Letter

Long-lived solitons and their signatures in the classical Heisenberg chain

Adam J. McRoberts,[1] Thomas Bilitewski,[1,2] Masudul Haque,[1,3,4] and Roderich Moessner[1]
[1]*Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, 01187 Dresden, Germany*
[2]*Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078, USA*
[3]*Department of Theoretical Physics, Maynooth University, County Kildare, Ireland*
[4]*Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany*

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Motivated by the Kardar-Parisi-Zhang (KPZ) scaling recently observed in the classical ferromagnetic Heisenberg chain, we investigate the role of solitonic excitations in this model. We find that the Heisenberg chain, although well known to be nonintegrable, supports a two-parameter family of long-lived solitons. We connect these to the exact soliton solutions of the integrable Ishimori chain with $\ln(1 + S_i \cdot S_j)$ interactions. We explicitly construct infinitely long-lived stationary solitons, and provide an adiabatic construction procedure for moving soliton solutions, which shows that Ishimori solitons have a long-lived Heisenberg counterpart when they are not too narrow and not too fast moving. Finally, we demonstrate their presence in thermal states of the Heisenberg chain, even when the typical soliton width is larger than the spin correlation length, and argue that these excitations likely underlie the KPZ scaling.

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Introduction. There has been renewed interest in understanding the long-time dynamics of classical many-body systems, in particular regarding the scope of anomalous, nondiffusive transport. A paradigmatic phenomenon is Kardar-Parisi-Zhang (KPZ) scaling [1], associated with (generalized) hydrodynamics [2–10] and integrability [9,11–24]. Recent theoretical developments have identified integrability and non-Abelian symmetry as key ingredients for KPZ physics [16,23–28]. Indeed, KPZ scaling is now established [9,16] in the integrable Ishimori chain [29,30], also known as the integrable lattice-Landau-Lifshitz model. Intriguingly, the simple *nonintegrable* nearest-neighbor classical Heisenberg chain was also found to host a long-lived regime of KPZ scaling at low temperature [18], and it was subsequently noted that KPZ scaling in the Ishimori chain persists under spin-symmetry-preserving perturbations [17]. While the classical Heisenberg spin chain is a widely studied system and a paradigmatic model of magnetism, it remains far from completely understood. For example, predictions of its hydrodynamics have involved ordinary diffusion [31–37] or different forms of anomalous behavior [38–41]. Our recent observation of KPZ behavior up to enormously large scales [18] thus raises the question: does the Heisenberg chain exhibit properties ordinarily associated with integrability? In particular, one might wonder if this phenomenology can be related to magnon dynamics or the existence of solitons, thought to be crucial for KPZ behavior both in quantum [24,26,42–44] and classical integrable one-dimensional (1D) spin systems [9,16,17,24].

In this Letter, we demonstrate the existence of long-lived solitons in the classical Heisenberg chain. The appellation soliton is justified by an explicit continuous connection to those of the Ishimori chain via an interpolating Hamiltonian. We provide a direct construction of stable (*infinitely* long-lived) *stationary* isolated solitons, as well as an adiabatic construction of *moving* solitons. A central result is the existence of a family of solitons which are stable over a broad parameter regime [see Fig. 1(a)]. This is, *a priori*, very surprising for a chain so far believed to be essentially generic. Beyond the isolated solitons, we study two-soliton scattering and observe behavior quite analogous to that of the integrable model. Finally, for low-temperature thermal states, we show that solitons are present and can be individually identified even when their density is high. Taken together, these observations provide a physical basis for the robust KPZ scaling observed at low temperatures in the Heisenberg chain.

Models. The classical Heisenberg Hamiltonian is

$$\mathcal{H} = -J \sum_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1} - 1), \tag{1}$$

where $\mathbf{S}_i$ are classical O(3) vectors at sites i of a chain, with nearest-neighbor ferromagnetic interaction strength J.

The integrable [25,29,30,45–48] Ishimori Hamiltonian,

$$\mathcal{H} = -2J \sum_i \ln\left(\frac{1 + \mathbf{S}_i \cdot \mathbf{S}_{i+1}}{2}\right), \tag{2}$$

possesses an extensive set of locally conserved charges, besides energy and magnetization, such as the torsion

$$\tau_i = \frac{\mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i-1})}{(1 + \mathbf{S}_i \cdot \mathbf{S}_{i+1})(1 + \mathbf{S}_i \cdot \mathbf{S}_{i-1})}. \tag{3}$$

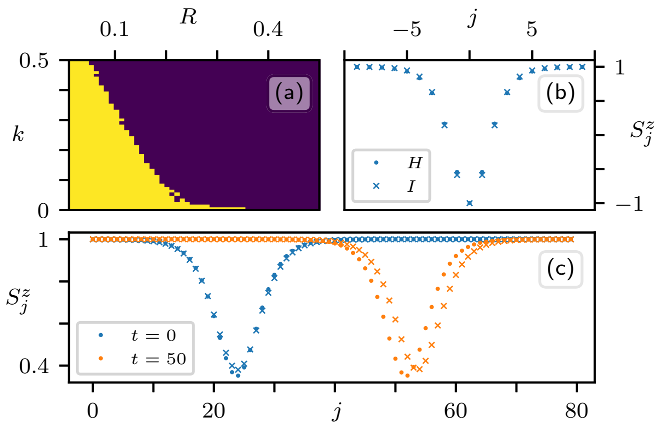

FIG. 1. Solitons in the classical Heisenberg chain. (a) Parameter space of Ishimori solitons for which we found an adiabatically connected soliton in the Heisenberg chain. (b),(c) Comparisons between adiabatically connected solitons (only z component shown) in the Heisenberg (H, •) and Ishimori (I, ×) chains. The solitons in (b) are stationary ($R = 0.25, k = 0$); and in (c), they move ($R = 0.1, k = 0.15$). The Heisenberg soliton moves at a slower velocity, but both preserve their initial profile.

We interpolate smoothly between the chains,

$$\mathcal{H} = -2J\gamma^{-1}\sum_i \ln\left(1+\gamma\frac{\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}-1}{2}\right), \tag{4}$$

with the Ishimori chain corresponding to $\gamma = 1$, and the Heisenberg chain to the limit $\gamma \to 0$, preserving SO(3) symmetry throughout. We set $J = 1$ in the following.

The classical equations of motion follow from

$$\dot{\boldsymbol{S}}_i = \frac{\partial\mathcal{H}}{\partial\boldsymbol{S}_i}\times\boldsymbol{S}_i, \tag{5}$$

from which we obtain the dynamics of Eq. (4),

$$\dot{\boldsymbol{S}}_i = 2\boldsymbol{S}_i\times\left(\frac{\boldsymbol{S}_{i-1}}{2-\gamma+\gamma\boldsymbol{S}_i\cdot\boldsymbol{S}_{i-1}}+\frac{\boldsymbol{S}_{i+1}}{2-\gamma+\gamma\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}}\right). \tag{6}$$

One-soliton solutions. In the Ishimori chain [29], these are indexed by two physical parameters: An inverse-width $R \in (0, \infty)$ and a wave number $k \in [-\pi/2, \pi/2)$; see the Supplemental Material (SM) [49] for explicit expressions and their properties.

The Heisenberg chain (1), by contrast, is not integrable. We next provide exact (though not closed-form) expressions for stationary solitons in the form of an (implicit) solution to the nonlinear equations of motion of the Heisenberg model.

For this, we use canonical coordinates, $z_i = S^z_i$, $\phi_i = \arctan(S^y_i/S^x_i)$. Our ansatz is based on the structure of the stationary ($k = 0$) Ishimori solitons. We assume (i) stationarity of the z components, i.e., $\dot{z}_i = 0$, $\forall i$, (ii) spatially uniform azimuthal angles ϕ_i (except for a discontinuity of π across the center), and (iii) a uniform rotation frequency of the in-plane spin components, i.e., $\phi_i(t) = \phi_i(0) + \omega t$, $\forall i$. This ansatz reduces the equations of motion to a set of consistency equations for the z_i,

$$\begin{aligned}\dot{\phi}_i = \omega = J\frac{z_i}{\sqrt{1-z_i^2}}\left(\sqrt{1-z_{i+1}^2}+\sqrt{1-z_{i-1}^2}\right)\\ - J(z_{i+1}+z_{i-1}),\end{aligned} \tag{7}$$

which, for a chosen frequency ω, may be solved numerically to arbitrary precision (see SM [49]).

This yields stable stationary solitons of arbitrary width ($\sim 1/R$), implying that the existence diagram in Fig. 1(a) extends to infinity on the x axis. An example of a soliton obtained from the solution of these equations is shown in Fig. 1(b). This constitutes an exact soliton in the Heisenberg model. We also note that this construction extends more generally to 1D spin chains (see SM [49]).

Adiabatic connection. We next connect these stationary solitons to those in the Ishimori chain by continuously tuning the interpolating Hamiltonian (4) between the two via a C^∞-smooth interpolation,

$$\gamma(t) = \begin{cases} 1-\frac{e^{-t_A/t}}{e^{-t_A/t}+e^{-t_A/(t_A-t)}}, & 0 < t < t_A\\ 0, & t \geqslant t_A,\end{cases} \tag{8}$$

from $\gamma = 1$ at $t = 0$ to $\gamma = 0$ at some long adiabatic time t_A. We evolve an initial Ishimori soliton (R, k) under the dynamics of Eq. (4), with this time-dependent $\gamma(t)$ given by Eq. (8), for some adiabatic time t_A; we then evolve up to some later time t_f under the Heisenberg dynamics (1).

This procedure continuously transforms stationary solitons of the Ishimori chain into stationary solitons of the Heisenberg chain with the same magnetization.

Moving solitons. As the connection between the stationary solitons of the two models does not guarantee the existence of moving soliton solutions of the Heisenberg chain, we next use our adiabatic procedure to extend the "existence diagram" in Fig 1(a) to finite k. We consider a resultant state a soliton of the Heisenberg chain if the following conditions are satisfied: (i) There is, for all times, a unique local minimum of $z_i(t)$; (ii) for $t > t_A$, the torsion $\tau = \sum_i \tau_i$ is constant in time; (iii) the unique local minimum propagates with a constant velocity. These conditions are examined in more detail in the SM [49]. An example of a thus constructed moving soliton is shown in Fig. 1(c) and compared to the original Ishimori soliton.

The resulting existence diagram [Fig. 1(a)] shows that the solitons are stable in the Heisenberg model over a remarkably large range of parameters (R, k), with the narrow solitons apparently becoming unstable first with increasing velocity ($\sim k$).

We find no indication of a finite lifetime of the single soliton states which are stable under this adiabatic procedure. Moreover, the torsion—generally not a conserved quantity of the Heisenberg chain—is conserved in these states, up to numerical accuracy.

While stationary solutions of nonlinear classical equations of motion are well known, stable moving solitons are not expected to exist in a generic system [50]. From this perspective, our finding of propagating objects with apparently unlimited stability is remarkable.

Next we consider how the properties of Ishimori solitons are modified in the Heisenberg model. Figure 2 shows that

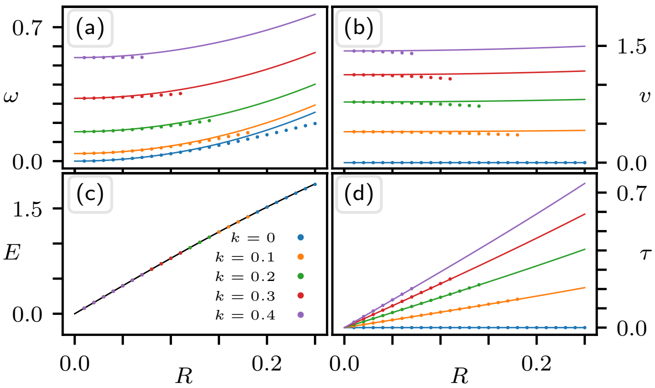

FIG. 2. Physical properties of the Ishimori solitons (solid lines) compared to Heisenberg solitons [dotted lines: up to the boundary of the existence diagram; Fig. 1(a)], shown as a function of inverse-width R, for various values of the wave number k. (a)–(d) The internal frequency ω, the velocity v, the energy E (measured by the Heisenberg Hamiltonian), and the torsion τ, respectively.

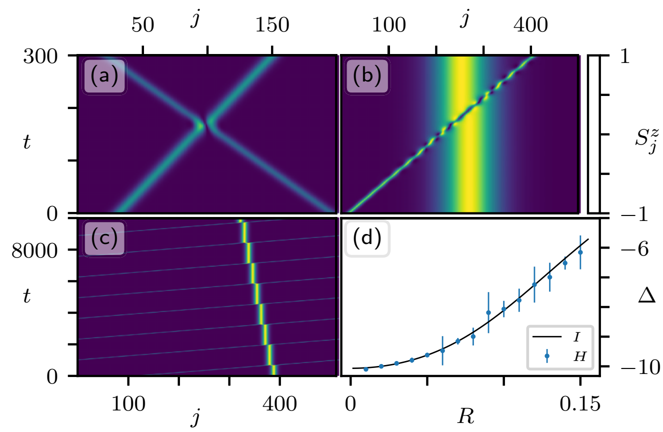

FIG. 3. Soliton scattering in the Heisenberg chain. Color scale shows the z components and is the same for (a)–(c). (a) Single scattering event between two solitons with parameters $(R, k) = (0.1, 0.1)$ and $(R', k') = (0.1, -0.15)$. (b) Screening of the magnetization transported by a narrower soliton as it moves through a wider soliton. (c) Repeated scattering of two solitons $[(R, k) = (0.1, 0.1)$ and $(R', k') = (0.1, 0)]$ under periodic boundary conditions. (d) Comparison of the scattering phase shift $\Delta(R, k = 0; 0.1, 0.1)$ in the Ishimori chain (solid line) and in the Heisenberg chain, for stationary target solitons. Phase shifts in the Heisenberg chain obtained by averaging over 10 scattering events [cf. (c)] and over the relative phases of the solitons—the error bars are the standard deviation with respect to the relative phases.

the internal frequency (the frequency with which the in-plane spin components rotate) and velocity of a Heisenberg soliton are suppressed. The energy (measured in both cases by the Heisenberg Hamiltonian) is only slightly reduced, while the torsion is very slightly higher for the Heisenberg solitons (see, also, Fig. S2 in the SM [49]). Overall, we note a remarkable similarity between the one-soliton properties in the Ishimori and Heisenberg chains. Differences increase when approaching the boundary of the stability region. i.e., for narrower solitons.

Two-soliton scattering. We now turn to interactions between the solitons. To set the stage, we briefly recall scattering in the Ishimori chain. As a fully integrable model, interactions are completely described by the two-soliton phase shifts, even for thermal multisoliton states [45,46]. When two solitons collide, the asymptotic result (compared to two separate one-soliton solutions) is unchanged, except that the solitons are displaced by a so-called phase shift [45],

$$\begin{aligned}\Delta(R, k; R', k') &= \operatorname{sgn}[v(R, k) - v(R', k')] \\ &\quad \times \frac{1}{2R} \ln\left\{\frac{\cosh[2(R + R')] - \cos[2(k - k')]}{\cosh[2(R - R')] - \cos[2(k - k')]}\right\}, \qquad (9)\end{aligned}$$

experienced by the soliton (R, k), due to a collision with the soliton (R', k').

Figure 3 displays the scattering of two Heisenberg solitons. The solitons survive scattering essentially unchanged [Fig. 3(a)], akin to the fully integrable model. While the collisions do leave the solitons unchanged asymptotically, the magnetization of a moving soliton is “screened” during the collision with a larger soliton, as seen in Fig. 3(b). Importantly, solitons survive multiple collisions [Fig. 3(c)], with the change to their trajectories apparently given by simple consecutive phase shifts.

However, there exist some important differences between the Heisenberg and Ishimori cases. First, absent integrability, scattering in the Heisenberg chain is not expected to be perfectly lossless. Indeed, there is a very small amount of radiation emitted during the collision (approximately $\delta S^z \sim 10^{-6}$ in magnitude) in Fig. 3(a). Second, scattering from narrow solitons at small k (where the existence diagram is wider in R) can emit significant amounts of radiation, although, curiously, the modified solitons that emerge appear to be stable to subsequent collisions. In addition, the phase shift Δ appears not to depend only on the soliton parameters R, k, R', k'. We extract the phase shift in Fig. 3(d) by averaging over 10 scattering events. They are also averaged over the relative phase (azimuthal angles) of the solitons at the moment of collision. In the integrable case, this has no effect—in the Heisenberg case, however, in particular for larger R, the phase shift depends on the relative phase (see SM [49]).

Despite these differences, collisions over a large parameter regime in the Heisenberg model strongly resemble the scattering in the Ishimori case. Importantly, while the phase shift Δ acquires some fluctuations, the velocities of the solitons remain unaffected by the collisions.

Solitons in thermal states. While the Heisenberg chain supports solitons as stable solitary waves, which suffer only very weak dissipation in scattering events, the imperfect nature of the scattering implies the existence of a thermal timescale on which they eventually decay. The question arises, then, as to what extent these solitons exert their influence on the hydrodynamics and transport properties: thermal states are not in any sense a dilute soliton gas and it would not be unreasonable to expect solitons to experience so many scattering events that, unprotected by integrability, they collapse too swiftly to generate a discernible superdiffusive contribution to the transport of spin or energy.

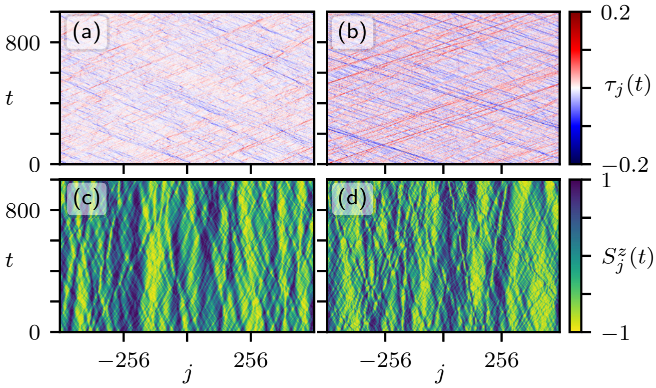

FIG. 4. Solitons in the thermal state at $T = 0.1J$ for the Heisenberg (left) and Ishimori (right) chains. Upper and lower panels show torsion $\tau_j(t)$, and $S_j^z(t)$, respectively. Ballistic trajectories are clearly visible in the torsion plots, both in the Ishimori and in the Heisenberg chains. These ballistic trajectories can also be seen in the plots of S^z, where the magnetization carried by a soliton changes as it moves through the chain—the mechanism preventing ballistic spin transport.

We find that the torsion (3) allows us to track the trajectories of solitons through a thermal background: Figs. 4(a) and 4(b) show the spacetime profile of the torsion $\tau(j, t)$ for a low-temperature thermal state of both the Heisenberg and Ishimori chains. The expected ballistic trajectories of the solitons are clearly observed in the Ishimori chain. Remarkably, very long-lived ballistic trajectories are also observed in the Heisenberg chain. These trajectories can also be seen in the z-spin component $S^z(j, t)$ [Figs. 4(c) and 4(d)]—though, since the magnetization changes as they propagate, spin is not transported ballistically. In a complementary approach, for both the Ishimori and the Heisenberg chain, the thermal solitons can also be isolated by surrounding an initial thermal state with a fully polarized state $\boldsymbol{S} = \hat{z}$, and allowing the thermal state to expand into this vacuum during the subsequent dynamics (see SM [49]).

Having established the existence and nature of the almost integrable behavior of the Heisenberg chain, we now address the KPZ scaling observed at low temperatures [18]. In the Ishimori chain, KPZ—rather than ballistic—spin transport emerges as follows. As smaller, faster solitons move through larger, slower solitons, they rotate to the "local vacuum" within the larger soliton. Thus, in any thermal state, the magnetization carried by the smaller solitons is screened on a timescale set by the rate at which they encounter larger solitons. This argument is qualitatively the same as that for KPZ scaling in the quantum $S = 1/2$ Heisenberg chain that appears in [26,27,42,43]. Since long-lived solitons are also present in the classical Heisenberg chain at low temperature, this provides a qualitative picture of and explanation for the KPZ regime in spin transport.

Conclusions. Our work clearly establishes the existence of a family of solitons in the nonintegrable ferromagnetic Heisenberg chain in terms of those known to exist in the integrable Ishimori chain. Furthermore, these solitons are shown to exist and be relevant for the dynamics of thermal low-temperature states of the Heisenberg model. This then allows us to explain the observation of KPZ scaling as a direct consequence of the nearly integrable scattering behavior of long-lived solitons and the screening of magnetization during collisions.

The fact that these solitons actually survive and determine the hydrodynamic behavior at finite temperatures, where correlation lengths are only a few lattice sites, seems truly remarkable, in particular considering that, in that case, the *adiabatically stable solitons are larger than the correlation lengths*. The crossover from this situation to the increasingly normal diffusive transport at higher temperatures is an obvious object for future studies.

Our work contributes to the broader study of the role of approximate integrability in many-body systems, a question of sustained interest in both quantum [51–57] and classical [58–64] contexts. For this, the Heisenberg chain provides a suitable setting as, besides the proximity to the Ishimori chain exploited here, it can also be thought of a lattice version of the integrable continuum Landau-Lifshitz model, via which route a family of approximate mobile solitons can be obtained by discretization [65], while in the limit of low temperatures, magnon-type excitations exhibit the usual "emergent integrability" of weakly interacting quasiparticles. Our work in particular raises the question about a wider applicability of these ideas about anomalous transport and the existence of solitons in classical spin models with SO(3) spin symmetry (see SM [49]). It certainly illustrates the point that even in the very simplest settings, many-body dynamics still holds many surprises awaiting discovery.

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[1] M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic Scaling of Growing Interfaces, Phys. Rev. Lett. **56**, 889 (1986).
[2] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in Out-of-Equilibrium *XXZ* Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. **117**, 207201 (2016).
[3] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Solvable Hydrodynamics of Quantum Integrable Systems, Phys. Rev. Lett. **119**, 220604 (2017).
[4] B. Doyon, T. Yoshimura, and J.-S. Caux, Soliton Gases and Generalized Hydrodynamics, Phys. Rev. Lett. **120**, 045301 (2018).
[5] B. Doyon, Lecture notes on generalised hydrodynamics, SciPost Phys. Lect. Notes **18** (2020).
[6] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium, Phys. Rev. X **6**, 041065 (2016).

[7] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Bethe-Boltzmann hydrodynamics and spin transport in the *XXZ* chain, Phys. Rev. B **97**, 045407 (2018).
[8] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, Generalized Hydrodynamics on an Atom Chip, Phys. Rev. Lett. **122**, 090601 (2019).
[9] A. Das, K. Damle, A. Dhar, D. A. Huse, M. Kulkarni, C. B. Mendl, and H. Spohn, Nonlinear fluctuating hydrodynamics for the classical *XXZ* spin chain, J. Stat. Phys. **180**, 238 (2020).
[10] B. Doyon, H. Spohn, and T. Yoshimura, A geometric viewpoint on generalized hydrodynamics, Nucl. Phys. B **926**, 570 (2018).
[11] C. B. Mendl and H. Spohn, Searching for the Tracy-Widom distribution in nonequilibrium processes, Phys. Rev. E **93**, 060101(R) (2016).
[12] C. B. Mendl and H. Spohn, Equilibrium time-correlation functions for one-dimensional hard-point systems, Phys. Rev. E **90**, 012147 (2014).
[13] M. Kulkarni, D. A. Huse, and H. Spohn, Fluctuating hydrodynamics for a discrete Gross-Pitaevskii equation: Mapping onto the Kardar-Parisi-Zhang universality class, Phys. Rev. A **92**, 043612 (2015).
[14] J. De Nardis, D. Bernard, and B. Doyon, Hydrodynamic Diffusion in Integrable Systems, Phys. Rev. Lett. **121**, 160603 (2018).
[15] Z. Chen, J. de Gier, I. Hiki, and T. Sasamoto, Exact Confirmation of 1D Nonlinear Fluctuating Hydrodynamics for a Two-Species Exclusion Process, Phys. Rev. Lett. **120**, 240601 (2018).
[16] A. Das, M. Kulkarni, H. Spohn, and A. Dhar, Kardar-Parisi-Zhang scaling for an integrable lattice Landau-Lifshitz spin chain, Phys. Rev. E **100**, 042116 (2019).
[17] D. Roy, A. Dhar, H. Spohn, and M. Kulkarni, Robustness of Kardar-Parisi-Zhang scaling in a classical integrable spin chain with broken integrability, arXiv.2205.03858.
[18] A. J. McRoberts, T. Bilitewski, M. Haque, and R. Moessner, Anomalous dynamics and equilibration in the classical Heisenberg chain, Phys. Rev. B **105**, L100403 (2022).
[19] S. Lepri, R. Livi, and A. Politi, Too Close to Integrable: Crossover from Normal to Anomalous Heat Diffusion, Phys. Rev. Lett. **125**, 040604 (2020).
[20] H. Spohn, Nonlinear fluctuating hydrodynamics for anharmonic chains, J. Stat. Phys. **154**, 1191 (2014).
[21] S. G. Das, A. Dhar, K. Saito, C. B. Mendl, and H. Spohn, Numerical test of hydrodynamic fluctuation theory in the Fermi-Pasta-Ulam chain, Phys. Rev. E **90**, 012124 (2014).
[22] O. Gamayun, Y. Miao, and E. Ilievski, Domain-wall dynamics in the Landau-Lifshitz magnet and the classical-quantum correspondence for spin transport, Phys. Rev. B **99**, 140301(R) (2019).
[23] Z. Krajnik and T. Prosen, Kardar-Parisi-Zhang physics in integrable rotationally symmetric dynamics on discrete space-time lattice, J. Stat. Phys. **179**, 110 (2020).
[24] V. B. Bulchandani, Kardar-Parisi-Zhang universality from soft gauge modes, Phys. Rev. B **101**, 041411(R) (2020).
[25] T. Prosen and B. Žunkovič, Macroscopic Diffusive Transport in a Microscopically Integrable Hamiltonian System, Phys. Rev. Lett. **111**, 040602 (2013).
[26] J. De Nardis, S. Gopalakrishnan, R. Vasseur, and B. Ware, Stability of Superdiffusion in Nearly Integrable Spin Chains, Phys. Rev. Lett. **127**, 057201 (2021).
[27] E. Ilievski, J. De Nardis, S. Gopalakrishnan, R. Vasseur, and B. Ware, Superuniversality of Superdiffusion, Phys. Rev. X **11**, 031023 (2021).
[28] M. Dupont and J. E. Moore, Universal spin dynamics in infinite-temperature one-dimensional quantum magnets, Phys. Rev. B **101**, 121106(R) (2020).
[29] Y. Ishimori, An integrable classical spin chain, J. Phys. Soc. Jpn. **51**, 3417 (1982).
[30] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, New York, 1987).
[31] R. W. Gerling and D. P. Landau, Comment on Anomalous Spin Diffusion in Classical Heisenberg Magnets, Phys. Rev. Lett. **63**, 812 (1989).
[32] R. W. Gerling and D. P. Landau, Time-dependent behavior of classical spin chains at infinite temperature, Phys. Rev. B **42**, 8214 (1990).
[33] M. Böhm, R. W. Gerling, and H. Leschke, Comment on "Breakdown of Hydrodynamics in the Classical 1D Heisenberg Model", Phys. Rev. Lett. **70**, 248 (1993).
[34] N. Srivastava, J.-M. Liu, V. Viswanath, and G. Müller, Spin diffusion in classical Heisenberg magnets with uniform, alternating, and random exchange, J. Appl. Phys. **75**, 6751 (1994).
[35] V. Oganesyan, A. Pal, and D. A. Huse, Energy transport in disordered classical spin chains, Phys. Rev. B **80**, 115104 (2009).
[36] D. Bagchi, Spin diffusion in the one-dimensional classical Heisenberg model, Phys. Rev. B **87**, 075133 (2013).
[37] P. Glorioso, L. V. Delacrétaz, X. Chen, R. M. Nandkishore, and A. Lucas, Hydrodynamics in lattice models with continuous non-Abelian symmetries, SciPost Phys. **10**, 015 (2021).
[38] G. Müller, Anomalous Spin Diffusion in Classical Heisenberg Magnets, Phys. Rev. Lett. **60**, 2785 (1988).
[39] O. F. de Alcantara Bonfim and G. Reiter, Breakdown of Hydrodynamics in the Classical 1D Heisenberg Model, Phys. Rev. Lett. **69**, 367 (1992).
[40] O. F. de Alcantara Bonfim and G. Reiter, De Alcantara Bonfim and Reiter Reply, Phys. Rev. Lett. **70**, 249 (1993).
[41] J. De Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, Universality Classes of Spin Transport in One-Dimensional Isotropic Magnets: The Onset of Logarithmic Anomalies, Phys. Rev. Lett. **124**, 210605 (2020).
[42] J. De Nardis, S. Gopalakrishnan, E. Ilievski, and R. Vasseur, Superdiffusion from Emergent Classical Solitons in Quantum Spin Chains, Phys. Rev. Lett. **125**, 070601 (2020).
[43] S. Gopalakrishnan and R. Vasseur, Kinetic Theory of Spin Diffusion and Superdiffusion in *XXZ* Spin Chains, Phys. Rev. Lett. **122**, 127202 (2019).
[44] M. Ljubotina, M. Žnidarič, and T. Prosen, Kardar-Parisi-Zhang Physics in the Quantum Heisenberg Magnet, Phys. Rev. Lett. **122**, 210602 (2019).
[45] N. Theodorakopoulos, Nontopological thermal solitons in isotropic ferromagnetic lattices, Phys. Rev. B **52**, 9507 (1995).
[46] N. Theodorakopoulos, Semiclassical excitation spectrum of an integrable discrete spin chain, Phys. Lett. A **130**, 249 (1988).
[47] E. K. Sklyanin, Classical limits of the SU(2)-invariant solutions of the Yang-Baxter equation, J. Sov. Math. **40**, 93 (1988).
[48] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation, Functional Analysis and Its Applications **16**, 263 (1983).

[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.106.L062202, which contains additional details on the single-soliton solutions of the Ishimori chain, the construction of the stationary solitons of the Heisenberg chain, the adiabatic procedure, soliton scattering and phase shifts, and solitons in thermal states of the Heisenberg chain.

[50] S. Flach and A. V. Gorbach, Discrete breathers—Advances in theory and applications, Phys. Rep. **467**, 1 (2008).

[51] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, Prethermalization and Thermalization in Models with Weak Integrability Breaking, Phys. Rev. Lett. **115**, 180601 (2015).

[52] F. Lange, Z. Lenarčič, and A. Rosch, Pumping approximately integrable systems, Nat. Commun. **8**, 15767 (2017).

[53] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Thermalization near Integrability in a Dipolar Quantum Newton's Cradle, Phys. Rev. X **8**, 021030 (2018).

[54] M. Žnidarič, Weak Integrability Breaking: Chaos with Integrability Signature in Coherent Diffusion, Phys. Rev. Lett. **125**, 180605 (2020).

[55] J. Durnin, M. J. Bhaseen, and B. Doyon, Nonequilibrium Dynamics and Weakly Broken Integrability, Phys. Rev. Lett. **127**, 130601 (2021).

[56] A. Bastianello, A. D. Luca, and R. Vasseur, Hydrodynamics of weak integrability breaking, J. Stat. Mech.: Theory Expt. (2021) 114003.

[57] V. B. Bulchandani, D. A. Huse, and S. Gopalakrishnan, Onset of many-body quantum chaos due to breaking integrability, Phys. Rev. B **105**, 214308 (2022).

[58] J. Ford, The Fermi-Pasta-Ulam problem: Paradox turns discovery, Phys. Rep. **213**, 271 (1992).

[59] G. P. Berman and F. M. Izrailev, The Fermi-Pasta-Ulam problem: Fifty years of progress, Chaos Interdisc. J. Nonlin. Sci. **15**, 015104 (2005).

[60] F. Jin, T. Neuhaus, K. Michielsen, S. Miyashita, M. Novotny, M. I. Katsnelson, and H. De Raedt, Equilibration and thermalization of classical systems, New J. Phys. **15**, 033009 (2013).

[61] Z. Dong, R. Moessner, and M. Haque, Classical dynamics of harmonically trapped interacting particles, J. Stat. Mech.: Theory Expt. (2018) 063106.

[62] X. Cao, V. B. Bulchandani, and J. E. Moore, Incomplete Thermalization from Trap-Induced Integrability Breaking: Lessons from Classical Hard Rods, Phys. Rev. Lett. **120**, 164101 (2018).

[63] C. Danieli, T. Mithun, Y. Kati, D. K. Campbell, and S. Flach, Dynamical glass in weakly nonintegrable Klein-Gordon chains, Phys. Rev. E **100**, 032217 (2019).

[64] T. Mithun, C. Danieli, Y. Kati, and S. Flach, Dynamical Glass and Ergodization Times in Classical Josephson Junction Chains, Phys. Rev. Lett. **122**, 054102 (2019).

[65] H.-J. Schmidt, C. Schröder, and M. Luban, Modulated spin waves and robust quasi-solitons in classical Heisenberg rings, J. Phys.: Condens. Matter **23**, 386003 (2011).