Letter

Thermodynamic uncertainty relation from involutions

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(Received 10 November 2022; accepted 9 December 2022; published 22 December 2022)

The thermodynamic uncertainty relation (TUR) is a lower bound for the variance of a current (over the mean squared) as a function of the average entropy production. Depending on the assumptions, one obtains different versions of the TUR. For instance, from the exchange fluctuation theorem, one obtains a corresponding exchange TUR. Alternatively, we show that TURs are a consequence of a very simple property: Every process s has only one conjugate $s' = m(s)$, where m is an involution, $m(m(s)) = s$. This property allows the derivation of a general TUR without using any fluctuation theorem. As applications, we obtain the exchange TUR, the hysteretic TUR, a fluctuation-response inequality and a lower bound for the entropy production in terms of other nonequilibrium metrics.

DOI: 10.1103/PhysRevE.106.L062104

Introduction. In nonequilibrium thermodynamics, physical observables fluctuate in time, such as currents ϕ of particles or heat [1–10]. These fluctuations are fully encoded in the probability density function $p(\sigma, \phi)$, where σ is the entropy production, which is usually system dependent and time dependent in transient regimes. However, there is something apparently universal in the fluctuations of any current ϕ: The variance of ϕ (normalized by $\langle\phi\rangle^2$) is lower bounded by a function of the average entropy production. This is called the thermodynamic uncertainty relation (TUR) [11–19], usually written as

$$\frac{\langle\phi^2\rangle - \langle\phi\rangle^2}{\langle\phi\rangle^2} \geqslant f(\langle\sigma\rangle), \tag{1}$$

where $f(x)$ is a function that depends on some characteristics of the system. The TUR was first derived for Markov jump processes [12], where $f(x) = 2/x$. Alternatively, if the system satisfies the exchange fluctuation theorem (XFT), $p(\phi\,\sigma) = e^{\sigma} p(-\phi, -\sigma)$, then $f(x)$ has another particular form [15,16].

In systems that are not time symmetric, XFT does not hold, but a different form of asymmetric TUR was obtained from the definition of entropy production, $p(\sigma, \phi) = e^{\sigma}\overline{p}(-\sigma, -\phi)$ [18], where $\overline{p}$ is the probability function in the backward experiment. In a similar setup, a hysteretic TUR was derived [19], and a tighter form was proposed in Ref. [20].

In both cases, some form of fluctuation theorem (FT) was assumed explicitly. In this paper, we ask if there is anything more fundamental than the FT behind the derivation of the TURs. We found that TURs can be understood as a consequence of the fact that each process s has a single conjugate $s' := m(s)$, which gives $m(m(s)) = s$. This map m, which is called an involution, can be explored to create the involution TUR (iTUR). In the most simple form, it reads

$$\frac{\langle\phi^2\rangle - \langle\phi\rangle^2}{\langle\phi\rangle^2} \geqslant f(D(P|P')), \tag{2}$$

valid for any asymmetric current, $\phi(s') = -\phi(s)$, and any involution m, where $P'(s) := P(m(s))$ and $D(P|P') = \sum_s P(s)\ln(P(s)/P'(s))$ is the Kullback-Leibler (KL) divergence, $f(x) = \sinh^{-2}(g(x)/2)$, and $g(x)$ the inverse of $h(x) = x\tanh(x/2)$, for $x \geqslant 0$.

The paper is organized as follows. First, we present the formalism and our main result. Then, prove the main result based only on the involution property. After that, we discuss the result and apply it in the context of the exchange FT to obtain exchange TURs. We also apply it to the framework of stochastic thermodynamics with asymmetric protocols, obtaining a tight version of the hysteretic TUR. Finally, we show how the result is useful to connect the KL divergence with the total variation distance and a general relation for the moment generating function. We also obtain a fluctuation-response inequality and the Crámer-Rao bound in a limiting case.

Formalism. We denote s the elements of a set S. In the context of thermodynamics, each s is a process that is observed with probability $P(s) \in [0, 1]$. Additionally, we consider an involution $m : S \to S$, such that $s' := m(s)$ and

$$m(m(s)) = s. \tag{3}$$

It means that each process s has a single conjugate s', a property that is behind the notion of time-reversal symmetry. We define $P' : S \to [0, 1]$ as

$$P'(s) := P(s') = P(m(s)), \tag{4}$$

using a notation introduced in Ref. [21]. Similarly, $Q : S \to [0, 1]$ is another probability function and $Q'(s) = Q(s') = Q(m(s))$. Normalization reads $\sum_s P(s) = \sum_s P'(s) = \sum_s Q(s) = \sum_s Q'(s) = 1$. Let $\phi : S \to \mathbb{R}$ be any current with property

$$\phi(m(s)) = -\phi(s), \tag{5}$$

which makes it odd under the involution. Consider the usual notation for the mean, $\langle\phi\rangle_P = \sum_s P(s)\phi(s)$, $\langle\phi\rangle_Q = \sum_s Q(s)\phi(s)$. Similarly, we have the variances defined as

$\langle(\phi-\langle\phi\rangle)^2\rangle_{P,Q}$, with respect to P and Q. Define the KL divergence $D(P|Q)=\sum_s P(s)\ln P(s)/Q(s)$ (defined if $Q(s)=0\to P(s)=0$ for any s, a property called absolute continuity).

Main result. Our main result is the iTUR with respect to any P and Q, which reads

$$\frac{\langle(\phi-\langle\phi\rangle_P)^2\rangle_P+\langle(\phi-\langle\phi\rangle_Q)^2\rangle_Q}{(1/2)(\langle\phi\rangle_P+\langle\phi\rangle_Q)^2} \geqslant f\left(\frac{D(P|Q')+D(Q'|P))}{2}\right), \quad (6)$$

with $f(x)=\sinh^{-2}(g(x)/2)$ and $g(x)$ is the inverse function of $h(x)=x\tanh(x/2)$, for $x\geqslant 0$, where P', Q' are defined as (4) for any involution (3) and any current (5) such that $\langle\phi\rangle_P+\langle\phi\rangle_Q\neq 0$.

Proof. Consider any probabilities P and Q, any involution m (3), with corresponding $P'$and Q' (4) such that the pairs P,Q' and Q,P' are absolute continuous. Let ϕ be any current (5). Define $p:S\to[0,1]$ as

$$p(s):=\frac{P(s)+Q'(s)}{2}. \quad (7)$$

Note that $\sum_s p(s)=1$ follows from the normalization of P and Q'. Using Eqs. (7), (3), and (5), one gets the average current,

$$\overline{\phi}:=\langle\phi\rangle_p=\sum_s\phi(s)\frac{(P(s)+Q(s'))}{2}=\frac{\langle\phi\rangle_P-\langle\phi\rangle_Q}{2}. \quad (8)$$

Combining Eqs. (7) and (8), we get a relation for the variance,

$$4\langle(\phi-\overline{\phi})^2\rangle_p=2\langle(\phi-\langle\phi\rangle_P)^2\rangle_P +2\langle(\phi-\langle\phi\rangle_Q)^2\rangle_Q+(\langle\phi\rangle_P+\langle\phi\rangle_Q)^2. \quad (9)$$

Additionally, for every s, we have for $P(s)+Q'(s)>0$,

$$\phi(s)\frac{(P(s)-Q'(s))}{2}=\phi(s)\frac{P(s)-Q'(s)}{P(s)+Q'(s)}p(s), \quad (10)$$

and after summing all s on both sides of Eq. (10), using Eqs. (3) and (5), it results in

$$\frac{\langle\phi\rangle_P+\langle\phi\rangle_Q}{2}=\langle\phi\frac{P-Q'}{P+Q'}\rangle_p=\left\langle(\phi-\overline{\phi})\frac{P-Q'}{P+Q'}\right\rangle_p, \quad (11)$$

where one should interpret P,Q' inside the averages as $P(s),Q'(s)$ and use $\langle(P-Q')/(P+Q')\rangle_p=0$. Using Cauchy-Schwarz inequality, one has

$$\left\langle(\phi-\overline{\phi})\frac{P-Q'}{P+Q'}\right\rangle_p^2\leqslant\langle(\phi-\overline{\phi})^2\rangle_p\left\langle\left(\frac{P-Q'}{P+Q'}\right)^2\right\rangle_p. \quad (12)$$

Now combining Eqs. (11) and (12), one obtains

$$\frac{(\langle\phi\rangle_P+\langle\phi\rangle_Q)^2}{4}\leqslant\langle(\phi-\overline{\phi})^2\rangle_p\left\langle\left(\frac{P-Q'}{P+Q'}\right)^2\right\rangle_p. \quad (13)$$

Then, note that

$$\frac{P-Q'}{P+Q'}=\frac{P/Q'-1}{P/Q'+1}=\tanh\left(\frac{1}{2}\ln\frac{P}{Q'}\right). \quad (14)$$

Now we use Jensen's inequality, for any $x:S\to\mathbb{R}$ and any probability function, one has

$$\left\langle\tanh\left(\frac{x}{2}\right)^2\right\rangle=\left\langle\tanh\left(\frac{g(h(x))}{2}\right)^2\right\rangle\leqslant\tanh\left(\frac{g(\langle h(x)\rangle)}{2}\right)^2, \quad (15)$$

where $h(x):=x\tanh(x/2)$ and $g(x)$ is the inverse function of h for $x\geqslant 0$, $g(h(x))=x$, since $d^2w(h)/dh^2<0$, for $w(h)=\tanh(g(h)/2)^2$, which is a property already used in other contexts [22–24]. Replacing $x=\ln(P/Q')$ and $\langle\rangle=\langle\rangle_p$ in Eq. (15) and using Eq. (14), it results in

$$\left\langle\left(\frac{P-Q'}{P+Q'}\right)^2\right\rangle_p\leqslant\tanh\left(\frac{1}{2}g\left(\left\langle h\left(\ln\frac{P}{Q'}\right)\right\rangle_p\right)\right)^2, \quad (16)$$

where the term $\langle h(\ln P/Q')\rangle_p$ can be simplified to

$$\left\langle h\left(\ln\left(\frac{P}{Q'}\right)\right)\right\rangle_p=\sum_s\ln\left(\frac{P(s)}{Q'(s)}\right)\frac{P(s)-Q'(s)}{P(s)+Q(s)}p(s) \quad (17)$$

$$=\frac{1}{2}(D(P|Q')+D(Q'|P)). \quad (18)$$

We finally obtain from Eqs. (13), (16), and (18),

$$\frac{(\langle\phi\rangle_P+\langle\phi\rangle_Q)^2}{4} \leqslant\langle(\phi-\overline{\phi})^2\rangle_p\tanh\left(\frac{1}{2}g\left(\frac{D(P|Q')+D(Q'|P)}{2}\right)\right)^2. \quad (19)$$

Inverting Eq. (19) and using $\tanh(x/2)^{-2}=1+\sin(x/2)^{-2}$, we get

$$\frac{4\langle(\phi-\overline{\phi})^2\rangle_p}{(\langle\phi\rangle_P+\langle\phi\rangle_Q)^2}\geqslant 1+f\left(\frac{D(P|Q')+D(Q'|P))}{2}\right). \quad (20)$$

Using Eq. (9) in Eq. (20), we obtain our main result (6).

Discussion. In the derivation of Eq. (6), we did not use any FT. Instead, the involution (3) was the only assumption needed. If one thinks of s as a process, the assumption is simply stating the existence of a single conjugate process $m(s)$, such that $m(m(s))=s$, which is the simplest property behind a sequence of events: If you flip the sequence twice, you get the original direction.

In the derivation above, we used several ideas from recent literature on TURs. Specifically, an analogous of definition (8) appeared in asymmetric TURs [18–20,25,26], and the tightest Jensen's inequality in Eq. (15) was also used before [16,21,24]. Our contribution was to write a general TUR solely in terms of the involution m instead of any FT, which is a mathematical statement about the object (S,P,Q) for any m and ϕ. The relation between iTUR and the physical TURs (in terms of entropy productions) will be explored below. We also noted a general expression similar to Eq. (6) appeared recently [27], where the equivalence can be obtained considering in our notation $S=\mathbb{R}$, and for any $s=x\in\mathbb{R}$, $\phi(x)=x$, $m(x)=-x$, with $P(s)=p(x)$ and $Q'(s)=q(x)$.

Application 1: exchange TURs. Consider $p(\sigma,\phi)$ as the probability of observing the entropy production σ and the current ϕ. In this case, the exchange fluctuation theorem reads

$$p(\sigma, \phi) = p(-\sigma, -\phi)e^{\sigma}, \tag{21}$$

valid for a series of setups [16,28,29], also called the strong detailed fluctuation theorem [2,30], the Evan-Searles fluctuation theorem [31], and Gallavotti-Cohen relation [8]. To obtain the corresponding TUR from the main result (6), we use $S = \{(\phi, \sigma)|\phi, \sigma \in \mathbb{R}\}$, also $Q = P$ and $m(\phi, \sigma) = (-\phi, -\sigma)$ as the involution, which makes $D(P|Q') = D(Q'|P) = D(P|P') = \langle\sigma\rangle$ from Eq. (21). In this case, Eq. (6) results in the following exchange thermodynamic uncertainty relation:

$$\frac{\langle \phi - \overline{\phi}\rangle^2}{\langle\phi\rangle^2} \geqslant f(\langle\sigma\rangle), \tag{22}$$

famously derived from the exchange FT (21) [15,16,30]. Again, note that Eq. (22) followed immediately from Eq. (6), where the role of the FT was only to assign $D(P|P') = \langle\sigma\rangle$.

Alternatively, we might have a situation more general than Eq. (21), given by

$$p(\phi, \sigma_I)/\overline{p}(-\phi, -\sigma_I) = e^{\sigma_I}, \tag{23}$$

where $\overline{p} \neq p$ is the distribution of a backward experiment. Equation (23) appears in situations with measurement and feedback [18], where $\sigma_I = \sigma + I$ is a general entropy production that contains an information term. In this case, our result (6) reads

$$\frac{\langle(\phi - \langle\phi\rangle_p)^2\rangle_p + \langle(\phi - \langle\phi\rangle_{\overline{p}})^2\rangle_{\overline{p}}}{(1/2)(\langle\phi\rangle_p + \langle\phi\rangle_{\overline{p}})^2} \geqslant f\left(\frac{\langle\sigma_I\rangle_p + \langle\sigma_I\rangle_{\overline{p}}}{2}\right), \tag{24}$$

with the same definitions of $s = (\phi, \sigma)$ and $m(\phi, \sigma) = (-\phi, -\sigma)$ as before, using $P(s) = p(\phi, \sigma_I)$ and $Q(s) = \overline{p}(\phi, \sigma_I)$, where $\langle\sigma_I\rangle = \int \sigma_I p(\phi, \sigma) d\phi d\sigma$ and $\langle\sigma_I\rangle_B = \int \sigma_I \overline{p}(\phi, \sigma_I) d\phi d\sigma$. Examples of systems satisfying (23) and the resulting TUR (24) include a quantum-dot coupled to a fermionic reservoir and a Szillard engine [18,32,33]. We remark that Eq. (24) is a tighter form of the TURs derived recently [18,20].

Application 2: hysteretic TURs. In stochastic thermodynamics, one has a trajectory $\Gamma = \{x_0, x_1, , ..., x_N\}$ of observed states x_i at time t_i. In this context, we consider the definition of a general stochastic variable [29]:

$$\sigma(\Gamma) := \ln P(\Gamma)/Q(\Gamma^\dagger), \tag{25}$$

in terms of any probabilities P and Q, where where $\Gamma^\dagger = \{x_N, ..., x_0\}$. For the specific case where the system is in contact with a single heat reservoir and satisfies local detailed balance, definition (25) captures the physical cumulative entropy production along a trajectory (see the path description in [10]) for $P(\Gamma) = P_F(\Gamma)$ and $Q(\Gamma) = P_B(\Gamma)$, where $P_F(\Gamma)$ is the probability of observing Γ in an experiment (forward) with a time dependent controllable parameter $\lambda = \{\lambda_0, ..., \lambda_N\}$. Similarly, $P_B(\Gamma)$ represents the backward experiment, using $\lambda' = \{\lambda_N, ..., \lambda_0\}$ and using the final state of the forward experiment as the initial state of the backward experiment.

The application of Eq. (6) is straightforward by considering $s = \Gamma$, the involution $m(\Gamma) = \Gamma^\dagger$, and probabilities $P(s) = P_F(\Gamma)$, $Q(s) = P_B(\Gamma)$. In this case, $Q'(s) = Q(m(s)) = P_B(\Gamma^\dagger)$. We obtain from Eq. (6) for any current $\phi(\Gamma) = -\phi(\Gamma^\dagger)$:

$$\frac{\langle(\phi - \langle\phi\rangle_F)^2\rangle_F + \langle(\phi - \langle\phi\rangle_B)^2\rangle_B}{(1/2)(\langle\phi\rangle_F + \langle\phi\rangle_B)^2} \geqslant f\left(\frac{\overline{\sigma} + \overline{\sigma}_B}{2}\right), \tag{26}$$

which is a form of hysteretic TUR [19], where $\sigma(\Gamma) := \ln(P_F(\Gamma)/P_B(\Gamma^\dagger))$, $\sigma_B(\Gamma) := -\sigma(\Gamma^\dagger)$, $\overline{\sigma} := \langle\sigma(\Gamma)\rangle_F = \sum_\Gamma P_F(\Gamma)\ln[P_F(\Gamma)/P_B(\Gamma^\dagger)]$, $\overline{\sigma}_B := \langle-\sigma(\Gamma^\dagger)\rangle_B = \sum_{\Gamma^\dagger} P_B(\Gamma^\dagger)\ln[P_B(\Gamma^\dagger)/P_F(\Gamma)]$ and the subscripts F, B are the averages over $P_F(\Gamma)$, $P_B(\Gamma)$.

As in the case of the exchange TUR, the role of the FT was to assign $\overline{\sigma} = D(P_F|P_B')$ and $\overline{\sigma}_B = D(P_B'|P_F) = D(P_B|P_F')$, where $P_F'(\Gamma) := P_F(\Gamma^\dagger)$, $P_B'(\Gamma) := P_B(\Gamma^\dagger)$. TURs for the form (26) were the subject of recent results [18–20], notably because they are applicable to asymmetric protocols. Note that the symmetric case $F = B$ recovers a form similar to the exchange TUR, Eq. (22).

Application 3: total variation bound. We find a bound for the KL divergence in terms of the total variation distance. This application has impact in the lower bound for apparent violations of the second law [22] as follows. Consider the exchange fluctuation theorem (21). Although $\langle\sigma\rangle \geqslant 0$, the stochastic nature of σ makes $\sigma < 0$ a possible outcome. Thus, the probability $P(\sigma < 0)$ is the probability that the system outputs a negative entropy production. As it turns out, the total variation can be written as $\Delta(P, P') = P(\sigma > 0) - P(\sigma < 0)$ for systems with $P(|\sigma|) > P(-|\sigma|)$, which is the case for Eq. (21). Therefore, any relation $D(P|P') \geqslant B(\Delta(P, P')/\sqrt{1 - P_0})$ combined with the normalization $1 = P(\sigma < 0) + P(\sigma > 0) + P_0$, for $P_0 := P(\sigma = 0)$, results in a bound for the apparent violation of the second law $P(\sigma < 0) \geqslant V(P_0, \langle\sigma\rangle)$ for $V(P_0, \langle\sigma\rangle) := 1/2 - [P_0 + B^{-1}(\langle\sigma\rangle)\sqrt{1 - P_0}]/2$. The term $\sqrt{1 - P_0}$ in the argument of B was added for convenience as shown below. The bound means that the second law is often "violated" at trajectory level at least by $V(P_0, \langle\sigma\rangle)$, a function of P_0 and the average entropy production, $\langle\sigma\rangle$.

For this application, we consider any set S, probabilities $Q = P$ and involution m for the particular current

$$\phi(s) := \mathrm{sgn}[P(s) - P(s')]. \tag{27}$$

From Eq. (27), we have $\phi(s') = \mathrm{sgn}[P(s') - P(s)] = -\phi(s)$, as expected. The mean of ϕ is given by

$$\langle\phi\rangle_P = \sum_s \mathrm{sgn}[P(s) - P(s')]P(s) = \frac{1}{2}\sum_s |P(s) - P(s')|, \tag{28}$$

where we used $\sum_s \phi(s)P(s) = (1/2)\sum_s \phi(s)(P(s) - P(s'))$, since $s' = m(s)$ is an involution. Note that $\Delta(P, P') = (1/2)\sum_s |P(s) - P(s')|$ is the definition of the total variation distance between P and P'. The variance of ϕ is

$$\langle\phi^2\rangle_P - \langle\phi\rangle_P^2 = 1 - P_0 - \Delta(P, P')^2, \tag{29}$$

where we defined the parameter

$$P_0 := \sum_s P(s)\theta(P(s) - P(s')), \tag{30}$$

for $\theta(0)=1$ (and $\theta(x)=0$ if $x\neq 0$), which one might understand intuitively as the probability of drawing a process from equilibrium ($P(s)=P(s')$, or detailed balance). Using Eqs. (27) and (30) in Eq. (6) for $P=Q$, one obtains

$$\frac{1-P_0-\Delta(P,P')^2}{\Delta(P,P')^2}\geqslant f(D(P|P')), \tag{31}$$

which inverts to

$$D(P|P')\geqslant 2\frac{\Delta(P,P')}{\sqrt{1-P_0}}\tanh^{-1}\left(\frac{\Delta(P,P')}{\sqrt{1-P_0}}\right), \tag{32}$$

which is a lower bound for the Kullback-Leibler divergence for any P, when P and P' are connected by any involution, $P'(s)=P(m(s))$. We also note that $D(P|P')\geqslant B(\Delta(P,P')/\sqrt{1-P_0})\geqslant B(\Delta(P,P'))$, for $B(x):=2x\tanh^{-1}(x)$, since $dB(x)/dx>0$, for $x>0$ and from Eq. (29) $1\geqslant\Delta(P,P')/\sqrt{1-P_0}\geqslant\Delta(P,P')$, which improves on the lower bound of Ref. [21] when P_0 is known.

Application 4: moment generation function. We apply the iTUR to obtain a general relation for the moment generating function (mgf) of the random variable $\ln(P(s)/P(s'))$. First, consider again $P=Q$ and any involution m. Define the following moment generating function for the variable $\ln(P(s)/P'(s))$ given by $G(\alpha)=\langle\exp(\alpha\ln(P/P'))\rangle_P=\langle(P/P')^\alpha\rangle_P$. The first and second derivatives of $G(\alpha)$ yield $G'_0:=G'(0)=\langle\ln P/P'\rangle_P=D(P|P')$. and $G''(0)=\langle(\ln P/P')^2\rangle$. The iTUR for the current $\phi(s)=\ln P(s)/P'(s)$ yields the bound

$$\frac{G_0'^2}{G''(0)}\leqslant\tanh\left(\frac{1}{2}g(G'_0)\right)^2. \tag{33}$$

However, one gets a more general relation for the mgf $G(\alpha)$ considering the current

$$\phi(s):=\left(\frac{P(s)}{P'(s)}\right)^\alpha-\left(\frac{P'(s)}{P(s)}\right)^\alpha, \tag{34}$$

for $P(s),P(s')\neq 0$ and $\phi(s)=0$ otherwise. Check that $\phi(s')=-\phi(s)$ as expected. The first and second moments of ϕ read

$$\langle\phi\rangle_P=\sum_s\phi(s)P(s)=G(\alpha)-G(-\alpha), \tag{35}$$

$$\langle\phi^2\rangle_P=\sum_s\phi(s)^2P(s)=G(2\alpha)+G(-2\alpha)-2. \tag{36}$$

Inserting Eqs. (35) and (36) in Eq. (6), we obtain

$$\frac{[G(\alpha)-G(-\alpha)]^2}{[G(2\alpha)+G(-2\alpha)-2]}\leqslant\tanh^2\left(\frac{1}{2}g(G'_0)\right). \tag{37}$$

In the specific case of $\alpha\to 0$, Eq. (37) yields Eq. (33). However, Eq. (37) is more general, as it holds for finite values of α. Also note that $(1-\alpha)D_\alpha(P|P')=\ln(G(\alpha-1))$ defines the Rényi divergence $D_\alpha(P|P')$.

Application 5: fluctuation-response bound. Let $p(x|\theta)$ be a probability density function in $\mathbb{R}$ for some parameter θ. We define $S=\mathbb{R}$, $m(x)=-x$, $P(x)=p(x|\theta+\epsilon)$ and $Q(x)=p(-x|\theta)$, so that any asymmetric current, $\phi(-x)=-\phi(x)$, results in the averages $\langle\phi\rangle_P=\int\phi(x)p(x|\theta+\epsilon)dx:=\langle\phi\rangle_{\theta+\epsilon}$ and $\langle\phi\rangle_Q=\int\phi(x)p(-x|\theta)dx=-\langle\phi\rangle_\theta$. In this case the iTUR (6) reads

$$\frac{(1/2)(\langle\phi\rangle_{\theta+\epsilon}-\langle\phi\rangle_\theta)^2}{\langle(\phi-\langle\phi\rangle_{\theta+\epsilon})^2\rangle_{\theta+\epsilon}+\langle(\phi-\langle\phi\rangle_\theta)^2\rangle_\theta}\leqslant\sinh\left(\frac{g(\Sigma(p_{\theta+\epsilon},p_\theta))}{2}\right)^2, \tag{38}$$

where we defined $\Sigma(p_{\theta+\epsilon},p_\theta):=(1/2)\int[p(x|\theta+\epsilon)-p(x|\theta)]\ln(p(x|\theta+\epsilon)/p(x|\theta))dx\geqslant 0$.

Note that Eq. (38) is a type of fluctuation-response inequality [34]. In the limiting case $\epsilon\to 0$, it reduces to the Crámer-Rao inequality as follows: Expanding Eq. (38) in ϵ, one obtains $\langle\phi\rangle_{\theta+\epsilon}-\langle\phi\rangle_\theta=\partial_\theta\langle\phi\rangle_\theta\epsilon+\mathcal{O}(\epsilon^2)$ and $\langle(\phi-\langle\phi\rangle_{\theta+\epsilon})^2\rangle_{\theta+\epsilon}=\langle(\phi-\langle\phi\rangle_\theta)^2\rangle_\theta+\mathcal{O}(\epsilon)$. We also have $\Sigma(p_{\theta+\epsilon},p_\theta)=(\epsilon^2/2)\int(\partial_\theta p(x|\theta))^2/p(x|\theta)dx+\mathcal{O}(\epsilon^3)=(\epsilon^2/2)I(\theta)+\mathcal{O}(\epsilon^3)$, where $I(\theta)$ is the Fisher information. Using $g(x)\approx\sqrt{2x}$ and $\sinh(x)\approx x$ for $x\approx 0$, one has $\sinh(g(\Sigma)/2)^2\approx\Sigma/2\approx\epsilon^2I(\theta)/4$, and Eq. (38) results in

$$\frac{(\partial_\theta\langle\phi\rangle_\theta)^2}{\langle(\phi-\langle\phi\rangle_\theta)^2\rangle_\theta}\leqslant I(\theta), \tag{39}$$

in the limit $\epsilon\to 0$, which is the famous Crámer-Rao bound [35,36].

Conclusions. We showed that a set S equipped with an involution m is able to produce the involution thermodynamic uncertainty relation (iTUR) for any pair of probabilities P,Q and any asymmetric current $\phi(m(s))=-\phi(s)$. Remarkably, the FT was not used in the derivation, which might sound unusual at first glance when compared to other TURs. With that result, we argue that the origin of the TUR is better understood as a consequence of the involution, where each event s has a single conjugate s'. This apparently naive property holds the key to the iTUR and the underlying applications. This is a purely nonequilibrium result, as a system in equilibrium [$P(s)=Q'(s)$, equivalent to detailed balance] would collapse the relations obtained in this paper.

It is interesting to note that for non-Markovian systems, the bound (26) is also true and the currents in the LHS are physical quantities, but the RHS contains $\overline{\sigma}=D(P|Q')$ which is not a cumulative entropy production along a trajectory anymore in the sense of Ref. [10]. So non-Markovian systems satisfy Eq. (26), but it makes the bound dependent on a divergence $(D(P_F|P'_B)+D(P_B|P'_F))$ that might not be the physical entropy production. In summary, Markovian property is not needed for the mathematical result, but it might be needed in the applications to assign KL to a physical entropy production.

As applications, we showed how the result implies the exchange fluctuation theorem and a tight form of the asymmetric fluctuation theorem, both important results in nonequilibrium thermodynamics. We also showed how the iTUR is related to a connection between $D(P|P')$ and other statistics of P and P', such as the total variation distance $\Delta(P,P')$, the mgf $\langle\exp(\alpha\ln(P/P'))\rangle$, and a fluctuation-response inequality, combining different results from nonequilibrium thermodynamics under the same framework.

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