Letter

# Random walk with multiple memory channels

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A class of one-dimensional, discrete-time random walk models with memory, termed "random walk with $n$ memory channels" (RW$n$MC), is proposed. In these models the information of $n$ ($n \in \mathbb{Z}$) previous steps from the walker's entire history is needed to decide a future step. Exact calculation of the mean and variance of position of the RW2MC ($n = 2$) has been done, which shows that it can lead to asymptotic diffusive and superdiffusive behavior in different parameter regimes. A connection between RW$n$MC and a Pólya-type urn model evolving by drawing $n$ balls at a time has also been reported. This connection for the RW2MC is discussed in detail and suggests the applicability of RW2MC in many population dynamics models with multiple competing species.

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The term "long-range memory," which is sometimes called "long-term persistence," implies that there is a non-negligible dependence between the present and the points in the past in a dynamical process. Long-range memory plays a significant role in many fields such as astrophysics [1,2], atmospheric science [3], genomics [4], financial markets [5,6], complex networks [7,8], geophysics [9,10], etc.

In this Research Letter we present a non-Markovian discrete-time random walk model where the random increment at time step $t$ depends on the complete history of the process. We term this walk a "random walk with $n$ memory channels" (RW$n$MC), where $n$ is an integer which indicates the number of memory channels. At time $t$, the walker can choose $n$ ($n \geqslant 1$) previous steps from its entire history with equal *a priori* probabilities, based on which it makes the decision in the next step by following certain rules.

RW2MC, presented here, offers the great advantage of analytical tractability. Exact calculation of the first and second moments of position shows that it exhibits asymptotic diffusive and superdiffusive behavior in different parameter regimes. Within the superdiffusive regime a nontrivial ballistic behavior can be seen for a certain parameter range, which is noteworthy for a random walk with complete memory of its history.

We have also shown that RW2MC has identical distribution to that of a proposed Pólya-type urn model, discussed in detail later in this Research Letter. In brief, the evolution of the composition of the urn can be transformed into statements about the evolution of the position of the RW2MC. We then generalize this connection to the urn model for $n > 2$.

*Construction of the RWnMC.* Let us define a RW2MC in a one-dimensional (1D) lattice. At each step the walker has three options: It can move to the nearest-neighbor site to its right, it can move to the nearest-neighbor site on its left, or it can remain at its present location. Denoting the position of the walker at time $t$ as $X_t$, one can write a probabilistic recurrence relation,

$$X_{t+1} = X_t + \sigma_{t+1}, \tag{1}$$

where $\sigma_{t+1}$ is a random number which can take one of the values $-1$, 0, or $+1$ according to the following rule:

$$\sigma_{t+1} = \mu^{(1)}_{t+1}\lambda^{(1)}_{t+1} + \mu^{(2)}_{t+1}\lambda^{(2)}_{t+1}. \tag{2}$$

Here, $\lambda^{(1)}_{t+1}$ and $\lambda^{(2)}_{t+1}$ are random numbers which depend on the entire history of the walk $\{\sigma_t\} = (\sigma_1, \dots, \sigma_t)$ as follows: Two random previous times $k_1$ and $k_2$ ($k_1 \neq k_2$) between 1 and $t$ are chosen with uniform probability such that $\lambda^{(1)}_{t+1} = \sigma_{k_1}$ and $\lambda^{(2)}_{t+1} = \sigma_{k_2}$.

$\mu^{(1)}_{t+1}$ and $\mu^{(2)}_{t+1}$ in Eq. (2) are two random variables which can take two possible values, $+1$ or $-1$. We define a parameter $p$ as the probability of occurrence of a positive value of $\mu^{(1)}_{t+1}$ and $\mu^{(2)}_{t+1}$. If by following the above equation the value of $\sigma_{t+1}$ becomes higher (lower) than $+1$ ($-1$), then it is made equal to $+1$ ($-1$). The process is initiated as follows: At $t = 1$ and 2, we allow the walker to take steps to the right with probability $s$ and to the left with probability $1 - s$. So, the first two steps exclude the possibility that the walker may not move. Hence, at $t = 2$, possible positions are $\pm 2$ and 0. If the walker starts from initial position $X_0 = 0$, then the position of the walker at time $t > 0$ is given by

$$X_t = \sum_{k=1}^{t} \sigma_k. \tag{3}$$

One can construct a RW$n$MC ($n > 2$) model by increasing the terms on the right-hand side of Eq. (2), i.e., the walker can choose uniformly at random $n$ ($n > 2$) steps from its entire history $\{\sigma_1, \sigma_2, \dots, \sigma_t\}$ for making $\sigma_{t+1}$ such that

$$\sigma_{t+1} = \sum_{i=1}^{n} \mu^{(i)}_{t+1}\lambda^{(i)}_{t+1}, \tag{4}$$

where $\sigma_{t+1}$ is again taken to be within $\pm 1$.

*Equivalent urn model.* Let us now introduce a discrete-time urn model with balls of three colors. Assume the three colors to be black ($B$), white ($W$), and gray ($G$). Suppose one

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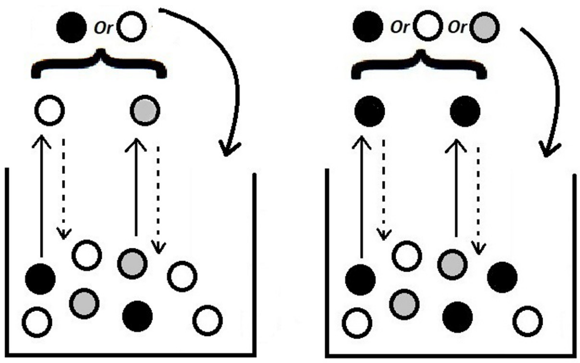


FIG. 1. Pólya-type urn containing balls of three different colors and evolving by drawing two balls at a time according to the mean replacement matrix (5). The solid upward straight arrows represent random drawing of balls in each step, and the dashed downward arrows represent replacement of the drawn balls. The curved downward arrows denote addition of a new ball according to the mean replacement matrix.

randomly draws two balls at a time (Fig. 1). The possible outcomes of any draw are $[B, B]$, $[B, W]$, $[W, W]$, $[G, G]$, $[G, W]$, and $[G, B]$. After observing, the drawn pair are replaced into the urn and a $B$, $W$, or $G$ ball is added into it. Which ball will be added into the urn depends on the drawn pair according to the following mean replacement matrix:

$$\begin{array}{c} \\ BB \\ BW \\ BG \\ WW \\ WG \\ GG \end{array} \begin{array}{c} \begin{array}{ccc} B & W & G \end{array} \\ \begin{pmatrix} a^2 & b^2 & 2ab \\ ab & ab & a^2+b^2 \\ a & b & 0 \\ b^2 & a^2 & 2ab \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{5}$$

Here, in general, $a + b = 1$. The elements of the matrix represent the conditional probability of the colored ball added in each step. The urn described above belongs to the Pólya-type urns evolving by drawing two balls at a time with randomized replacement rules [11]. If we associate $+1$ with black balls, $-1$ with white balls, and 0 with gray balls, then the mean replacement matrix (5) can be obtained for the choice $a = p$ and $b = 1 - p$ from Eq. (2).

The composition of the urn at time $t$ is given by a set $N_t = (N_t^B, N_t^W, N_t^G)$, where $N_t^B$, $N_t^W$, and $N_t^G$ count the number of black, white, and gray balls, respectively. We restrict ourselves to selected starting compositions with balls of two colors only such that at time $t = 0$, $N_0$ can take values of either (1,1,0), (2,0,0), or (0,2,0). This restriction is necessary to establish a connection with the RW2MC discussed earlier.

One can show that for a particular $p$, if $X_t$ is the position of a walker that started from $X_0 = 0$ such that $X_2 = N_0^B - N_0^W$, then

$$X_{t+2} =_d N_t^B - N_t^W, \tag{6}$$

where $=_d$ implies equality in law or equality in distribution. That is to say, the difference between the number of black and white balls in the urn at time $t$ follows the same distribution as the position of the RW2MC at time $t + 2$ with position at $t = 2$ equaling $N_0^B - N_0^W$.

Now generalizing the above process, one can verify that the urn model equivalent to RW$n$MC ($n > 2$) will be a Pólya-type urn model containing balls of three colors and evolving by drawing $n$ balls at a time with suitable initial conditions. One can obtain the corresponding mean replacement matrix from Eq. (4) by following the same procedure.

*Calculation of moments*. The mean displacement and the mean-square displacement as functions of time characterize the nature of the motion of the walker. For RW2MC, both of them can be calculated analytically. However, here we include the possibility of $k_1 = k_2$, i.e., the walker can randomly select the same two steps from its history. Since the probability of occurrence of such an event is very low ($\frac{1}{t}$), inclusion of such a possibility does not cause any notable difference in the asymptotic limit ($t \gg 1$) from the case $k_1 \neq k_2$. For analytical calculations we will apply the initial condition only at the first time step. So, the walk can be initiated as follows.

At $t = 1$ the walker moves to the right with probability $s$ and to the left with probability $1 - s$. Let us introduce shorthand notations $\gamma_{t+1}$ and $\gamma'_{t+1}$, where $\gamma_{t+1} = \mu_{t+1}^{(1)}\lambda_{t+1}^{(1)}$ and $\gamma'_{t+1} = \mu_{t+1}^{(2)}\lambda_{t+1}^{(2)}$.

Suppose $f_1(p,t)$, $f_{-1}(p,t)$, and $f_0(p,t)$ are the fraction of $+1$, $-1$, and 0 steps, respectively, in $\{\sigma_t\}$. Then one must have

$$f_1(p,t) + f_{-1}(p,t) + f_0(p,t) = 1. \tag{7}$$

For a given history $\{\sigma_t\}$, the conditional probability that $\gamma_{t+1} = \pm 1, 0$, for $t > 2$, can be written as

$$P(\gamma_{t+1} = \pm 1) = \frac{1}{2t}\sum_{k=1}^{t}\left(1 - \left(1 - \sigma_k^2\right) + (2p-1)\sigma_k\gamma_{t+1}\right),$$

$$P(\gamma_{t+1} = 0) = \frac{1}{2t}\sum_{k=1}^{t} 2\left(1 - \sigma_k^2\right).$$

Similar equations can be written for $P(\gamma'_{t+1} = \pm 1)$ and $P(\gamma'_{t+1} = 0)$. Now for a given history $\{\sigma_t\}$ and $\gamma'_{t+1}$ the conditional probabilities that the increment $\sigma_{t+1}$ takes the values $\pm 1$ and 0 are

$$P(\sigma_{t+1} = 1|\{\sigma_t\}, \gamma'_{t+1}) = \left(1 - \frac{\gamma'_{t+1}}{2}\right)(1 + \gamma'_{t+1}) \times [P(\gamma_{t+1} = +1) + \gamma'_{t+1}P(\gamma_{t+1} = 0)], \tag{8}$$

$$P(\sigma_{t+1} = -1|\{\sigma_t\}, \gamma'_{t+1}) = \left(1 + \frac{\gamma'_{t+1}}{2}\right)(1 - \gamma'_{t+1}) \times [P(\gamma_{t+1} = -1) - \gamma'_{t+1}P(\gamma_{t+1} = 0)]. \tag{9}$$

If $\gamma'_{t+1} = \pm 1$ or 0, then

$$P(\sigma_{t+1} = 0|\{\sigma_t\}, \gamma'_{t+1}) = P(\gamma_{t+1} = -\gamma'_{t+1}). \tag{10}$$

This follows from the dynamics (2) of the process. So, the conditional mean increment is given by

$$\langle\sigma_{t+1}|\{\sigma_t\}, \gamma'_{t+1}\rangle = P(\sigma_{t+1} = 1|\{\sigma_t\}, \gamma'_{t+1}) - P(\sigma_{t+1} = -1|\{\sigma_t\}, \gamma'_{t+1}). \tag{11}$$

Averaging over $\gamma'_{t+1}$, one can obtain

$$\langle\sigma_{t+1}|\{\sigma_t\}\rangle = \frac{(2p-1)(1 + f_0(p,t))}{t}X_t. \tag{12}$$

One can also obtain

$$\begin{aligned}\langle \sigma_{t+1}^2 | \{\sigma_t\}\rangle &= P(\sigma_{t+1}=1|\{\sigma_t\}) + P(\sigma_{t+1}=-1|\{\sigma_t\}) \\ &= \frac{1}{2} - \frac{3}{2} f_0^2(p,t) + f_0(p,t) + \frac{(2p-1)^2}{2}\frac{X_t^2}{t^2}.\end{aligned} \tag{13}$$

The left-hand side of Eq. (13) must have a positive value. So, one can write

$$\frac{1}{2} - \frac{3}{2} f_0^2(p,t) + f_0(p,t) + \frac{(2p-1)^2}{2}\frac{X_t^2}{t^2} \geqslant 0. \tag{14}$$

Since for $p = 1$ one obtains a trivial ballistic walk with a maximum possible value of $\frac{\langle X_t^2\rangle}{t^2} = 1$, it is expected to be either $\frac{\langle X_t^2\rangle}{t^2} \sim 0$ or constant (>0) for $p < 1$ and $t \gg 1$.

So, for $t \gg 1$ one obtains

$$(1 - 3f_0)\frac{df_0}{dt} \geqslant 0, \tag{15}$$

which gives rise to two possibilities: Either $\frac{df_0}{dt}$ is equal to zero or it is not. When $\frac{df_0}{dt} \neq 0$, one can find that $|f_0 - \frac{1}{3}| \to 0$. So, when time is sufficiently large, the expectation value $\langle f_0(p,t)X_t\rangle$ can be written as $\langle f_0(p,t)\rangle$ times $\langle X_t\rangle$ assuming that the quantities $f_0(p,t)$ and $X_t$ are uncorrelated.

Therefore averaging over all the histories gives the following expression for the mean value:

$$\langle \sigma_{t+1}\rangle = \frac{\eta}{t}\langle X_t\rangle, \tag{16}$$

where

$$\eta = (2p-1)(1 + \langle f_0(p,t)\rangle). \tag{17}$$

Equation (16) gives rise to the following recursion for the mean displacement:

$$\langle X_{t+1}\rangle = \left(1 + \frac{\eta}{t}\right)\langle X_t\rangle. \tag{18}$$

From the initial condition, one can obtain $\langle X_1\rangle = 2s - 1$ and $\langle X_2^2\rangle = 1$. So, the solution of (18) is obtained by iteration:

$$\langle X_t\rangle = \langle X_1\rangle \frac{\Gamma(t+\eta)}{\Gamma(1+\eta)\Gamma(t)} \sim \frac{(2s-1)}{\Gamma(1+\eta)} t^{\eta} \quad \text{for } t \gg 1. \tag{19}$$

For $s = 1$ and $p = 1$ one must get the trivial ballistic walk, i.e., $\langle X_t\rangle = t$. So, the maximum value of $\eta$ is 1.

For the second moment of the displacement one can obtain the following recursion:

$$\langle X_{t+1}^2\rangle = \left(1 + \frac{2\eta}{t} + \frac{(2p-1)^2}{2t^2}\right)\langle X_t^2\rangle + D(p), \tag{20}$$

where $D(p) = \frac{1}{2} - \frac{3}{2}\langle f_0(p,t)^2\rangle + \langle f_0(p,t)\rangle$.

The solution of the recursion (20) leads to the mean-square displacement for $t \gg 1$

$$\langle X_t^2\rangle = G t^{2\eta} - \frac{R}{2\eta - 1} t, \tag{21}$$

where $G = \frac{1}{1-\delta_{\eta,1}\frac{(2p-1)^2}{4\eta(2\eta-1)\Gamma(2\eta)}}\left(\frac{1}{\Gamma(2\eta+1)} + \frac{D(p)}{2\eta(2\eta-1)\Gamma(2\eta)}\right)$,

$R = D(p) + \delta_{\eta,1}\frac{(2p-1)^2 G}{2}$, and $\delta_{\eta,1}$ is the Kronecker delta. Now imposing the condition $\frac{df_0(p,t)}{dt} = 0$, we obtain from (16) and (13) that $\eta$ must be equal to 1. For $p \geqslant 0.88$, $f_0(p) \sim \frac{1}{3}$ gives $\eta > 1$, which is not possible. In this case there is no other alternative than $\frac{df_0(p,t)}{dt} = 0$ and $\eta = 1$. So, in (21) the first term will be the dominant term for $p \geqslant 0.88$, which gives the Hurst exponent $H = 2\eta = 2$. On the other hand, for $p < 0.88$, if the value of $\eta$ is 1, it will violate Eq. (7). In this case, one must have $\frac{df_0(p,t)}{dt} \neq 0$ and $f_0(p,t) \sim \frac{1}{3}$ for large $t$, and therefore we obtain from (17) that $H = \frac{8}{3}(2p-1)$. However, for $p \leqslant 0.68$ (corresponding to $\eta < 0.5$), $2\eta$ becomes less than 1. In this case the second term in (21) is the dominant term, and $\mathrm{Var}(X_t) = \langle X_t^2\rangle - \langle X_t\rangle^2$ increases linearly with time. So, asymptotically, one has

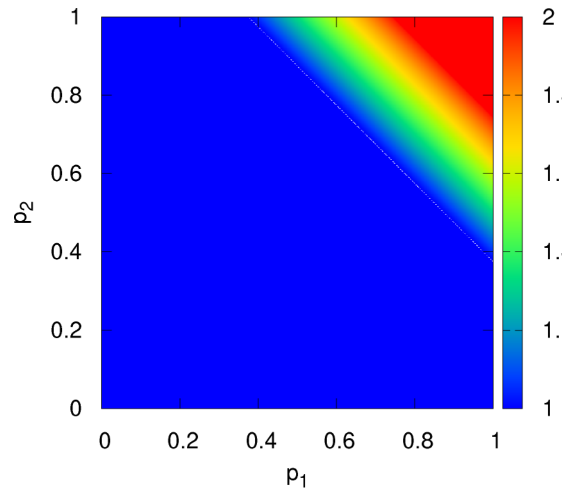


FIG. 2. The Hurst exponent is shown as a function of $p_1$ and $p_2$. Following the line $p_1 = p_2$, one can get the phase diagram for single-parameter dynamics of RW2MC.

$$\mathrm{Var}(X_t) \propto t, \quad p \leqslant 0.68. \tag{22}$$

For $0.68 < p < 0.88$ (corresponding to $\eta > 0.5$) the mean-square displacement increases as $\sim t^{2\eta}$, where $\eta = \frac{4}{3}(2p-1)$, which is of the same order as the square of the mean but the prefactor is different. Therefore the variance becomes superdiffusive. So, asymptotically, one can write

$$\mathrm{Var}(X_t) \propto t^{2\eta}, \quad 0.68 < p < 0.88. \tag{23}$$

For $p \geqslant 0.88$ (corresponding to $\eta = 1$) the mean-square displacement increases ballistically with a prefactor $G$ where $0 < G \leqslant 1$. The variance still remains superdiffusive, but here the internal dynamics is different. Asymptotically, one has $f_0(p,t) = \frac{2(1-p)}{(2p-1)}$ and $\frac{df_0(p,t)}{dt} = 0$. So, asymptotically, one has

$$\mathrm{Var}(X_t) \propto t^2, \quad p \geqslant 0.88. \tag{24}$$

If we assume two different parameters $p_1$ and $p_2$ in place of a single parameter $p$, i.e., $\mu_t^{(1)}$ and $\mu_t^{(2)}$ can take positive values (+1) with probability $p_1$ and $p_2$, respectively, then following the same procedure it can be obtained that $\eta = (p_1 + p_2 - 1)(1 + \langle f_0(p,t)\rangle)$. The variation of the Hurst exponent with respect to $p_1$ and $p_2$ is shown in Fig. 2. We have also performed numerical simulations, and the results are given in the Supplemental Material [12].

In summary, we have formulated a non-Markovian discrete-time random walk model, RW$n$MC, subjected to long-range time memory correlations. The case for $n = 2$ is discussed in detail, and the first two moments are exactly calculated. The problem can be studied with two parameters, $p_1$ and $p_2$, for which the Hurst exponent is obtained exactly and the phase diagram in the $p_1$-$p_2$ plane is fully characterized. The transitions from diffusive to superdiffusive behavior are also described in detail. In the context of this Research Letter it may be noted that a classical random walk with long-term memory has been considered before. Such a walk was termed an "Elephant random walk" (ERW), which is perhaps the first model to contain memory of the entire history of the walker [13]. After that, many modifications of the ERW either in the memory pattern or in the decision-making process have been studied [14–17]. Recently, a quantum walk with long-range memory, named an "elephant quantum walk," has also been proposed [18]. In ERW the walker can choose only one step from its entire history with equal *a priori* probability; that is, it has only one memory channel. Hence the RW$n$MC may be regarded as a generalization of the ERW. In RW$n$MC a paradigm emerges where the memory pattern of different channels can be varied independently. Following this line, one can define an Alzheimer's walk [16,17,19] with multiple memory channels, which will obviously further inspire research on the effect of memory loss. We have constructed a RW2MC in such a way that the dynamics can be expressed by a simple algebraic equation (2). However, one can construct a RW2MC in different ways also.

A connection between the elephant random walk (ERW) and a Pólya-type urn model containing balls of two colors and evolving by drawing one ball at a time was established earlier [20]. Here we have shown that the urn model equivalent to the RW$n$MC presented here is a Pólya-type urn containing balls of three colors and evolving by drawing multiple balls at a time.

The urn model of RW2MC presented here has interactions of the type $A + B \rightarrow A + B + C$; that is, an interaction between two drawn balls gives as a product the same two balls plus a new ball. The color of the new ball is determined by a disorder parameter $p$. For $p = 1$ (no disorder) one can find some similarity with the urn scheme inspired by the famous rock-paper-scissors game [21]. Interactions of the type $A + B \rightarrow kC$ ($k \in \mathbb{Z}$) or $A + B \rightarrow C + D$, where reactants can annihilate, occur in many population dynamics models with multiple competing species. It will be interesting to see whether such a model can be described by a RW2MC [22]. The equivalence with an urn model implies that practical applications of RW$n$MC do not necessarily require living systems with memory (a human, animal, insect, etc.) as a walker. A complex physical system made up of nonliving elements that have no memory can also produce this kind of random walk. So, this work can shed light on the origin of long-range memory in complex physical systems. In some earlier works [23,24], walks inspired by dynamical models have been studied leading to some intriguing results. It will be interesting to see whether, for example, an opinion dynamics model can be conceived from the present study and to compare the results.

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