Stiffening or softening of elastic media: Anomalous elasticity near phase transitions

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We present the general theory of Ising transitions in isotropic elastic media with vanishing thermal expansion. By constructing a minimal model with appropriate spin-lattice couplings, we show that in two dimensions near a continuous transition the elasticity is anomalous in unusual ways: the system either significantly stiffens with a hitherto unknown unique, positional order logarithmically stronger than quasi-long-range order, or, as the inversion asymmetry of the order parameter in its coupling with strain increases, it destabilizes when system size L exceeds a finite threshold. In three dimensions, stronger inversion-asymmetric couplings induce instability to the long-range positional order for all L. Sufficiently strong order parameter-displacement couplings can also turn the phase transition first order at all dimensions, concomitant with finite jumps in the elastic modulii across the transition. Our theory establishes a one-to-one correspondence between the order of the phase transitions and anomalous elasticity near the transitions.

Elastic media, e.g., crystals, are paradigms of broken continuous symmetry phases with positional order (PO) in condensed matter systems. The interplay between the broken symmetry Goldstones modes (i.e., phonons) and the long-range correlated order parameter close to continuous phase transitions in broken symmetry phases can significantly affect the material properties of systems, e.g., binary alloy crystals, polymerized, or tethered networks in binary fluids [1,2], magnetic crystals, and even *in vivo* systems like biological cells [3]. Previous studies [4,5] suggested that the universal scaling properties of the second-order transitions are unaffected by the elasticity, although the elastic modulii could get depressed and display anomaly at the onset of supersolid transition [5] as a result of the elasticity-superfluid order coupling. In a seminal study, Ref. [6] discussed the generic instability of an elastic solid near a continuous Ising transition, except when $dT_c/dV = 0$, where T_c and V are the Ising transition temperature and system volume, respectively. In this case, Ref. [6] argued that the spin and the elastic degrees of freedom decouple, leaving each one unaffected by the other, with vanishing or zero thermal expansion (ZTE) in the thermodynamic limit.

In this letter we specifically study continuous Ising transitions in isotropic elastic media, e.g., a gel, with ZTE and investigate the measurable macroscopic properties near the phase transition. To this end, we present the general theory of Ising transitions in isotropic elastic media with ZTE, by constructing a suitable minimal model that includes generic spin-lattice interactions while maintaining the ZTE conditions. Unexpectedly in this model, we find anomalous elasticity, belonging to a hitherto unstudied universality class,

making two-dimensional (2D) elastic media either stiffer or unstable near the transitions. This shows that in an elastic solid coupled to Ising spins, the position and spin fluctuations are not independent of each other in the ZTE limit, or $dT_c/dV = 0$. Our theory revises the conclusions of Ref. [6] for an isotropic elastic media at $dT_c/dV = 0$.

For our work, we conceptualize a schematic, purpose-built minimal spin-lattice model with interactions near the ZTE limit. We consider a conceptual spring network model for gels, with each node carrying an Ising spin interacting with its nearest neighbor ferromagnetically, to illustrate the generic macroscopic properties near the phase transitions. We delineate the interactions from the physical considerations that both the spin-spin exchange interaction strengths and the spring constants should be locally affected by the interactions, which are chosen to maintain ZTE. For simplicity, we assume that the spin-spin interaction rises quadratically with the local strains for small strains. We further allow a local spin-lattice coupling that depends quadratically on the strain but linearly on the spin, and provides "local magnetic field-like" contributions. This manifestly breaks the Ising symmetry, being asymmetric or selective on the local spin state. Although such an Ising symmetry breaking term is unexpected in a ZTE system with magnetic or electronic origin, it, however, can in principle exist in potential soft-matter realizations of the system [7]. Quadratic dependence on strains of the interactions ensures vanishing thermal average of the strain, ensuring ZTE. Alternatively, we could imagine the spring constants of the gel to depend both linearly and bilinearly on the nearest-neighbor spins. Instead of studying the discrete lattice model directly, we take a Landau-Ginzburg (LG) continuum approach for this system, which is particularly suitable to extract large-scale universal properties [8]. In this approach, we describe the system by a continuous Ising order parameter ϕ and symmetric strain tensor $u_{ij} \equiv (\partial_i u_j + \partial_j u_i + \partial_i u_m \partial_j u_m)/2$ [8],

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where $\mathbf{u}(\mathbf{x})$ is the displacement of position $\mathbf{x}$ in the undistorted system [8,9]. We develop a generic and testable theory of the system as a thin, 2D sheet and in the bulk (dimension $d > 2$). Our most surprising result is that in two dimensions close to a second-order transition at temperature $T = T_c$, the elasticity is anomalous: The system either significantly stiffens or softens, controlled by the order parameter-strain couplings. When the order parameter-strain couplings are weakly asymmetric in ϕ, the elastic modulii diverge in the long wavelength limit, and the system considerably stiffens *vis-à-vis* away from T_c. Indeed, we find for a system of size L and microscopic cutoff (of the order of lattice constants) a_0,

$$\Delta_u^a \equiv \langle \mathbf{u}^a(\mathbf{x})^2 \rangle \propto [\ln(L/a_0)]^{2/3}, \ a = T, L, \quad (1)$$

near T_c, which although it grows indefinitely, it does so considerably slower than the usual $\ln(L/a_0)$ dependence on L in a 2D crystal without any transition. Here $\mathbf{u}^a(\mathbf{x})$ is the inverse Fourier transform of $\mathbf{u}^a(\mathbf{q})$, the component of $\mathbf{u}(\mathbf{q})$ transverse ($a = T$), and longitudinal ($a = L$) to the wave vector $\mathbf{q}$; $\mathbf{u}^T(\mathbf{x}) \cdot \mathbf{u}^L(\mathbf{x}) = 0$. Further, the correlation function

$$\langle [\mathbf{u}^a(\mathbf{x}) - \mathbf{u}^a(\mathbf{x}')]^2 \rangle \propto [\ln(r/a_0)]^{2/3}, \quad (2)$$

for large $r \equiv |\mathbf{x} - \mathbf{x}'|$ near T_c, which is significantly slower growing with r than the more conventional $\ln(r/a_0)$ scaling in an ordinary 2D crystal. Equations (1) and (2) illustrate a hitherto unknown novel positional order logarithmically stronger than the conventional quasi-long-range order (QLRO), which we name SQLRO, and constitute an entirely new, heretofore unstudied universality class for these systems. This is not the only state of the system near T_c. Increasing the degree of the inversion asymmetry of ϕ in its coupling with u_{ij} destabilizes the system with only short-range order (SRO) as soon as the system size L exceeds a finite threshold. The order parameter-strain couplings can turn the phase transition, which is otherwise second order, into a first-order one. In that case the elastic modulii are anomalous though in a different way: They do not diverge; instead, they and hence Δ_u^a (which now shows QLRO) display finite jumps across the transition.

For bulk three-dimensional (3D) samples, the elasticity is nonanomalous for weak inversion asymmetry; the system displays positional long-range order (LRO) near T_c, indistinguishable from its behavior without any transition. Stronger asymmetry can make the elastic modulii vanish for all L and hence destabilizes the system for any L, unlike two dimensions. Similar to two dimensions, ϕ-u_{ij} couplings can turn the phase transition a first order, across which the elastic modulii and hence Δ_u^a display finite jumps.

We now outline the derivation of these results; more details and additional results are available in the associated long paper (ALP) [10].

The LG free energy functional $\mathcal{F}$ of our minimal model, obtained phenomenologically by gradient expansions of the fields, must be rotationally and translationally invariant [8,9]. Hence, its dependence on $\mathbf{u}(\mathbf{x})$ must be through u_{ij}. The form of $\mathcal{F}$, after dropping irrelevant terms, valid near a second-order transition of ϕ, and for length scales much larger than

a_0 is

$$\mathcal{F} = \int d^d x \left[\frac{\tilde{r}}{2}\phi^2 + \frac{1}{2}(\nabla\phi)^2 + v\phi^4 + \mu(\nabla_i u_j^T)^2 + \frac{\tilde{\lambda}}{2}(\nabla_i u_j^L)^2 \right. $$
$$\left. + (g_1\phi^2 + \overline{g}_1\phi)(\nabla_i u_j^T)^2 + (g_2\phi^2 + \overline{g}_2\phi)(\nabla_i u_j^L)^2 \right], \quad (3)$$

Here $\tilde{r} \equiv T - T_c$, $v > 0$ for thermodynamic stability. The first three terms on the r.h.s. of (3), together identical to $\mathcal{F}(\mathbf{u} = 0)$, constitute the LG free energy for the Ising model near its critical point in a rigid uniform lattice in the absence of any external field [8]. The μ and $\tilde{\lambda}$ terms give the elastic free energy of a crystal with μ, $\tilde{\lambda}$ being the shear and bulk elastic modulii [8]. There is no bilinear term in $\mathbf{u}^L$, $\mathbf{u}^T$ due to these being mutually orthogonal. Each of the mixed anharmonic terms in (3) carries simple physical interpretations consistent with the interactions in the schematic spin-lattice microscopic model above. For instance, ϕ-dependent corrections to the local elastic modulii are given by $\mu(\phi) = \mu + g_1\phi^2 + \overline{g}_1\phi$, $\tilde{\lambda}(\phi) = \lambda + 2g_2\phi^2 + 2\overline{g}_2\phi$. Alternatively, we could define a local critical temperature $T_c^* = T_c - 2[g_1(u_{ij})^2 + g_{20}(u_{ii})^2]$ and a term formally analogous to "local magnetic field" $h_\phi = -[\overline{g}_1(u_{ij})^2 + \overline{g}_{20}(u_{ii})^2]$. Couplings g_1, $g_2 \geqslant 0$ due to thermodynamic stability reasons; $\overline{g}_1$, $\overline{g}_2$, the "asymmetry" parameters, have arbitrary signs and violate the Ising symmetry of ϕ. Such couplings which are odd in ϕ, are frequently used in mixed soft matter systems, where ϕ represents the local concentration difference in the two components [7], although these odd in ϕ-anharmonic terms vanish in systems with Ising symmetry at the microscopic level, e.g., in a magnetic crystal. In order to generalize the scope of our work, we include the Ising symmetry-breaking spin-lattice interactions and study the model. The form of $\mathcal{F}$ applies to systems with vanishing strain in the zero-stress states [11]: $\langle u_{ij} \rangle = 0$ identically in the absence of any external stress implying vanishing thermal expansion. Furthermore, (3) implies $dT_c/dV = 0$ [10]. The ϕ-u_{ij} anharmonic terms in (3) are not considered in Ref. [6] even when $dT_c/dV = 0$.

In the absence of the anharmonic effects, $\mathcal{F}$ implies

$$\langle \mathbf{u}^T(\mathbf{x})^2 \rangle = \frac{T}{2\mu} \ln\left(\frac{L}{a_0}\right), \ \langle \mathbf{u}^L(\mathbf{x})^2 \rangle = \frac{T}{\tilde{\lambda}} \ln\left(\frac{L}{a_0}\right) \quad (4)$$

at all temperatures T. Near T_c, ϕ fluctuations are scale invariant and thus can "connect" distant parts of the system, creating a correlated background for the elastic deformations. The combination of thermal fluctuations and anharmonic effects can substantially modify the scaling behaviors given in (4). To study this, we perform a one-loop momentum-space renormalization group (RG) analysis [8] of the model (3). Dimensional analysis gives that v has a critical dimension of 4, whereas 2 is that of g_1, g_2, $\overline{g}_1$, $\overline{g}_2$ [10]; see also [5]. Therefore, the phase transition of ϕ is unaffected by (assumed small) bare or unrenormalized g_1, g_2, $\overline{g}_1$, $\overline{g}_2$. The RG calculation is performed by integrating over the short wavelength Fourier modes of $\phi(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$, followed by rescaling of lengths and the long wavelength parts of ϕ and $\mathbf{u}$ [8,12]. From the structure of (3), $\mathbf{u}^T$ and $\mathbf{u}^L$ are mutually independent. We discuss the calculation for the renormalized correlation function of $\mathbf{u}^T$; the same for $\mathbf{u}^L$ is done analogously. The
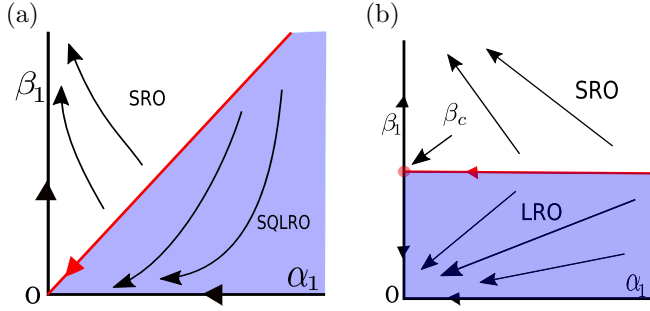
FIG. 1. RG flow diagram in the α_1-β_1 plane. (a) In two dimensions, the inclined red line is the separatrix given by Eq. (10). (b) $d > 2$: The horizontal red line is the separatrix given by $\beta = \epsilon/2$. The small circle on the β_1 axis is the unstable fixed point $(0, \beta_c)$. Other arrows indicate the flow directions.

perturbative RG procedure outlined above gives the following differential recursion relations for μ, g_1, $\bar{g}_1$:

$$\frac{d\mu}{dl} = \mu\alpha_1 - \mu\frac{\beta_1}{2}, \tag{5}$$

$$\frac{dg_1}{dl} = -\epsilon g_1 - 2\frac{T_c g_1^2}{\mu}\frac{S_d \Lambda^\epsilon}{(2\pi)^d} - \frac{T_c \bar{g}_1^4}{8\mu^3}\frac{S_d \Lambda^\epsilon}{(2\pi)^d}, \tag{6}$$

$$\frac{d\bar{g}_1}{dl} = -\frac{\epsilon}{2}\bar{g}_1 + \frac{T_c \bar{g}_1^3}{2\mu^2}\frac{S_d \Lambda^\epsilon}{(2\pi)^d} - \frac{2T_c g_1 \bar{g}_1}{\mu}\frac{S_d \Lambda^\epsilon}{(2\pi)^d}. \tag{7}$$

Here S_d is the surface area of a d-dimensional unit sphere, $\epsilon \equiv d - 2$, $\exp(l)$ is the length rescaling factor, and $\Lambda = 2\pi/a_0$ is an upper wave-vector cutoff. Flow Eqs. (5)–(9) in turn gives the RG flow equations for two effective dimensionless couplings $\alpha_1 \equiv \frac{T_c g_1 S_d}{(2\pi)^d \mu}\Lambda^\epsilon$, $\beta_1 \equiv \frac{T_c \bar{g}_1^2 S_d}{(2\pi)^d \mu^2}\Lambda^\epsilon$:

$$\frac{d\alpha_1}{dl} = -\epsilon\alpha_1 - 3\alpha_1^2 - \frac{\beta_1^2}{8} + \frac{\alpha_1 \beta_1}{2}, \tag{8}$$

$$\frac{d\beta_1}{dl} = -\epsilon\beta_1 + 2\beta_1^2 + 6\alpha_1 \beta_1. \tag{9}$$

In two dimensions, $\epsilon = 0$; the only fixed point of (8) and (9) is $\alpha_1 = 0$, $\beta_1 = 0$. Its stability property is intriguing: It is stable (i.e., attractive) along the α_1 axis, but unstable (i.e., repulsive) along the β_1 axis. Thus, starting from any initial value $(\alpha_1(l = 0), \beta_1(l = 0)) = (\alpha_{10}, 0)$, the system flows to the origin $(0, 0)$ implying stability, whereas starting from any initial value $(0, \beta_{10})$ the system flows away from the origin, indicating instability. The separatrix, an invariant manifold under RG in the (α_1, β_1) plane, that separates the stable phase from instability can be obtained by using (8) and (9) along with the condition $d(\beta_1/\alpha_1)/dl = 0$ [13]. This, for small α_1, β_1, is a straight line

$$\beta_1 = \Gamma_{1c}\alpha_1, \quad \Gamma_{1c} \equiv \tfrac{1}{2}(-12 + \sqrt{240}); \tag{10}$$

see Fig. 1(a). Separatrix (10) is repulsive: by expanding around (10), (8), and (9) give $\beta_1(l)/\alpha_1(l) \sim 1/l \to 0$, when α_{10}, β_{10} lie in the stable region, i.e., below the line (10) in the α_1-β_1 plane in the thermodynamic limit. Therefore, systems whose starting parameters lie below the separatrix not only flow to the origin, they also move away from the separatrix with increasing length scale; see Fig. 1(a). Thus, in the long wavelength limit, the system is effectively identi-

cal with those having no inversion asymmetry; the inversion symmetry of ϕ, although absent at small scales, appears as an emergent symmetry in the thermodynamic limit. On the stable side of the separatrix, using (5) we find the renormalized, scale-dependent shear modulus $\mu(l) \sim l^{1/3}$, which implies wave-vector-dependent $\mu(q) \approx \mu_R[\ln(\Lambda/q)]^{1/3}$, in the long wavelength limit, where μ_R is the amplitude of the renormalized shear modulus. This means the elasticity is anomalous, analogous to the well-known anomalous elasticity found in 3D smectics [14]. This in turn gives, as shown in detail in the ALP [10],

$$\langle|u^T(\mathbf{q})|^2\rangle \approx \frac{T_c}{2\mu_R[\ln(\Lambda/q)]^{1/3}q^2}, \tag{11}$$

for small q close to T_c. Inverse Fourier transform of (11) gives (1) and (2). Thus, both $\langle|u^T(\mathbf{q})|^2\rangle$ and $\langle[u^T(\mathbf{x}) - u^T(\mathbf{x}')]^2\rangle$ are significantly suppressed in the long wavelength limit vis-à-vis their values away from T_c, a hallmark of SQLRO. Results (11) together with (2) illustrate the novel state of the elastic media near T_c and define a universality class with unique features that has not been studied before. Thus, in a ZTE system with microscopic Ising symmetry, i.e., of magnetic or electronic origin, $\beta_1 = 0$ identically, and the results (11) and (2) are generic and observable without any additional fine tuning of the control parameters. For ZTE systems of soft matter origin with nonzero microscopic Ising symmetry-breaking anharmonic spin-lattice interactions, β_1 is nonzero microscopically. Hence, to observe (11) and (2) additional tuning making the "RG initial values" $\alpha_1(l = 0)$, $\beta_1(l = 0)$ lying below the separatrix is required.

We now turn to the region above the separatrix in Fig. 1(a), which flows away from the origin. This region is accessible in potential soft matter realizations of ZTE systems by additional tuning to make the "RG initial values" $\alpha_1(l = 0)$, $\beta_1(l = 0)$ lie above the separatrix. In this region, $\beta_1(l)$ diverges, whereas $\alpha_1(l)$ vanishes as l exceeds a nonuniversal value of l, controlled by the microscopic model parameters. This in turn means that the system flows towards negative μ. While it is not possible to follow these flows all the way to $\mu = 0$ (since β_1 diverge there breaking down our perturbation theory), this signals breakdown of elasticity and loss of PO in large enough systems. This region of the parameter space therefore corresponds to a phase with only SRO.

Whether or not the system destabilizes depends surprisingly upon the small-scale or unrenormalized values of the model parameters. The above results can be used to show that for $\Gamma_1 < (>)\Gamma_{1c} \equiv \bar{g}_1^2/(\mu g_1)$, a combination of the unrenormalized parameters, SQLRO ensues (gets unstable with SRO). Phase diagrams in (1) $\bar{g}_1$-μ for a fixed g_1 and (2) $\bar{g}_1$-g_1 for a fixed μ in Fig. 2(a) and 2(b), respectively, follow directly from this threshold relation. Both phase boundaries are parabolas, as can be seen from the instability threshold.

On the unstable side of the separatrix, $\beta_1(l) \ll \alpha_1(l)$ for large l; (5) reduces to $\frac{d\mu}{dl} \approx -\mu\frac{\beta_1}{2}$. Solving, we obtain

$$\mu(l) \approx \mu(l = 0)\left(\frac{1}{2\beta_{10}}\right)^{1/4}\left[l - \frac{1}{2\beta_{10}}\right]^{1/4} \tag{12}$$
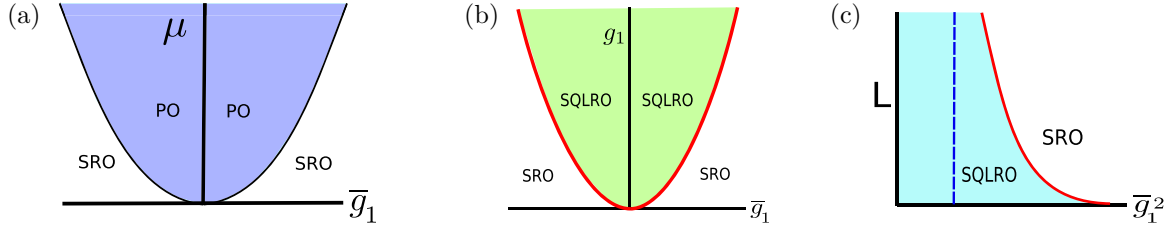
FIG. 2. Schematic phase diagram in the (a) $\bar{g}_1$-μ plane. The blue shaded region with $\mu > 0$ corresponds to positional SQLRO at 2D and LRO at 3D near T_c. The white region outside has SRO. (b) $\bar{g}_1$-g_1 plane. The middle light green region indicates SQLRO in two dimensions. The region outside has SRO. (c) $\bar{g}_1^2$-L plane in two dimensions near T_c. The red curved line demarcates regions with SQLRO and SRO. The region left to the vertical broken blue line corresponds to SQLRO for any large L. The region between the vertical blue line and the curved red line corresponds to systems having a finite $L < \xi$, a threshold for PO; for $L > \xi$ only SRO is possible.

for large l. Of course, we cannot use (12) all the way to $\mu(l) \approx 0$ as $\beta_1(l)$ diverges and the RG scheme breaks down. Instead, as shown in the ALP [10], we can conveniently use the bare perturbation theory to define a persistence or correlation length ξ that is finite on the unstable side of the separatrix, given by $\mu[l = \ln(\xi/a_0)] = 0$. This gives, as shown in ALP [10],

$$\xi = a_0 \exp\left\{2\pi\mu/\left[T_c\left(\bar{g}_1^2/2\mu - g_1\right)\right]\right\}, \quad (13)$$

as one approaches the instability threshold from the unstable side [15]. Thus, as $\bar{g}_1^2 \to 2\mu g_1$, ξ diverges as an essential singularity, surprisingly reminiscent of the behavior of the correlation length near the Kosterlitz-Thouless transition of the 2D XY model [8,16]. Physically, it is clear that as one moves from the stable region to the unstable region crossing the separatrix in the α_1-β_1 plane, the system undergoes a structure phase transition from a phase with positional SQLRO to a phase with only SRO. Unsurprisingly, ξ is infinite on the stable side of the separatrix. Equating ξ with L, we get from (13) the maximum linear size that can show PO; see Fig. 2(c) for a plot of L versus $\bar{g}_1^2$.

We now consider bulk systems with $d > 2$, $d = 3$; i.e., $\epsilon = 1$ is the physically relevant dimension. We focus on the fluctuations of $\mathbf{u}^T$, and an equivalent analysis for $\mathbf{u}^L$ can be done exactly in the same manner; see ALP [10]. Flow Eqs. (8) and (9) give two fixed points for $\epsilon > 0$: (1) ($\alpha_1 = 0$, $\beta_1 = 0$), i.e., the origin, which is linearly stable, and (2) ($\alpha_1 = 0$, $\beta_1 = \epsilon/2$), unstable along the β_1 direction, but stable along the α_1 direction; see Fig. 1(b). In fact, separatrix $\beta_1 = \epsilon/2$, a straight line parallel to the α_1 axis, determines that systems with microscopic model parameters in the region below the system flow towards the origin rapidly. As shown in the ALP, at this stable fixed point, μ does not diverge in the thermodynamic limit unlike in two dimensions but is a constant. Thus, elasticity is nonanomalous, with $\langle\mathbf{u}^T(\mathbf{x})^2\rangle$ being bounded, i.e., a constant independent of L, which is a telltale signature of positional LRO, and is indistinguishable from its behavior away from T_c. On the other hand, systems whose starting parameters lie above the separatrix flow towards the β_1 axis but away from the origin, making $\beta_1(l)$ diverge and $\alpha_1(l)$ vanish again rapidly. Correspondingly, $\mu(l)$ flows to zero for all L. When β_1 increases beyond $\epsilon/2$, corresponding to $\bar{g}_1$ rising above a threshold, the system undergoes a structural phase transition in which it loses LRO and displays just SRO, akin to liquids. As shown in ALP [10], bare perturbation theory reveals that as soon as $\beta_{10} > 2 + 2\alpha_{10}$, fluctuation-corrected μ becomes

negative, giving complete loss of LRO, independent of L, unlike in two dimensions. Indeed, ξ for which $\mu(\xi) = 0$ abruptly changes from very large to zero (or very small), as one crosses the separatrix from the stable to the unstable sides. See Fig. 2 (left) for a schematic phase diagrams in the $\bar{g}_1$-μ plane for $d > 2$.

For compressible systems with bulk modulus $\tilde{\lambda}$ and longitudinal displacement u^L, exactly analogous results and phase diagrams for both two and three dimensions exist; see the ALP [10].

So far we have tacitly assumed that the second-order phase transition of ϕ is unaffected by the displacement-order parameter couplings. How correct is that? We first consider the case with microscopic Ising symmetry, i.e., the couplings $\bar{g}_1$, $\bar{g}_2$ vanish. In this case, in order to have a second-order transition, it is required that under mode elimination, v_e, the fluctuation-corrected v at any intermediate scale never turns negative. This may not hold true for sufficiently strong order parameter-strain couplings. In the anticipation that v_e can actually turn negative, we extend $\mathcal{F}$ by adding a $v_6\phi^6$ term in it with $v_6 > 0$ for thermodynamic stability reasons. We consider the inhomogeneous fluctuation corrections to v that originate from $g_1, \bar{g}_1, g_2, \bar{g}_2$, which themselves do not depend upon v explicitly. These contributions are finite but negative: we get [10]

$$\beta_c v_e \equiv \beta_c v - 2dT_c^2\left(\frac{\beta_c^2 g_1^2}{4\mu^2} + \frac{\beta_c^2 g_2^2}{\tilde{\lambda}^2}\right)\frac{\Lambda^d}{(2\pi)^d}, \quad (14)$$

valid for all $d \geq 2$. Now, for $v_e > 0$, the $v_6\phi^6$ term is unnecessary. The phase transition of ϕ is unaffected by the order parameter-strain couplings and remains a continuous transition belonging to the Ising universality class. If, however, $v_e < 0$, then a $v_6\phi^6$ term must be taken into account for reasons of thermodynamic stability. In that case, ϕ now undergoes a first-order transition with the order parameter $m \equiv \langle\phi\rangle$ jumping of magnitude $[|v_e|/(2v_6)]^{1/2}$. We thus conclude that in ZTE systems with microscopic Ising symmetry, sufficiently strong spin-lattice couplings necessarily turn the second-order transition into a first-order one.

For nonzero inversion symmetry breaking or the selectivity parameters, additional tuning is necessary to access a second-order transition. This is because fluctuations generate a $g\phi^3$ term in $\mathcal{F}$, where g is a coupling constant of arbitrary sign. This allows us to generalize the Landau-Ginzburg free energy $\mathcal{F}$ to $\tilde{\mathcal{F}} = \mathcal{F} + \int d^d x g_e \phi^3$, giving a generic liquid-gas-like first-order transition [8,10] with an order parameter

jump $m = -g_e/(2v)$ at a transition temperature $T^* = T_c + 9g_e^2/(16v)$ [8]. To proceed further, we integrate out the strains in $\tilde{\mathcal{F}}$ perturbatively in the coupling constants, generating a "dressed" free energy in terms of the dressed parameters, which depends only on ϕ. By shifting $\phi \rightarrow \phi + \phi_0$ and choosing ϕ_0 appropriately the effective coefficient of ϕ^3 can be made to vanish in a manner analogous to the liquid-gas transition; see the ALP [10] for more details. In terms of the shifted ϕ, $\tilde{\mathcal{F}}$ has the same form as $\mathcal{F}$, albeit with shifted model parameters. The requirement of ZTE remains satisfied due to the absence of terms in the free energy functional that are odd in the displacement **u**. Note also that in this case βv_e in (14) is to be supplemented by an additional finite negative contribution $2dT_c^4 \left(\frac{\beta_c^4 \bar{g}_1^4}{32\mu^4} + \frac{\beta_c^4 \bar{g}_2^4}{\bar{\lambda}^4} \right) \frac{\Lambda^d}{(2\pi)^d}$ [10]. Again, if $v_e > 0$ a second-order Ising transition follows, and a $v_6 \phi^6$ term is not necessary. In the vicinity of this transition, our above results on the anomalous elasticity then ensue. On the other hand, if $v_e < 0$, a first-order transition ensues with an order parameter jump $m = [|v_e|/(2v_6)]^{1/2}$. Thus, even in the presence of Ising-symmetry breaking spin-lattice coupling terms, although the transition is generically first order, a second-order Ising transition can be accessed by tuning the model parameters reminiscent of the second-order transition in liquid-gas systems. This second-order transition can get converted into a different first-order one for sufficiently strong spin-lattice interactions. Across such first-order transitions, the elastic modulii are finite, but still anomalous in the sense given below.

Near a first-order transition, fluctuations of ϕ do not have long-range correlations, and as a result, all the corrections to μ at 2D are finite (and small). In a mean-field description that suffices near a first-order transition we get, as shown in ALP,

$$\mu_{T<T^*} = \mu_{T>T^*} + \left(g_1 - \bar{g}_1^2/\mu \right) m^2 \neq \mu_{T>T^*}, \quad (15)$$

valid at all dimensions, where T^* is the first-order transition temperature. Thus, depending upon the relative magnitudes of

g_1 and $\bar{g}_1$, $\mu(T < T^*)$ can be larger or small than $\mu(T > T^*)$, with a finite jump in its value at T^*, related to the jump in the order parameter. Therefore at 2D, $\langle \mathbf{u}^T(\mathbf{x})^2 \rangle$ should scale as $\ln(L/a_0)$ corresponding to QLRO with an amplitude that shows the jump. In contrast, across a second-order transition, the elastic modulii do not display any jump; instead they either continuously increase or decrease (and approach zero for large enough systems) as T_c is approached from either side. In three dimensions, $\langle \mathbf{u}^T(\mathbf{x})^2 \rangle$ shows conventional LRO on both sides of T^*, differing only by a finite jump. At any dimension, for very large $\bar{g}_1$, $\mu(T < T^*)$ can even turn negative, signaling instability, and SRO. A similar relation exists for $\tilde{\lambda}$. We thus establish a one-to-one correspondence between the order of phase transitions and anomalous elasticity around the transition temperature. Detailed derivation of all these results in addition to a plethora of others are available in the ALP [10].

We have thus developed the theory of Ising transitions in isotropic elastic media with vanishing thermal expansion. This theory predicts anomalous elasticity with the system either stiffening or softening near the transitions, being controlled by the strain-order parameter couplings. These are in contrast to Ref. [6] due to the absence of the spin-lattice anharmonic interaction terms there even at $dT_c/dV = 0$. Our theory should be a guideline to theoretically study of ZTE materials of diverse origin, including electronic magnetism and soft matter systems [17], which are in great demand in high-precision applications [18,19]. Experiments on purpose-built synthetic ZTE systems [18–21] in the future should help to verify our results.

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