Finite-time Lyapunov fluctuations and the upper bound of classical and quantum out-of-time-ordered expansion rate exponents

Miguel A. Prado Reynoso[*]

Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, 62210, Cuernavaca, Morelos, Mexico

Guilherme J. Delben [†]

Departamento de Ciências Naturais e Sociais, Universidade Federal de Santa Catarina, 89520-000 Curitibanos, Brazil

Martin Schlesinger[‡]

TraceTronic GmbH, Stuttgarter Strasse 3, 01189 Dresden, Germany

Marcus W. Beims [§]

Departamento de Física, Universidade Federal do Paraná, 81531-980 Curitiba, Paraná, Brazil

This Letter demonstrates for chaotic maps [logistic, classical, and quantum standard maps (SMs)] that the exponential growth rate (Λ) of the out-of-time-ordered four-point correlator is equal to the classical Lyapunov exponent (λ) plus fluctuations ($\Delta^{(\text{fluc})}$) of the one-step finite-time Lyapunov exponents (FTLEs). Jensen's inequality provides the upper bound $\lambda \leqslant \Lambda$ for the considered systems. Equality is restored with $\Lambda = \lambda + \Delta^{(\text{fluc})}$, where $\Delta^{(\text{fluc})}$ is quantified by k-higher-order cumulants of the (covariant) FTLEs. Exact expressions for Λ are derived and numerical results using $k = 20$ furnish $\Delta^{(\text{fluc})} \sim \ln{(\sqrt{2})}$ for all maps (large kicking intensities in the SMs).

Introduction. Interest in the quantum-classical correspondence of classically chaotic systems has been renewed in the last years due to the conjecture that puts a bound on the exponential growth rate $\Lambda \leqslant 2\pi T$ (T is the temperature) of an out-of-time-ordered four-point correlator (OTOC) [1]. Introduced in the context of the theory of superconductivity [2], Λ is closely associated with the largest positive asymptotic Lyapunov exponent (LE) of the classical chaotic system for times shorter than the Ehrenfest time ($t < t_E$), for which quantum interference effects did not have time to become relevant. Besides serving as a tool to understand the fundamentals in the quantum-classical relation of classically regular [3–5], quasiregular [6,7], and chaotic [6,8] systems, the behavior of the OTOC attracted considerable attention in many-body systems [9–17] and in experiments [18–21]. In this context, we refer the readers to the recent interesting review about semiclassical many-body quantum chaos [22].

In general, it is known that even though related, Λ obtained from classical and quantum OTOCs and the LE (λ) are not precisely equal due to the distinct order of averaging. While Λ is proportional to $\ln{(\mathbb{E}[X])}$, λ is proportional to $\mathbb{E}[\ln{(X)}]$, where $X = \{x_1, x_2, \ldots, x_N\}$ is a random variable related to the local finite-time Lyapunov exponent (FTLE) $\lambda^{(t)}$, calculated

at time t (for $t \to \infty$, $\lambda^{(\infty)} = \lambda$). Here, $\mathbb{E}[\cdot]$ is the average over all points in the phase space. In some cases, the relation between both is written as $\Lambda = \lambda + \Delta$. For example, in the completely chaotic region of the kicked rotator it was observed numerically that $\Delta \approx \ln{\sqrt{2}}$ [6] and in the many-body Dicke model, $\Delta \approx 0.015$ [17]. Even though the distinct order of averaging seems to be a mere mathematical property, it has deep, interesting physical consequences. The mathematical background of our findings lies in Jensen's inequality (JI) $\mathbb{E}[\varphi(X)] \geqslant \varphi(\mathbb{E}[X])$, where $\varphi(X)$ is a convex function. Equality is restored when all variables in $X = \{x_1, x_2, \ldots, x_N\}$ are equal or when higher moments of X are taken into account, which is the case considered here. An elucidating example [23] of JI in another context is the entropy gain $\Delta S = CN \ln{(\frac{\overline{T}_A}{\overline{T}_G})}$ obtained after N reservoirs at constant capacity C, and initial temperature T_i, are put in thermal contact. $\overline{T}_A = \frac{1}{N} \sum_i^N T_i$ and $\overline{T}_B = (\prod_i^N T_i)^{1/N}$ are, respectively, the arithmetic and geometric mean. Positive entropy gain implies in JI [using $\varphi(T) = -\ln{(T)}$], namely, $\ln{(\overline{T}_A)} > \ln{(\overline{T}_B)} = \frac{1}{N} \sum_i^N \ln{(T_i)}$. Thus, the logarithm of the arithmetic mean of T_i is larger than the arithmetic mean of the logarithm of T_i. In other words, the kind of fluctuation, or distribution of T_i until equilibrium is reached, is responsible for the distinction before and after log averaging.

This Letter demonstrates that fluctuations of the FTLEs lead to the distinction between Λ and λ. In general, the concept, relevance, and ability of the FTLEs in describing dynamical systems have been demonstrated since the 1980s in

―――――――
[*]prado_angel92@hotmail.com
[†]guilherme.delben@ufsc.br
[‡]martin-schlesinger@gmx.de
[§]mbeims@fisica.ufpr.br

distinct physical situations [24–42]. Specifically, fluctuations of the *local one-step* FTLEs are well-known properties in dynamical systems (see, for example, Refs. [43,44]). Consequently, our results establish that for times $t < t_E$, the classical and quantum exponential growth rates Λ contain features of classical fluctuations not visible in the asymptotic λ itself. Analytical and numerical results are shown for the chaotic logistic and tend maps and the classical and quantum standard maps. The classical OTOC is defined as

$$C_{\mathrm{cl}}(t) = \mathbb{E}[\{x(t), p(0)\}^2] = \mathbb{E}\left[\left(\frac{\partial p(t)}{\partial x(0)}\right)^2\right]. \qquad (1)$$

For chaotic systems it is expected that $C_{\mathrm{cl}}(t) \sim e^{2\Lambda t}$ and the exponential growth rate is determined through the numerical computation of $\Lambda = 1/2 \lim_{t\to\infty} \lim_{\Delta x(0)\to 0} \ln[C_{\mathrm{cl}}(t+1)/C_{\mathrm{cl}}(t)]$, where $\Delta x(0)$ are small initial displacements. For one-dimensional systems $X = J(t)^2 = e^{2t\lambda^{(t)}}$, where $J(t)$ is the Jacobian at time t. Applying JI to $-\ln(X)$ (convex) we have $\ln(\mathbb{E}[X]) \geqslant \mathbb{E}[\ln(X)]$, which provides the upper bound for the LE, namely that $\lambda \leqslant \Lambda$.

Chaotic logistic map (LM). The map is defined as $x_{n+1} = 4x_n(1 - x_n)$, with discrete times $n = 1, 2, \ldots$. The LM has an invariant density $\rho_{\mathrm{LM}}(x) = 1/[\pi\sqrt{x(1-x)}]$ and the asymptotic LE is $\lambda_{\mathrm{LM}} = \ln(2)$. It has been shown that the probability distribution function (PDF) in this case has a cusp form [36,44]. Using the definition of the OTOC for the map, we have

$$C_{\mathrm{cl}}^{\mathrm{LM}}(n) = \mathbb{E}\left[\left(\frac{\partial x_n}{\partial x_0}\right)^2\right] = \mathbb{E}[(e^{2\sum_n \lambda_{\mathrm{LM}}^{(n)}})], \qquad (2)$$

where $\lambda_{\mathrm{LM}}^{(n)} = \ln|J_n|$ are the local one-step FTLEs and $J_n = \partial x_n/\partial x_{n-1} = 4 - 8x_{n-1}$ is the Jacobian of the map at time n. Note that $\partial x_n/\partial x_0 = J_n J_{n-1} \cdots J_2 J_1$. The local one-step FTLEs are fluctuating quantities and are responsible for the emergence of a nontrivial probability density, as observed previously for the LM [44], that asymptotically converges to a delta centered at $\lambda_{\mathrm{LM}}(=\lim_{N\to\infty} 1/N \sum_n^N \lambda_{\mathrm{LM}}^{(n)})$. Thus, fluctuations of the FTLEs are expected to be relevant when determining Eq. (2).[1]

Concerning the term in the middle of Eq. (2), it is easy to show that $C_{\mathrm{cl}}^{\mathrm{LM}}(1) = \int_0^1 dx_0 \rho_{\mathrm{LM}}(x_0)(\partial x_1/\partial x_0)^2 = 8$, $C_{\mathrm{cl}}^{\mathrm{LM}}(2) = 64$, $C_{\mathrm{cl}}^{\mathrm{LM}}(3) = 512$, $C_{\mathrm{cl}}^{\mathrm{LM}}(4) = 4096, \ldots$. From this we obtain the *exact* OTOC exponent $\Lambda_{\mathrm{LM}}^{(\mathrm{exact})} = \ln[C_{\mathrm{cl}}^{\mathrm{LM}}(2)/C_{\mathrm{cl}}^{\mathrm{LM}}(1)]/2 = \ln[C_{\mathrm{cl}}^{\mathrm{LM}}(3)/C_{\mathrm{cl}}^{\mathrm{LM}}(2)]/2 = \ln[C_{\mathrm{cl}}^{\mathrm{LM}}(4)/C_{\mathrm{cl}}^{\mathrm{LM}}(3)]/2 = 3\ln(2)/2$. These are two-iteration processes, namely from $n = 0 \to n = 2$, $n = 1 \to n = 3$, and $n = 2 \to n = 4$, respectively. Observe that $\Lambda_{\mathrm{LM}}^{(\mathrm{exact})} = \lambda_{\mathrm{LM}} + \ln(\sqrt{2})$, so that $\Delta_{\mathrm{LM}}^{(\mathrm{exact})} = \ln(\sqrt{2})$ gives the exact gap between both exponents.

To demonstrate the fluctuation properties of the gap, we use a numerical method to determine $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})}$. Motivated by the exact results obtained above for the two-iteration processes in

the determination of the middle term from Eq. (2), it is reasonable to use only two iterations to attain knowledge about the fluctuations of the one-step FTLEs which are relevant to $C_{\mathrm{cl}}^{\mathrm{LM}}(n)$. This simplifies enormously our task since we only need two iterations for the calculation of the last term on the right-hand side of Eq. (2). Thus, in terms of the one-step FTLEs, we have

$$\Lambda_{\mathrm{LM}}^{(1\to 2)} = \frac{1}{2}\ln\mathbb{E}\left[\left(\frac{\partial x_2}{\partial x_0}\right)^2\right] - \frac{1}{2}\ln\mathbb{E}\left[\left(\frac{\partial x_1}{\partial x_0}\right)^2\right],$$
$$= \frac{1}{2}\ln\mathbb{E}[(e^{2(\lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)})})] - \frac{1}{2}\ln\mathbb{E}[(e^{2\lambda_{\mathrm{LM}}^{(1)}})]. \quad (3)$$

It crucial to realize that $\lambda_{\mathrm{LM}}^{(1)}$ and $\lambda_{\mathrm{LM}}^{(2)}$ are one-step FTLEs from $n = 0 \to 1$ and from $n = 1 \to 2$, respectively. In order to connect Eq. (3) with the fluctuations of the one-step FTLEs, we apply the generating function of the cumulants

$$\ln\left(\mathbb{E}[\exp(2\lambda_T^{(f)})]\right) = \sum_{k=1}^{\infty} \tilde{\kappa}_k^{(f)}\frac{(2)^k}{k!}, \qquad (4)$$

to both terms in Eq. (3), being $\lambda_T^{(f=2)} = \lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)}$ and $\lambda_T^{(f=1)} = \lambda_{\mathrm{LM}}^{(1)}$, so that $\tilde{\kappa}_k^{(f=2)}$ are the k-order cumulants of the sum $\lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)}$, and $\tilde{\kappa}_k^{(f=1)}$ are the k-order cumulants of $\lambda_{\mathrm{LM}}^{(1)}$. Note that using the one-step FTLEs, the time n does not appear explicitly in Eq. (4) since it is incorporated in the sum $\lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)}$. Therefore

$$\Lambda_{\mathrm{LM}}^{(1\to 2)} = \tilde{\kappa}_1^{(2)} - \tilde{\kappa}_1^{(1)} + \frac{1}{2}\sum_{k=2}^{\infty}(\tilde{\kappa}_k^{(2)} - \tilde{\kappa}_k^{(1)})\frac{(2)^k}{k!}, \qquad (5)$$

where $\tilde{\kappa}_k^{(1)}$ are the cumulants related to $\lambda_{\mathrm{LM}}^{(1)}$, and $\tilde{\kappa}_k^{(2)}$ are the cumulants related to the *sum* $\lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)}$, namely the joint cumulants of $\lambda_{\mathrm{LM}}^{(2)}$ and $\lambda_{\mathrm{LM}}^{(1)}$,[2] so that

$$\Lambda_{\mathrm{LM}}^{(1\to 2)} \sim \left\{\lambda_{\mathrm{LM}} + \left(\mathbb{E}[\lambda_{\mathrm{LM}}^{(1)}\lambda_{\mathrm{LM}}^{(2)}] - \lambda_{\mathrm{LM}}^2 - \kappa_2^{(1)}\right) \right.$$
$$+ \left(\frac{3}{2}\mathbb{E}[\lambda_{\mathrm{LM}}^{(1)}(\lambda_{\mathrm{LM}}^{(2)})^2] + \frac{3}{2}\mathbb{E}[(\lambda_{\mathrm{LM}}^{(1)})^2\lambda_{\mathrm{LM}}^{(2)}]\right.$$
$$\left.\left. - 2\lambda_{\mathrm{LM}}^3 - \kappa_3^{(1)}\right) + \cdots\right\},$$

where we used $\kappa_1^{(2)} = 2\kappa_1^{(1)} = 2\lambda_{\mathrm{LM}}$ and $\tilde{\kappa}_k^{(1,2)} = \kappa_k^{(1,2)}n^k$ (with $n = 1$). Taking into account the first $k = 20$ cumulants we determine (see below)

$$\Lambda_{\mathrm{LM}}^{(1\to 2)} \sim [\lambda_{\mathrm{LM}} + 0.347\,500]. \qquad (6)$$

Thus, the contribution of the fluctuations leads to $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})}(20) \sim 0.347\,500$, which is close to the analytical gap $\Delta_{\mathrm{LM}}^{(\mathrm{exact})} = \log(\sqrt{2}) \sim 0.346\,574$. To determine $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})}(k)$

---

[1] For the present analysis it is not adequate to use $\mathbb{E}[(e^{2\sum_n \lambda_{\mathrm{LM}}^{(n)}})] = \mathbb{E}[(e^{2N\lambda_{\mathrm{LM}}})]$ due to the relevance of each one-step FTLE.

[2] Here, we use the joint cumulant of the variables $\lambda_{\mathrm{LM}}^{(2)}$ and $\lambda_{\mathrm{LM}}^{(1)}$, namely $\tilde{\kappa}_1^{(2)} = \mathbb{E}[\lambda_{\mathrm{LM}}^{(1)}\lambda_{\mathrm{LM}}^{(2)}] - \mathbb{E}[\lambda_{\mathrm{LM}}^{(1)}]\mathbb{E}[\lambda_{\mathrm{LM}}^{(2)}] = \mathbb{E}[\lambda_{\mathrm{LM}}^{(1)}\lambda_{\mathrm{LM}}^{(2)}] - \lambda_{\mathrm{LM}}^2$, and so on for higher moments.
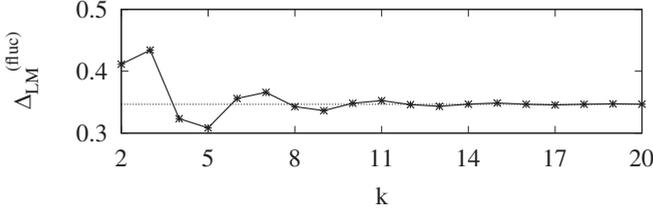
FIG. 1. $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})}$ as a function of the higher-order cumulants for the two-iteration case. The dashed line shows $\ln(\sqrt{2})$ for reference.

we integrate numerically the central moments $\Upsilon_{\mathrm{LM}}^k(N)$,[3]

$$\Upsilon_{\mathrm{LM}}^k(1) = \int_0^1 dx_n\, \rho_{\mathrm{LM}}(x_n)\big[\lambda_{\mathrm{LM}}^{(1)} - \lambda_{\mathrm{LM}}\big]^k,$$

$$\Upsilon_{\mathrm{LM}}^k(2) = \int_0^1 dx_n\, \rho_{\mathrm{LM}}(x_n)\big[\lambda_{\mathrm{LM}}^{(2)} + \lambda_{\mathrm{LM}}^{(1)} - 2\lambda_{\mathrm{LM}}\big]^k.$$

Results for $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})}(k)$ are presented in Fig. 1 as a function of the cumulants' order k, and it shows that it converges to 0.347 500. Thus, we expect that for $k \to \infty$ cumulants, the gap converges to $\Delta_{\mathrm{LM}}^{(\mathrm{fluc})} \to \log\sqrt{2} \sim 0.346\,573\,5$. It is worth mentioning that the cumulant expansion of the individual terms in Eq. (3) increases without bounds, but the difference between them converges following Fig. 1. Furthermore, Eqs. (3) and (5) could be generalized to $\Lambda_{\mathrm{LM}}^{(n \to n+1)}$, however, the determination of $\Upsilon_{\mathrm{LM}}^k(N)$ becomes harder and harder as n increases and does not provide new relevant information.

Tend map (TM). For the tend map, defined as $x_{n+1} = 2x_n$ for $x_n < 1/2$, and $x_{n+1} = 2(1 - x_n)$ for $x_n \geqslant 1/2$, the asymptotic LE is $\lambda_{\mathrm{TM}} = \ln(2)$. The logistic and tend maps have the same LE [45]. The OTOC growth rate is determined exactly as $\Lambda_{\mathrm{TM}}^{(\mathrm{exact})} = \ln[C_{\mathrm{cl}}^{\mathrm{TM}}(n)/C_{\mathrm{cl}}^{\mathrm{TM}}(1)]/[2(n-1)] = \lambda_{\mathrm{TM}}$. Since for the TM the invariant density is $\rho_{\mathrm{TM}}(x) = 1$, the FTLEs are independent of the ICs, and no fluctuations are expected, so the corresponding central moments are exactly zero, leading to $\Delta_{\mathrm{TM}}^{(\mathrm{fluc})} = 0$. This trivial example establishes that when fluctuations of the FTLEs are absent, the OTOC and Lyapunov exponents are identical.

Expressing the OTOC in terms of covariant Lyapunov vectors (CLVs). Before discussing results for the classical and quantum SMs, we present an expression for the classical OTOC in terms of CLVs in the two-dimensional continuous case. CLVs were proposed [46] to recover the directions of the LEs, after being corrupted by the repeated Gram-Schmidt orthogonalization. We write the right-hand side of Eq. (1) as a function of quantities related to the evolution in the tangent space $T_{\mathbf{x}}M \equiv \mathbb{R}^2$, namely in the CLV basis, $\{v_{\mathbf{x}}\} = \{v_{\mathbf{x}}^{(u)}, v_{\mathbf{x}}^{(s)}\}$, which generates the Oseledec unstable $\{E_{\mathbf{x}}^{(u)}\}$ and stable $\{E_{\mathbf{x}}^{(s)}\}$ subspaces with the properties $D_{\mathbf{x}}f^t v_{i,\mathbf{x}}^{(i)} = \gamma_{i,\mathbf{x}}^{(t)}\, v_{i,\mathbf{x}+t}^{(t)}$, $\angle(E_{i,\mathbf{x}}, E_{j,\mathbf{x}}) \neq 0$ (for $i \neq j$), and $\lim_{t\to\infty} \frac{1}{t}\log\|D_{\mathbf{x}}f^t v_{i,\mathbf{x}}^{(t)}\| =$

$\lambda_i^{(\infty)}$ being the magnitude of the asymptotic Lyapunov exponent, with $i = u$ or $i = v$. It is possible to show, after straightforward manipulation, that

$$C_{\mathrm{cl}}^{(\mathrm{CLV})}(t) = \mathbb{E}\big[f_{u,\mathbf{x}}^{(t)} f_{u,\mathbf{x}}^{(t)} e^{2t\lambda_{u,\mathbf{x}}^{(t)}} + f_{s,\mathbf{x}}^{(t)} f_{s,\mathbf{x}}^{(t)} e^{2t\lambda_{s,\mathbf{x}}^{(t)}}\big]$$
$$- 2\mathbb{E}\big[f_{u,\mathbf{x}}^{(t)} f_{s,\mathbf{x}}^{(t)} e^{t(\lambda_{u,\mathbf{x}}^{(t)} + \lambda_{s,\mathbf{x}}^{(t)})}\big], \tag{7}$$

with the amplitudes

$$f_{u,\mathbf{x}}^{(t)} = \cos(\phi_{\mathbf{x}} + \theta_{\mathbf{x}}/2)\cos(\phi_{f^t(\mathbf{x})} - \theta_{f^t(\mathbf{x})}/2)/\sin\theta_{\mathbf{x}},$$

$$f_{s,\mathbf{x}}^{(t)} = \cos(\phi_{\mathbf{x}} - \theta_{\mathbf{x}}/2)\cos(\phi_{f^t(\mathbf{x})} + \theta_{f^t(\mathbf{x})}/2)/\sin\theta_{\mathbf{x}}.$$

Equation (7) furnishes explicitly the quantity $C_{\mathrm{cl}}^{(\mathrm{CLV})}(t)$ as a function of the finite-time CLV $\lambda_{u,\mathbf{x}}^{(t)}$, related to the unstable manifold, the finite-time CLV $\lambda_{s,\mathbf{x}}^{(t)}$ related to the stable manifold, the angle $\theta_{\mathbf{x}}$ between both manifolds, their time derivative $\theta_{f^t(\mathbf{x})}$, the angle $\phi_{\mathbf{x}}$, which is the angle between $\theta_{\mathbf{x}}/2$ and the horizontal axis, and $\phi_{f^t(\mathbf{x})}$, its time derivative. We notice that the CLVs $\lambda_{u,\mathbf{x}}^{(t)}$ and $\lambda_{s,\mathbf{x}}^{(t)}$ are calculated for finite times t, and only for $t \to \infty$ do they lead to the usual asymptotic LEs $\lambda_u^{(\infty)}$ and $\lambda_s^{(\infty)}$, respectively. In other words, for short times, the local values of $\lambda_{u,\mathbf{x}}^{(t)}$ and $\lambda_{s,\mathbf{x}}^{(t)}$, and their fluctuations, are essential for the behavior of the OTOC. Furthermore, the amplitudes of the exponents provide a clear contribution to the underline dynamics. For example, for $\sin\theta_{\mathbf{x}} \to 0$ an alignment between CLVs occurs and relevant contributions from the amplitudes of Eq. (7) are expected.[4] Recent works in other contexts focus on the role of prefactors to the OTOC [47].

The standard map (SM). The classical dissipative map is defined as [48] $p_{n+1} = \gamma\, p_n + \frac{K}{2\pi}\sin(2\pi q_n)(\mathrm{mod}\,1)$, and $q_{n+1} = q_n + p_{n+1}(\mathrm{mod}\,1)$, where (p_n, q_n) are conjugate variables, $n = 1, 2, \ldots$ the discrete time, γ is the dissipation parameter, and K is the nonlinear parameter. For the analytical analysis of the SM, we use Eq. (1) in the form

$$C_{\mathrm{cl}}^{\mathrm{SM}}(n) = \mathbb{E}\left[\left(\frac{\partial p_n}{\partial q_0}\right)^2\right] = \int_0^1 \int_0^1 dq_0\, dp_0 \left(\frac{\partial p_n}{\partial q_0}\right)^2, \tag{8}$$

where the integration is over all phase-space initial conditions.

The conservative case $(\gamma = 1)$. The analytical LE can be estimated from $\lambda_{\mathrm{SM}}^{(\mathrm{exact})} = \int_0^1 dq \ln|L(q)|$, with $L(q) = 1 + k(q)/2 + \mathrm{sgn}[k(q)]\sqrt{k(q)[1 + k(q)/4]}$ and $k(q) = K\cos(2\pi q)$ [48]. Fluctuations of the stability exponents in the SM have been already studied in another context [49]. It is known that for $K > 4$ only one chaotic component lives in the phase space [45]. Therefore, for large values of K, a completely chaotic motion is observed, and the asymptotic LE is $\lambda_{\mathrm{SM}} = \ln(K/2)$. Equation (8) furnishes exact expressions, namely $C_{\mathrm{cl}}^{\mathrm{SM}}(1) = K^2/2$ and $C_{\mathrm{cl}}^{\mathrm{SM}}(2) = K^2 + K^4/4$, so that

$$\Lambda_{\mathrm{SM}}^{(1 \to 2)} = \frac{1}{2}\ln\left[\frac{C_{\mathrm{cl}}^{\mathrm{SM}}(2)}{C_{\mathrm{cl}}^{\mathrm{SM}}(1)}\right] = -\ln\sqrt{2} + \frac{\ln(4 + K^2)}{2}. \tag{9}$$

Therefore, the gap is $\Delta_{\mathrm{SM}}^{(\mathrm{exact})} = \Lambda_{\mathrm{SM}}^{(1 \to 2)} - \lambda_{\mathrm{SM}} = \ln\sqrt{2} + \frac{1}{2}\ln(4 + K^2) - \ln(K)$, which for $K^2 \gg 4$ reduces to $\Delta_{\mathrm{SM}}^{(\mathrm{exact})} \approx \ln(\sqrt{2})$. Amazingly, this is the same gap $\Delta_{\mathrm{LM}}^{(\mathrm{exact})}$

---

[3]The cumulants in terms of the central moments can be obtained from expansions of the incomplete Bell polynomials. As an example, the first terms are $\kappa_2 n = \Upsilon^{(2)}$, $\kappa_3 n^2 = \Upsilon^{(3)}$, $\kappa_4 n^3 = \Upsilon^{(4)} - 3(\Upsilon^{(2)})^2$, $\ldots, \kappa_5 n^4 = \Upsilon^{(5)} - 10\Upsilon^{(3)}\Upsilon^{(2)}, \ldots$. To not confuse the readers we did not use $n = 1$ here.

[4]More details of these contributions will be considered elsewhere.
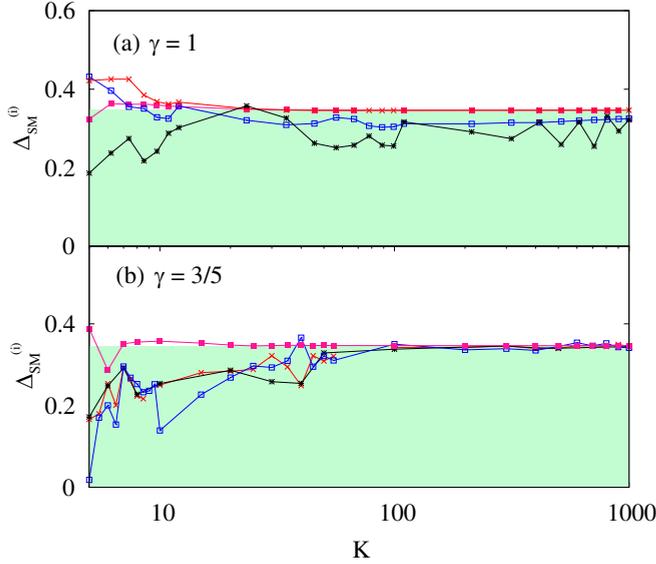
FIG. 2. $\Delta_{\mathrm{SM}}^{(i)} = \Lambda_{\mathrm{SM}}^{(i)} - \lambda_{\mathrm{SM}}^{(\mathrm{exact})}$ for the distinct calculated $\Lambda_{\mathrm{SM}}^{(i)}$, namely the analytical result $\Lambda_{\mathrm{SM}}^{(i)} = \Lambda_{\mathrm{SM}}^{(1 \to 2)}$ (black line), $\Lambda_{\mathrm{SM}}^{(i)} = \Lambda_{\mathrm{SM}}^{(\mathrm{CLV})}$ (red crosses), $\Lambda_{\mathrm{SM}}^{(i)} = \Lambda_{\mathrm{SM}}^{(\mathrm{fluc})}$ (blue square), and $\Lambda_{\mathrm{SM}}^{(i)} = \Lambda_{\mathrm{SM}}^{(Q)}$ (black stars). (a) For the conservative case and (b) for the dissipative case. The upper borders of the light green rectangles show $\ln(\sqrt{2})$ for reference to be compared when K is large.

obtained for the LM, which is a dissipative system. We present results for the SM with $K \geqslant 4$, since for smaller values of K the dynamic is mixed (regular and chaotic), and the classical and quantum averages lead to additional difficulties which, besides being of general interest, are not essential for the goal of the present Letter.

For the numerical results, we initially compared the time evolution of $C_{\mathrm{cl}}^{(\mathrm{CLV})}(n)$ from Eq. (7) with $C_{\mathrm{cl}}(n)$ from Eq. (1), obtained using some small initial displacements $\Delta x(0)$. Both results are in full agreement. However, the $C_{\mathrm{cl}}^{(\mathrm{CLV})}(n)$ from Eq. (7) is more superior in terms of the stability for longer iterations, since it does not depend on $\Delta x(0)$. Thus, to calculate the exponent $\Lambda_{\mathrm{SM}}^{(\mathrm{CLV})}(n)$ we use Eq. (7). To obtain the fluctuations of the CLVs, we determine numerically the distributions of $\lambda_{u,\mathbf{x}}^{(t)}$ at the Ehrenfest time,[5] and obtain the $k = 20$ first central moments directly from these distributions. Such higher moments lead again to $\Delta_{\mathrm{SM}}^{(\mathrm{fluc})}$, whose convergences are similar to those obtained for the LM in Fig. 1(a). The distributions keep changing shape for short times and strongly depend on K. When possible, values of the first cumulants were compared to previous studies [49] and have shown to be consistent. As pointed out in Ref. [49], distributions of FTLEs converge very slowly to a Gaussian when $n \to \infty$ and probably only the first two cumulants are nonzero. However, for short iterations, higher-order momenta for the cumulant expansions become essential. Figure 2(a) shows results for $\Delta_{\mathrm{SM}}^{(\mathrm{fluc})}$ (blue squares), $\Delta_{\mathrm{SM}}^{(\mathrm{CLV})} = \Lambda_{\mathrm{SM}}^{(\mathrm{CLV})} - \lambda_{\mathrm{SM}}^{(\mathrm{exact})}$ (red crosses) for distinct values of K, together with the exact results $\Delta_{\mathrm{SM}}^{(\mathrm{exact})}$

(dark pink squares). For $K \geqslant 8$, $\Delta_{\mathrm{SM}}^{(\mathrm{CLV})}$ and $\Delta_{\mathrm{SM}}^{(\mathrm{exact})}$ are indistinguishable, and approach the value 0.346 for $K = 1000$. The values of $\Delta_{\mathrm{SM}}^{(\mathrm{fluc})}$ are very close, even though a bit smaller. The discrepancy between distinct curves for smaller values of K is surely a consequence of the larger amount of dynamical fluctuations due to sticky motion [45], which strongly depend on appropriate averages.

Before we proceed to the quantum analysis, some information must be given. The quantum OTOC is obtained numerically from $C_{Q}(n) = \mathbb{E}\{[\hat{q}(n), \hat{p}(0)]^2\}$, where $(\hat{q}, \hat{p})$ are the corresponding position and momentum operators, and $[,]$ denotes the commutator. The quantum SM problem is described using the kicked Hamiltonian operator (dimensionless units) $\hat{\mathcal{H}} = \hat{p}/2 + K/(4\pi^2) \cos(2\pi\hat{q}) \sum_{m=0}^{\infty} \delta(t - m\tau)$, and $C_{Q}^{(\mathrm{SM})}(n)$ is obtained from the numerical integration of the corresponding Schrödinger equation. The associated OTOC exponent is named $\Lambda_{\mathrm{SM}}^{(Q)}$. We use individual angular-momentum eigenstates $|\Psi(0)\rangle = \sum_{n=-\infty}^{\infty} a_n^{(0)} |n\rangle$ and Gaussian wave packets $a_n^{(0)} = \exp(-\frac{\hbar_{\mathrm{eff}}^2 (n - n_0)^2}{2\sigma^2})$, where $n_0 = \frac{p_0}{\hbar_{\mathrm{eff}}}$. For the numerical integration we use $p_0 = 0$, $\sigma = 4$, and $|\Psi\rangle$, represented in a finite basis of eigenstates $|n\rangle$, $n \in [-N, N-1]$. Functions of $\hat{p}$ are applied on this basis, and functions of $\hat{q}$ are applied in the Fourier-transformed representation. We use an adaptive grid with $2\hbar_{\mathrm{eff}}N \in [2^{12}, 2^{16}]$ [6]. The quantum and classical OTOC exponents for the conservative SM were already considered recently [6], and our exponents are in good agreement with those presented in Fig. 2 from Ref. [6], for the considered K values. We choose not to repeat such a figure, but instead display results for Δ_{SM}, our main interest. Numerical results for $\Delta_{\mathrm{SM}}^{(Q)} = \Lambda_{\mathrm{SM}}^{(Q)} - \lambda_{\mathrm{SM}}^{(\mathrm{exact})}$ as a function of K are shown as black stars in Fig. 2(a). We notice that, except for specific values of K, $\Delta_{\mathrm{SM}}^{(Q)}$ and $\Delta_{\mathrm{SM}}^{(\mathrm{CLV})}$ are in relatively good agreement. Differences between both gaps are related to quantum averages and the number of eigenstates for each value of K. The determination of $\Lambda_{\mathrm{SM}}^{(Q)}$ revealed to be a difficult numerical issue. Summarizing, Fig. 2(a) demonstrates that the gaps between the distinct OTOCs and the classical LE are close to each other, and all quantities approach $\sim \ln(\sqrt{2})$ for large K values, as accurately explained by the fluctuations of the finite-time CLVs.

The dissipative case ($\gamma = 3/5$). We could not obtain an analytical expression for the OTOC exponent using arbitrary values of γ. However, for $\gamma = 3/5$, we attain

$$\Lambda_{\mathrm{SM}}^{(1 \to 2)} = \frac{1}{2} \log \left\{ \frac{34}{25} + \frac{K^2}{2} - \frac{\sqrt{5 - \sqrt{5}}[KJ_0(K) - J_1(K)]}{K\pi\sqrt{2}} \right.$$
$$\left. + \frac{5\sqrt{5 - \sqrt{5}}[KJ_0(2K) + (K^2 - 1)J_1(2K)]}{48K\pi\sqrt{2}} \right\},$$

with $J_i(K)$ $(i = 0, 1)$ being the Bessel function of the first type. Other specific values of γ could be used. Furthermore, Eq. (7) is used to calculate $\Lambda_{\mathrm{SM}}^{(\mathrm{CLV})}$, and the FTLE $\lambda_{\mathrm{SM}}^{(\mathrm{num})}$ is determined numerically, as usual. Dissipation in the quantum model is introduced between the kicks by coupling the main system to a zero-temperature environment. The density operator is determined as an ensemble mean over pure states obtained from the quantum state diffusion [50] Ito-stochastic Schrödinger equation $|d\psi\rangle = -\hat{\mathcal{H}}|\psi\rangle dt + \sum_k (L_k - $

---

[5]The Ehrenfest time depends on K. For details and specific values of the Ehrenfest times, we refer the reader to Ref. [6].

$\langle L_k \rangle)|\psi\rangle d\xi_k - 1/2 \sum_k (L_k^\dagger L_k - 2\langle L_k^\dagger \rangle)L_k + |\langle\psi\rangle|^2|\psi\rangle dt$. L_k are the Lindblad operators with $k = 1, 2$ and $\langle\cdot\rangle$ stands for the expectation value. The Lindblad operators induce a damping $-\nu\langle\hat{p}\rangle$, and the dissipation parameter becomes $\gamma = e^{-\nu\tau}$, where the kicking time $\tau = \hbar_{\text{eff}}$ is the effective Planck's constant [50]. For details, we refer to Ref. [51]. Figure 2(b) summarizes our results for the dissipative case with $\gamma = 3/5$. We plot $\Delta_{\text{SM}}^{(i)} = \Lambda_{\text{SM}}^{(i)} - \lambda_{\text{SM}}^{(\text{num})}$ for the distinct calculated $\Lambda_{\text{SM}}^{(i)}$, namely the analytical result $\Lambda_{\text{SM}}^{(i)} = \Lambda_{\text{SM}}^{(\text{exact})}$ (dark pink squares), $\Lambda_{\text{SM}}^{(i)} = \Lambda_{\text{SM}}^{(\text{CLV})}$ (red crosses), $\Lambda_{\text{SM}}^{(i)} = \Lambda_{\text{SM}}^{(\text{fluc})}$ (blue square), and $\Lambda_{\text{SM}}^{(i)} = \Lambda_{\text{SM}}^{(Q)}$ (black stars). As for the conservative case, all quantities lead to a gap $\Delta_{\text{SM}} \sim \ln(\sqrt{2})$ for larger K values, nicely explained by the fluctuations of the one-step finite-time CLVs.

Conclusions. Time fluctuations of the one-step FTLEs in the LM and the one-step finite-time CLVs in the SM are demonstrated to be the origin of the distinction between the classical and quantum OTOC exponential growth rate (Λ) and the classical LE (λ). The fluctuations are quantified by higher-order cumulant expansions corrections $\Delta^{(\text{fluc})}$, so that the upper bound $\Lambda = \lambda + \Delta^{(\text{fluc})}$ is reached. Comparing the LM, and the SM for $K \geqslant 4$, the correction is $\Delta_{\text{LM}}^{(\text{fluc})} \sim \Delta_{\text{SM}}^{(\text{fluc})} \sim \ln(\sqrt{2})$. Such an approximated equality is intriguing: The statistical properties of the one-step FTLEs

from the dissipative chaotic attractor of the LM are equal to those (CLVs) of the chaotic component of the conservative and dissipative SMs. For the tend map, no fluctuations of the FTLEs are observed, leading to $\Delta_{\text{TM}}^{(\text{fluc})} = 0$. Thus, the quantum-classical correspondence regarding the exponential growth of instabilities in the SMs becomes clear and uniquely described for $t < t_E$ when taking into account the dynamical fluctuations of the one-step finite-time CLVs. Because the tend, logistic, and standard maps are paradigmatic models describing a huge number of dynamical systems in distinct physical contexts, we are confident that the finite-time Lyapunov fluctuations producing the gap $\Delta^{(\text{fluc})}$ should be a generic property. Finally, it would be interesting to investigate the relation of higher-order fluctuations of the FTLEs, and the gap due to Jensen's inequality, in many-body systems [22], in the context of Loschmidt echo [52], the oceanic sound propagation through a structured ocean [38], and in disordered systems, with possible applications to Anderson localization [40].

[1] J. Maldacena, S. H. Shenker, and D. Stanford, J. High Energy Phys. 08 (2016) 106.

[2] A. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **55**, 2262 (1969).

[3] E. B. Rozenbaum, L. A. Bunimovich, and V. Galitski, Phys. Rev. Lett. **125**, 014101 (2020).

[4] S. Pilatowsky-Cameo, J. Chávez-Carlos, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Phys. Rev. E **101**, 010202(R) (2020).

[5] K. Hashimoto, K.-B. Huh, K.-Y. Kim, and R. Watanabe, J. High Energy Phys. 11 (2020) 068.

[6] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. **118**, 086801 (2017).

[7] T. Goldfriend and J. Kurchan, Phys. Rev. E **102**, 022201 (2020).

[8] I. Garcia Mata, M. Saraceno, R. A. Jalabert, A. J. Roncaglia, and D. A. Wisniacki, Phys. Rev. Lett. **121**, 210601 (2018).

[9] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A **94**, 040302(R) (2016).

[10] N. Y. Yao, F. Grusdt, B. Swingle, M. D. Lukin, D. M. Stamper-Kurn, Joel E. Moore, and E. A. Demler, arXiv:1607.01801.

[11] J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016).

[12] S. Cotler, D. Ding, and G. R. Penington, Ann. Phys. **396**, 318 (2018).

[13] H. Shen, P. Zhang, R. Fan, and H. Zhai, Phys. Rev. B **96**, 054503 (2017).

[14] A. Bohrdt, C. B. Mendl, M. Endres, and M. Knap, New J. Phys. **19**, 063001 (2017).

[15] D. Bagrets and A. Altland, Nucl. Phys. B **921**, 727 (2017).

[16] J. Rammensee, J. D. Urbina, and K. Richter, Phys. Rev. Lett. **121**, 124101 (2018).

[17] J. Chávez-Carlos, B. Lopez-del-Carpio, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Phys. Rev. Lett. **122**, 024101 (2019).

[18] M. Gartner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nat. Phys. **13**, 781 (2017).

[19] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, Phys. Rev. X **7**, 031011 (2017).

[20] K. X. Wei, C. Ramanathan, and P. Cappellaro, Phys. Rev. Lett. **120**, 070501 (2018).

[21] M. Niknam, L. F. Santos, and D. G. Cory, Phys. Rev. Res. **2**, 013200 (2020).

[22] K. Richter, J. D. Urbina, and S. Tomsovic, J. Phys. A: Math. Theor. **55**, 453001 (2022).

[23] P. T. Landsberg, Phys. Lett. A **67**, 1 (1978).

[24] P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).

[25] R. Badii and A. Politi, Phys. Rev. A **35**, 1288 (1987).

[26] T. Tél, Phys. Rev. A **36**, 2507 (1987).

[27] P. Grassberger and A. Politi, J. Stat. Phys. **51**, 135 (1988).

[28] M. A. Sepúlveda, R. Badii, and E. Pollak, Phys. Rev. Lett. **63**, 1226 (1989).

[29] S. Vaienti, J. Stat. Phys. **56**, 403 (1989).

[30] F. J. Romeiras, C. Grebogi, and E. Ott, Phys. Rev. A **41**, 784 (1990).

[31] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, UK, 1993).

[32] A. Crisanti and A. Vulpiani, *Products of Random Matrices* (Spring, Berlin, 1993).

[33] B. Echkhardt and D. Yao, Physica D **65**, 100 (1993).

[34] C. Amitrano and R. S. Berry, Phys. Rev. E **47**, 3158 (1993).

[35] A. Adrover and M. Giona, Physica A **253**, 143 (1998).

[36] A. Prasad and R. Ramaswamy, Phys. Rev. E **60**, 2761 (1999).

[37] F. K. Diakonos, D. Pingel, and P. Schmelcher, Phys. Rev. E **62**, 4413 (2000).

[38] M. A. Wolson and S. Tomsovic, J. Acoust. Soc. Am. **109**, 2693 (2001).

[39] H. Yamada and T. Okabe, Phys. Rev. E **63**, 026203 (2001).

[40] H. Schomerus and M. Titov, Phys. Rev. E **66**, 066207 (2002).

[41] C. Manchein, M. Beims, and J. Rost, Chaos **22**, 033137 (2012).

[42] R. M. da Silva, C. Manchein, M. W. Beims, and E. G. Altmann, Phys. Rev. E **91**, 062907 (2015).

[43] H. Fujisaka, Prog. Theor. Phys. **70**, 1264 (1983).

[44] C. Anteneodo, Phys. Rev. E **69**, 016207 (2004).

[45] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer, New York, 1992).

[46] F. Ginelli, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, Phys. Rev. Lett. **99**, 130601 (2007).

[47] Y. Gu and A. Kitaev, J. High Energy Phys. 02 (2019) 075.

[48] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).

[49] S. Tomsovic and A. Lakshminarayan, Phys. Rev. E **76**, 036207 (2007).

[50] N. O. Gisin and I. C. Percival, J. Phys. A: Math. Gen. **25**, 5677 (1992).

[51] M. W. Beims, M. Schlesinger, C. Manchein, A. Celestino, A. Pernice, and W. T. Strunz, Phys. Rev. E **91**, 052908 (2015).

[52] R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001).