let us imagine dividing the space of the uncontrolled degrees of freedom into infinitesimal elements $\delta\mathbf{x}$, and labeling these elements by a positive integer index n. As event $\mathcal{E}_n$, let us consider the following: "At the time t_f, the system is inside the nth element." The associate inverse event $\tilde{\mathcal{E}}_n$ is thus, "At the time zero, the system is inside the nth element." The events $\mathcal{E}_n$ are clearly mutually exclusive because the system can only be in one element at t_f. Similarly, the events $\tilde{\mathcal{E}}_n$ are mutually exclusive. We can thus apply Eq. (5) to the total set of events comprising $\mathcal{E}_n$ and the conjugate $\tilde{\mathcal{E}}_n$. Specifically, in place of $P_F(\text{only } \mathcal{E}_n)$ we have $\rho_F(\mathbf{x}, t_f)\delta\mathbf{x}$, and in place of $P_B(\text{only } \tilde{\mathcal{E}}_n)$ we have $\rho_{\rm eq,1}(\mathbf{x})\delta\mathbf{x}$, where $\rho_{\rm eq,1}(\mathbf{x})$ is the equilibrium distribution at the beginning of the backward process. In addition, the second addend on the right-hand side of Eq. (5) is absent since the elements cover the whole $\mathbf{x}$ space [hence $1 - \sum_n P_F(\text{only } \mathcal{E}_n) = 0$]. The summation on n becomes an integral which corresponds to the Kullback-Leibler divergence [51]:

$$\mathcal{D}\big(\rho_F(\mathbf{x}, t_f)||\rho_{\rm eq,1}(\mathbf{x})\big) \leqslant \beta\langle w_{{\rm diss},F}\rangle. \tag{36}$$

This is a well-known result originally obtained by Vaikuntanathan and Jarzynski [52] and later resumed by Seifert in Ref. [7]. The same relation, along with a more stringent inequality involving the time derivatives of the quantities on both sides, has also been derived within the framework of the nonstationary Fokker-Planck equation [53]. The derivation of Eq. (36) given here is similar to the one presented in Sec. 7.2 of Ref. [7].

1. Bounds on conditional probabilities for driven forward-backward processes

Let us suppose we know that, starting from equilibrium, the system was subjected to a driven process, and that a certain event $\mathcal{E}_s$ took place but it is not known if the process was actually conducted in the F or B direction (according to our identification of F and B). For the sake of simplicity, let us consider the case of symmetric events as defined in Sec. III A. In figurative terms, we might imagine "hearing a click" when $\mathcal{E}_s$ takes place. Just knowing that $\mathcal{E}_s$ occurred, can we guess if the process was F or B? What information do we need to make the guess?

The answer can be given only in probabilistic terms, namely, by expressing the conditional probability, or likelihood, $P(F|\mathcal{E}_s)$ [with $P(B|\mathcal{E}_s) = 1 - P(F|\mathcal{E}_s)$]. Let us tackle the problem by adopting a reasoning similar to the one used by Jarzynski in Ref. [6] for inferring the arrow of time. The approach consists in combining the FT with Bayes' theorem of statistical inference.

If the F and B modalities have equal probability *a priori*, Bayes' theorem leads to [54]

$$P(F|\mathcal{E}_s) = \frac{P_F(\mathcal{E}_s)}{P_F(\mathcal{E}_s) + P_B(\mathcal{E}_s)}. \tag{37}$$

By simply writing $a = P_F(\mathcal{E}_s)$ and $b = P(F|\mathcal{E}_s)$ [with the obvious bound $b \geqslant a/(1+a)$ obtainable by replacing $P_B(\mathcal{E}_s)$ with 1], the combination of Eq. (37) with $\mathcal{C}_a^{\rm FB} \leqslant \beta\langle w_{{\rm diss},F}\rangle$ yields

$$a \ln\left(\frac{b}{1-b}\right) + (1-a)\ln\left[\frac{b(1-a)}{b(1+a)-a}\right] \leqslant \beta\langle w_{{\rm diss},F}\rangle. \tag{38}$$

By exchanging forward with backward, an analogous relation is obtained:

$$a\left(\frac{1-b}{b}\right)\ln\left(\frac{1-b}{b}\right) + \left[\frac{b(1+a)-a}{b}\right]\ln\left[\frac{b(1+a)-a}{b(1-a)}\right] \leqslant \beta\langle w_{{\rm diss},B}\rangle. \tag{39}$$

Taken separately, inequalities (38) and (39) set a nontrivial mutual bound between b and a. The bound becomes more stringent if both constraints are considered together. Giving a value to $P_F(\mathcal{E}_s)$, the conditional probability $P(F|\mathcal{E}_s)$ is comprised between a minimum and a maximum value. What is required is the knowledge of $\langle w_{{\rm diss},F}\rangle$ and/or $\langle w_{{\rm diss},B}\rangle$.

Figure 6 shows the allowed regions obtained by checking numerically the simultaneous fulfillment of Eqs. (38) and (39) for some values of $\beta\langle w_{{\rm diss},F}\rangle$ and $\beta\langle w_{{\rm diss},B}\rangle$. The lowest dotted profile corresponds to the trivial bound $a/(1+a)$. We can see that, as the average dissipation in the F and B directions increases, the lower bound approaches the trivial bound, whereas the upper bound becomes ever less stringent.

It can be proved [55] that, for any $\langle w_{{\rm diss},F}\rangle$ and $\langle w_{{\rm diss},B}\rangle$, the upper branch in Fig. 6 monotonically decreases as $P_F(\mathcal{E}_s)$ increases, while the lower branch monotonically increases. This has a relevant implication. Suppose we only know that $P_F(\mathcal{E}_s) \geqslant \epsilon$ for the event of interest. From the monotonic behavior of the branches, it follows that $P(F|\mathcal{E}_s)$ is comprised between the two values that are identified by the intersection with the vertical line placed at ϵ. This implies that the same profiles shown in Fig. 6 are also valid with ϵ on the abscissa and the conditional probability $P(F|\mathcal{E}_s$ with $P(\mathcal{E}_s) \geqslant \epsilon)$ on the ordinate.

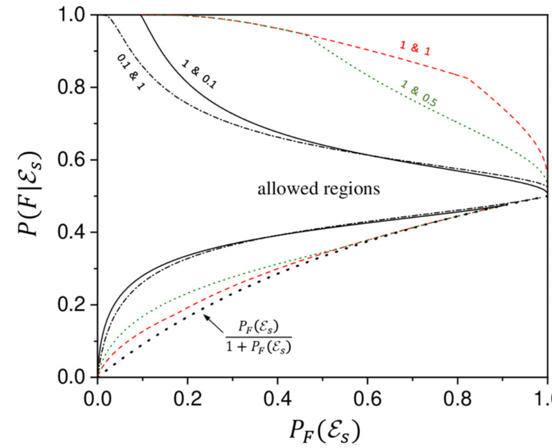

FIG. 6. Finite-time processes with time-dependent energy modulation starting from equilibrium. For a symmetric event $\mathcal{E}_s$, the figure shows the upper and lower bounds on the conditional probability $P(F|\mathcal{E}_s)$ versus $P_F(\mathcal{E}_s)$ for some pairs of average dissipations $\beta\langle w_{{\rm diss},F}\rangle$ and $\beta\langle w_{{\rm diss},B}\rangle$. The lowest dotted profile corresponds to the indicated trivial bound. As pointed out in the text, the same profiles are valid if on the abscissa we put a value ϵ and on the ordinate we put $P(F|\mathcal{E}_s$ with $P(\mathcal{E}_s) \geqslant \epsilon)$.

Here we have considered symmetric events for simplicity. The same approach can be applied, although with a bit more complex elaboration, to the general case of nonsymmetric events. Equation (37) is still valid with $\mathcal{E}$ or $\tilde{\mathcal{E}}$ in place of $\mathcal{E}_s$. This allows us to express $P(F|\mathcal{E})$ in terms of $P_F(\mathcal{E})$ and $P_B(\mathcal{E})$, and $P(F|\tilde{\mathcal{E}})$ in terms of $P_F(\tilde{\mathcal{E}})$ and $P_B(\tilde{\mathcal{E}})$. Ultimately, one arrives at a set of four inequalities involving $P(F|\mathcal{E})$, $P(F|\tilde{\mathcal{E}})$, $P_F(\mathcal{E})$, and $P_F(\tilde{\mathcal{E}})$.

VI. EXAMPLES

In order to illustrate some of the outcomes, let us consider a simple overdamped one-dimensional rotor. The stochastic variable x is the angle expressed in radians. Let $V_0(x)$ be the bare energy, which we take to be of the form $\beta V_0(x) = 2[1-\cos(x)]$, featuring a single energy barrier of $4k_BT$ at π. Three situations are considered: Case A, in which the system, initially at equilibrium, is subjected to a finite-time cyclic process at the end of which the initial system's energy is restored; case B, in which the system, starting from equilibrium, is taken into a periodic steady state under the action of an external periodic drive; and case C, in which the system, initially at equilibrium, is taken into a steady state under the action of a constant force that induces a net drift.

The dynamics are modeled by means of the Langevin equation with constant diffusion coefficient D (set equal to 1 in some units of inverse-of-time) and Gaussian white noise. Specifically, the evolution rule is $x(t+\Delta t) = x(t) + \beta D F(x,t)\Delta t + \sqrt{2D\Delta t}\, s(t)$, where Δt is the propagation time step, $F(x,t)$ the deterministic force (possibly time dependent and to be specified case by case), and $s(t)$ is a value randomly drawn from the Gaussian distribution with zero

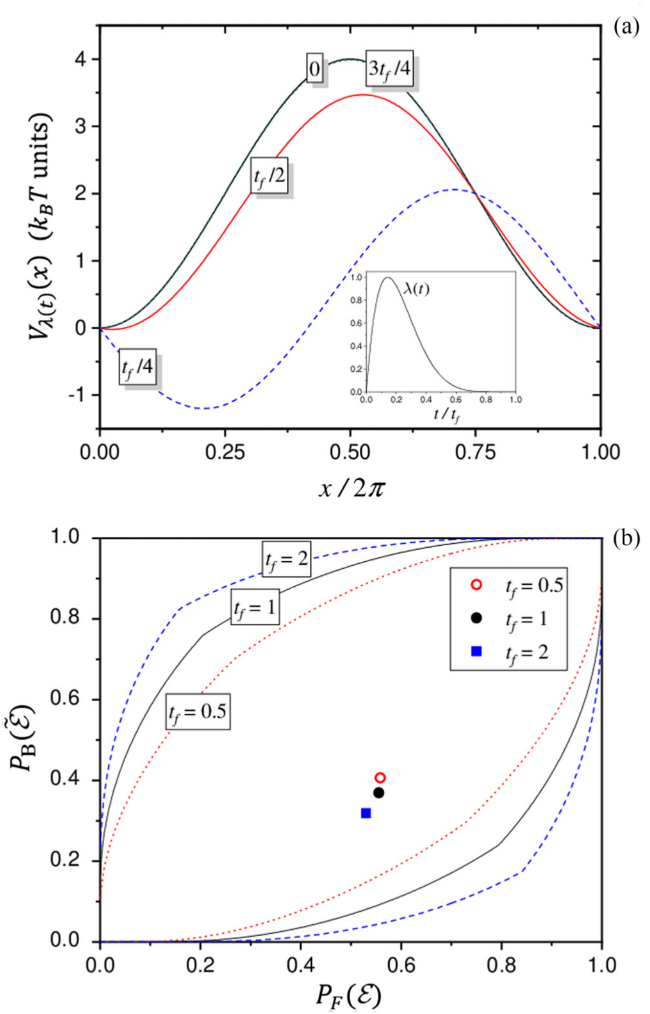

FIG. 7. One-dimensional overdamped rotor subjected to cyclic forward and backward driven processes starting from equilibrium. The bare energy has a periodic profile (case A; see text for details). (a) Time modulation of the system's energy at some fractions of the process of duration t_f. The inset shows the temporal profile of the controlled parameter. (b) The allowed regions for the probabilities $P_F(\mathcal{E})$ and $P_B(\tilde{\mathcal{E}})$ for three values of t_f. The symbols indicate the actual collocation of the outcomes from the simulations. The event $\mathcal{E}$ of interest is the net positive rotation with respect to the initial location.

mean and unit variance. Computational details are given in Ref. [56]. In the simulations, the variable x is left unconstrained so as to keep track of multiple rotations in the positive and negative sense.

In case A, the time-dependent energy is modeled as $V_{\lambda(t)}(x) = [1 - \lambda(t)]V_0(x) + \lambda(t)V_1(x)$, with $\beta V_1(x) = -2\sin(x)$ and $\lambda(t) = (7^7/6^6)(1 - t/t_f)^6 t/t_f$, where t_f is the duration of the process. The value of the controlled parameter $\lambda(t)$ rises from zero, reaches a maximum equal to 1 at the time $t_f/7$, and then decreases and vanishes at t_f. Hence, the energy profile starts from $V_0(x)$, becomes $V_1(x)$ at $t_f/7$, and returns to $V_0(x)$ at the end of the process. The evolution of the energy profile is displayed in Fig. 7(a), where the inset shows the profile of $\lambda(t)$. The deterministic force is given by $F(x,t) = -\partial V_{\lambda(t)}(x)/\partial x$. In the backward process, the protocol, which is not time symmetric, has to be inverted. As event $\mathcal{E}$, we simply consider the net positive rotation with respect to the initial location, i.e., the realization of $x(t_f) \geqslant x(0)$. The inverse event $\tilde{\mathcal{E}}$ is thus complementary to $\mathcal{E}$. From a number of repeated simulations of the process in the forward and backward directions, the average works and the probabilities of occurrence of the events were determined for three values of t_f. The results are $\beta\langle w_F\rangle = 0.398\,(0.010)$, $\beta\langle w_B\rangle = 0.405\,(0.009)$, $P_F(\mathcal{E}) = 0.558\,(0.005)$, and $P_B(\tilde{\mathcal{E}}) = 0.406\,(0.004)$ for $t_f = 0.5$; $\beta\langle w_F\rangle = 0.681\,(0.010)$, $\beta\langle w_B\rangle = 0.706\,(0.008)$, $P_F(\mathcal{E}) = 0.556\,(0.005)$, and $P_B(\tilde{\mathcal{E}}) = 0.369\,(0.006)$ for $t_f = 1$; and $\beta\langle w_F\rangle = 1.060\,(0.013)$, $\beta\langle w_B\rangle = 1.086\,(0.011)$, $P_F(\mathcal{E}) = 0.530\,(0.005)$, and $P_B(\tilde{\mathcal{E}}) = 0.319\,(0.003)$ for $t_f = 2$. The numbers within brackets, used here and below, represent the uncertainties expressed as one standard deviation [56]. The average work does not generally have a monotonic dependence on the duration of a given driven process. In the present case, for the relative short t_f here considered, we see that both $\langle w_F\rangle$ and $\langle w_B\rangle$ increase as t_f increases. Given the average work values, the allowed regions in the plane $P_B(\tilde{\mathcal{E}})$ versus $P_F(\mathcal{E})$ are specified by Eq. (17). The results are displayed in Fig. 7(b). The symbols correspond to the specific outcomes which, as we see, fall within the corresponding allowed regions. Note that all points fall slightly below the diagonal, meaning that $\mathcal{E}$ in the forward process is only slightly more probable than $\tilde{\mathcal{E}}$ in the backward process. As the duration t_f increases, the two events become statistically more distinct but, at the same time, the average dissipation increases and the bounds become less stringent (the allowed regions become wider).

In case B, the energy is again expressed as $V_{\lambda(t)}(x) = [1 - \lambda(t)]V_0(x) + \lambda(t)V_1(x)$, where now $V_1(x)$ has a sawtooth profile. Specifically, we adopt $\beta V_1(x) = 4\sum_{k=1}^{n} \alpha_k(n)\sin[k(x + 1.5)]/k$ with $n = 20$ and $\alpha_k(n) = \binom{2n}{n-k}/\binom{2n}{n}$. Similarly to case A, the deterministic force is given by $F(x,t) = -\partial V_{\lambda(t)}(x)/\partial x$. The protocol is here taken to be $\lambda(t) = \sin^2(\omega t/2)$ with $\omega = 2\pi/\tau$ being τ the period. The simulations were done for $\tau = 1$. Since the protocol is time symmetric, we can use the results applicable under periodic steady-state conditions. The evolution of the energy profile is shown in Fig. 8(a) at the beginning of a cycle [i.e., $V_0(x)$], at 1/4 of the cycle, and halfway through the cycle [i.e., $V_1(x)$], the subsequent half of the cycle being symmetric. The inset shows the profile of $\lambda(t)$ in a cycle. The form of $V_1(x)$ has been tuned in such a way as to promote an average drift in the positive sense of rotation. Indeed, during the transformation of $V_0(x)$ into $V_1(x)$, the system is taken on an energy slope which induces a clockwise motion towards angles beyond the energy barrier of $V_0(x)$. The drift appears clearly in Fig. 8(b), which shows three trajectories starting from points drawn at random from $\rho_{\mathrm{eq},0}(x) \propto \exp\{-\beta V_0(x)\}$. Figure 8(c) shows $\beta\langle w_\tau\rangle$, cycle by cycle, up to 50 cycles. It appears that the periodic steady-state conditions are quickly reached after about four cycles. Figure 8(d) shows the work distribution function $\rho^{\mathrm{ss}}(w_\tau)$ obtained by means of a histogram construction. Let us note the considerable portion which appears at negative work values. The average work turns out to be $1.725\,(0.022)k_BT$ units and $P^{\mathrm{ss}}(w_\tau < 0) = 0.282\,(0.004)$.

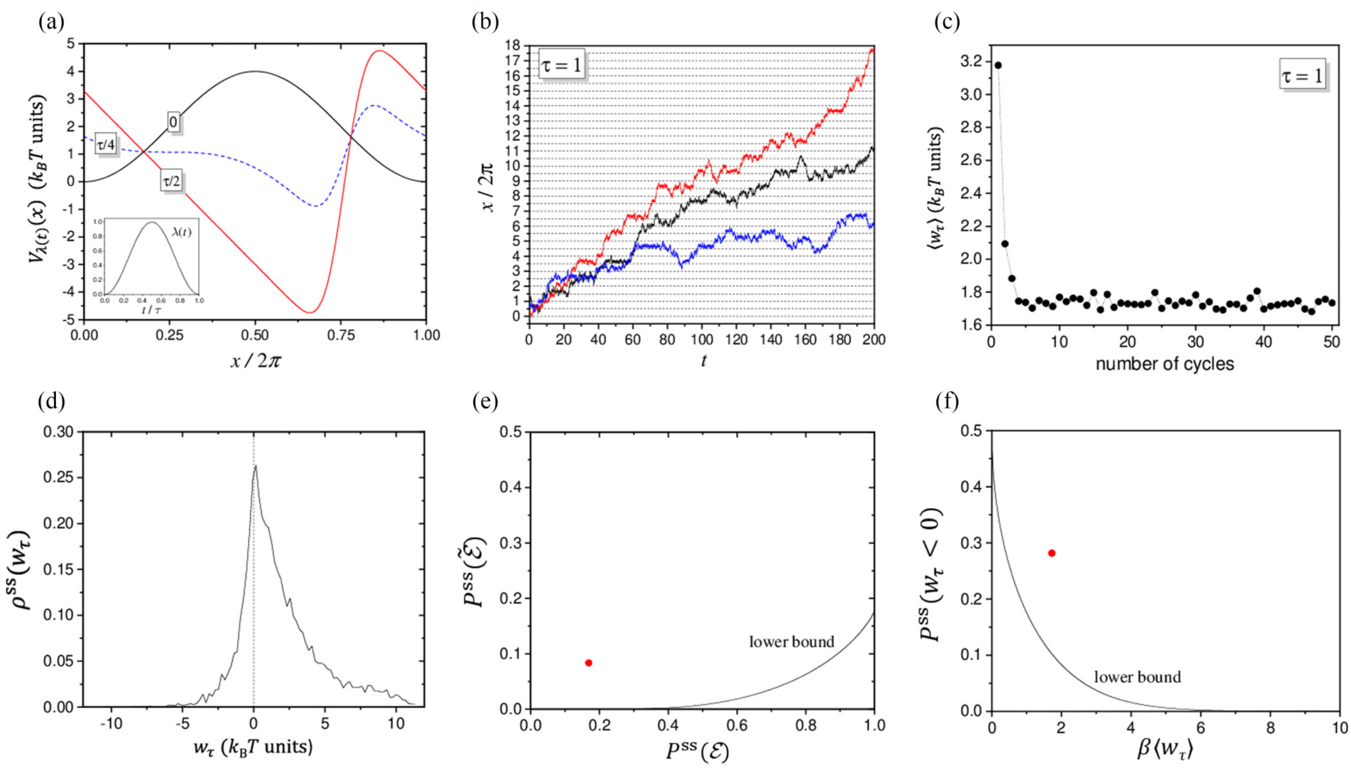

FIG. 8. The same one-dimensional overdamped rotor of Fig. 7, here subjected to a time-symmetric periodic drive (case B; see text for details). (a) Time modulation of the system's energy at the beginning, at 1/4 and at 1/2 of the period. The inset shows the temporal profile of the controlled parameter. The simulations were done for $\tau = 1$. (b) Three trajectories which develop in the course of 200 cycles. (c) Average work per cycle for the first 50 cycles. (d) Distribution function of the work values in a period under periodic steady-state conditions (the chosen cycle is the 50th one). (e) Lower bound on $P^{ss}(\tilde{\mathcal{E}})$ versus $P^{ss}(\mathcal{E})$, where event $\mathcal{E}$ is the net positive rotation of at least $\pi/2$ in a time τ under periodic steady-state conditions. The solid circle corresponds to the actual outcome from the simulations. (f) Lower bound on the probability of doing negative work in a time τ, $P^{ss}(w_\tau < 0)$, versus the average work per cycle in k_BT units. The circle is the actual outcome.

As event $\mathcal{E}$ of interest, let us consider the positive rotation of at least $\pi/2$ in a time window equal to τ. The outcomes are $P^{ss}(\mathcal{E}) = 0.169\,(0.004)$ and $P^{ss}(\tilde{\mathcal{E}}) = 0.0836\,(0.0017)$. Figure 8(e) shows only the lower bound on $P^{ss}(\tilde{\mathcal{E}})$ versus $P^{ss}(\mathcal{E})$ because, as a result of the induced drift, the upper bound can be certainly set to 0.5. The circle, which falls inside the region, corresponds to the outcome from the simulations. Finally, Fig. 8(f) is the equivalent of Fig. 5, adapted here to the specific case of realization of negative work. Since $P^{ss}(w_\tau < 0) \leqslant 0.5$, only the lower bound is shown.

In case C, the deterministic force acting on the rotor is time independent and given by $F(x) = -dV_0(x)/dx + f_{\text{ext}}$, where f_{ext} is an external contribution which induces a drift. By "unfolding" the angular variable, the dynamics can be visualized as a motion on the tilted potential $V_0(x) - f_{\text{ext}}\,x$. After an initial transient phase, the system reaches a steady state in the presence of the external force. The system was monitored over a time window of duration $t_{\text{obs}} = 1$ at the steady state. The simulations were performed for $f_{\text{ext}} = 1.75$, slightly lower than the critical value 2 beyond which the tilted potential has a monotonically decreasing profile. For $f_{\text{ext}} = 1.75$, the average work done in the time $t_{\text{obs}} = 1$ results to be comparable with $\langle w_\tau \rangle$ for $\tau = 1$ of case B above. Some trajectories are shown in Fig. 9(a); the inset shows the corresponding tilted potential. Similarly to case B, the event of interest was the positive rotation of at least $\pi/2$ in a time window of duration t_{obs}. The outcomes are $\beta\langle w_{t_{\text{obs}}} \rangle = 1.726\,(0.020)$ and $P^{ss}(\mathcal{E}) = 0.3166\,(0.0050)$, $P^{ss}(\tilde{\mathcal{E}}) = 0.0271\,(0.0018)$. The results are displayed in Fig. 9(b). Again, the circle falls inside the allowed region. By comparing Fig. 9(b) with Fig. 8(e), we note that at about the same average dissipation in a time window of equal duration, the direct action of the external force is more effective in promoting event $\mathcal{E}$ against $\tilde{\mathcal{E}}$ under steady conditions. Clearly, the two different kinds of steady conditions can be compared by making other choices. For instance, in Ref. [57], in a setup similar to the present one (diffusion dynamics on a ring), the two steady conditions were compared in terms of average dissipation rate at fixed time-averaged current and nonequilibrium distribution.

As a whole, the above outcomes are certainly not striking since the bounds are not particularly stringent. Such model cases only serve to illustrate the fulfillment of some of the theoretical expectations. As already stressed, the value of the inequalities lies more in the fact that they allow us to state what *cannot* be realized (no-go conditions) for a given physical setup rather than what can be realized.

VII. REMARKS AND PERSPECTIVES

The breakdown of the statistical equivalence between direct and inverse dynamical outputs (events) under nonequi-

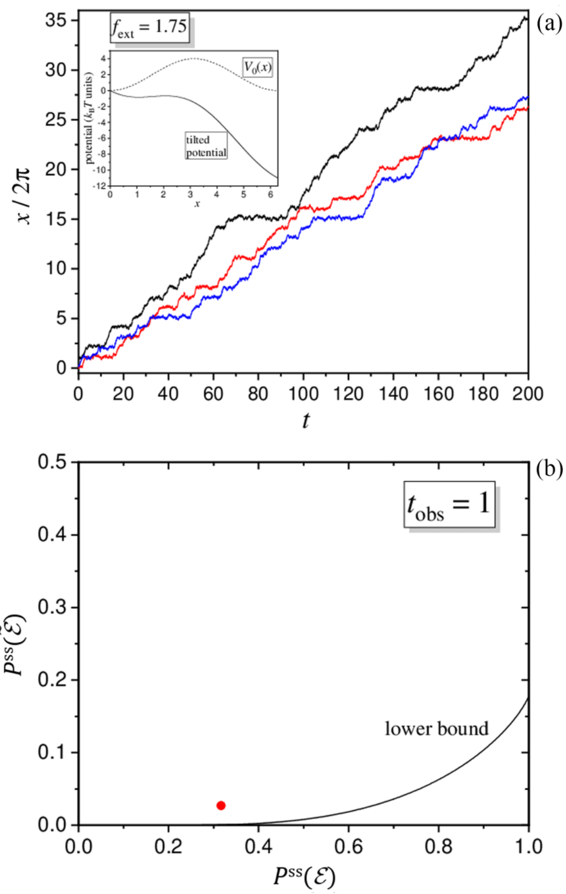

FIG. 9. The same one-dimensional overdamped rotor of Figs. 7 and 8, here subjected to a constant external force $f_{\rm ext} = 1.75$ (case C; see text for details). (a) Three trajectories up to time 200. The inset shows the potential $V_0(x)$ and the tilted potential $V_0(x) - f_{\rm ext}x$. (b) The lower bound on $P^{\rm ss}(\tilde{\mathcal{E}})$ versus $P^{\rm ss}(\mathcal{E})$, where event $\mathcal{E}$ is the net positive rotation of at least $\pi/2$ in the observation time $t_{\rm obs} = 1$ under steady-state conditions. The solid circle corresponds to the actual outcome from the simulations.

librium conditions is a key feature in any "active" scenario in which some average directionality is important. This is particularly relevant for nanoscale systems, where a bias needs to be guaranteed in the presence of fluctuations.

In this work we started from the fluctuation theorem (FT) in an attempt to derive a set of inequalities concerning the probabilities of occurrence of pair-conjugate direct and inverse events, $\mathcal{E}$ and $\tilde{\mathcal{E}}$, in a bidirectional setup. The general result of Eq. (12) was derived by bundling the trajectories (in the forward and backward directions) under the requisite of occurrence of the events, and then exploiting the Jensen inequality. Such a general relation sets the scene to obtain, or re-obtain, a number of specific results.

The general result was then applied to some relevant nonequilibrium conditions, namely, to forward-backward processes, to periodic steady states reached under time-symmetric protocols, and to steady states under external constant forces. In all these situations, the processes start from the system at thermal equilibrium. The specific results presented here, which are collected in Sec. V, mainly have the value of *no-go* conditions since they delimit forbidden regions within which the outcomes *cannot* fall.

The bounds here derived are not stringent because, given a pair of conjugate events $\mathcal{E}$ and $\tilde{\mathcal{E}}$, it is required to operate far enough from equilibrium in order to have an appreciable differentiation between their probabilities of occurrence. This, in turn, causes the allowed region to rapidly become ever wider (like in Fig. 3) or, equivalently, the forbidden region reduces more and more. The results are interesting when we consider a collection of mutually exclusive events, or when elaborating the statistics of additive properties. This led us to obtain the lower bound in Eq. (35) concerning the work distribution function under steady conditions, and find the already known generalized thermodynamic uncertainty relations, Eqs. (31) and (33).

Furthermore, in Sec. VI we reversed the viewpoint turning to the inference about the forward or backward direction of a driven process if it is known that an event did take place. Although this is an ancillary application of the more general results, the perspective might offer hints for further studies.

It is worth noting that the average work done in a driven transformation can be related to average responses of the system and with intrinsic dynamical properties of the system itself. For instance, in cyclic transformations starting from equilibrium, the average work (hence, the average dissipation) sets a lower bound on the average polarization that can be realized on periodic degrees of freedom of the system [58,59]. Also, in the limit of sufficiently small perturbations from equilibrium, the average work is related to the intrinsic modes and rates of fluctuation [60]. By exploiting these results, it would be interesting to elaborate inequalities that *directly* connect the probability of occurrence of events with average responses and intrinsic dynamical features of the system at equilibrium.

Finally, it must be stressed that the focus here was on systems subjected to an active external drive, which may be a controlled energy modulation or the application of an external force. The outcomes can easily be generalized including the situation of systems kept under nonequilibrium steady states by kinetic processes with broken detailed balance. For instance, this would be the natural setup of autonomous molecular motors and machines [61,62], in which the stochastic dynamics on continuous degrees of freedom is coupled with jumps among different energy landscapes. Even in such cases, the system can be observed for a given time, and the probabilities of seeing the direct or inverse event can still be expressed and mutually bounded. The key quantity "on the right-hand side of the inequalities" would now be the average dissipated energy (in k_BT units) which, at the steady state, linearly increases with the observation time. For instance, an interesting variant of case B treated in Sec. VI could be a "flashing ratchet model" in which the system stochastically switches between two energy profiles. This kind of model has been widely studied from the dynamical point of view [61] and, later, also from the stochastic thermodynamics perspective [63]. It would be interesting to inspect such a system from the viewpoint of the unbalancing between direct and inverse events at the steady state.

ACKNOWLEDGMENTS

I am grateful to T. Nishiyama for our insightful discussions on the lower bound of the Kullback-Leibler divergence, and for having indirectly led me to consider the context of the generalized thermodynamic uncertainty relations.

APPENDIX A: ENERGETICS IN THE NONEQUILIBRIUM PROCESSES

1. Time-dependent energy modulation

The deterministic change of a parameter λ results in a time-dependent energy $V_{\lambda(t)}(\mathbf{x})$, meaning that if the drive were stopped at some time t and the controlled parameter were held fixed, the system would freely relax towards the "underlying" equilibrium distribution $\rho_{\text{eq},\lambda(t)}(\mathbf{x}) \propto \exp\{-\beta V_{\lambda(t)}(\mathbf{x})\}$.

In the forward processes, during which the controlled parameter is varied from λ_0 to λ_1 according to a chosen protocol $\lambda_F(t)$ with $0 \leqslant t \leqslant t_f$, the system is taken out of equilibrium, meaning that the actual distribution differs from the underlying equilibrium distribution. In particular, at the time t_f the system will be in one of the possible microstates with distribution $\rho_F(\mathbf{x}, t_f)$ which depends on the specific protocol, on the initial distribution $\rho_0(\mathbf{x})$, and on the dynamical response of the system. Similarly, in the backward process, starting from microstates picked from $\rho_1(\mathbf{x})$, the final distribution will be $\rho_B(\mathbf{x}, t_f)$. Note that, in general, $\rho_F(\mathbf{x}, t_f) \neq \rho_1(\mathbf{x})$ and that $\rho_B(\mathbf{x}, t_f) \neq \rho_0(\mathbf{x})$.

Along single trajectories γ and with reference to the forward process $[\lambda(t) = \lambda_F(t)]$, the work done between two generic times t_1 and t_2 is given by [10]

$$w_F(t_1, t_2) = \int_{t_1}^{t_2} dt\, \frac{\partial V_{\lambda(t)}(\mathbf{x})}{\partial t}\bigg|_{\mathbf{x}=\mathbf{x}(t)} \tag{A1}$$

while the exchanged heat is

$$q_F(t_1, t_2) = [V_{\lambda(t_2)}(\mathbf{x}(t_2)) - V_{\lambda(t_1)}(\mathbf{x}(t_1))] - w_F(t_1, t_2). \tag{A2}$$

Analogous expressions hold for the backward process. If the whole time window of interest is taken into account, the above expressions yield $w_F(\gamma)$ and $q_F(\gamma)$.

2. Time-independent external forces

If only an external nonconservative force directly acts on the system, the energetics remains unaltered: As soon as the external action is stopped, the force disappears and the system relaxes to $\rho_{\text{eq},0}(\mathbf{x}) \propto \exp\{-\beta V_0(\mathbf{x})\}$ with $V_0(\mathbf{x})$ being the bare energy. Let $\mathbf{f}_{\text{ext}}(\mathbf{x})$ be the external force, which is here assumed to be time independent but possibly dependent on the system's microstate $\mathbf{x}$.

In such a scenario, the F and B directions can be collectively differentiated from each other only by the different initial distributions $\rho_0(\mathbf{x})$ (for F) and $\rho_1(\mathbf{x})$ (for B). At the single-trajectory level, indeed, the applied external force is always $\mathbf{f}_{\text{ext}}(\mathbf{x})$. Along a single trajectory, the work performed between two instants t_1 and t_2 is now expressed by

$$w(t_1, t_2) = \int_{t_1}^{t_2} dt\, \mathbf{f}_{\text{ext}}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \tag{A3}$$

and the exchanged heat is

$$q(t_1, t_2) = [V_0(\mathbf{x}(t_2)) - V_0(\mathbf{x}(t_1))] - w(t_1, t_2). \tag{A4}$$

In the whole time window of interest, the above expressions yield $w(\gamma)$ and $q(\gamma)$. If $\rho_0(\mathbf{x})$ and $\rho_1(\mathbf{x})$ were identical, for instance, equal to the thermal-equilibrium distribution, then there would be no distinction between F and B (and such labels would be superfluous). With reference to the general case, though, we still write $\langle w_F\rangle$, $\langle w_B\rangle$, $\langle q_F\rangle$, and $\langle q_B\rangle$ for the average quantities.

APPENDIX B: EXPRESSIONS OF $\langle \Phi_F \rangle$

1. General expression

The average of $\Phi_F(\gamma)$ over the ensemble of trajectories γ turns out to be expressible as

$$\langle \Phi_F\rangle = -\beta\langle q_F\rangle + \frac{S[\rho_F] - S[\rho_0]}{k_B} + \mathcal{D}(\rho_F||\rho_1), \tag{B1}$$

where

$$S[\rho] = -k_B \int d\mathbf{x}\, \rho(\mathbf{x}) \ln \rho(\mathbf{x}) \tag{B2}$$

is the Shannon entropy of a distribution $\rho(\mathbf{x})$, and where

$$\mathcal{D}(\rho_F||\rho_1) = \int d\mathbf{x}\, \rho_F(\mathbf{x}, t_f) \ln \frac{\rho_F(\mathbf{x}, t_f)}{\rho_1(\mathbf{x})} \tag{B3}$$

is the Kullback-Leibler divergence (or relative entropy) [51] of $\rho_F(\mathbf{x}, t_f)$ with respect to $\rho_1(\mathbf{x})$.

To see this, let us indicate the second term on the right-hand side of Eq. (2) as $[\cdots] = -\ln \rho_1(\tilde{\mathbf{x}}(t_f)) + \ln \rho_0(\mathbf{x}(0))$. Its average over the ensemble of trajectories γ in the F process is $\langle[\cdots]\rangle = \int d\mathbf{x}(0) \int d\mathbf{x}(t_f)\, \rho_F(\mathbf{x}(0), \mathbf{x}(t_f))[\cdots]$, where $\rho_F(\mathbf{x}(0), \mathbf{x}(t_f))$ is the joint distribution with marginals $\int d\mathbf{x}(t_f)\, \rho_F(\mathbf{x}(0), \mathbf{x}(t_f)) = \rho_0(\mathbf{x}(0))$ and $\int d\mathbf{x}(0)\, \rho_F(\mathbf{x}(0), \mathbf{x}(t_f)) = \rho_F(\mathbf{x}(t_f), t_f)$. By inserting the explicit form of $[\cdots]$ it follows that $\langle[\cdots]\rangle = \int d\mathbf{x}(0)\, \rho_0(\mathbf{x}(0)) \ln \rho_0(\mathbf{x}(0)) - \int d\mathbf{x}(t_f)\, \rho_F(\mathbf{x}(t_f), t_f) \ln \rho_1(\tilde{\mathbf{x}}(t_f))$. The first term corresponds to $-S[\rho_0]/k_B$ while the second, including the minus sign, can be decomposed as $\mathcal{D}(\rho_F||\rho_1) + S[\rho_F]/k_B$. Equation (B1) is then obtained by also including the contribution $-\beta\langle q_F\rangle$ coming from the first term on the right-hand side of Eq. (2).

Let us now derive the explicit expressions of $\langle \Phi_F\rangle$ for the three cases described in Sec. II C.

2. Finite-time processes starting from equilibrium

Let us consider finite-time processes of duration t_f starting from equilibrium. Under a time-dependent drive, this means that $\rho_0(\mathbf{x}) \equiv \rho_{\text{eq},0}(\mathbf{x})$ and $\rho_1(\mathbf{x}) \equiv \rho_{\text{eq},1}(\mathbf{x})$, where $\rho_{\text{eq},0}(\mathbf{x}) \propto \exp\{-\beta V_{\lambda_0}(\mathbf{x})\}$ and $\rho_{\text{eq},1}(\mathbf{x}) \propto \exp\{-\beta V_{\lambda_1}(\mathbf{x})\}$ are the canonical distributions at thermal equilibrium. Accordingly (see the derivation in Ref. [64]),

$$\langle \Phi_F\rangle = \frac{\Delta S_{\text{tot}}}{k_B} \equiv \beta(\langle w_F\rangle - \Delta A), \tag{B4}$$

where ΔS_{tot} is the average variation of the entropy of system plus environment also including the free relaxation phase after the driven process, and ΔA is the variation of the system's Helmholtz free energy when the equilibrium distribution changes from $\rho_{\mathrm{eq},0}(\mathbf{x})$ to $\rho_{\mathrm{eq},1}(\mathbf{x})$. The quantity $\langle w_F\rangle - \Delta A$ represents the average dissipated work in the active part of the process and, when divided by the temperature, gives the average entropy production in the environment also including the contribution of the final free relaxation phase. In Eq. (B4), $\langle w_F\rangle$ has to be evaluated by averaging the work $w_F(\gamma)$ expressed according to Eq. (A1).

Under a time-independent external force, we have instead that $\rho_0(\mathbf{x}) = \rho_1(\mathbf{x}) \equiv \rho_{\mathrm{eq},0}(\mathbf{x}) \propto \exp\{-\beta V_0(\mathbf{x})\}$, where, let us remember, $V_0(\mathbf{x})$ is the unaltered energy. In this case, labels F and B are superfluous since there is no distinction between forward and backward directions. Thus, $\langle w_F\rangle$ is simply $\langle w\rangle$ and Eq. (B4) is replaced by (see Ref. [65])

$$\langle \Phi_F\rangle = \frac{\Delta S_{\mathrm{ext}}}{k_B} \equiv \beta\langle w\rangle, \tag{B5}$$

where ΔS_{ext} is the average variation of the environment's entropy up to time t_f. In Eq. (B5), $\langle w\rangle$ refers to the time window of duration t_f and has to be evaluated by averaging the work $w(\gamma)$ expressed according to Eq. (A3).

3. Periodic steady states with time-symmetric drive

Let us consider a system permanently subjected to a periodic external drive of period τ, and let the external drive be time symmetric as specified in Sec. II C. In this situation, we are free to take two time windows of duration τ, one for the forward and one for the backward direction, such that, under periodic steady-state conditions, all the distributions $\rho_0(\mathbf{x})$, $\rho_1(\mathbf{x})$, $\rho_F(\mathbf{x},\tau)$, and $\rho_B(\mathbf{x},\tau)$ coincide. This implies that (see Ref. [66])

$$\langle \Phi_F\rangle = \frac{\Delta S_{\mathrm{ext,cycle}}}{k_B} \equiv \beta\langle w_\tau\rangle, \tag{B6}$$

where $\Delta S_{\mathrm{ext,cycle}}$ is the average entropy variation of the environment per cycle, and $\langle w_\tau\rangle$ is the average work in a time window of duration equal to the period τ. This average value is independent of the collocation of the time window on the timeline.

4. Steady states under time-independent forces

Let us consider steady-state conditions eventually reached, starting once again from equilibrium, under the persistent action of an external force. By choosing a time window of generic duration t_{obs}, we have that $\rho_0(\mathbf{x})$, $\rho_1(\mathbf{x})$, $\rho_F(\mathbf{x},t_{\mathrm{obs}})$, and $\rho_B(\mathbf{x},t_{\mathrm{obs}})$ are all identical to the steady-state distribution $\rho^{\mathrm{ss}}(\mathbf{x})$. Let us note that, also in this case, labels F and B are immaterial. In addition, $\langle q_{t_{\mathrm{obs}}}\rangle = -\langle w_{t_{\mathrm{obs}}}\rangle$, where t_{obs} as subscript refers to the duration of the monitoring. With these positions, from Eq. (B1) it follows that

$$\langle \Phi_F\rangle = \beta\langle w_{t_{\mathrm{obs}}}\rangle. \tag{B7}$$

The average work can also be expressed as

$$\beta\langle w_{t_{\mathrm{obs}}}\rangle \equiv \frac{\sigma^{\mathrm{ss}}\, t_{\mathrm{obs}}}{k_B}, \tag{B8}$$

where σ^{ss} is the average entropy production rate at the steady state.

APPENDIX C: DERIVATION OF EQ. (12)

Let us first consider the case of a single event $\mathcal{E}$ and its inverse $\tilde{\mathcal{E}}$. Let us start from the main fluctuation relation, Eq. (1). On the left-hand side, let us take the summation over all trajectories γ along which, in the forward process, $\Phi_F(\gamma) = \Phi_F$ *and* $\mathcal{E}$ occurs. Correspondingly, on the right-hand side the summation is taken over the conjugate trajectories $\tilde{\gamma}$ along which, in the backward process, $\Phi_B(\tilde{\gamma}) = -\Phi_F$ *and* $\tilde{\mathcal{E}}$ occurs. Here, Φ_F is some fixed value. These summations yield

$$\rho_F(\Phi_F,\mathcal{E})\, e^{-\Phi_F} = \rho_B(-\Phi_F,\tilde{\mathcal{E}}). \tag{C1}$$

The joint distributions can be written as $\rho_F(\Phi_F,\mathcal{E}) = P_F(\mathcal{E})\,\rho_F(\Phi_F|\mathcal{E})$ and $\rho_B(-\Phi_F,\tilde{\mathcal{E}}) = P_B(\tilde{\mathcal{E}})\,\rho_B(-\Phi_F|\tilde{\mathcal{E}})$, where "$|\mathcal{E}$" and "$|\tilde{\mathcal{E}}$" stand for conditions to be fulfilled. Plugging these forms into Eq. (C1) and integrating over Φ_F leads to $P_F(\mathcal{E})\,\langle e^{-\Phi_F}\rangle_{\mathcal{E}} = P_B(\tilde{\mathcal{E}})$, where we have introduced the conditioned average $\langle f(\Phi_F)\rangle_{\mathcal{E}} \equiv \int d\Phi_F\, f(\Phi_F)\,\rho_F(\Phi_F|\mathcal{E})$ for a generic function of Φ_F. By exploiting the convexity of the exponential function, Jensen inequality yields $\langle e^{-\Phi_F}\rangle_{\mathcal{E}} \geqslant e^{-\langle \Phi_F\rangle_{\mathcal{E}}}$. Thus, the following inequality is obtained:

$$\ln\frac{P_F(\mathcal{E})}{P_B(\tilde{\mathcal{E}})} \leqslant \langle \Phi_F\rangle_{\mathcal{E}}. \tag{C2}$$

The same reasoning can be applied by replacing event $\mathcal{E}$ with its negation "no $\mathcal{E}$," and $\tilde{\mathcal{E}}$ with "no $\tilde{\mathcal{E}}$." This leads to

$$\ln\frac{P_F(\text{no }\mathcal{E})}{P_B(\text{no }\tilde{\mathcal{E}})} \leqslant \langle \Phi_F\rangle_{\text{no }\mathcal{E}} \tag{C3}$$

with $P_F(\text{no }\mathcal{E}) = 1 - P_F(\mathcal{E})$ and $P_B(\text{no }\tilde{\mathcal{E}}) = 1 - P_B(\tilde{\mathcal{E}})$. By multiplying both members of Eq. (C2) by $P_F(\mathcal{E})$, and both members of Eq. (C3) by $1 - P_F(\mathcal{E})$, and then summing the two expressions, it follows that

$$P_F(\mathcal{E})\ln\frac{P_F(\mathcal{E})}{P_B(\tilde{\mathcal{E}})} + (1-P_F(\mathcal{E}))\ln\left(\frac{1-P_F(\mathcal{E})}{1-P_B(\tilde{\mathcal{E}})}\right) \leqslant \langle \Phi_F\rangle, \tag{C4}$$

where it has been taken into account that $P_F(\mathcal{E})\langle \Phi_F\rangle_{\mathcal{E}} + (1 - P_F(\mathcal{E}))\langle \Phi_F\rangle_{\text{no }\mathcal{E}} = \langle \Phi_F\rangle$.

The left-hand side of Eq. (C4) corresponds to the $\mathcal{C}_a^{\mathrm{FB}}$ defined in Eq. (4). By referring back to Eq. (B1) for $\langle \Phi_F\rangle$, the specific form of Eq. (12) with $\mathcal{C} = \mathcal{C}_a^{\mathrm{FB}}$ is obtained.

In order to elaborate the case with $\mathcal{C} = \mathcal{C}_b^{\mathrm{FB}}$, let us start by considering Eq. (C2) written for "only $\mathcal{E}_i$," i.e., $\ln[P_F(\text{only }\mathcal{E}_i)/P_B(\text{only }\tilde{\mathcal{E}}_i)] \leqslant \langle \Phi_F\rangle_{\text{only }\mathcal{E}_i}$. By multiplying each member by $P_F(\text{only }\mathcal{E}_i)$ and then summing on the index i which labels the events of the set, we get

$$\sum_{i=1}^{N} P_F(\text{only }\mathcal{E}_i)\ln\frac{P_F(\text{only }\mathcal{E}_i)}{P_B(\text{only }\tilde{\mathcal{E}}_i)} \leqslant \sum_{i=1}^{N} P_F(\text{only }\mathcal{E}_i)\langle \Phi_F\rangle_{\text{only }\mathcal{E}_i}. \tag{C5}$$

Let us now introduce the event $\mathcal{E}_0$ specified by the following: "Either none of the events takes place, or two or more different events occur in the given time window." The inverse event $\tilde{\mathcal{E}}_0$ is correspondingly specified. Note that $\mathcal{E}_0$ is the negation of

"Only $\mathcal{E}_1$, or only $\mathcal{E}_2, \ldots$, or only $\mathcal{E}_N$ takes place." Thus,

$$P_F(\mathcal{E}_0) = 1 - \sum_{i=1}^{N} P_F(\text{only } \mathcal{E}_i),$$
$$P_B(\tilde{\mathcal{E}}_0) = 1 - \sum_{i=1}^{N} P_B(\text{only } \tilde{\mathcal{E}}_i). \tag{C6}$$

From the analog of Eq. (C2) written for $\mathcal{E}_0$, it follows that

$$P_F(\mathcal{E}_0) \ln \frac{P_F(\mathcal{E}_0)}{P_B(\tilde{\mathcal{E}}_0)} \leqslant P_F(\mathcal{E}_0)\langle \Phi_F \rangle_{\mathcal{E}_0}. \tag{C7}$$

Finally, inequality (12) with $\mathcal{C} = \mathcal{C}_b^{\mathrm{FB}}$ is obtained by summing member with member Eqs. (C5) and (C7), by inserting Eqs. (C6), and finally considering that

$$\sum_{i=1}^{N} P_F(\text{only } \mathcal{E}_i)\ \langle \Phi_F \rangle_{\text{only } \mathcal{E}_i} + P_F(\mathcal{E}_0)\langle \Phi_F \rangle_{\mathcal{E}_0} = \langle \Phi_F \rangle. \tag{C8}$$

The other forms of Eq. (12), with $\mathcal{C}$ equal to $\mathcal{C}_c^{\mathrm{FB}}$ and $\mathcal{C}_d^{\mathrm{FB}}$, follow directly as explained in Sec. III C. Since the initial distributions in the F and B processes are general, the derivation is also valid when $\mathcal{C}$ corresponds to $\mathcal{C}_a^{\mathrm{ss}}$, $\mathcal{C}_b^{\mathrm{ss}}$, $\mathcal{C}_c^{\mathrm{ss}}$, or $\mathcal{C}_d^{\mathrm{ss}}$. What changes is $\langle \Phi_F \rangle$ on the right-hand side of Eq. (12), which must be expressed according to the specific kind of steady conditions.

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[24] Equation (1) can be derived from the following relation which holds for stochastic Markov dynamics (it is also required that the system's energetics should be invariant under the replacement of $\mathbf{x}$ with $\tilde{\mathbf{x}}$) [5]: $P_F(\gamma|\mathbf{x}_{\text{init}} = \mathbf{x}(0)) = P_B(\tilde{\gamma}|\mathbf{x}_{\text{init}} = \tilde{\mathbf{x}}(t_f)) \times e^{-\beta q_F(\gamma|\mathbf{x}_{\text{init}}=\mathbf{x}(0))}$, where $\gamma|\mathbf{x}_{\text{init}} = \mathbf{x}(0)$ specifies that the trajectory γ starts from the microstate $\mathbf{x}(0)$, and $\tilde{\gamma}|\mathbf{x}_{\text{init}} = \tilde{\mathbf{x}}(t_f)$ specifies that the conjugate trajectory $\tilde{\gamma}$ starts from the terminal microstate $\mathbf{x}(t_f)$ with reversed momenta if these are included among the degrees of freedom; $q_F(\gamma|\mathbf{x}_{\text{init}} = \mathbf{x}(0))$ is the total heat exchanged, up to time t_f in the forward process, between the system, which follows the trajectory γ, and the thermal bath. This key relation holds for systems either subjected to an energy modulation or under external forces, on condition that the heat is evaluated accordingly. A simple way to understand this is to follow the steps involved in Crooks' approach (see Ref. [5]) by considering that the microreversibility is violated when external forces are at play. [Specifically, F and B evolutions are discretized into sequences of elemental jumps, each one being the mirrored version of the other. Given two vicinal microstates $\mathbf{x}^{(a)}$ and $\mathbf{x}^{(b)}$, and the conjugate $\tilde{\mathbf{x}}^{(a)}$ and $\tilde{\mathbf{x}}^{(b)}$, the conditional probabilities of inverse and direct jumps are related by $P(\tilde{\mathbf{x}}^{(a)} \leftarrow \tilde{\mathbf{x}}^{(b)}|\tilde{\mathbf{x}}^{(b)}) = P(\mathbf{x}^{(a)} \rightarrow \mathbf{x}^{(b)}|\mathbf{x}^{(a)}) \times e^{\beta\delta q_{a\rightarrow b}}$, where $\delta q_{a\rightarrow b}$ is the infinitesimal amount of heat exchanged in the $\mathbf{x}^{(a)} \rightarrow \mathbf{x}^{(b)}$ jump. Such a relation holds for both time-dependent energy modulations and external forces, on condition that $\delta q_{a\rightarrow b}$ is expressed, respectively, according to Eq. (A2) or Eq. (A4).] Then, $P_F(\gamma) = \rho_0(\mathbf{x}(0))P_F(\gamma|\mathbf{x}_{\text{init}} = \mathbf{x}(0))$ and $P_B(\tilde{\gamma}) = \rho_1(\tilde{\mathbf{x}}(t_f))P_B(\tilde{\gamma}|\mathbf{x}_{\text{init}} = \tilde{\mathbf{x}}(t_f))$ are the probabilities of observing the two trajectories, respectively, in the F and B processes. By plugging these expressions into the above relation, it

follows that $P_F(\gamma)e^{\beta q_F(\gamma)}e^{[-\ln\rho_0(\mathbf{x}(0))+\ln\rho_1(\tilde{\mathbf{x}}(t_f))]} = P_B(\tilde{\gamma})$. Equation (1) is then obtained by recalling the definition of $\Phi_F(\gamma)$ given in Eq. (2).

[25] In this work, any thermodynamic state function of a system (e.g., Helmholtz free energy A, entropy S) refers to that system in equilibrium conditions.

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[29] For instance, $\lambda_F(t) = \lambda_0 \sin(\omega t + \phi)$, with ϕ being a certain phase, fulfills Eq. (3) by setting $t^* = (\pi - 2\phi)/\omega$. Instead, Eq. (3) is not satisfied for a non-time-symmetric periodic protocol like $\lambda_F(t) = \lambda_0 \sin(\omega t)\exp\{\sin(2\omega t)\}$.

[30] Implicitly, by adopting such a viewpoint, one refers to an individual system. If, instead, one refers to an ensemble of replicas of the same system all evolving independently and in parallel, or equivalently if one refers to an infinite number of repetitions of the process, then the probabilities become statistical distributions on the outcomes.

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[32] Let us focus on probability distributions on countable sets. The total variation (TV) between two distributions $\{P_i\}$ and $\{Q_i\}$ is defined as $\mathrm{TV} = 2^{-1}\sum_i |P_i - Q_i|$, which follows from a more general definition of TV [67]. The following inequality holds [31]: $\mathrm{TV} \leqslant \sqrt{1 - e^{-\mathcal{D}}}$ where $\mathcal{D} = \sum_i P_i \ln(P_i/Q_i)$ is the Kullback-Leibler divergence. In our specific applications, the distributions are defined on two-element sets. Let us take $\{P_F(\mathcal{E}), 1 - P_F(\mathcal{E})\}$ as set $\{P_i\}$, and take $\{P_B(\tilde{\mathcal{E}}), 1 - P_B(\tilde{\mathcal{E}})\}$ as set $\{Q_i\}$. The left-hand sides of the inequalities in Eq. (17) correspond to the Kullback-Leibler divergence of $\{P_i\}$ with respect to $\{Q_i\}$ (see first inequality), and of $\{Q_i\}$ with respect to $\{P_i\}$ (see second one). In both cases, the total variation is simply $\mathrm{TV} = 2^{-1}[|P_F(\mathcal{E}) - P_B(\tilde{\mathcal{E}})| + |1 - P_F(\mathcal{E}) - (1 - P_B(\tilde{\mathcal{E}}))|] = |P_F(\mathcal{E}) - P_B(\tilde{\mathcal{E}})|$. By considering the two upper bounds given in Eq. (17) simultaneously, then Eq. (18) is readily obtained by taking the most stringent bound.

[33] Indeed, by summing on the left-hand side of Eq. (1) over the direct trajectories along which the value of the cumulative variable falls between $\chi - \delta\chi/2$ and $\chi + \delta\chi/2$, and on the right-hand side over the conjugate inverse trajectories for which the value falls between $-\chi - \delta\chi/2$ and $-\chi + \delta\chi/2$, we end up with $\ln(\rho_F(\chi)/\rho_B(\chi)) = \ln\langle e^{\Phi_F}\rangle_\chi \leqslant \langle\Phi_F\rangle_\chi$, where χ at subscript means that the ensemble average is taken under the constraint of χ being fixed, and where Jensen's inequality has been applied for the last step. The multiplication by $\rho_F(\chi)$ at both members, and the subsequent integration over χ, finally yields Eq. (27) by taking into account that $\int d\chi \langle\Phi_F\rangle_\chi \rho_F(\chi) \equiv \langle\Phi_F\rangle$ and by recalling Eq. (B4).

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$P(B|\mathcal{E}_s) = 1$, the combination of the two expressions leads to $P(\mathcal{E}_s) = [P_F(\mathcal{E}_s) + P_B(\mathcal{E}_s)]/2$ and, ultimately, to Eq. (37).

[55] Let us first consider only Eq. (38). For brevity, let us indicate with $F(a, b)$ the function on the left-hand side, and let $c = \beta\langle w_{\mathrm{diss},F}\rangle$. The two branches, for Eq. (38) alone, would correspond to the functions $b_c(a)$ that are solutions of $F(a, b_c(a)) = c$. From this we get that $db_c(a)/da = -(\partial F(a, b)/\partial a\,|_{b=b_c(a)})/(\partial F(a, b)/\partial b\,|_{b=b_c(a)})$. With some algebraic steps, the two partial derivatives are found to be $\partial F(a, b)/\partial a = \alpha - \ln\alpha - 1$ and $\partial F(a, b)/\partial b = \vartheta(2b - 1)$, where we have introduced $\alpha = [b(1 + a) - a]/[(1 - a)(1 - b)]$ and $\vartheta = a/\{b(1 - b)[b(1 + a) - a]\}$. Both α and ϑ are non-negative. Since $\ln\alpha \geqslant \alpha - 1$ for all $\alpha > 0$, it follows that $\partial F(a, b)/\partial a\,|_{b=b_c(a)}> 0$ for any c and a. For $b > 1/2$ (upper branch), $\partial F(a, b)/\partial b\,|_{b=b_c(a)}> 0$ for any c and a. Thus, $db_c(a)/da \leqslant 0$ for the upper branch which, therefore, has a decreasing profile. For $b < 1/2$ (lower branch), the situation is reversed (increasing profile). The same kind of result is obtained working only with Eq. (39). By combining both inequalities, that is, by taking the most restrictive constraint, the monotonicity of the upper and lower branches is still ensured.

[56] In all cases, the time step Δt was set in the order of 10^{-4} (the exact value was tuned in order to have an integer number of steps in the time window of interest). The drawing of $x(0)$ from the initial equilibrium distribution was done by employing von Neuman's sampling method [see Appendix G.6 in M. P. Allen and D. J. Tildesley, *Computer Simulation of Liquids* (Claredon Press, Oxford, U.K., 1987)]. In all cases, the number of realizations in a simulation was 10^4. The simulations were repeated ten times to obtain the reported average values and standard deviations. In case B, the independence of the outcome from the choice of the starting time of monitoring, t_0, was checked by repeating the simulations for different fixed values of t_0. After that, the simulations were performed by generating t_0 at random (for each trajectory) within a prescribed cycle. For the determination of the probabilities $P^{\mathrm{ss}}(\mathcal{E})$ and $P^{\mathrm{ss}}(\tilde{\mathcal{E}})$ in case B, the chosen cycle was the 50th one. The values of w_τ in all cycles between the 41st and the 50th were considered to determine $\langle w_\tau\rangle$ and $P^{\mathrm{ss}}(w_\tau < 0)$. In case C, the initial acquisition time was set equal to $t_0 = 50$. It was found that $|\,V_0(x(t_0 + t_{\mathrm{obs}})) - V_0(x(t_0))\,|$ was at most 0.15% of the average work done in the same time window, thus ensuring that steady conditions were reached. In cases A and B, the work done in the course of the process was computed by summing up the contributions $\delta w = V(x(t_n), t_{n+1}) - V(x(t_n), t_n)$ with $t_{n+1} = t_n + \Delta t$. For case B, the work distribution $\rho^{\mathrm{ss}}(w_\tau)$ was determined from 10^4 values w_τ in the 50th cycle. In case C, the work along a trajectory was computed as $f_{\mathrm{ext}} \times [x(t_0 + t_{\mathrm{obs}}) - x(t_0)]$.

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[64] Let us consider the system's energetics associated with the underlying equilibrium distribution at completion of the forward driven process: $\beta V_{\lambda_1}(\mathbf{x}) = -\ln\rho_{\mathrm{eq},1}(\mathbf{x}) - \ln Z_1$, where Z_1 is the canonical partition function. The average variation of such energy in the free relaxation phase (after the time t_f) is $\langle\Delta V\rangle_{\mathrm{relax}} = \int d\mathbf{x}V_{\lambda_1}(\mathbf{x})\rho_{\mathrm{eq},1}(\mathbf{x}) - \int d\mathbf{x}V_{\lambda_1}(\mathbf{x})\rho_F(\mathbf{x}, t_f)$. Explicitly, $\beta\langle\Delta V\rangle_{\mathrm{relax}} = [-\int d\mathbf{x}\rho_{\mathrm{eq},1}(\mathbf{x})\ln\rho_{\mathrm{eq},1}(\mathbf{x}) - \ln Z_1] - [-\int d\mathbf{x}\rho_F(\mathbf{x}, t_f)\ln\rho_{\mathrm{eq},1}(\mathbf{x}) - \ln Z_1]$. Since no work is performed in this phase, $\langle\Delta V\rangle_{\mathrm{relax}} \equiv \langle q_F\rangle_{\mathrm{relax}}$, where $\langle q_F\rangle_{\mathrm{relax}}$ is the average heat exchanged between system and thermal bath in the course of the free relaxation. By bringing back to mind the definitions of Shannon's entropy and Kullback-Leibler divergence, it follows that $\beta\langle q_F\rangle_{\mathrm{relax}} = -\{(S[\rho_F] - S[\rho_1])/k_B + \mathcal{D}(\rho_F||\rho_1)\}$, where, in this situation, $\rho_1(\mathbf{x}) \equiv \rho_{\mathrm{eq},1}(\mathbf{x})$. By using this result in Eq. (B1) it follows that $\langle\Phi_F\rangle = -\beta\langle q_F\rangle_{\mathrm{tot}} + (S[\rho_1] - S[\rho_0])/k_B$ (with $\rho_0(\mathbf{x}) \equiv \rho_{\mathrm{eq},0}(\mathbf{x})$), where $\langle q_F\rangle_{\mathrm{tot}} = \langle q_F\rangle + \langle q_F\rangle_{\mathrm{relax}}$ is the average total heat exchanged in the whole process (i.e., active phase plus free relaxation). The average variations of the environment's entropy and of the system's entropy are given respectively by $\Delta S_{\mathrm{ext}} = -\langle q_F\rangle_{\mathrm{tot}}/T$ and $\Delta S = S[\rho_1] - S[\rho_0]$; hence $\langle\Phi_F\rangle = \Delta S_{\mathrm{tot}}/k_B$ where $\Delta S_{\mathrm{tot}} = \Delta S + \Delta S_{\mathrm{ext}}$ is the average variation of the system-plus-environment entropy.

[65] Equation (B5) can be proved by following the reasoning in Ref. [64]. In the present case, the system's energy at the end of the process is $V_0(\mathbf{x})$, and $\langle\Delta V\rangle_{\mathrm{relax}} = \int d\mathbf{x}V_0(\mathbf{x})\rho_{\mathrm{eq},0}(\mathbf{x}) - \int d\mathbf{x}V_0(\mathbf{x})\rho(\mathbf{x}, t_f)$. By inserting $\beta V_0(\mathbf{x}) = -\ln\rho_{\mathrm{eq},0}(\mathbf{x}) - \ln Z_0$, and considering that $\langle q\rangle_{\mathrm{relax}} \equiv \langle\Delta V\rangle_{\mathrm{relax}}$, one gets $\beta\langle q\rangle_{\mathrm{relax}} = -\{(S[\rho] - S[\rho_0])/k_B + \mathcal{D}(\rho||\rho_0)\}$, where, in this situation, $\rho_0(\mathbf{x}) \equiv \rho_{\mathrm{eq},0}(\mathbf{x})$. By using Eq. (B1) it follows that $\langle\Phi_F\rangle = -\beta\langle q\rangle_{\mathrm{tot}} = \Delta S_{\mathrm{tot}}/k_B$. In this case, $\Delta S_{\mathrm{tot}} \equiv \Delta S_{\mathrm{ext}}$ because $\Delta S = 0$. More simply, Eq. (B5) follows from Eq. (B4) by setting $\Delta A = 0$.

[66] Since in this case $\rho_F \equiv \rho_1$ and $S[\rho_F] = S[\rho_0]$, from Eq. (B1) we get $\langle\Phi_F\rangle = -\beta\langle q_\tau\rangle$ in any time window of duration equal to the period of the cycle at the periodic steady state (either in the forward or backward mode). Now let us consider that $\langle\Delta V\rangle_{\mathrm{cycle}} = \langle q_\tau\rangle + \langle w_\tau\rangle = 0$ since there is no average variation of the system's energetics in a cycle, and that $\Delta S_{\mathrm{ext,cycle}}/k_B = -\langle q_\tau\rangle/T$. It follows that $\langle\Phi_F\rangle = \Delta S_{\mathrm{ext,cycle}}/k_B = \beta\langle w_\tau\rangle$.

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