

## Joint measurability in nonequilibrium quantum thermodynamics

Konstantin Beyer<sup>1</sup>, Roope Uola<sup>2</sup>, Kimmo Luoma<sup>1,3</sup> and Walter T. Strunz<sup>1</sup>

<sup>1</sup>*Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany*

<sup>2</sup>*Department of Applied Physics, University of Geneva, 1211 Geneva, Switzerland*

<sup>3</sup>*Turku Center for Quantum Physics, Department of Physics and Astronomy, University of Turku, FI-20014 Turun Yliopisto, Finland*



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In this Letter we investigate the concept of quantum work and its measurability from the viewpoint of quantum measurement theory. Very often, quantum work and fluctuation theorems are discussed in the framework of projective two-point measurement (TPM) schemes. According to a well-known no-go theorem, there is no work observable which satisfies both (i) an average work condition and (ii) the TPM statistics for diagonal input states. Such projective measurements represent a restrictive class among all possible measurements. It is desirable, both from a theoretical and experimental point of view, to extend the scheme to the general case including suitably designed unsharp measurements. This shifts the focus to the question of what information about work and its fluctuations one is able to extract from such generalized measurements. We show that the no-go theorem no longer holds if the observables in a TPM scheme are jointly measurable for any intermediate unitary evolution. We explicitly construct a model with unsharp energy measurements and derive bounds for the visibility that ensure joint measurability. In such an unsharp scenario a single work measurement apparatus can be constructed that allows us to determine the correct average work and to obtain free energy differences with the help of a Jarzynski equality.

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**Introduction.** A suitable definition of work in quantum systems has proven to be elusive, as witnessed by a vast amount of different proposals in the last two decades [1–18]. Among them the two-point measurement (TPM) scheme [2,19–22] has become one of the standard approaches. Its popularity is certainly related to its ability to directly recover fluctuation theorems known from classical nonequilibrium thermodynamics, such as the ones by Jarzynski and Crooks [7,23–30]. Despite the formal agreement of the quantum fluctuation theorems with their classical counterparts, several conceptual differences remain. Most prominently, the work obtained by two projective quantum energy measurements is generally not consistent with the average energy change in a closed system [2,31,32]. Rather recently, it was rigorously shown that this discrepancy is fundamental and cannot be solved by a more clever definition of fluctuating work: The no-go theorem of Ref. [33] (see also Ref. [34]) states that there is no quantum work observable which satisfies the work fluctuation statistics of the projective TPM scheme and also reproduces the average change of energy.

A projective TPM scenario consists of the following consecutive operations. First, a closed quantum system is initialized in a state  $\rho$ . Then an energy measurement with respect to an initial Hamiltonian  $\hat{H}_A = \sum_a E_a \Pi_a$  is performed and an outcome  $a$  is recorded. A unitary process  $U_{A \rightarrow B}$  generated by a time-dependent system Hamiltonian is applied to the output state and a second energy measurement with respect to the final Hamiltonian  $\hat{H}_B = \sum_b E'_b \Pi'_b$  is implemented, yielding outcome  $b$ . The energy change in the closed system is interpreted as work  $w(a, b) = E'_b - E_a$ . If we are only interested

in the work values  $w$  but not in the initial and final energies  $E_a$  and  $E'_b$  themselves, we may ask if the TPM protocol can be replaced by a single quantum measurement  $\mathcal{W}$  which directly yields  $w$ .

Work should be defined for any process, but its value is process dependent. Thus, we look for an observable  $\mathcal{W}$  that exists for any process  $U_{A \rightarrow B} = U$  but may depend on it. The no-go theorem is a result of the clash of physically well-motivated constraints set for such a work observable  $\mathcal{W}$ . (I) Averaging over all outcomes  $w$  should reproduce the average energy change:  $\text{Tr}[\mathcal{W}\rho] = \text{Tr}[\hat{H}_B U \rho U^\dagger] - \text{Tr}[\hat{H}_A \rho]$ . (II) For initial states  $\rho_\Delta$ , diagonal in the basis of the first energy measurement  $\{\Pi_a\}$ , the statistics of  $\mathcal{W}$  should agree with the statistics of the sequential measurement of  $\hat{H}_A$  and  $\hat{H}_B$ :  $p_{\mathcal{W}}(w = E'_b - E_a) = p_{\text{TPM}}(a, b)$ . Here one presumes the nondegeneracy of  $w$ , which is necessary to prove the no-go theorem [33]. Later we will replace this assumption by a more general one.

The above constraints are motivated by classical thermodynamics. The first constraint (I) states that the average work is equal to the average change of internal energy in a closed system. If the initial state is a Gibbs state, the second constraint (II) ensures that the work satisfies Jarzynski's fluctuation theorem well known from classical stochastic thermodynamics.

Projective energy measurements are an idealization. In the past it has often proven insightful to depart from this severe limitation and to consider generalized measurements described by a *positive operator valued measure* (POVM), which can be seen as a projective measurement on an enlarged Hilbert space. Most prominently and relevant for this

paper, the compatibility of two POVMs is no longer determined by their commutativity as we know it for projective measurements [35]. In certain scenarios POVMs perform better than projective measurements. For instance, the optimal measurement for quantum state tomography is a POVM [36] and POVMs can lead to stronger violations of some Bell inequalities [37]. POVMs are also of experimental relevance. It has been shown that an ideal projective measurement would require infinite resources [38]. Real experiments are often indirect and the actual measurement is indeed better described by a POVM with unsharp elements.

Using POVMs, the question arises of how to estimate the desired quantity from the measurement outcomes. In the projective case it is clear that a given detector click can be associated with an eigenvalue of the Hamiltonian. Once we deviate from the projective scenario we have to carefully analyze what we can actually learn from the measurement outcomes. These last are the only quantities we can access in an experiment.

TPM schemes beyond the projective case have been proposed, for example, in Refs. [39–42]. There, the authors studied different generalized energy measurements, based, for instance, on Gaussian pointers, and analyzed their thermodynamic implications, in particular with respect to fluctuation theorems. In the present work we add an aspect to the study of generalized TPM schemes: Are given generalized energy measurements able to provide the correct average work and satisfy a fluctuation theorem? What are the fundamental limitations for a generalized TPM scheme to allow for a (unsharp) work observable  $\mathcal{W}$ ? These questions are not only of theoretical interest, but also experimentally relevant since nonprojective measurements are often easier to implement, especially if the measurement has to be non-destructive, as it is the case for the first measurement in a TPM scheme. Instead of projective energy measurements given by the Hamiltonians  $\hat{H}_{A,B}$ , we consider two general POVMs  $\mathbf{A}$  and  $\mathbf{B}$  at the beginning and at the end of the protocol, respectively.  $\mathbf{A}$  and  $\mathbf{B}$  can be thought of as unsharp versions of the projective energy measurements. However, we keep them fully general for the moment and only later investigate their properties for a widely used class of unsharp POVMs.

In a projective measurement each projector can unambiguously be associated with a corresponding eigenvalue  $\Pi_a \rightarrow E_a$ . Crucially, such a correspondence does not exist for POVMs, as the latter are not given by the spectral decomposition of a Hermitian operator. The energy associated with a certain POVM element  $\mathbf{A}_a$  can in general only be an estimate based on additional assumptions. To reflect and emphasize this ambiguity, we introduce energy assignment functions that yield an energy estimate for the specific outcomes of the POVM  $\mathbf{A}_a \rightarrow f(a)$  and  $\mathbf{B}_b \rightarrow g(b)$ . Only in the limiting case of projective energy measurements their ambiguity vanishes and  $f(a) = E_a$  and  $g(b) = E_b$ .

Refining the constraints (I) and (II) for the case of general observables, we find that the existence of  $\mathcal{W}$  is closely related to the concept of joint measurability [35,43–45]. We show that the no-go theorem can be extended to all energy POVMs  $\mathbf{A}$  and  $\mathbf{B}$  that are incompatible for at least some intermediate evolution  $U$ . In turn, if  $\mathbf{A}$  and  $\mathbf{B}$  are jointly measurable for any process  $U$ , the no-go statement generally does not apply.

We illustrate that with a specific model for unsharp energy measurements.

*Definitions.* We restrict ourselves to measurements with discrete, i.e., countably many outcomes. A discrete POVM is described by a set of *effects*  $\mathbf{A} = \{\mathbf{A}_a\}$ , with positive elements  $\mathbf{A}_a \geq 0$  and  $\sum_a \mathbf{A}_a = \mathbb{1}$ . The probability to obtain outcome  $a$  when measuring a system in state  $\rho$  is given by Born's rule:  $p(a) = \text{Tr}[\mathbf{A}_a \rho]$ .

As motivated above, the energy estimate for a specific outcome  $a$  is given by a function  $f(a)$ . We can then define a generalized (unsharp) energy observable  $\mathcal{A}$  as a tuple  $(\mathbf{A}, f)$ . Two observables  $\mathcal{A}$  and  $\mathcal{A}'$  that only differ in the function  $f$  can be measured with the same apparatus as they share a common set of effects  $\mathbf{A}$ . The average operator of  $\mathcal{A}$  is defined as  $\hat{A}_f = \sum_a f(a) \mathbf{A}_a$ .

Two POVMs  $\mathbf{A}$  and  $\mathbf{B}$  with  $m$  and  $n$  outcomes, respectively, are said to be *jointly measurable* if they are the marginals of a third POVM  $\mathbf{W}$  with  $m \times n$  outcomes, i.e.,  $\mathbf{A}_a = \sum_b \mathbf{W}_{ab}$ ,  $\mathbf{B}_b = \sum_a \mathbf{W}_{ab}$  [46]. Two measurements that are not jointly measurable are *incompatible*. For projective measurements, joint measurability reduces to commutativity [43,44,47].

The effects of  $\mathbf{A}$  only determine the probabilities of the measurement outcomes. In a TPM scheme we also have to specify the postmeasurement state of measurement  $\mathbf{A}$ . This can be done with the concept of *quantum instruments* [43,48]. An  $\mathbf{A}$ -compatible instrument  $\mathcal{I}^{\mathbf{A}}$  is a set of completely positive maps  $\{\mathcal{I}_a^{\mathbf{A}}\}$  that satisfy  $\text{Tr}[\mathcal{I}_a^{\mathbf{A}}(\rho)] = \text{Tr}[\mathbf{A}_a \rho]$ ,  $\forall \rho$ . Any  $\mathbf{A}$ -compatible instrument can be written as  $\mathcal{I}_a^{\mathbf{A}}(\rho) = \mathcal{E}_a[\mathbf{A}_a^{1/2} \rho \mathbf{A}_a^{1/2}]$ , where  $\mathcal{E}_a$  is a completely positive trace-preserving (CPT) map which may depend on the outcome of the measurement. If  $\mathcal{E}_a = \mathbb{1}$ ,  $\forall a$ , the instrument is called the *Lüders instrument* of  $\mathbf{A}$  [43,46]. Note that in the projective TPM scheme the Lüders instrument is usually tacitly assumed.

*General TPM scheme.* A *general TPM* (GTPM) scheme is depicted in Fig. 1. An observable  $\mathcal{A} = (\mathbf{A}, f)$ , whose effects  $\mathbf{A}$  are implemented by an instrument  $\mathcal{I}^{\mathbf{A}}$ , is measured on a system initialized in state  $\rho$ . After a unitary evolution  $U$ , a second observable  $\mathcal{B} = (\mathbf{B}, g)$  is measured. The probability to obtain an outcome sequence  $(a, b)$  is given by

$$p_{\text{GTPM}}(a, b) = \text{Tr}[\mathbf{B}_b U \mathcal{I}_a^{\mathbf{A}}(\rho) U^\dagger] = \text{Tr}[\mathbf{B}_b^U \mathcal{I}_a^{\mathbf{A}}(\rho)], \quad (1)$$

where we absorbed the intermediate evolution to the second measurement with the Heisenberg picture operators  $\mathbf{B}_b^U = U^\dagger \mathbf{B}_b U$ .

*Work observable  $\mathcal{W}$ .* Can the GTPM scheme be replaced by a generalized work measurement? We consider an observable of the form  $\mathcal{W}^U = [\mathbf{W}^U, w(a, b) = g(b) - f(a)]$ , where the superscript  $U$  indicates the dependence on the intermediate evolution. The following two conditions shall apply.

(1) *Average condition:* The average operator of  $\mathcal{W}^U$  agrees with the difference of the average operators of  $\mathcal{B}^U$  and  $\mathcal{A}$  for all choices of  $f$  and  $g$ :

$$\hat{W}_w^U = \hat{B}_g^U - \hat{A}_f, \quad \forall f, g.$$

(2) *Fluctuation condition:* For states  $\rho_\Delta$  diagonal in the initial energy eigenbasis the statistics given by  $\mathbf{W}^U$  agrees with the sequentially measured GTPM statistics:

$$\text{Tr}[\mathbf{W}_{ab}^U \rho_\Delta] = \text{Tr}[\mathbf{B}_b^U \mathcal{I}_a^{\mathbf{A}}(\rho_\Delta)].$$

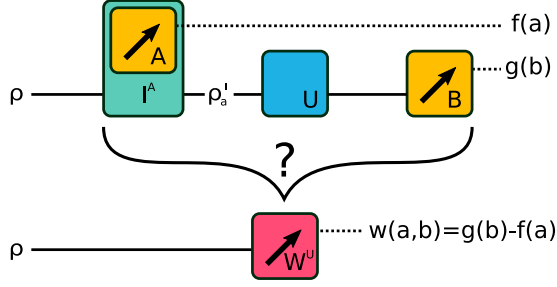


FIG. 1. In a general TPM scheme a system is prepared in a state  $\rho$  and measured by an initial measurement  $\mathbf{A}$ . This first measurement is implemented by an instrument  $\mathcal{I}^{\mathbf{A}}$  that outputs a postmeasurement state  $\rho'_a$  depending on the outcome  $a$ . After an arbitrary unitary evolution  $U$  a final measurement  $\mathbf{B}$  is implemented and outcome  $b$  is obtained. For each outcome  $a$  and  $b$ , an energy estimate  $f(a)$  and  $g(b)$  is assigned and a work value  $w(a, b) = g(b) - f(a)$  can be determined. In the case of projective measurements, the estimates  $f(a)$  and  $g(b)$  can uniquely be identified with the energy eigenvalues  $E_a$  and  $E'_b$ . For general POVMs such a correspondence is missing. We investigate whether such a scheme can be replaced by a single observable  $\mathcal{W}^U$  that (i) reproduces the correct average work for any input state as well as (ii) the full statistics for input states diagonal in the initial energy eigenbasis of the system (see main text for precise formulations). For the projective case it is known that such an observable cannot exist. In contrast, if  $\mathbf{A}$  and  $\mathbf{B}$  are jointly measurable POVMs for any  $U$ , the answer can be positive.

We note that in the first condition, the subscripts refer to energy assignment functions, whereas the subscripts in the second condition refer to measurement outcomes. These conditions can be seen as generalizations of the conditions (I) and (II) in the projective case.

**Lemma.** An observable  $\mathcal{W}^U$  satisfying the average condition (i) can be found for any choice of  $f$  and  $g$  and any intermediate evolution  $U$  if and only if  $\mathbf{A}$  and  $\mathbf{B}^U$  are jointly measurable for any  $U$ .

**Proof.** For the if-direction we assume that  $\mathcal{W}^U$  is a joint POVM of  $\mathbf{A}$  and  $\mathbf{B}^U$ :

$$\begin{aligned} \hat{W}_w^U &= \sum_{ab} \mathcal{W}_{ab}^U w(a, b) \\ &= \sum_b \mathcal{B}_b^U g(b) - \sum_a \mathcal{A}_a^U f(a) \\ &= \hat{B}_g^U - \hat{A}_f^U. \end{aligned} \quad (2)$$

For the only-if-direction we fix an outcome  $a'$  and set  $f(a') = 1$ ,  $f(a \neq a') = g(b) = 0$ . Plugging this choice into the average condition we get  $\sum_b \mathcal{W}_{ab}^U = \mathcal{A}_{a'}^U$ . Applying the same reasoning to all POVM elements  $\mathcal{B}_b^U$  and  $\mathcal{A}_a^U$ , we can conclude that

$$\sum_a \mathcal{W}_{ab}^U = \mathcal{B}_b^U, \quad \sum_b \mathcal{W}_{ab}^U = \mathcal{A}_a^U. \quad (3)$$

Thus,  $\mathcal{W}^U$  is a joint POVM of  $\mathbf{A}$  and  $\mathbf{B}^U$ .

**Corollary.** A work observable does *not* exist if the measurements  $\mathbf{A}$  and  $\mathbf{B}^U$  are incompatible for some unitary  $U$ . A no-go result for projective measurements immediately follows from this condition since joint measurability is equivalent

to commutativity in the projective case and two nontrivial projective observables cannot commute for every  $U$ .

The fluctuation condition (ii) depends [contrary to condition (i)] on the choice of the instrument  $\mathcal{I}^{\mathbf{A}}$  implementing the first measurement and, thus, further restricts the form of the joint POVM  $\mathcal{W}^U$ . However, in the following paragraph we show first that such an unsharp observable can indeed exist in a nontrivial and physically well-motivated quantum thermodynamical scenario. Then we investigate the role of the energy assignments  $f, g$  and how they could be chosen based on different physical arguments.

**Model for a joint observable of unsharp energies.** We construct an unsharp observable  $\mathcal{W}^U$  and show that both conditions (i) and (ii) can be satisfied. Let  $\hat{H}_A = \sum_{a=1}^d E_a \Pi_a$  and  $\hat{H}_B = \sum_{b=1}^d E'_b \Pi'_b$  be two Hamiltonians of a  $d$ -dimensional quantum system. Let us consider an apparatus which measures a mixture of the projectors instead of the perfectly sharp elements themselves. Thus, the effects have the form

$$\mathbf{A}_a = \sum_i R_{ai} \Pi_i, \quad \mathbf{B}_b = \sum_j S_{bj} \Pi'_j, \quad (4)$$

where  $R$  and  $S$  are bistochastic matrices. The simplest one-parameter model of this form, often considered in the literature, is given by symmetric matrices  $R_{a=i} = \lambda + (1 - \lambda)/d$ ,  $R_{a \neq i} = (1 - \lambda)/d$  and  $S_{b=j} = \gamma + (1 - \gamma)/d$ ,  $S_{b \neq j} = (1 - \gamma)/d$  [35,44,49]. (This choice is basically equivalent to a depolarizing noise which is omnipresent in many experimental setups [50,51].) We will stick to this model here since it allows for a fully analytical treatment for any dimension  $d$ . Thus, the unsharp effects are given by the original projectors mixed with the identity (sometimes referred to as adding white noise).

$$\mathbf{A}_a = \lambda \Pi_a + \frac{1 - \lambda}{d} \mathbb{1}, \quad \mathbf{B}_b = \gamma \Pi'_b + \frac{1 - \gamma}{d} \mathbb{1}, \quad (5)$$

where  $0 < \lambda < 1$  and  $0 < \gamma < 1$  are *visibility* parameters interpolating between the sharp energy measurements ( $\lambda, \gamma = 1$ ) and the trivial POVM ( $\lambda, \gamma = 0$ ). The first measurement is implemented by a Lüders instrument, i.e.,  $\mathcal{I}_a^{\mathbf{A}}(\rho) = \mathbf{A}_a^{1/2} \rho \mathbf{A}_a^{1/2}$ . As an ansatz for the joint POVM  $\mathcal{W}^U$  we propose the following form for its effects:

$$\mathcal{W}_{ab}^U = \mathbf{A}_a^{1/2} \mathbf{C}_b^U \mathbf{A}_a^{1/2}, \quad (6)$$

where  $\mathbf{C}^U$  is an auxiliary POVM which is constructed in the following. From the average condition (i) we know that the marginals of  $\mathcal{W}_{ab}^U$  must agree with the POVMs  $\mathbf{A}$  and  $\mathbf{B}^U$ , so we have  $\mathcal{B}_b^U = \sum_a \mathbf{A}_a^{1/2} \mathbf{C}_b^U \mathbf{A}_a^{1/2} = \mathcal{I}_A(\mathbf{C}_b^U)$  and, therefore,

$$\mathbf{C}_b^U = \mathcal{I}_A^{-1}(\mathcal{B}_b^U) = \mathcal{I}_A^{-1}[\Psi_\gamma(U^\dagger \Pi'_b U)], \quad (7)$$

with the depolarizing channel  $\Psi_\gamma(X) = \gamma X + (1 - \gamma)/d \mathbb{1}$ .

Since measurement  $\mathbf{A}$  is implemented by a Lüders instrument, the map  $\mathcal{I}_A$  is unital. The inverse map (which exists for any  $\lambda, \gamma \neq 1$ ) is unital also and, thus, also the marginal condition for  $\mathbf{A}$  is satisfied:  $\sum_b \mathbf{A}_a^{1/2} \mathcal{I}_A^{-1}(\mathcal{B}_b^U) \mathbf{A}_a^{1/2} = \mathbf{A}_a$ .

One can verify that the operators defined by Eqns. (6, 7) fulfill the fluctuation condition (ii) for any input state diagonal in the initial energy basis  $\{\Pi_a\}$  (see the Supplemental Material [52]). Since the average condition (i) is satisfied by construction, it only remains to establish bounds for the

visibility parameters  $\lambda$  and  $\gamma$  that guarantee positivity of  $\mathbf{W}^U$  for any intermediate evolution  $U$ .

It is well known that any two given POVMs  $\mathbf{A}$  and  $\mathbf{B}$  become jointly measurable if enough white noise is added as in Eq. (5) [44]. In our case the situation is slightly different because we require joint measurability of  $\mathbf{A}$  with the Heisenberg  $\mathbf{B}^U$  for *arbitrary*  $U$ . Thus, while the unsharp energy measurements  $\mathbf{A}$  and  $\mathbf{B}$  are fixed, the joint POVM  $\mathbf{W}^U$  has to depend on the evolution. Therefore, we are looking for parameters  $\lambda$  and  $\gamma$  such that  $\mathbf{W}_{ab}^U \geq 0, \forall a, b, U$ .

By construction,  $\mathbf{W}_{ab}^U$  is positive if  $\mathbf{C}_b^U$  is positive. Thus, from Eq. (7) we see that the concatenation of the depolarizing channel with the inverse instrument  $\Xi = \mathcal{I}_A^{-1} \circ \Psi_\gamma$  needs to be a positive (but not necessarily completely positive) map. A map  $\Xi$  is positive if its Choi state has a nonnegative expectation value for any product state [53]. In our case, this condition can be checked analytically for any dimension  $d$  (see the Supplemental Material [52]). The map  $\Xi$  is positive, and therefore  $\mathbf{W}^U$  is a valid POVM if the following relation for the two visibility parameters  $\lambda$  and  $\gamma$  holds:

$$\gamma \leq \frac{2\kappa}{d + 2\kappa - d\kappa}, \quad (8)$$

with

$$\kappa = 2\sqrt{\lambda + \frac{1-\lambda}{d}} \sqrt{\frac{1-\lambda}{d}} + (d-2)\frac{1-\lambda}{d}. \quad (9)$$

Thus, for  $\lambda, \gamma$  satisfying Eq. (8) the GTPM scheme can be replaced by a single work observable  $\mathcal{W}^U$ .

For  $d > 2$  the visibilities  $\lambda$  and  $\gamma$  are strictly smaller than the best known bounds for the joint measurability of white noise affected measurements [49]. This is expected since condition (ii) imposes an additional constraint. The ansatz for the effects  $\mathbf{W}_{ab}^U$  in Eq. (6) is *a priori* not the most general one [54]. However, numerical simulations suggest that our visibility bounds are tight and we conjecture that our construction is optimal to satisfy condition (ii) for the given model Eq. (5) (see Supplemental Material for details [52]).

*Energy assignments.* Up to now we kept  $f$  and  $g$  arbitrary. To establish the connection to thermodynamic quantities we will now consider possible assignments of energies to the measurement outcomes. The visibility bounds derived above ensure that a measurement apparatus exists that can replace the GTPM scheme. The measurement outcomes of such an apparatus can then be used to determine different quantities of interest depending on the chosen assignments  $f$  and  $g$ . For the following physically motivated choices we always require that the energy eigenvalues are obtained in the limit of maximal visibility  $f(a) \xrightarrow{\lambda \rightarrow 1} E_a, g(b) \xrightarrow{\gamma \rightarrow 1} E'_b$ .

*Average work.* The first obvious choice are the energy eigenvalues themselves  $f(a) = E_a, g(b) = E'_b$ . The experimenter assigns energies as if the measurements were projective. Clearly, such an assignment does not yield the correct energy expectation values of the Hamiltonians  $\hat{H}_A$  and  $\hat{H}_B$ . Since the measured effects are unsharp, the determined energy is shifted towards the average energy of a fully mixed state  $\bar{E}_{A,B} = 1/d \text{Tr}[H_{A,B}]$ . Thus, the experimenter compensates for

this shift by assigning corrected energies

$$f(a) = \frac{1}{\lambda} E_a - \frac{1-\lambda}{\lambda} \bar{E}_A, \quad g(b) = \frac{1}{\gamma} E'_b - \frac{1-\gamma}{\gamma} \bar{E}_B.$$

This choice leads to the correct average energies of the marginal observables. Condition (i) then ensures that the average of the joint observable  $\mathcal{W}^U$  agrees with the average work  $\hat{W}_w^U = \hat{H}_B - \hat{H}_A = \hat{W}_{\text{avg}}$ . Thus, even though the measurements are unsharp, by correctly choosing  $f$  and  $g$  the work observable  $\mathcal{W}$  yields the accurate value for the average work in the system for any input state. This might seem surprising at first glance. However, the unsharp POVMs are informationally equivalent to the projective energy measurements in the sense that the first can be used to reconstruct the statistics of the second.

*Free energy difference.* We say a GTPM scheme satisfies a generalized Jarzynski equation (JE) if the following equation holds for any unitary  $U$ :

$$\sum_{a,b} \text{tr}[\mathbf{B}_b U \mathcal{I}_a^A(\rho_0) U^\dagger] e^{-\beta[g(b)-f(a)]} = e^{-\beta\Delta\tilde{F}}, \quad (10)$$

where  $\Delta\tilde{F}$  is real and independent of the unitary  $U$  on the left-hand side. This can, in general, only be satisfied for any  $U$  if we demand

$$\sum_a e^{+\beta f(a)} \mathcal{I}_a^A[\rho_0] = \alpha \mathbb{1}, \quad (11)$$

where  $\alpha$  is a positive real number [55]. Here we only consider  $\rho_0 = \rho_{\text{Gibbs}}$ . In the limit of projective Lüders measurements ( $\mathbf{A}_a^{1/2} = \Pi_a$ ), the standard JE is immediately recovered if we set  $\alpha = 1/Z_A$ , where  $Z_A = \sum_a \exp[-\beta E_a]$ , and we get  $\Delta\tilde{F} = \Delta F$  on the right-hand side of Eq. (10). Thus, as is well known, a projective TPM scheme can be used to determine the free energy difference  $\Delta F$  [2]. However, the projective measurements cannot be replaced by a work observable  $\mathcal{W}^U$ . Accordingly, the correct average energies cannot be obtained from the same experimental setup.

Interestingly, Eq. (11) can also be satisfied by unsharp measurements if a suitable assignment  $f$  is chosen

$$f(a) = \frac{1}{\beta} \ln \left[ \frac{\alpha Z_A}{\lambda} \left( e^{+\beta E_a} - \frac{1-\lambda}{d} \sum_i e^{+\beta E_i} \right) \right]. \quad (12)$$

We can set again  $\alpha = 1/Z_A$ . Demanding  $\Delta\tilde{F} = \Delta F$ , we then find the assignment  $g(b) = E'_b$ .

Therefore, if we choose visibilities satisfying Eq. (8), condition (ii) guarantees that a joint observable  $\mathcal{W}^U$  exists such that

$$\text{Tr} \left[ \sum_{a,b} e^{-\beta w(a,b)} \mathbf{W}_{ab}^U \rho_{\text{Gibbs}} \right] = \frac{Z_B}{Z_A} = e^{-\beta\Delta F}. \quad (13)$$

Accordingly, we can use the POVM  $\mathbf{W}^U$  to also obtain the free energy difference  $\Delta F$  between the initial and final Hamiltonian.

It is worth noting that the GTPM scheme, unlike the projective one, generally yields fluctuations also in the trivial case of  $\hat{H}_A = \hat{H}_B$  and  $U = \mathbb{1}$  due to the inherent nonrepeatability of unsharp measurements [43,56]. Importantly, the vanishing



average work and free energy difference is, of course, correctly obtained also for this trivial scenario.

*Conclusion.* The idealized projective TPM scenario rules out the existence of a work observable  $\mathcal{W}^U$  that satisfies both an average and a fluctuation condition. Our results show that this is no longer true for the general scenario using unsharp measurements. The answer to the question of whether a work observable can exist is richer in this case and demands the consideration of joint measurability. The no-go theorem from Ref. [33] can be extended to the case of general energy POVMs if the two measurements **A** and **B** are incompatible for some intermediate unitary evolution  $U$ . However, if **A** and  $\mathbf{B}^U$  are jointly measurable for every  $U$ , the no-go theorem can be overcome and a work observable  $\mathcal{W}^U$  may exist. By explicit construction of  $\mathcal{W}^U$  for a specific model of unsharp measurements we showed that a general TPM scheme can indeed be replaced by a single measurement in a physically well-motivated scenario.

For nonprojective measurements a unique correspondence between POVM elements and associated energies is missing. Energy assignment functions  $f(a)$  and  $g(b)$  have to be specified. The existence of  $\mathcal{W}^U$  is independent of that choice.

However,  $f$  and  $g$  are determined by the quantity one would like to estimate from the measurement outcomes and depend on the effects of the POVM. For the specific model we show that a single measurement apparatus implementing the POVM  $\mathbf{W}^U$  can be used to determine both the correct average work  $W_{\text{avg}}$  for any input state and the correct free energy difference  $\Delta F$  between the Hamiltonians  $\hat{H}_A$  and  $\hat{H}_B$  if the system starts in a Gibbs state.

Theoretical investigations on quantum work and fluctuations should embrace the rich possibilities offered by quantum measurement theory. Our work highlights the central role of joint measurability for TPM schemes. We show yet again that unsharp measurements enable to achieve goals that are unreachable with projective ones, here with important applications in quantum thermodynamics. Joint measurability offers an operationally motivated approach to the quantum-to-classical transition and we showed that this transition is closely related to the existence of a work observable. In the future it will be interesting to explore how other operational characterizations of quantumness, such as inherent measurement disturbance, emerge in thermodynamic settings.

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