Exact analytical soliton solutions of N-component coupled nonlinear Schrödinger equations with arbitrary nonlinear parameters

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Exact analytical soliton solutions play an important role in soliton fields. Soliton solutions were obtained with some special constraints on the nonlinear parameters in nonlinear coupled systems, but they usually do not hold in real physical systems. We successfully release all usual constrain conditions on nonlinear parameters for exact analytical vector soliton solutions in N-component coupled nonlinear Schrödinger equations. The exact soliton solutions and their existence condition are given explicitly. Applications of these results are discussed in several present experimental parameter regimes. The results would motivate experiments to observe more novel vector solitons in nonlinear optical fibers, Bose-Einstein condensates, and other nonlinear coupled systems.

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I. INTRODUCTION

Analytical soliton solutions can describe the dynamics of localized waves in many nonlinear systems well, and some of them have even been used to direct experimental observations [1–7]. Analytical solutions usually contain exact solutions and approximate solutions. Exact analytical soliton solutions have been paid much more attention due to their beauty and convenience for uncovering underlying physics [8–13]. Unfortunately, exact analytical solutions are usually limited to integrable cases [14], which can be derived by the Bäcklund transformation [15,16], inverse scattering method [17], and Hirota bilinear method [18]. However, those methods usually fail to derive analytic solutions for nonintegrable cases. For nonintegrable cases, many methods have been proposed to derive approximate analytic solutions, such as the perturbation method [19], multi-scale expansion method [20,21], and Lagrangian variational method [22–27]. The Lagrangian variational method is a well-known method for deriving approximate soliton solutions [22], whose precision usually depends on trial functions. Some exact soliton solutions can still be derived by the method with proper trial functions [28,29]. This provides possibilities to derive exact soliton solutions for nonintegrable cases and even more general cases.

N-component coupled nonlinear Schrödinger equations play an important role in soliton fields due to their simplicity and wide applications [30–33]. They admit scalar solitons ($N = 1$) and vector solitons ($N > 1$). The exact scalar bright and dark soliton solutions were first derived by the methods for integrable cases (noting that $N = 1$ cases are always integrable), and then they were also obtained by the Lagrangian variational method [28,29]. However, there are many nonlinear parameters when $N > 1$, which makes the coupled systems usually no longer integrable. Most vector soliton solutions were obtained with some special constraints on these nonlinear parameters, e.g., the Manakov model, with all of them being equal [14]. Many efforts have been made to develop variational methods to address the cases for which the special constraint conditions are violated [34–39]. However, exact soliton solutions are still lacking for the general N-component coupled model, partly because the form of the trial functions affects the precision of the approximation.

In this paper, we derive exact soliton solutions for the general N-component coupled model with arbitrary nonlinear parameters. We suggest that the soliton width in each component should be introduced independently first, although they are usually equal. The proper trial functions and modified Lagrangian variational method enable us to obtain exact soliton solutions and clarify their existence condition. These exact analytic soliton solutions motivate experiments to look for more novel vector solitons.

This paper is organized as follows. In Sec. II, we give the exact soliton solutions and their existence condition explicitly based on our proper trial functions and the modified Lagrangian variational method. As an example, we show two- and three-component vector soliton solutions and their existence region in Sec. III. We discuss the stability and collision characters of two-component solitons in Sec. IV. Applications of these results are discussed in Bose-Einstein condensates with present experimental parameter regimes in Sec. V. Finally, our conclusions are given in Sec. VI.

II. THE EXACT VECTOR SOLITON SOLUTIONS AND LAGRANGIAN METHOD

Coupled nonlinear Schrödinger equations have been used to describe nonlinear wave dynamics in many different physical systems [32,40,41], such as Bose–Einstein

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condensates [41] and nonlinear optical fibers [30,42]. We consider general N-component coupled nonlinear Schrödinger equations (NLSEs) with arbitrary nonlinear parameters, which can be written as

$$\mathrm{i}\partial_t \mathbf{\Psi} = \left(-\frac{1}{2}\partial_{xx} + \sum_{j=1}^{N} g_{ij}|\psi_j|^2\right)\mathbf{\Psi}, \tag{1}$$

where $\mathbf{\Psi} = (\psi_1, \cdots, \psi_i, \cdots, \psi_N)^T$ and where ψ_j stands for the wave function of the jth component. x is the distribution axis for wave function, and its domain is $(-\infty, +\infty)$. The nonlinear parameter g_{ij} is the intra-species interaction in one component (interspecies interaction between two components) for $i = j$ $(i \neq j)$, and $g_{ij} = g_{ji}$. When all these nonlinear parameters are equal, the model becomes the well-known Manakov model [14]. The soliton solutions of the Manakov model have motivated many experiments to observe dark-dark (DD) solitons [43,44], dark-bright (DB) solitons [3], etc. Considering that real experiments usually do not satisfy integrable conditions [3,4] and that even five-component Bose-Einstein condensates have been prepared in experiments [45–48], we wish to find exact analytic solutions for more general conditions.

The Lagrangian variational method is a well-known method for deriving approximate solutions, whose precision usually depends on trial functions. The Gaussian profile is usually used for localized wave packets due to its simplicity and easy calculation. However, this form usually fails to obtain exact solutions in nonlinear systems. It was further suggested that the sech- (tanh-)type ansatz could be more accurate than the Gaussian ansatz [34]. For bright or dark solitons in the ith component, we introduce the trial wave function as

$$\begin{aligned}\psi_{iD} &= \left(i\sqrt{a_i^2 - f_i^2(t)} + f_i(t)\tanh\{w_i(t)[x - b(t)]\}\right)e^{i\theta_i(t)},\\ \psi_{iB} &= f_i(t)\mathrm{sech}\{w_i(t)[x - b(t)]\}e^{i\{\xi_i(t)+[x-b(t)]\phi_i(t)\}},\end{aligned} \tag{2}$$

where ψ_{iD} (ψ_{iB}) denotes the wave function of the dark (bright) soliton, f_i and w_i describe the amplitude and width of the dark (bright) soliton, respectively, a_i is the background of the dark component, the central position of the soliton is $b(t)$, θ_i, and ξ_i are the time-dependent phases of the dark and bright components, respectively, and ϕ_i is related to the velocity of the bright soliton.

We introduce the Lagrangian density as $\mathcal{L} = \sum_{i=1}^{N}[\frac{i}{2}(\tilde{\psi}_i^*\partial_t\tilde{\psi}_i - \tilde{\psi}_i\partial_t\tilde{\psi}_i^*)(1 - \frac{a_i^2}{|\tilde{\psi}_i|^2}\delta_{D,i}) - \frac{1}{2}|\partial_x\tilde{\psi}_i|^2 - \sum_{j=1}^{N}\frac{g_{ij}}{2}(|\tilde{\psi}_j|^2 - a_j^2\delta_{D,j})(|\tilde{\psi}_i|^2 - a_i^2\delta_{D,i})]$, where

$$\delta_{D,i} = \begin{cases} 0, & \tilde{\psi}_i \text{ denotes bright soliton,} \\ 1, & \tilde{\psi}_i \text{ denotes dark soliton,} \end{cases}$$

$$\begin{aligned}L = \sum_i^N \Bigg\{&\left[2f_i^2(t)\phi_i(t)\frac{b'(t)}{w_i(t)} - 2f_i^2(t)\frac{\xi_i'(t)}{w_i(t)} - \frac{1}{3}f_i^2(t)w_i(t) - f_i^2(t)\frac{\phi_i^2(t)}{w_i(t)} - \frac{2}{3}g_{ii}\frac{f_i^4(t)}{w_i(t)}\right]\tilde{\delta}_{D,i} + \Bigg\{-2f_i(t)\sqrt{a_i^2 - f_i^2(t)}b'(t)\\ &+ 2a_i^2\arcsin\left[\frac{f_i(t)}{a_i}\right]b'(t) - \frac{2}{3}f_i^2(t)w_i(t) - \frac{2}{3}g_{ii}\frac{f_i^4(t)}{w_i(t)}\Bigg\}\delta_{D,i} - \sum_{j(j\neq i)}^{N} g_{ij}f_i^2(t)f_j^2(t)G_{ij}\Bigg\}, \quad (\tilde{\delta}_{D,i} = |\delta_{D,i} - 1|),\end{aligned} \tag{3}$$

to obtain Euler-Lagrangian equations. In particular, the terms $1 - a_i^2/|\tilde{\psi}_i|^2\delta_{D,i}$ and $|\tilde{\psi}_i|^2 - a_i^2\delta_{D,i}$ were introduced in the Lagrangian density for dark solitons [29]. Notably, the modified $\mathcal{L}$ corresponds to dynamical equation $\mathrm{i}\partial_t\tilde{\psi}_i = -\frac{1}{2}\partial_{xx}\tilde{\psi}_i + [\sum_{j=1}^{N} g_{ij}(|\tilde{\psi}_j|^2 - a_j^2\delta_{D,j})]\tilde{\psi}_i$ $(i = 1, \cdots, N)$. This is different from Eq. (1), but the solutions of the two dynamical equations can be transformed to each other by the phase factor $\psi_i/\tilde{\psi}_i = e^{i\theta_i(t)}$, where $\theta_i(t) = -\sum_{j=1}^{N} g_{ij}a_j^2 t\delta_{D,j}$. We obtain the Lagrangian (L) by substituting Eq. (2) into $\mathcal{L}$ and integrating over space from $-\infty$ to $+\infty$, which can be simplified as Eq. (3).

Specifically, the soliton width parameter w_i is set to be different for different components, which is in contrast to the previous trial functions for vector solitons [34–39]. In most previous studies, the trial functions were chosen as the Gaussian (sech- or tanh-type) ansatz for the unequal (equal) width parameter setting [34–39], partly because the Gaussian ansatz with different widths is much easier to calculate than the sech (tanh) type. This makes the Lagrangian variational results only give an approximate solution for the vector model [34–39]. However, one encounters the problem that it is difficult to accommodate the factor $G_{ij} = \int_{-\infty}^{+\infty}(\pm\,\mathrm{sech}^2\{w_i(t)[x - b(t)]\})(\pm\,\mathrm{sech}^2\{w_j(t)[x - b(t)]\})dx$, and the sign $-(+)$ corresponds to the dark (bright) component. We take these width parameters independently when deriving the Euler-Lagrangian equations. After obtaining the equations of motion by the Euler-Lagrangian formula, we finally calculate G_{ij}, $\frac{\partial G_{ij}}{\partial w_i}$, and $\frac{\partial G_{ij}}{\partial w_j}$ by setting $w_i = w_j = w$ (see details in Appendix A). These operations bring us more constraint conditions on soliton parameters and finally enable us to obtain exact analytical soliton solutions.

Based on all simplified Euler-Lagrangian equations, we have the essential constraint equations of f_i and w as

$$\begin{pmatrix} g_{11} & \cdots g_{1j} \cdots & g_{1N} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{i1} & \cdots g_{ij} \cdots & g_{iN} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N1} & \cdots g_{Nj} \cdots & g_{NN} & 1 \end{pmatrix}\begin{pmatrix} \pm f_1^2 \\ \vdots \\ \pm f_i^2 \\ \vdots \\ \pm f_N^2 \\ w^2 \end{pmatrix} = 0. \tag{4}$$

The sign $-(+)$ corresponds to the dark (bright) component. The other parameters can be obtained as $b(t) = vt$, $\phi_i = v$, $\xi_i(t) = \frac{1}{2}(w^2 + v^2)t + \theta_i(t)$, and the soliton velocity

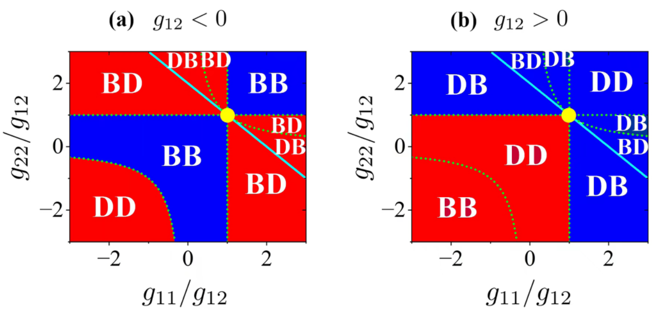

FIG. 1. Phase diagram for different vector solitons in nonlinear parameter space. (a) and (b) represent the cases of $g_{12} < 0$ and $g_{12} > 0$, respectively. The green dashed lines are the boundary lines for different vector soliton regions. The cyan lines denote the condition for spin solitons. The yellow dot denotes the usual integrable case. The blue regions represent that the vector solitons can evolve stably against weak white noises, and red regions mean that the vector solitons are not stable (details in Sec. IV). Our variational method greatly extends the existence region for exact vector soliton solutions.

$v = w\sqrt{a_i^2 - f_i^2}/f_i$. The constraint equation on backgrounds of dark components is $a_i/a_j = f_i/f_j$. In this way, we obtain many exact analytic vector soliton solutions when the nonlinear parameters can be arbitrary. Different types of vector solitons usually exist in different regions in nonlinear parameter space. Their existence regions can also be clarified by the constraint conditions on soliton parameters. The above equations can be simplified as $g(\pm f_1^2 \pm f_2^2 \cdots \pm f_N^2) + w^2 = 0$ for integrable cases ($g_{ij} = g$), which can be used to derive all previously reported vector soliton solutions [13,49–59]. We choose two- and three-component systems as examples to show the variational results explicitly based on vector soliton experiments in Refs. [3–6].

III. THE EXACT SOLUTION OF TWO- AND THREE-COMPONENT VECTOR SOLITON

A. Two-component coupled system

For two-component systems ($N = 2$), the condition for vector soliton solutions can be derived from Eq. (4), which is simplified as

$$\alpha_2 f_2^2 = \frac{g_{11} - g_{12}}{g_{22} - g_{12}}\alpha_1 f_1^2,\ w^2 = \frac{g_{12}^2 - g_{11}g_{22}}{g_{22} - g_{12}}\alpha_1 f_1^2, \quad (5)$$

where $\alpha_i = \pm 1$ and where the sign $-$ (+) is chosen for the dark (bright) soliton. Then, we clarify the existence regimes for different vector solitons by analyzing the soliton parameters of Eq. (5). The phase diagram for different vector solitons is summarized in Fig. 1 ($g_{12} \neq 0$), in which (a) and (b) show the results for the $g_{12} < 0$ and $g_{12} > 0$ cases, respectively. When $g_{12} = 0$, the model decouples into two scalar models. Our variational method extends the existence region of exact vector soliton solutions from the "yellow dot" (Manakov model) to the whole parameter plane. The vector solitons can still be classified into four families, similar to the integrable model [13,49–52]. However, there are many additional constraints on the soliton parameters and nonlinear parameters, which are absent for integrable models [13,49–52].

For bright-bright (BB) solitons, Eq. (5) gives the conditions $\frac{g_{11}-g_{12}}{g_{22}-g_{12}} > 0$ and $\frac{g_{12}^2-g_{11}g_{22}}{g_{22}-g_{12}} > 0$ on nonlinear parameters. Then, the region for the BB soliton can be given in Fig 1. The region for DD solitons can be given in a similar way. We note that there is an additional constraint on the background amplitudes for the DD soliton ($a_2/a_1 = f_2/f_1$) in nonintegrable cases. Namely, density dip's depth is closely related with dark soliton's background density. This is in sharp contrast to the Manakov model, for which there is no constraint on the ratio of background amplitudes for DD solitons [50,51].

For dark-bright (DB) or bright-dark (BD) solitons, the regions for them are shown in Fig. 1. The DB and BD solitons are defined by the total density of the two components, which admits a dip (for DB) or hump (for BD). In particular, the total density can be uniform when $2g_{12} = g_{11} + g_{22}$, and the vector soliton in this case becomes the spin soliton [60] (denoted by cyan lines in Fig. 1). Note that the amplitude of the bright soliton can be larger than the dark soliton's background amplitude; i.e., BD soliton can exist even though the nonlinear interactions are all repulsive (all nonlinear parameters are positive). This characteristic is absent for the Manakov case [13], for which only DB soliton exists for repulsive interactions. Similarly, DB soliton can still exist with the nonlinear interactions all attractive (all nonlinear parameters are negative) for nonintegrable cases. Our DB soliton solution with zero velocity can be reduced to the ones given in Ref. [61]. The exact BB and BD (DD and DB) soliton solutions cannot coexist in the given nonlinear parameters for the nonintegrable cases (see Fig. 1), in contrast to the integrable case [13,49–52].

Our variational results can be reduced to a previously known exact soliton solution in the two-component model [13,49–52,60]. The general exact vector soliton solutions of two-component cases are provided explicitly in Appendix B, which partly overlap with the ones given by the periodic wave expansion method [62,63]. We emphasize that our variational method here can be extended more conveniently to arbitrary N-component cases. Experiments have been performed on three-component systems and even five-component Bose-Einstein condensates [4,31,45–48], but the exact analytic soliton solutions for them are still absent. These results motivate us to derive soliton solutions explicitly for more components cases. Next, we apply our variational result to three-component systems.

B. Three-component coupled system

The exact soliton solution of three-component systems has been widely studied for integrable models [53–57]. Our variational result greatly widens the existence region of vector soliton solutions, which is very meaningful for soliton experiments [4]. The essential conditions for soliton solutions can be given as

$$\alpha_2 f_2^2 = \frac{P_{123}}{P_{213}}\alpha_1 f_1^2,\ \alpha_3 f_3^2 = \frac{P_{132}}{P_{213}}\alpha_1 f_1^2,\ w^2 = \frac{Q}{P_{213}}\alpha_1 f_1^2, \quad (6)$$

where $Q = 2g_{12}g_{13}g_{23} - g_{12}^2 g_{33} - g_{13}^2 g_{22} - g_{11}(g_{23}^2 - g_{22}g_{33})$, $P_{lmn} = g_{ln}^2 - g_{ll}g_{nn} + g_{lm}(g_{nn} - g_{ln}) + g_{mn}(g_{ll} - g_{ln})$, $(l, m, n = 1, 2, 3)$, and $\alpha_i = \pm 1$ $(i = 1, 2, 3)$. The exact soliton solutions can be given from Eq. (4) with $N = 3$, whose

TABLE I. Parameter conditions of different soliton solutions for the three-component coupled systems; "+" and "−" denote the parameters being positive and negative, respectively.

Soliton type	Parameter region		
	P_{123}/P_{213}	P_{132}/P_{213}	Q/P_{213}
BBB	+	+	+
DBB	−	−	−
DDB	+	−	−
DDD	+	+	−

explicit expressions are given in Appendix B. The vector soliton can be classified into four families, and their existence conditions are summarized in Table I. We cannot give explicit conditions of existence for different soliton solutions, as done in the two-component cases, since there are many more nonlinear parameters in the three-component cases. To show the physical meaning of the conditions in Table I, we discuss the existence conditions of different vector solitons with setting $g_{12} = 1$, $g_{13} = 2$, and $g_{23} = 3$. With the repulsive interspecies interaction, the exact bright-bright-bright (BBB) soliton solution can still exist when $g_{11} < 0$, $g_{22} < 1/g_{11}$, and $g_{33} < (-12 + 9g_{11} + 4g_{22})/(-1 + g_{11}g_{22})$. By further tuning the intra-interactions, we can obtain the existence region for dark-bright-bright (DBB), dark-dark-bright (DDB), and dark-dark-dark (DDD) solitons. Recently, the numerical method for obtaining the stationary vector solitons in N-component NLSEs was reported in Ref. [64], which could be also used to find soliton profiles predicted by our analytic soliton solutions.

IV. THE STABILITY AND COLLISION CHARACTERS OF VECTOR SOLITONS

The stability of these vector solitons can be investigated by numerical simulations. As an example, we discuss the stability of them in the two-component case. We perform numerical simulations from the initial conditions given by the above exact analytical solutions with adding weak white noises [64–66]. The results are shown in Fig. 1. The blue regions represent that the vector solitons can evolve stably against weak white noises, and red regions means that the vector solitons are not stable. These characters are supported by our numerous numerical simulations. For examples, the evolution of a BB soliton in the blue region are shown in Figs. 2(a1) and (a2), and the result for a BD soliton in the blue region is shown in Figs. 2(b1) and (b2). It is seen that these solitons can be stable over 300 time units. We show the DD and BB soliton in red regions in Figs. 3(a1), (a2), (b1), and (b2), respectively. It is seen that they are not stable and they can split with radiations, which is similar to the ones reported in nonintegrable conditions [67].

Then we can numerically investigate the collision behavior of stable vector solitons, for which the initial states can be given by linear superposition of the above soliton solutions. Our numerical simulations indicate that their collisions can be elastic and inelastic, which are related to the relative phase, relative velocity, and nonlinear parameters. For inelastic collision cases, solitons's velocities or profiles can change, and

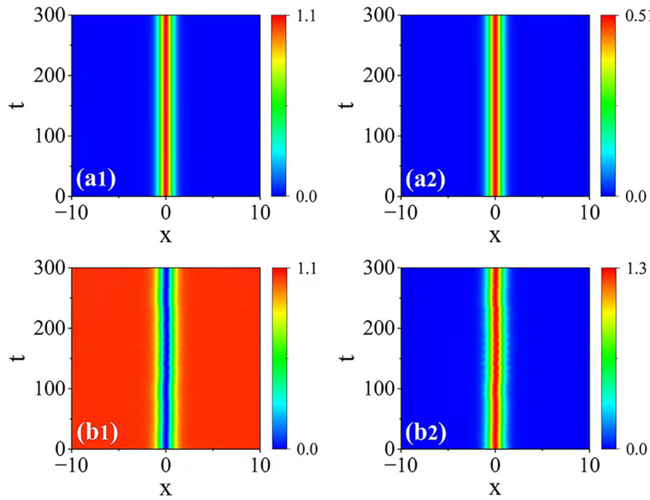

FIG. 2. (a1) and (a2): The numerical evolution of a BB soliton. (b1) and (b2): The numerical evolution of a BD soliton. 3% random noises are added to initial states given by related exact soliton solutions. It is seen that these solitons can be stable over 300 time units. The parameters are $g_{11} = -1.5, g_{12} = -1, g_{22} = -2, f_1 = 1, f_2 = \frac{\sqrt{2}}{2}, w = \sqrt{2}, v = 0$ for (a1) and (a2). For (b1) and (b2), the parameters are $g_{11} = 2.9, g_{12} = 1, g_{22} = -0.5, f_1 = 1, f_2 = 1.13, w = 1.28, a_1 = 1, v = 0$.

there can be obvious radiation after collision. For example, the collision of BB solitons are shown in Figs. 4(a1) and (a2). We can see that the density can redistribute between the two bright solitons, which is similar to the ones in the Manakov model [56,68,69]. But the velocity of solitons can change greatly after collision, in contrast to the Manakov model. The collision of two BD solitons are shown in Figs. 4(b1) and (b2).

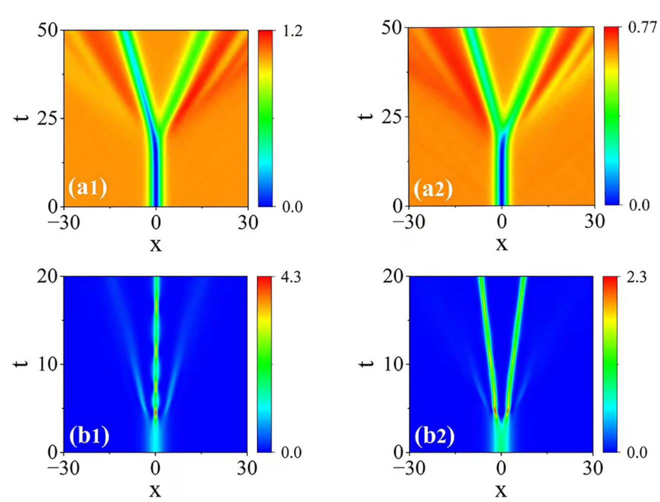

FIG. 3. (a1) and (a2): The numerical evolution of a DD soliton. (b1) and (b2): The numerical evolution of a BB soliton. 3% random noises are added to the initial states. It is seen that they are not stable and they can split with radiations. The parameter settings are: $g_{11} = 1, g_{12} = -1, g_{22} = 2, f_1 = 1, f_2 = \frac{\sqrt{6}}{3}, w = \frac{\sqrt{3}}{3}, a_1 = 1, a_2 = \frac{\sqrt{6}}{3}, v = 0$ for (a1) and (a2); $g_{11} = -1, g_{12} = 1, g_{22} = -1.3, f_1 = 1, f_2 = 0.93, w = 0.36, v = 0$ for (b1) and (b2).

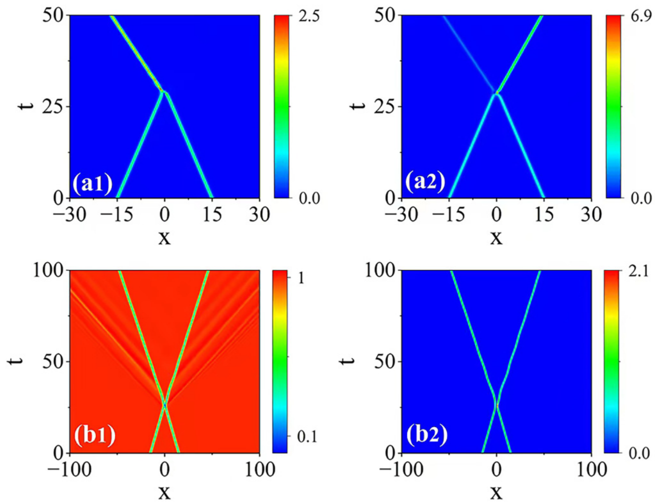

FIG. 4. (a1) and (a2): The collision of two BB solitons. We can see that the density can redistribute between the two bright solitons, and the velocity of solitons can change greatly after collision. The initial states are $\psi_1 = f_1\text{sech}[w(x+15)]\exp[iv(x+15)-i\phi] + f_1\text{sech}[w(x-15)]\exp[-iv(x-15)+i\phi]$, $\psi_2 = f_2\text{sech}[w(x+15)]\exp[iv(x+15)] + f_2\text{sech}[w(x-15)]\exp[-iv(x-15)]$, where $g_{11} = -2, g_{12} = -1$, $g_{22} = -1.5$, $f_1 = 1, f_2 = \sqrt{2}$, $w = 2, v = 0.5$, $\phi = 1$. (b1) and (b2): The collision of two BD solitons. It is seen that the dark solitons oscillate with obvious radiations, and bright solitons just oscillate after collision. The initial states are $\psi_1 = \{i\sqrt{1-f_1^2} + f_1 \tanh[w(x+15)]\}\{i\sqrt{1-f_1^2} - f_1\tanh[w(x-15)]\}$, $\psi_2 = f_2\text{sech}[w(x+15)]\exp[iv(x+15)] + f_2\text{sech}[w(x-15)]\exp[-iv(x-15)]$, where $g_{11} = 2.9, g_{12} = 1, g_{22} = -0.5, f_1 = 0.9, f_2 = 1.01, w = 1.15, v = 0.557$.

It is seen that the dark solitons oscillate with obvious radiation, and bright solitons just oscillate after collision. These collision characters are much more complicated than the ones in integrable cases, which still need further research.

V. APPLICATIONS IN EXPERIMENTS

We first discuss the applications of our solutions in a two-component Bose-Einstein condensate of ^{87}Rb [3,70] with hyperfine states $|1,-1\rangle$ and $|2,0\rangle$ (denoted by ψ_1 and ψ_2). The scattering lengths are $a_{11} = 100.86a_0$, $a_{12} = 98.98a_0$, and $a_{22} = 94.57a_0$, where $a_0 = 5.29\times10^{-11}$ m is the Bohr radius. The nonlinear parameters in our rescaling model satisfy $g_{11} : g_{12} : g_{22} = 1 : 0.981 : 0.938$, and $g_{11} = 0.0114$. The dynamical equations in the mean-field approximation become nonintegrable. The theoretical analysis of the experimental results is usually performed based on an integrable model, ignoring the small differences between the nonlinear parameters [3]. This is partly because exact analytical solutions are very rare with the parameters in real experiments. We discuss the properties of vector solitons based on our variational results with the real nonlinear parameters and identical atom density $5.8\times10^{13}\text{cm}^{-3}$ [3]. Because of asymmetry interaction parameters, as $g_{11} - g_{12} > 0$, $g_{22} - g_{12} < 0$, $g_{12}^2 - g_{11}g_{22} > 0$, the solution predicts that dark solitons exist only in the ψ_1 component [$\alpha_1 = -1, \alpha_2 = 1$ from Eq. (5)] and that they cannot exchange with the second component, in contrast to the prediction of the integrable model. The sound speed is predicted to be 0.766 mm/s by our soliton solutions, which is smaller than the sound velocity 1 mm/s given by the integrable results [3]. Moreover, the constraint conditions on nonlinear parameters predict that there is no exact analytical DD soliton solution in this case but that the DD soliton solution can be given approximately by the integrable model. These characteristics may inspire experiments to check these differences.

There are many theoretical works in three-component systems [53–57]. DDB and DBB solitons were realized experimentally in [4,71]. They obtained the approximate solution of DDB and DBB solitons by a multiscale expansion method. With taking the nonlinear parameter setting in [4,71] and ignoring the spin exchange effects, we predict that only exact DDD soliton solutions exist from Table I, but exact DDB and DBB solitons can not be given by the above variational results. We further test the stability of our DDD soliton numerically with taking weak spin exchange effects in the experiments. Our results indicate that the DDD solitons can exist stably even with white noises, which could inspire experiments to observe them.

VI. CONCLUSION

Our work successfully releases all usual constrain conditions on nonlinear parameters for exact analytical vector soliton solutions in N-component coupled nonlinear Schrödinger equations, which have played important roles in soliton fields. Our explicit soliton solutions and their existence condition could motivate experiments to observe vector solitons in Bose-Einstein condensates [4–6,31] and nonlinear optical fibers [30,42], especially when the nonlinear parameters deviate from the usual integrable conditions. They could provide an important supplement for the experimental observation of vector solitons, which are usually around integrable conditions [3–6]. The general vector soliton solutions are helpful to discuss the soliton dispersion relation and soliton transport for more general cases [60]. Our attempt will stimulate more efforts to derive exact soliton solutions of other nonlinear models [72–74], which can describe real nonlinear systems better.

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APPENDIX A: THE DERIVATION OF N-COMPONENT VECTOR SOLITON SOLUTIONS

We consider a general N-component coupled nonlinear Schrödinger equation, which can be written as

$$i\partial_t\mathbf{\Psi} = \left(-\frac{1}{2}\partial_{xx} + \sum_{j=1}^{N} g_{ij}|\psi_j|^2\right)\mathbf{\Psi}, \tag{A1}$$

where $\mathbf{\Psi} = (\psi_1\cdots,\psi_i,\cdots\psi_N)^T$. The nonlinear parameters g_{ij}s are usually equal for exact soliton solutions.

We would like to derive exact soliton solutions for arbitrary nonlinear parameters by the Lagrangian variational method. We introduce the Lagrangian density as, $\mathcal{L}=\sum_{i=1}^{N}[\frac{i}{2}(\tilde{\psi}_i^{*}\partial_t\tilde{\psi}_i-\tilde{\psi}_i\partial_t\tilde{\psi}_i^{*})(1-\frac{a_i^2}{|\tilde{\psi}_i|^2}\delta_{D,i})-\frac{1}{2}|\partial_x\tilde{\psi}_i|^2-\sum_{j=1}^{N}\frac{g_{ij}}{2}(|\tilde{\psi}_j|^2-a_j^2\delta_{D,j})(|\tilde{\psi}_i|^2-a_i^2\delta_{D,i})]$ to obtain Euler-Lagrangian equations, where

$$\delta_{D,i}=\begin{cases}0, & \tilde{\psi}_i \text{ denotes bright soliton,}\\ 1, & \tilde{\psi}_i \text{ denotes dark soliton.}\end{cases}$$

In particular, the terms $1-\frac{a_i^2}{|\tilde{\psi}_i|^2}\delta_{D,i}$ and $|\tilde{\psi}_i|^2-a_i^2\delta_{D,i}$ were introduced in the Lagrangian density for dark soliton [29,75], mainly because a dark soliton is a density dip with a striking phase jump, in sharp contrast to a bright soliton. Notably, the modified $\mathcal{L}$ corresponds to the following dynamical equations:

$$\mathrm{i}\partial_t\tilde{\mathbf{\Psi}}=\left[-\frac{1}{2}\partial_{xx}+\sum_{j=1}^{N}g_{ij}\left(|\tilde{\psi}_j|^2-a_j^2\delta_{D,j}\right)\right]\tilde{\mathbf{\Psi}}, \tag{A2}$$

where $\tilde{\mathbf{\Psi}}=(\tilde{\psi}_1\cdots,\tilde{\psi}_i,\cdots\tilde{\psi}_N)^T$. This is different from Eq. (A1), but the solutions of the two dynamical equations can be transformed to each other by the phase factor $\psi_i/\tilde{\psi}_i=e^{i\theta_i(t)}$, where $\theta_i(t)=-\sum_{j=1}^{N}g_{ij}a_j^2t\delta_{D,j}$. After testing many different trial functions with the Lagrangian method and the direct numerical simulations, we find that the ideal trial wave function of the ith component should be written as

$$\tilde{\psi}_{iD}=i\sqrt{a_i^2-f_i^2(t)}+f_i(t)\tanh\{w_i(t)[x-b(t)]\},$$
$$\tilde{\psi}_{iB}=f_i(t)\mathrm{sech}\{w_i(t)[x-b(t)]\}e^{i\{\xi_i(t)+[x-b(t)]\phi_i(t)\}}, \tag{A3}$$

where $\tilde{\psi}_{iD}$ ($\tilde{\psi}_{iB}$) denotes the wave function of dark (bright) soliton. f_i and w_i describe the amplitude and width of dark (bright) soliton, respectively. a_i is the background of dark soliton components, and $b(t)$ denotes the soliton center's position. ξ_i is the time-dependent phase of the bright soliton component, and ϕ_i is related to the velocity of the bright soliton. Specifically, we take these width parameters independently when deriving the Euler-Lagrangian equations. After obtaining the equations of motion by the Euler-Lagrangian formula, we finally calculate G_{ij}, $\frac{\partial G_{ij}}{\partial w_i}$, and $\frac{\partial G_{ij}}{\partial w_j}$ by setting $w_i=w_j=w$. These operations are in sharp contrast to the previously used ansatz form, which brings us more constrain conditions on soliton parameters, and finally enables us to obtain exact analytical soliton solutions. This is our main point for developing variational methods.

The Lagrangian (L) can be obtained by substituting Eq. (A3) into $\mathcal{L}$ and integrating over space from $-\infty$ to $+\infty$, which is

$$\begin{aligned}L=\sum_{i}^{N}\Bigg\{&\left[2f_i^2(t)\phi_i(t)\frac{b'(t)}{w_i(t)}-2f_i^2(t)\frac{\xi_i'(t)}{w_i(t)}-\frac{1}{3}f_i^2(t)w_i(t)-f_i^2(t)\frac{\phi_i^2(t)}{w_i(t)}-\frac{2}{3}g_{ii}\frac{f_i^4(t)}{w_i(t)}\right]\tilde{\delta}_{D,i}+\Bigg\{-2f_i(t)\sqrt{a_i^2-f_i^2(t)}b'(t)\\&+2a_i^2\arcsin\left[\frac{f_i(t)}{a_i}\right]b'(t)-\frac{2}{3}f_i^2(t)w_i(t)-\frac{2}{3}g_{ii}\frac{f_i^4(t)}{w_i(t)}\Bigg\}\delta_{D,i}-\sum_{j(j\neq i)}^{N}g_{ij}f_i^2(t)f_j^2(t)G_{ij}\Bigg\},\quad(\tilde{\delta}_{D,i}=|\delta_{D,i}-1|).\end{aligned} \tag{A4}$$

The G_{ij} denotes $\int_{-\infty}^{+\infty}(\pm\mathrm{sech}^2\{w_i(t)[x-b(t)]\})(\pm\mathrm{sech}^2\{w_j(t)[x-b(t)]\})dx$, where the sign $-(+)$ corresponds to dark (bright) component. The integral is hard to be calculated with different width parameters. We will deal with this problem after deriving the Euler-Lagrangian equations.

We obtain the Euler-Lagrangian equations from $\frac{d}{dt}[\frac{\partial L}{\partial\alpha'(t)}]=\frac{\partial L}{\partial\alpha(t)}$, where $\alpha(t)$ denotes the variational parameters $f_i(t)$, $w_i(t)$, $w_j(t)$, $\phi_i(t)$, $\xi_i(t)$, and $b(t)$, respectively. If the ith component admits a dark soliton, the nontrivial equations can be written as

$$\alpha(t)=f_i(t),\quad 2g_{ii}f_i^2+\tfrac{3}{2}g_{ij}f_j^2w_iG_{ij}+w_i^2-\frac{3f_iw_ib'(t)}{\sqrt{a_i^2-f_i^2}}=0, \tag{A5}$$

$$\alpha(t)=w_i(t),\quad -g_{ii}f_i^2+\tfrac{3}{2}g_{ij}f_j^2w_i^2\frac{\partial G_{ij}}{\partial w_i}+w_i^2=0. \tag{A6}$$

If the ith component admits a bright soliton, the nontrivial equations can be written as

$$\alpha(t)=\phi_i(t),\quad \phi_i(t)=b'(t) \tag{A7}$$

$$\alpha(t)=f_i(t),\quad 4g_{ii}f_i^2+3g_{ij}f_j^2w_iG_{ij}+w_i^2-3b'(t)^2+6\xi_i'(t)=0, \tag{A8}$$

$$\alpha(t)=w_i(t),\quad -2g_{ii}f_i^2+3g_{ij}f_j^2w_i^2\frac{\partial G_{ij}}{\partial w_i}+w_i^2+3b'(t)^2-6\xi_i'(t)=0. \tag{A9}$$

We further get the constrain conditions on the amplitude f_i and w_i for the N-component soliton from Eqs. (A6), (A8), and (A9), which are simplified as

$$\begin{aligned}
&\left[g_{11}f_1^2+\sum_{j=2}^{N}\left(\frac{3}{2}g_{1j}f_j^2w_1G_{1j}+\frac{3}{2}g_{1j}f_j^2w_1^2\frac{\partial G_{1j}}{\partial w_1}\right)+w_1^2\right]\tilde{\delta}_{D,1}+\left(-g_{11}f_1^2+\sum_{j=2}^{N}\frac{3}{2}g_{1j}f_j^2w_1^2\frac{\partial G_{1j}}{\partial w_1}+w_1^2\right)\delta_{D,1}=0,\\
&\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
&\left[g_{ii}f_i^2+\sum_{j\neq i}^{N}\left(\frac{3}{2}g_{ij}f_j^2w_1G_{ij}+\frac{3}{2}g_{ij}f_j^2w_i^2\frac{\partial G_{ij}}{\partial w_i^2}\right)+w_i^2\right]\tilde{\delta}_{D,i}+\left(-g_{ii}f_i^2+\sum_{j\neq i}^{N}\frac{3}{2}g_{ij}f_j^2w_i^2\frac{\partial G_{ij}}{\partial w_i}+w_i^2\right)\delta_{D,i}=0,\\
&\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
&\left[\sum_{j=1}^{N-1}\left(\frac{3}{2}g_{Nj}f_j^2w_NG_{Nj}+\frac{3}{2}g_{Nj}f_j^2w_N^2\frac{\partial G_{Nj}}{\partial w_N}\right)+g_{NN}f_N^2+w_N^2\right]\tilde{\delta}_{D,N}+\left(\sum_{j=1}^{N-1}\frac{3}{2}g_{Nj}f_j^2w_N^2\frac{\partial G_{Nj}}{\partial w_N}-g_{NN}f_N^2+w_N^2\right)\delta_{D,N}=0.
\end{aligned}\tag{A10}$$

The $\frac{\partial G_{ij}}{\partial w_i}$ is $-2\int_{-\infty}^{+\infty}\{\pm[x-b(t)]\mathrm{sech}^2\{w_i(t)[x-b(t)]\}\tanh\{w_i(t)[x-b(t)]\}\}(\pm\,\mathrm{sech}^2\{w_j(t)[x-b(t)]\})dx$, for which the sign $-(+)$ corresponds to the dark (bright) component. After obtaining the above Euler-Lagrangian equations, it is very critical to deal with the integral factors G_{ij} and $\frac{\partial G_{ij}}{\partial w_i}$.

It is indeed hard to analytically calculate the integral factors with different width parameters. Interestingly, we find that it is reasonable to calculate them with setting $w_i = w_j = w$, which enables us to obtain exact analytical soliton solutions. For example, we show the results for one case in which the ith (jth) component admits a dark (bright) soliton. The related integral results can be given as follows:

$$\begin{aligned}
G_{ij}&=\int_{-\infty}^{+\infty}(-\mathrm{sech}^2\{w_i(t)[x-b(t)]\})(\mathrm{sech}^2\{w_j(t)[x-b(t)]\})dx\big|_{w_i=w_j}=-\frac{4}{3w},\\
\frac{\partial G_{ij}}{\partial w_i}&=-2\int_{-\infty}^{+\infty}\{-[x-b(t)]\mathrm{sech}^2\{w_i(t)[x-b(t)]\}\tanh\{w_i(t)[x-b(t)]\}\}(\mathrm{sech}^2\{w_j(t)[x-b(t)]\})dx|_{w_i=w_j}=\frac{2}{3w^2},\\
\frac{\partial G_{ij}}{\partial w_j}&=-2\int_{-\infty}^{+\infty}(\mathrm{sech}^2\{w_i(t)[x-b(t)]\})\{-[x-b(t)]\mathrm{sech}^2\{w_j(t)[x-b(t)]\}\tanh\{w_j(t)[x-b(t)]\}\}dx|_{w_i=w_j}=\frac{2}{3w^2}.
\end{aligned}$$

The other cases for G_{ij} and $\frac{\partial G_{ij}}{\partial w_i}$ can be calculated in similar ways.

Finally, the constraint conditions on f_i, w can be simplified as

$$\begin{pmatrix} g_{11} & \cdots g_{1j}\cdots & g_{1N} & 1\\ \vdots & \vdots & \vdots & \vdots\\ g_{i1} & \cdots g_{ij}\cdots & g_{iN} & 1\\ \vdots & \vdots & \vdots & \vdots\\ g_{N1} & \cdots g_{Nj}\cdots & g_{NN} & 1\end{pmatrix}\begin{pmatrix}\pm f_1^2\\ \vdots\\ \pm f_i^2\\ \vdots\\ \pm f_N^2\\ w^2\end{pmatrix}=0.\tag{A11}$$

The other parameters can be obtained as $b(t)=vt$, $\phi_i=v$, $\xi_i(t)=\frac{1}{2}(w^2+v^2)t+\theta_i(t)$, and the soliton velocity $v=w\sqrt{a_i^2-f_i^2}/f_i$ from other Euler-Lagrangian equations. The constraint equation on backgrounds of dark components is $a_i/a_j=f_i/f_j$. It is emphasized that all width parameters become identical one w in the expressions for soliton solutions. The exact soliton solutions for N-component coupled nonlinear Schrödinger equations with arbitrary nonlinear parameters can be given by solving the constraint conditions. The expression of soliton solutions in any ith component for Eq. (A1) can be given as

$$\begin{aligned}
\psi_{iD}&=\left\{i\sqrt{a_i^2-f_i^2}+f_i\tanh[w(x-vt)]\right\}e^{i\theta_i(t)},\\
\psi_{iB}&=f_i\mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v+\theta_i(t)]}.
\end{aligned}\tag{A12}$$

Different types of vector solitons usually exist in different regions in nonlinear parameter spaces. Their existence regions can be also clarified by the constrain conditions on soliton parameters.

APPENDIX B: THE EXPLICIT EXPRESSIONS OF VECTOR SOLITONS IN TWO- AND THREE-COMPONENT SYSTEM

1. The solutions of two-component soliton

The vector solitons for two-component systems can be still classified in four families, similar to the integrable model. We would like to present the explicit expressions for them as follows:

(i) The exact solution of bright-bright soliton is

$$\psi_{1B} = f_1 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]},$$
$$\psi_{2B} = f_2 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]}, \quad \text{(B1)}$$

where $f_2 = \sqrt{\frac{g_{11}-g_{12}}{g_{22}-g_{12}}} f_1$, $w = \sqrt{\frac{g_{12}^2-g_{11}g_{22}}{g_{22}-g_{12}}} f_1$.

(ii) The exact solution for dark-bright (DB) or bright-dark (BD) soliton is

$$\psi_{1D} = \{i\sqrt{a_1^2-f_1^2} + f_1 \tanh[w(x-vt)]\}e^{-ig_{11}a_1^2 t},$$
$$\psi_{2B} = f_2 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]-ig_{21}a_1^2 t}, \quad \text{(B2)}$$

where $f_2 = \sqrt{\frac{g_{11}-g_{12}}{g_{12}-g_{22}}} f_1$, $w = \sqrt{\frac{g_{12}^2-g_{11}g_{22}}{g_{12}-g_{22}}} f_1$, and $v = \sqrt{\frac{g_{12}^2-g_{11}g_{22}}{g_{12}-g_{22}}}\sqrt{a_1^2-f_1^2}$. The difference between DB and BD is defined by total density of the two components, which admits a dip (for DB) or hump (for BD). Especially, the total density can be uniform when $2g_{12} = g_{11} + g_{22}$, the vector soliton in this case will become the spin soliton.

(iii) The exact solution of dark-dark soliton is

$$\psi_{1D} = \{i\sqrt{a_1^2-f_1^2} + f_1 \tanh[w(x-vt)]\}e^{-ig_{11}a_1^2 t - ig_{12}a_2^2 t},$$
$$\psi_{2D} = \{i\sqrt{a_2^2-f_2^2} + f_2 \tanh[w(x-vt)]\}e^{-ig_{21}a_1^2 t - ig_{22}a_2^2 t}, \quad \text{(B3)}$$

where $f_2 = \sqrt{\frac{g_{11}-g_{12}}{g_{22}-g_{12}}} f_1$, $w = \sqrt{\frac{g_{12}^2-g_{11}g_{22}}{g_{12}-g_{22}}} f_1$, $v = \sqrt{\frac{g_{12}^2-g_{11}g_{22}}{g_{12}-g_{22}}}\sqrt{a_1^2-f_1^2}$ and $\frac{a_2}{a_1} = \frac{f_2}{f_1}$.

2. The solutions of three-component soliton

We would like to present the explicit expressions for vector soliton solutions in three-component coupled systems as follows.

(i) The exact solution of bright-bright-bright soliton is

$$\psi_{1B} = f_1 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]},$$
$$\psi_{2B} = f_2 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]},$$
$$\psi_{3B} = f_3 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]}, \quad \text{(B4)}$$

where $f_2 = \sqrt{A_1} f_1$, $f_3 = \sqrt{A_2} f_1$, and $w = \sqrt{A_3} f_1$. The parameters A_1, A_2, and A_3 are as follows:

$$A_1 = \frac{g_{13}^2 - g_{11}g_{33} + g_{12}(g_{33}-g_{13}) + g_{23}(g_{11}-g_{13})}{g_{23}^2 - g_{22}g_{33} + g_{12}(g_{33}-g_{23}) + g_{13}(g_{22}-g_{23})},$$
$$A_2 = \frac{g_{12}^2 - g_{11}g_{22} + g_{13}(g_{22}-g_{12}) + g_{23}(g_{11}-g_{12})}{g_{23}^2 - g_{22}g_{33} + g_{12}(g_{33}-g_{23}) + g_{13}(g_{22}-g_{23})},$$
$$A_3 = \frac{2g_{12}g_{13}g_{23} - g_{13}^2 g_{22} - g_{12}^2 g_{33} - g_{11}(g_{23}^2 - g_{22}g_{33})}{g_{23}^2 - g_{22}g_{33} + g_{12}(g_{33}-g_{23}) + g_{13}(g_{22}-g_{23})}.$$

They hold for all vector soltions in three-component models.

(ii) The exact solution of dark-bright-bright soliton is

$$\psi_{1D} = \{i\sqrt{a_1^2-f_1^2} + f_1 \tanh[w(x-vt)]\}e^{-ig_{11}a_1^2 t},$$
$$\psi_{2B} = f_2 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]-ig_{21}a_1^2 t},$$
$$\psi_{3B} = f_3 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]-ig_{31}a_1^2 t}, \quad \text{(B5)}$$

where $f_2 = \sqrt{-A_1} f_1$, $f_3 = \sqrt{-A_2} f_1$, $w = \sqrt{-A_3} f_1$, and $v = \frac{w}{f_1}\sqrt{a_1^2-f_1^2}$.

(iii) The exact solution of dark-dark-bright soliton is

$$\psi_{1D} = \{i\sqrt{a_1^2-f_1^2} + f_1 \tanh[w(x-vt)]\}e^{-ig_{11}a_1^2 t - ig_{12}a_2^2 t},$$
$$\psi_{2D} = \{i\sqrt{a_2^2-f_2^2} + f_2 \tanh[w(x-vt)]\}e^{-ig_{21}a_1^2 t - ig_{22}a_2^2 t},$$
$$\psi_{3B} = f_3 \mathrm{sech}[w(x-vt)]e^{i[\frac{1}{2}(w^2+v^2)t+(x-vt)v]-ig_{31}a_1^2 t - ig_{32}a_2^2 t}, \quad \text{(B6)}$$

where $f_2 = \sqrt{A_1} f_1$, $f_3 = \sqrt{-A_2} f_1$, $w = \sqrt{-A_3} f_1$, $v = \frac{w}{f_1}\sqrt{a_1^2-f_1^2}$, and $\frac{a_2}{a_1} = \frac{f_2}{f_1}$.

(iv) The exact solution of dark-dark-dark soliton is

$$\psi_{1D} = \{i\sqrt{a_1^2-f_1^2} + f_1 \tanh[w(x-vt)]\}e^{-ig_{11}a_1^2 t - ig_{12}a_2^2 t - ig_{13}a_3^2 t},$$
$$\psi_{2D} = \{i\sqrt{a_2^2-f_2^2} + f_2 \tanh[w(x-vt)]\}e^{-ig_{21}a_1^2 t - ig_{22}a_2^2 t - ig_{23}a_3^2 t},$$
$$\psi_{3D} = \{i\sqrt{a_3^2-f_3^2} + f_3 \tanh[w(x-vt)]\}e^{-ig_{31}a_1^2 t - ig_{32}a_2^2 t - ig_{33}a_3^2 t}, \quad \text{(B7)}$$

where $f_2 = \sqrt{A_1} f_1$, $f_3 = \sqrt{A_2} f_1$, $w = \sqrt{-A_3} f_1$, $v = \frac{w}{f_1}\sqrt{a_1^2-f_1^2}$, and $a_1 : a_2 : a_3 = f_1 : f_2 : f_3$.

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