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Letter

Path integrals for fractional Brownian motion and fractional Gaussian noise

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Wiener's path integral plays a central role in the study of Brownian motion. Here we derive exact path-integral representations for the more general fractional Brownian motion (FBM) and for its time derivative process, fractional Gaussian noise (FGN). These paradigmatic non-Markovian stochastic processes, introduced by Kolmogorov, Mandelbrot, and van Ness, found numerous applications across the disciplines, ranging from anomalous diffusion in cellular environments to mathematical finance. Their exact path-integral representations were previously unknown. Our formalism exploits the Gaussianity of the FBM and FGN, relies on the theory of singular integral equations, and overcomes some technical difficulties by representing the action functional for the FBM in terms of the FGN for the subdiffusive FBM and in terms of the derivative of the FGN for the super-diffusive FBM. We also extend the formalism to include external forcing. The exact and explicit path-integral representations make inroads in the study of the FBM and FGN.

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Introduction. The importance of path integrals in theoretical physics is broadly recognized. Their application has proved to be rewarding not only as a computational tool, both analytical and numerical, but also as a powerful and versatile conceptual framework. The notion of path integrals was introduced by Wiener [1] for Brownian motion (BM). Since then it has helped uncover many nontrivial statistical properties of BM [2–6]. Feynman reinvented path integrals within his reformulation of quantum mechanics [7–9]. He is also credited with making path integrals an intrinsic part of physicist's toolbox [10–17].

Path-integral representations of stochastic processes and fields are especially useful in the studies of large-deviation statistics of physical quantities. Performing a saddle-point evaluation of the pertinent path integral (which relies on a problem-specific large parameter), one can determine the optimal, that is, the most likely, history of the system which dominates the statistics in question. This method of large-deviation analysis appears in different areas of physics under different names: the optimal fluctuation method, the instanton method, the weak-noise theory, the macroscopic fluctuation theory, the dissipative WKB approximation, etc. A full list of references on different applications of this method would exceed 100.

The key object of a path-integral representation of BM and its functionals is the probability density $P[x(t)]$ of a given realization of a Brownian trajectory $x(t)$, $P[x(t)] \sim \exp\{-S[x(t)]\}$, where the action functional $S[x(t)]$ is given by the Wiener formula [1]

$$S[x(t)] = \frac{1}{2}\int dt\, \dot{x}^2(t) \tag{1}$$

(the dot here and henceforth denotes the time derivative and we set the diffusion coefficient to $\frac{1}{2}$ for brevity). The local-in-time Wiener action (1) reflects the Markovian nature of BM. The past two decades have witnessed great interest in fractional Brownian motion (FBM), introduced by Mandelbrot and van Ness [18] and earlier by Kolmogorov [19]. The Mandelbrot–van Ness (MvN) FBM is a non-Markovian generalization of the Brownian motion which keeps the important properties of Gaussianity, stationarity of the increment, and dynamical scale invariance. For the two-sided (that is, prethermalized) FBM, time t is defined on the entire axis $|t| < \infty$. For the one-sided FBM $0 \leqslant t < \infty$. Here the process starts at $t = 0$ and there is no past. Both versions of the FBM are zero-mean Gaussian processes [for convenience we set $x(0) = 0$] and they are completely defined by their covariance functions

$$\begin{aligned}\kappa_2(t,t') &= \langle x(t)x(t')\rangle = \tfrac{1}{2}(|t|^{2H} + |t'|^{2H} - |t-t'|^{2H}),\\ \kappa_1(t,t') &= \langle x(t)x(t')\rangle = \tfrac{1}{2}(t^{2H} + t'^{2H} - |t-t'|^{2H}).\end{aligned} \tag{2}$$

Here the subscripts 1 and 2 stand for the one- and two-sided processes, respectively, the angular brackets denote ensemble averaging, and $0 < H < 1$ is the Hurst index which quantifies the dynamical scale invariance of the process [20] and its ruggedness. For $H < \frac{1}{2}$ the FBM is subdiffusive, i.e., the mean-square displacement $\langle x^2(t)\rangle = t^{2H}$ grows sublinearly with time. For $H > \frac{1}{2}$ the FBM is superdiffusive. In the borderline case $H = \frac{1}{2}$ one recovers the standard BM. Figure 1 presents examples of numerical stochastic realizations of FBM for $H = \frac{1}{4}, \frac{1}{2}$ (standard BM), and $\frac{3}{4}$.

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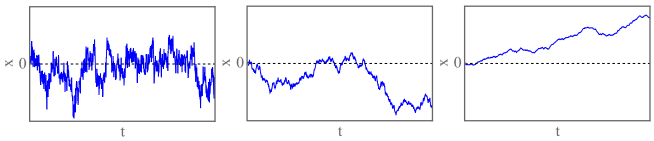

FIG. 1. Stochastic realizations of one-sided MvN FBM $x(t)$ for $H = \frac{1}{4}$ (left) and $H = \frac{3}{4}$ (middle). $H = \frac{1}{2}$ (right) corresponds to the standard BM.

The fractional Gaussian noise (FGN) was introduced by Mandelbrot and van Ness [18] as the time derivative of $x(t)$. That is, by definition, the FBM $x(t)$ obeys the Langevin equation $\dot{x}(t) = y(t)$, where $y(t)$ is the FGN. For $H < \frac{1}{2}$ the FGN is antipersistent (that is, it has negative autocorrelations). For $H > \frac{1}{2}$ it is positively correlated. For $H = \frac{1}{2}$ the δ-correlated white Gaussian noise is recovered. The subsequent analysis covers both sub- and superdiffusive cases.

Multiple physical processes have been successfully modeled as FBM. These include fluctuating interfaces [21] dynamics in crowded fluids [22,23], subdiffusive dynamics of bacterial loci in a cytoplasm [24], telomere diffusion in the cell nucleus [25,26], modeling of conformations of serotonergic axons [27], diffusion of a tagged bead of a polymer [28,29], translocation of a polymer through a pore [29–32], single-file diffusion in ion channels [33–35], etc. A review can be found in [36]. In turn, the FGN [18] is used to model antipersistent or persistent dependence structures in observed time series in many applications including hydrology [37], information theory [38], climate data analysis [39], and physiology [40], to mention a few.

By now the MvN FBM has become a standard model of anomalous diffusion in systems with memory. Still, a satisfactory path-integral representation of this process is unavailable.[1] This is in spite of the fact that, for non-Markovian but Gaussian processes, such as the MvN FBM, there is a straightforward path [13] to constructing an analog of Eq. (1). It involves the determination of a (highly singular) nonlocal kernel, inverse to the covariance function (2), via solving a singular integral equation [such as Eq. (5) below]. For MvN FBM this equation is hard to deal with analytically, which explains the scarcity of results on path-integral representations of the FBM.[2]

[1]In simple large-deviation problems, analyzed with the optimal fluctuation method, the explicit knowledge of the inverse kernel can be unnecessary. This happens when the ensuing nonlocal Euler-Lagrange equation (a linear integral equation) can be transformed into a form containing only the covariance function of the process [41–43]. For more complicated problems, however, no such transformation exists and the knowledge of the inverse kernel is indispensable.

[2]For the Riemann-Liouville FBM (a different mathematical model of anomalous diffusion, introduced by Lévy [44]), the action $S[x(t)]$ can be determined by employing fractional calculus, as was done in Refs. [45,46] and in Chap. 10 in Ref. [16]. Also, in Refs. [47,48] a path-integral representation was found for the fractional Lévy motion [47], still another mathematical model. All these results, however, are irrelevant to the MvN FBM studied in our work.

These technical obstacles were circumvented in early work [49], where a nonlocal analog of the Wiener action (1) was derived, by a different method, in the particular case of the dynamics of a tagged bead in an infinitely long prethermalized Rouse polymer [49]. Under some natural assumptions, this non-Markovian system is equivalent to a fluctuating interface in one dimension and the latter is known to be describable by the MvN FBM with the Hurst exponent $H = \frac{1}{4}$ [21]. The action, calculated in Ref. [49], is given, up to a constant factor, by the expression

$$S_{\text{bead}}[x(t)] \sim \iint \frac{dt_1 dt_2}{|t_1 - t_2|^{1/2}} \dot{x}(t_1)\dot{x}(t_2) \tag{3}$$

(see also Ref. [50]). We should also mention a series of works [51–56] aimed at determining $S[x(t)]$ for the one-sided MvN FBM in the form of a perturbation expansion around the Wiener action (1). By construction, such an expansion, based on the small parameter $|H - \frac{1}{2}| \ll 1$, is quite limited in its validity.

In this work we find exact and explicit nonlocal analogs of the Wiener action (1) for the MvN FBM: for arbitrary $0 < H < 1$ and for both two-sided and one-sided versions of the FBM. We also extend the path integrals to include overdamped motion of the particle under external force. We achieve these goals by seeking, from the start, the action functional for the FBM in terms of its time derivative processes: the first-derivative process (that is, the FGN) for the subdiffusive FBM and the second-derivative process for the superdiffusive FBM.[3] Our formalism fully exploits the Gaussianity of the FBM and relies on the well-established theory of singular integral equations (see, e.g., Refs. [57,58]). The resulting path integrals are convenient to work with, as they involve only mildly singular kernels. Finally, we extend the formalism to include external forcing.

General expressions and main results. Quite generally, the action functional $S[X(t)]$ of a Gaussian process $X(t)$ on a time interval Ω can be represented as [13]

$$S[X(t)] = \frac{1}{2}\int_\Omega dt_1 \int_\Omega dt_2 \mathcal{K}(t_1, t_2) X(t_1) X(t_2). \tag{4}$$

The kernel $\mathcal{K}(t_1, t_2)$ (a symmetric function of t_1 and t_2) is the inverse of the covariance function $\kappa(t_1, t_2)$ of the process $X(t)$:

$$\int_\Omega dt_1 \kappa(t_1, t_3)\mathcal{K}(t_1, t_2) = \delta(t_2 - t_3). \tag{5}$$

Once $\mathcal{K}(t_1, t_2)$ is known, the action functional (4) is completely defined, giving the probability density $P[X(t)]$ of a given realization of the process $X(t)$. Now we present our main results for the action functionals of the MvN FBM $x(t)$. They have different forms for the subdiffusive and superdiffusive FBM and for the two- and one-sided processes.

We start with the subdiffusion. For the two-sided subdiffusive ($0 < H < \frac{1}{2}$) FBM $x(t)$, the action $S = S[x(t)]$ is

[3]Indeed, although $x(t)$ is a Gaussian process, the Wiener action (1) is expressed through $\dot{x}(t)$, rather than through $x(t)$. Furthermore, one can rewrite Eq. (1) as $S[x(t)] = \frac{1}{2}\iint dt_1 dt_2 \delta(t_1 - t_2)\dot{x}(t_1)\dot{x}(t_2)$, with a δ-function kernel. In the x representation the kernel $\partial^2_{t_1 t_2}\delta(t_1 - t_2)$ is more singular and less convenient to work with.

given by

$$S = \frac{\cot(\pi H)}{4\pi H}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dt_1 dt_2}{|t_1-t_2|^{2H}}\dot{x}(t_1)\dot{x}(t_2). \tag{6}$$

For the one-sided subdiffusive FBM we obtain

$$S = \frac{\cot(\pi H)}{4\pi H}\int_{0}^{\infty}\int_{0}^{\infty}\frac{dt_1 dt_2 I_z\left(\frac{1}{2}-H,H\right)}{|t_1-t_2|^{2H}}\dot{x}(t_1)\dot{x}(t_2), \tag{7}$$

where $I_z(a,b)$ is the regularized incomplete beta function

$$I_z(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^z x^{a-1}(1-x)^{b-1}dx, \tag{8}$$

$\Gamma(\cdots)$ is the Gamma function, and $z = 4t_1t_2(t_1+t_2)^{-2}$.

As one can see, the action functionals (6) and (7) are nonlocal in time and written in terms of the FGN $\dot{x}(t)$ rather than in terms of $x(t)$ itself. The expression (7) for the one-sided case is more complicated than that for the two-sided one, Eq. (6). In particular, the two-sided kernel in Eq. (6) is a difference kernel, which reflects the stationarity in time of the two-sided derivative process, the FGN. The one-sided kernel (7) is not a difference kernel in spite of the stationarity of the FGN. The nonstationarity, however, is temporary, as it is caused by a transient created by the initial condition $x(t=0)=0$. Indeed, in the limit of $t_1, t_2 \to \infty$ and $t_1 - t_2 = \text{const}$, z tends to 1, the one-sided kernel coincides with the two-sided one, and the stationarity is restored.

In the limiting case $H = \frac{1}{2}$, the kernels in Eqs. (6) and (7) become δ functions and yield the classical Wiener formula (1), as we show in Ref. [59]. For $H = \frac{1}{4}$ Eq. (6) has the same functional form as the two-sided expression (3), as to be expected in view of the prethermalization of the Rouse polymer [49]. We also remark that Eq. (6) was postulated in Ref. [50] as an effective Hamiltonian of topologically stabilized polymers in melts, permitting one to cover various conformations ranging from ideal Gaussian coils to crumpled globules. Our derivation validates their approach.

Now we present our results for the superdiffusive FBM, $\frac{1}{2} < H < 1$. In the two-sided case we obtain

$$S = \frac{\sigma(H)}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1 dt_2 |t_1-t_2|^{2-2H}\ddot{x}(t_1)\ddot{x}(t_2), \tag{9}$$

where

$$\sigma(H) = -\frac{\cot(\pi H)}{4\pi H(1-H)(2H-1)}, \tag{10}$$

a positive function. For the one-sided case

$$S = \frac{\sigma(H)}{2}\int_{0}^{\infty}\int_{0}^{\infty}dt_1 dt_2 |t_1-t_2|^{2-2H} \times I_{z'}\left(\tfrac{3}{2}-H, 2H-2\right)\ddot{x}(t_1)\ddot{x}(t_2), \tag{11}$$

where $z' = \min(t_1,t_2)/\max(t_1,t_2)$. Again, the expressions in Eqs. (9) and (11) are nonlocal in time, but now they are written in terms of $\ddot{x}(t)$, that is, in terms of the first derivative of the FGN. The two-sided kernel is a difference kernel. The one-sided kernel is not, but it approaches the difference form following an initial transient. Also, the classical Wiener form (1) is recovered in the limit $H \to \frac{1}{2}$ [59].

Expressions (6)–(11), along with Eqs. (25) and (27) below, represent the main results of this work. Here we present derivations of Eqs. (6) and (9) for the two-sided subdiffusive and superdiffusive FBM, respectively. The derivation of the (a bit more bulky) one-sided expressions in Eqs. (7) and (11) is relegated to the Supplemental Material [59].

Subdiffusion. Here we work directly with the FGN. Its covariance function $c(t_1,t_2)$ can be readily calculated

$$c(t_1,t_2) = \langle y(t_1)y(t_2)\rangle = \langle \dot{x}(t_1)\dot{x}(t_2)\rangle = \frac{\partial^2}{\partial t_1 \partial t_2}\langle x(t_1)x(t_2)\rangle = \frac{d}{d\tau}(H|\tau|^{2H-1}\text{sgn}\tau), \tag{12}$$

where $\tau = t_1 - t_2$ and we used Eq. (2). Equation (12) holds for both the two-sided and the one-sided process and for all $0 < H < 1$. Notably, the FGN is a stationary process. For $H = \frac{1}{2}$ Eq. (12) gives $c(\tau) = \frac{1}{2}(d/d\tau)\text{sgn}\tau = \delta(\tau)$, as to be expected for white noise.

Let us denote by $\mathcal{C}(\tau)$ the kernel inverse to $c(\tau)$. For the two-sided process, $\mathcal{C}(\tau)$ is defined by the equation $\int_{-\infty}^{\infty} d\tau\, c(\tau - t)\mathcal{C}(\tau) = \delta(t)$ or in the explicit form

$$\int_{-\infty}^{\infty} d\tau\, \mathcal{C}(\tau)\frac{d}{d\tau}[|\tau - t|^{2H-1}\text{sgn}(\tau - t)] = \frac{1}{H}\delta(t). \tag{13}$$

Integrating by parts and assuming that the boundary terms are zero (as can be verified *a posteriori*), we arrive at the integral equation

$$\int_{-\infty}^{\infty} d\tau \frac{\text{sgn}(\tau - t)}{|\tau - t|^{1-2H}}\mathcal{D}(\tau) = -\frac{1}{H}\delta(t) \tag{14}$$

for the unknown function $\mathcal{D}(\tau) = d\mathcal{C}(\tau)/d\tau$. The solution can be found in Ref. [57]:

$$\mathcal{D}(\tau) = \frac{d\mathcal{C}(\tau)}{d\tau} = \frac{\cot(\pi H)}{2\pi H}\frac{d}{d\tau}\frac{1}{|\tau|^{2H}}. \tag{15}$$

Getting rid of the τ derivative and using the fact that the kernel must vanish at $|\tau| \to \infty$, we obtain

$$\mathcal{C}(\tau) = \frac{\cot(\pi H)}{2\pi H}\frac{1}{|\tau|^{2H}}. \tag{16}$$

The ensuing Gaussian action functional (4), written in terms of $X(t) = \dot{x}(t)$, yields Eq. (6).

Superdiffusion. Here we work with the second-derivative process $z(t) = \ddot{x}(t)$. Its covariance is

$$q(t_1,t_2) = \frac{d^3}{d\tau^3}[H|\tau|^{2H-1}\text{sgn}(\tau)]. \tag{17}$$

For the two-sided process the inverse kernel $\mathcal{Q}(t_1,t_2)$ is defied by the equation $\int_{-\infty}^{\infty} d\tau\, q(\tau - t)\mathcal{Q}(\tau) = \delta(t)$ or in the explicit form

$$\int_{-\infty}^{\infty} d\tau\, \mathcal{Q}(\tau)\frac{d^3}{d\tau^3}[|\tau - t|^{2H-1}\text{sgn}(\tau - t)] = \frac{1}{H}\delta(t). \tag{18}$$

Integrating three times by parts and assuming that the boundary terms are zero (as verified *a posteriori*), we arrive at the equation

$$\int_{-\infty}^{\infty} d\tau \frac{\text{sgn}(\tau - t)}{|\tau - t|^{1-2H}}\mathcal{Z}(\tau) = -\frac{1}{H}\delta(t), \tag{19}$$

where $\mathcal{Z}(\tau) = d^3\mathcal{Q}(\tau)/d\tau^3$. This is exactly the same equation as Eq. (14) but now $1 - 2H < 0$. It is convenient to rewrite this equation as

$$-\int_{-\infty}^{t} d\tau (t-\tau)^{2H-1}\mathcal{Z}(\tau) + \int_{t}^{\infty} d\tau (\tau - t)^{2H-1}\mathcal{Z}(\tau) = -\frac{1}{H}\delta(t) \tag{20}$$

and differentiate both sides of Eq. (20) with respect to t. The resulting equation

$$\int_{-\infty}^{\infty} d\tau \frac{\mathcal{Z}(\tau)}{|\tau - t|^{2-2H}} = \frac{1}{H(2H-1)}\delta'(t) \tag{21}$$

is solvable [57] and we obtain

$$\mathcal{Z}(\tau) = -\frac{\cot(\pi H)}{2\pi H(2H-1)}\frac{d^2}{d\tau^2}\frac{\mathrm{sgn}\tau}{|\tau|^{2H-1}}. \tag{22}$$

Integrating this expression over τ three times and taking into account the fact that the kernel must vanish at $|\tau| \to \infty$, we obtain the desired inverse kernel

$$\mathcal{Q}(\tau) = \sigma(H)|\tau|^{2(1-H)}, \tag{23}$$

where $\sigma(H)$ is defined in Eq. (10). The resulting Gaussian action functional (4), written in terms of $X(t) = z(t) \equiv \ddot{x}(t)$, yields Eq. (9).

External force. An important extension of this formalism deals with situations where the FBM of a particle is accompanied by its overdamped motion under an external force $f(x)$. A natural approach to modeling this situation employs the non-Markovian Langevin equation [61]

$$\dot{x}(t) = f[x(t)] + y(t), \tag{24}$$

where the noise term $y(t)$ describes FGN. When the external force $f(x)$ is confining, the x distribution approaches a steady state. This steady state, however, is a non-Boltzmann one. Therefore, not surprisingly, it violates the fluctuation-dissipation theorem [61]. As the FGN $y(t)$ is a Gaussian process, a natural path-integral representation for Eq. (24) is provided by the action functional

$$S[x(t)] = \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{C}(t_1 - t_2)\{\dot{x}(t_1) - f[x(t_1)]\} \times\{\dot{x}(t_2) - f[x(t_2)]\}, \tag{25}$$

where $\mathcal{C}(\tau)$ is the inverse kernel for the FGN, given by Eq. (16). Here we assumed a two-sided subdiffusive FBM.

For a superdiffusive FBM a suitable non-Markovian Langevin equation can be obtained by a formal differentiation of Eq. (24) with respect to time, leading to

$$\ddot{x}(t) = f'[x(t)]\dot{x}(t) + z(t), \tag{26}$$

where $f'(x) \equiv df(x)/dx$ and the noise term $z(t)$ is the time derivative of the FGN. The corresponding path integral for the two-sided process is given by the action functional

$$S[x(t)] = \tfrac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{Q}(t_1 - t_2) \times\{\ddot{x}(t_1) - f'[x(t_1)]\dot{x}(t_1)\}\{\ddot{x}(t_2) - f'[x(t_2)]\dot{x}(t_2)\}, \tag{27}$$

where $\mathcal{Q}(\tau)$ is given by Eq. (23). Expressions similar to Eqs. (25) and (27) but with the one-side kernels as in Eqs. (7) and (11) hold for the one-sided sub- and superdiffusive FBM, respectively.

Summary. We generalized the classical Wiener path integral for BM and found exact path-integral representations for the two-sided and one-sided MvN FBM for the whole range $0 < H < 1$ of the Hurst exponent. We also extended the formalism to include external forcing. The exact and explicit path-integral representations make inroads into analytical and numerical studies of FBM (an important paradigm of scale-invariant stochastic processes with memory) in a multitude of applications in natural sciences, technology, and finance.

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Correction: The caption to Figure 1 contained typographical errors and has been fixed.