Occupation time statistics of the fractional Brownian motion in a finite domain

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We study statistics of occupation times for a fractional Brownian motion (fBm), which is a typical model of a non-Markov process. Due to the non-Markovian nature, recurrence times to the origin depend on the history. Numerical simulations indicate that dependence on the sum of successive recurrence times becomes weak. As a result, the distribution of the occupation time in a finite domain follows the Mittag-Leffler distribution when the Hurst exponent of the fBm is close to $1/2$. We show this distributional behavior of a time-averaged observable by renewal theory. This result is an extension of the distributional limit theorem known as the Darling-Kac theorem in general Markov processes to non-Markov processes.

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I. INTRODUCTION

Statistical properties of occupation times in stochastic processes are well studied in mathematics [1–5] as well as physics [6–10]. The occupation time of a set A for a stochastic process $x(t)$ is defined as

$$T_A(t) \equiv \int_0^t V_A(x(\tau))\mathrm{d}\tau, \tag{1}$$

where $V(x)$ is the characteristic function of set A. There are two typical fluctuations in occupation times. One is the generalized arcsine distribution [2], and the other is the Mittag-Leffler distribution [1]. The arcsine law is known as fluctuations of occupation time of leads in coin tossing [4]. The generalized arcsine law is observed to a plethora of systems such as fluorescence of quantum dots [11], currents in stochastic thermodynamics [12], drift in anomalous diffusion [13], α-percentile options in stock prices [14,15], and leads in sports games [16]. On the other hand, the Mittag-Leffler distribution is known as a universal distribution for time averages of integrable observables in infinite ergodic theory [17]. Recently, this universal law is applied to nonstationary processes such as anomalous diffusion [18–26] and laser cooling processes [27–30].

Fractional Brownian motion (fBm) is proposed as a model of natural time series [31], which is a generalization of Brownian motion to a non-Markov process. The fBm $B_H(t)$ is defined as

$$B_H(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\int_{-\infty}^t (t-t')^{H-\frac{1}{2}}dB(t'), \tag{2}$$

where $dB(t)$ is an increment of the ordinary Brownian motion, $\Gamma(x)$ represents the Gamma function, and the parameter $H \in (0, 1)$ is the Hurst exponent. The dynamic equation of the Brownian motion $B(t)$ can be described by

$$\dot{B}(t) = \sqrt{2}\xi(t), \tag{3}$$

where $\xi(t)$ is a white Gaussian noise with variance 1, i.e., the diffusion coefficient of $B(t)$ is $D = 1$. The ordinary Brownian motion corresponds to the case of $H = 1/2$. By a simple calculation of Eq. (2), the covariance of the fBm is given by

$$\langle B_H(t)B_H(s)\rangle = t^{2H} + s^{2H} - |t-s|^{2H}. \tag{4}$$

The increment of the fBm is not independent but strongly correlated with the past increments. This interdependence of the increments is a characteristic of the fBm. This non-Markovian nature explains many varieties of natural processes such as anomalous diffusion in cells [32–38], viscoelastic motions of lipid molecules [32,39], and polymer translocation [40–42].

Two types of generalizations of occupation time statistics have been investigated. One is an aging extension of the distributional limit theorem [43–46], and the other is a non-Markovian extension of the generalized arcsine law [47,48]. As a generalization to non-Markov processes, the distribution of occupation times on a semi-infinite interval such as the positive or negative side is obtained for the fBm [47,48]. However, this type of generalization has not yet been done for occupation times in a finite domain. In Markov processes, Darling and Kac showed that the distribution of occupation times in a finite interval follows the Mittag-Leffler distribution [1]. Therefore, it is important to unravel the distribution of occupation times in a finite interval for non-Markov processes. The occupation time statistics for the fBm in a finite domain are relevant to trajectory-to-trajectory fluctuations of the occupation times of non-Markovian natural phenomena such as diffusion in a cell, polymer translocation, and water flows in hydrology in a specific finite domain [49].

The paper aims to obtain the distribution of trajectory-to-trajectory fluctuations of occupation times in a finite interval for the fBm. The rest of the paper is organized as follows. In Sec. II, we review some well-known results in Markov processes and derive the distribution of occupation times in a finite interval using renewal theory. In Sec. III, we conduct numerical simulations of the fBm and obtain the distribution of occupation times on a finite interval in the fBm. Sections IV

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and V are devoted to the discussion and the conclusion, respectively.

II. OCCUPATION TIME STATISTICS IN MARKOV PROCESSES

Here, we review the first passage time (FPT) statistics and derive a distributional limit theorem for the occupation time on a finite interval in the Brownian motion. The FPT is a time when the particle has first reached a target. To derive the FPT distribution of the Brownian motion, we consider the Brownian motion with the absorbing boundary condition at the origin. The Fokker-Planck equation (diffusion equation) is described by

$$\partial_t Z_+^{(0)}(x_0, x, t) = \partial_x^2 Z_+^{(0)}(x_0, x, t), \quad (5)$$

$$Z_+^{(0)}(x_0, x, 0) = \delta(x - x_0), \quad (6)$$

where $Z_+^{(0)}(x_0, x, t)$ is the propagator, i.e., the probability density function (PDF) of position x at time t with $B(0) = x_0 > 0$. By using the method of images, the propagator $Z_+^{(0)}(x_0, x, 0)$ can be obtained as

$$Z_+^{(0)}(x_0, x, t) = \frac{1}{\sqrt{4\pi t}}\left[e^{-(x-x_0)^2/(4t)} - e^{-(x+x_0)^2/(4t)}\right]. \quad (7)$$

The survival probability is obtained by integrating $Z_+^{(0)}(x_0, x, t)$ with respect to x from 0 to ∞:

$$S^{(0)}(x_0, t) = \int_0^\infty Z_+^{(0)}(x_0, x, t)\mathrm{d}x = \mathrm{erf}\left(\frac{x_0}{2\sqrt{t}}\right), \quad (8)$$

where $\mathrm{erf}(x)$ is the error function. It follows that the PDF of FPT t of a Brownian particle starting from x to the origin becomes

$$p_x(t) = \frac{|x|}{2\sqrt{\pi t^3}} \exp\left(-\frac{x^2}{4t}\right). \quad (9)$$

The elegant proof of the FPT distribution is also given in Ref. [50]. Therefore, the asymptotic behavior becomes $p_x(t) \propto t^{-3/2}$, which means that the mean first passage time diverges, i.e., $\langle t\rangle = \int_0^\infty t\, p_x(t)\mathrm{d}t = \infty$.

Next, we consider the occupation time of a set. By the self-similarity property of the Brownian motion, this set can be assumed to be arbitrary. For simplicity, we assume A = [-1, 1]. To apply renewal theory, we construct a dichotomous process from the Brownian motion (see Fig. 1). In particular, we define $R(t)$ as

$$R(t) = V_A(B(t)) = \begin{cases} 1, & B(t) \in A \\ 0, & B(t) \notin A \end{cases}, \quad (10)$$

where $V_A(x)$ is the characteristic function of set A. When the Brownian particle exits set A and reaches $1 + \Delta x$, the recurrence time to set A is the same as the FPT of the Brownian motion starting from Δx to the origin. Therefore, the PDF of the recurrence time to set A from $1 + \Delta x$ is given by $p_{\Delta x}(t)$. In the same way, the PDF of the recurrence time to set A from $-1 - \Delta x$ is given by $p_{\Delta x}(t)$. Moreover, the exit time, which is a time when a Brownian particle exits set A after entering set A, is an independent random variable with a finite mean because the increment of the Brownian motion does not depend

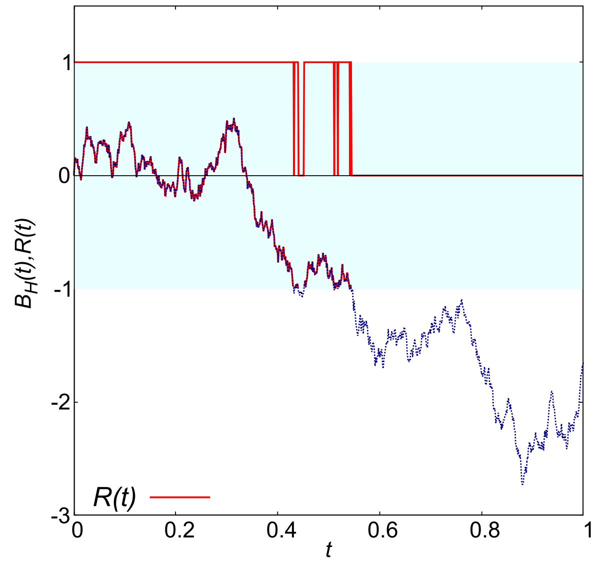

FIG. 1. Dichotomous process obtained from the Brownian motion. The dashed line represents a trajectory of the Brownian motion. The solid line corresponds to $R(t) = 1$, where $A = [-1, 1]$.

on the history, i.e., Markov property. It follows that duration times of a state $R(t) = 0$ and $R(t) = 1$ are independent and identically distributed (IID) random variables.

A dichotomous process of $R(t)$ is considered to be an alternating renewal process [51]. We denote the PDFs of durations of states $R(t) = 0$ or $R(t) = 1$ by $\psi_0(\tau)$ and $\psi_1(\tau)$, respectively, which are related to the PDFs for recurrence times and exit times. The occupation time of set A can be represented by

$$T_A(t) = \sum_{k=1}^{N_t} \tau_k, \quad (11)$$

where τ_k is the kth duration time for state $R(t) = 1$ and N_t is the number of changes from $R(t) = 0$ to $R(t) = 1$ until time t. Let S_r be the time at the rth renewal, we have

$$\mathrm{Prob}\{N_t < r\} = \mathrm{Prob}\{S_r > t\}. \quad (12)$$

The asymptotic behavior of $\psi_0(\tau)$ is the same as that of $p_\varepsilon(\tau)$, i.e., $\psi_0(\tau) \propto \tau^{-3/2}$ for $\tau \to \infty$. In what follows, we consider a general situation that the PDF $\psi_0(\tau)$ follows $\psi_0(\tau) \propto \tau^{-1-\alpha}$ ($\tau \to \infty$). Using Eq. (12) and $r = xt^\alpha$, we have

$$\mathrm{Prob}\left\{\frac{N_t}{t^\alpha} < x\right\} = \mathrm{Prob}\left\{\frac{S_r}{r^{1/\alpha}} > x^{-\frac{1}{\alpha}}\right\}. \quad (13)$$

By the generalized central limit theorem [52], we have the long-time limit

$$\mathrm{Prob}\left\{\frac{N_t}{t^\alpha} < x\right\} = \int_{x^{-\frac{1}{\alpha}}}^\infty g_\alpha(x')\mathrm{d}x', \quad (14)$$

where $g_\alpha(x)$ is the one-sided Lévy density [52]

$$g_\alpha(x) = -\frac{1}{\pi x}\sum_{k=1}^\infty \frac{\Gamma(1+k\alpha)}{k!}(-cx^{-\alpha})^k \sin(k\pi\alpha), \quad (15)$$

and c is a scale factor. The PDF of $N_t/t^\alpha \propto T_A(t)/t^\alpha$ is given by differentiating Eq. (14), i.e.,

$$f_\alpha(x) = \frac{g_\alpha(x^{-\frac{1}{\alpha}})}{\alpha x^{1+\frac{1}{\alpha}}}, \tag{16}$$

which is called the Mittag-Leffler (ML) distribution with order α. The case $\alpha = 1/2$ corresponds to Brownian motion, and the limit distribution converges to the half Gaussian. The all moments of the ML distribution are finite. Therefore, the expectation of $T_A(t)$ becomes

$$\langle T_A(t)\rangle \propto t^\alpha, \tag{17}$$

in the long-time limit. We consider the normalized occupation time defined as

$$\tilde{T}_A(t) = \frac{1}{u(t)}\int_0^t V_A(x(\tau))\mathrm{d}\tau, \tag{18}$$

where $u(t) = \langle T_A(t)\rangle \propto t^\alpha$. Thus, the ensemble average of $\tilde{T}_A(t)$ is $\langle \tilde{T}_A(t)\rangle = 1$ by definition, and the moments are

$$\lim_{t\to\infty}\langle \tilde{T}_A(t)^k\rangle = \frac{k!\Gamma(1+\alpha)}{\Gamma(\alpha k+1)}. \tag{19}$$

Therefore, the normalized occupation time converges in distribution. This result is known as the Darling-Kac theorem in general Markov processes [1].

III. OCCUPATION TIME STATISTICS IN A NON-MARKOV PROCESS

A. Survival probability of the fractional Brownian motion

Using the path integral method [47,48,53], the propagator $Z_+(x_0, x, t)$ of the fBm restricted on the positive side can be written as

$$Z(x_0, x, t) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]e^{-\mathcal{S}[x]}\Theta[x], \tag{20}$$

where $\Theta(x)$ is the indicator function, which is 1 if $x(t) > 0$ and 0 otherwise, and $\mathcal{S}[x]$ is the action

$$\mathcal{S}[x] = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \frac{1}{2} x(t_1)G(t_1,t_2)x(t_2), \tag{21}$$

where $G(t_1, t_2)$ is the kernel of the action given by

$$G^{-1}(t_1, t_2) = \langle x(t_1)x(t_2)\rangle. \tag{22}$$

In the following, we will calculate Eq. (20), using a perturbative approach for $H = 1/2 + \varepsilon$, given in [47,48].

By using perturbation theory, Eq. (21) can be written as

$$\begin{aligned}\mathcal{S}[x] &= \frac{1}{D}\left[\mathcal{S}^{(0)}[x] + \varepsilon\mathcal{S}^{(1)}[x] + O(\varepsilon^2)\right]\\ &= \frac{1}{D}\left[\frac{1}{4}\int_0^t \mathrm{d}t'(\partial_{t'}x(t'))^2 - \frac{\varepsilon}{2}\int_0^t \mathrm{d}t_1\int_{t_1+\omega}^t \mathrm{d}t_2 \frac{\partial_{t_1}x(t_1)\partial_{t_2}x(t_2)}{t_2-t_1} + O(\varepsilon^2)\right],\end{aligned} \tag{23}$$

where the diffusion constant D is given by

$$D = e^{\varepsilon 2(1+\ln\omega)+O(\varepsilon^2)}. \tag{24}$$

Note that the diffusion constant becomes $D = 1$ for $H = 1/2$. The small-time cutoff $\omega > 0$ is introduced to regularize the integrals in the action. The action Eq. (23) in Eq. (20) gives

$$\begin{aligned}Z(x_0, x, t) &= \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]e^{-\mathcal{S}_0}\Theta[x] + \varepsilon\int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x]\left(\frac{1}{2}\mathcal{S}_1 + 2\mathcal{S}_0(1+\ln\omega)\right)e^{-\mathcal{S}_0}\Theta[x] + o(\varepsilon^2)\\ &=: Z^{(0)}(x_0, x, t) + \varepsilon Z^{(1)}(x_0, x, t),\end{aligned} \tag{25}$$

where $Z^{(0)}(x_0, x, t)$ is the propagator of the Brownian motion and $Z^{(1)}(x_0, x, t)$ is a perturbative term in the fBm.

1. Semi-infinite domain

The propagator in the semi-infinite domain is calculated as Eq. (B1). Using Eq. (B1), the survival probability $S(x_0, t)$ is calculated in the same way as in the Brownian case [Eq. (8)] [53]. For $t \to \infty$, we have

$$S(x_0, t) \propto t^{-\frac{1}{2}}[1 + \varepsilon\ln(t)] = O(t^{-\alpha}), \tag{26}$$

where the first term, i.e., $t^{-\frac{1}{2}}$ originates from the Brownian and the second term is obtained from $B_2(x, t)$ in Appendix B. Using the small-ε expansion:

$$t^\varepsilon \cong 1 + \varepsilon\ln t, \tag{27}$$

we have the power law exponent α given by

$$\alpha = \tfrac{1}{2} - \varepsilon + O(\varepsilon^2) \cong 1 - H. \tag{28}$$

We confirmed numerically Eq. (28) (see Fig. 2).

2. Finite domain

One can obtain the propagator in a finite domain (see Appendix C). The survival probability $S_{\rm in}(t)$ in a finite domain is obtained by integrating Eq. (C4) with respect to x in the interval $[-L, L]$. The asymptotic behavior of the leading order becomes an exponential decay:

$$S_{\rm in}(t) \propto e^{-t}(1 + \varepsilon t\mathrm{Ei}(t)e^{-t}) \sim e^{-t}\ (t\to\infty), \tag{29}$$

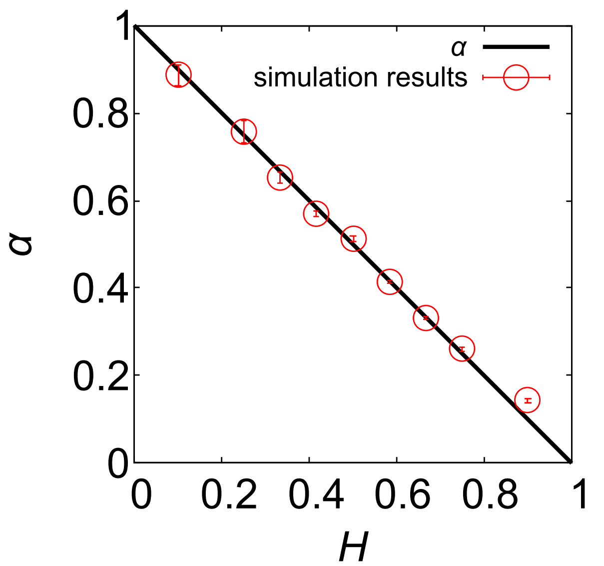

FIG. 2. Power-law exponent of the FPT distribution. Symbols are the results of numerical simulations. The solid line represents $\alpha = 1 - H$. We use 10^5 trajectories and the FPT is defined by the time until $B_H(t)$ with $B_H(0) = 0$ passes the origin for the first time.

where $\mathrm{Ei}(t)$ is the exponential integral. When t is large enough, the effect of the perturbation term can be ignored. Moreover, the mean survival time is finite.

B. Numerical simulations of correlations of durations of states in a dichotomous process

We consider recurrence times for the fBm. Here, we define the recurrence time as the time elapsed before the particle passes the origin again after passing the origin. Let τ_n be the nth recurrence time (see Fig. 3). In the case of $H = 1/2$, there is no correlation between successive recurrence times. On the other hand, when $H \neq 1/2$, there is a correlation of successive recurrence times due to the non-Markov property of the fBm. In fact, the occupation time statistics for the fBm differ from those for Markov processes due to the correlation. For a dichotomous process with a power-law waiting-time distribution, the PDF of the occupation time of one of the two states follows the generalized arcsine law [2,6], where the PDF has a peak at the middle when the power-law exponent α is greater than 1/2. On the other hand, for $H < 1/2$, which corresponds to $\alpha > 1/2$, the PDF of the occupation time for the positive side in the fBm does not have a peak [53]. This is evidence of a correlation between successive recurrence times. In particular, the successive recurrence time has a negative correlation. When the previous recurrence time is large, the recurrence time tends to be small. Therefore, the ratio of occupation time on the positive side to that on the negative side rarely takes one. In other words, the peak of the PDF does not appear.

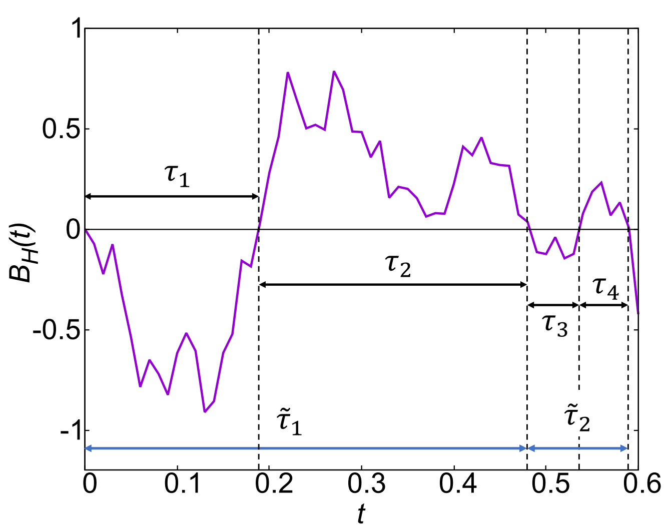

FIG. 3. A schematic figure of two recurrence times, i.e., τ_1 and $\tilde{\tau}_1$, where $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are given by $\tilde{\tau}_1 = \tau_1 + \tau_2$ and $\tilde{\tau}_2 = \tau_3 + \tau_4$. The solid line represents a trajectory of the Brownian motion with $\Delta t = 0.01$, i.e., $H = 1/2$. The trajectory passes the origin four times.

Here, we consider the sum of two consecutive recurrence times, i.e., $\tilde{\tau}_n = \tau_n + \tau_{n+1}$, instead of recurrence times (see Fig. 3). If τ_{n-1} depends only on τ_n, the correlation between $\tilde{\tau}_n$ and $\tilde{\tau}_{n+1}$ becomes smaller than the correlation between τ_n and τ_{n+1}. In Appendix D, we show that the correlation between $\tilde{\tau}_n$ and $\tilde{\tau}_{n+1}$ becomes weak when τ_{n-1} depends only on τ_n. In renewal processes, the waiting times are IID random variables. Therefore, we expect that the renewal theory can be applied to a dichotomous process constructed by the fBm.

C. Trajectory-to-trajectory fluctuations of occupation times

We investigate trajectory-to-trajectory fluctuations of occupation times in a finite domain for the fBm. Figure 4 shows the distribution of the occupation times in a finite domain for the fBms with different Hurst exponents. Numerical simulations indicate that the distribution converges to the ML distribution with order $\alpha = 1 - H$, where we calculate the distribution $P(\tilde{T}_A)$ of the normalized occupation time $\tilde{T}_A$ defined by $\tilde{T}_A \equiv T_A(t)/\langle T_A(t)\rangle$.

To quantify the trajectory-to-trajectory fluctuations of the occupation times, we calculate the ergodicity breaking (EB) parameter of the occupation time for different Hurst exponents [18]. The EB parameter is defined by the relative variance of the occupation time:

$$\mathrm{EB}(t) \equiv \frac{\langle T_A(t)^2\rangle - \langle T_A(t)\rangle^2}{\langle T_A(t)\rangle^2}. \tag{30}$$

If it converges to zero, it is ergodic in the sense that the time average of the occupation time converges to a constant in the long-time limit. By extensive numerical simulations, we confirm that the EB parameter converges to a nonzero value which is the same as that for the ML distribution (see Fig. 5). In particular, the EB parameters of $T_A(t)$ for the fBm converge to those of the ML distributions for $H = 5/12$ and $7/12$ ($|\varepsilon| = 1/12$), i.e., small $|\varepsilon|$. However, there are small deviations from the ML distribution for large $|\varepsilon|$. It follows that the occupation time distribution for a finite domain of the fBm converges to the ML distribution for small $|\varepsilon|$.

Here, we explain why the ML distribution is observed in the normalized occupation time distribution using renewal theory. The occupation time can be represented by the number N_t of changes of the values for $R(t)$ until time t and the mean

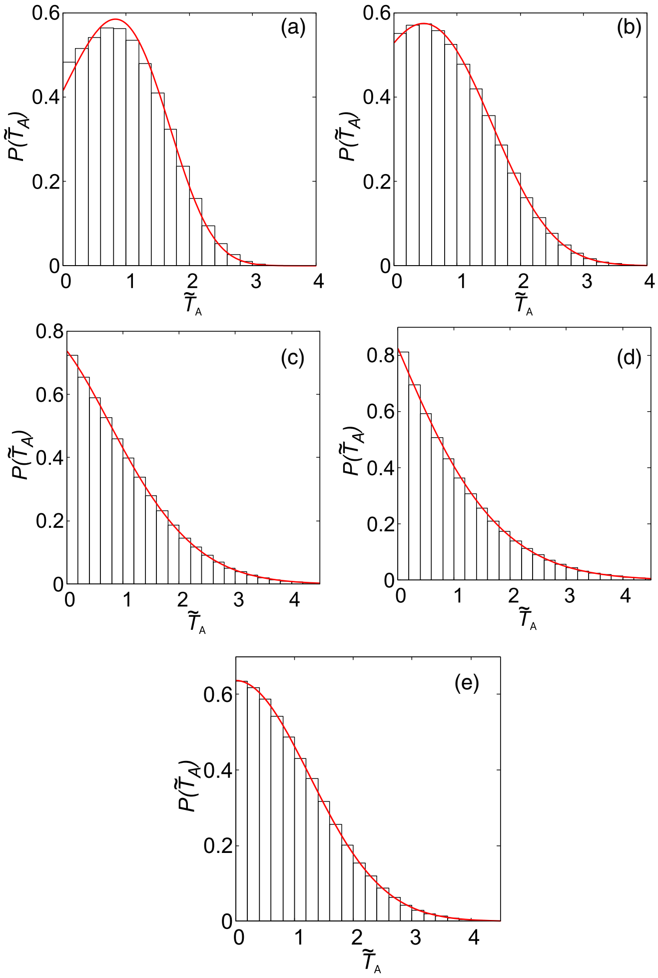

FIG. 4. Probability density function of the normalized occupation times for the fBm for (a) $H = 1/3$, (b) $H = 5/17$, (c) $H = 7/12$, (d) $H = 2/3$, and (e) $H = 1/2$. The histogram is the results of numerical simulations for finite domain $A = [-0.001, 0.001]$, where 2^{20} steps and 10^6 particles were used. The solid lines are the PDFs of the ML distributions, i.e., Eq. (16), with $\alpha = 1 - H$.

exit time $\langle \tau_{\rm in} \rangle$, i.e., the mean duration time of $R(t) = 1$:

$$T_A(t) \sim \frac{N_t}{2} \times \langle \tau_{\rm in} \rangle, \tag{31}$$

for $t \to \infty$. Because $\langle \tau_{\rm in} \rangle$ is finite, the distribution of $T_A(t)$ is equivalent to that of N_t. In Sec. II, we have shown that the distribution of N_t follows the ML distribution with order α when the PDF of durations, i.e., $\psi_0(\tau)$ follows a power-law distribution with exponent α. In the fBm, durations of a state outside the set A are power-law distributed but not independent random variables. However, the sum of the durations becomes independent as shown in the previous subsection. Therefore, a renewal theory with power-law distributed duration time with exponent $1 - H$ can be applied. It follows that the normalized occupation time distribution follows the ML distribution with order $1 - H$.

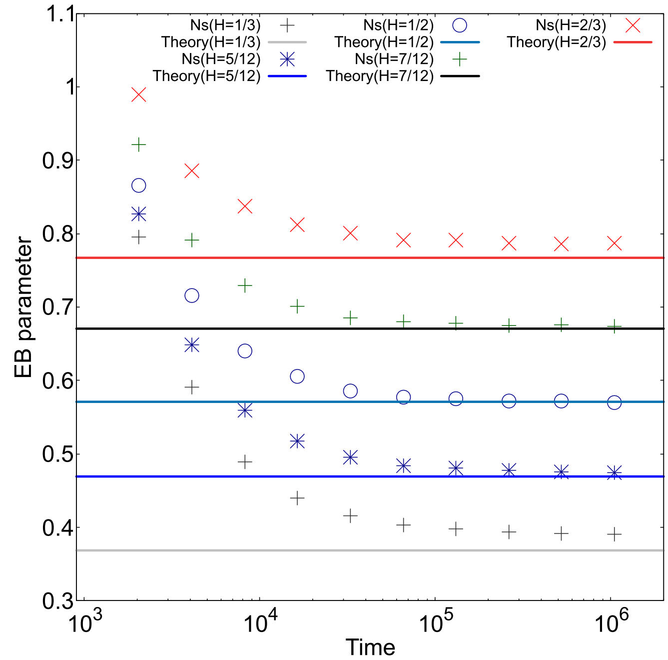

FIG. 5. The EB parameter of $T_A(t)$ as a function of the measurement time for different H (= 1/3, 5/12, 1/2, 7/12, and 2/3), where $A = [-0.001, 0.001]$ and 10^6 trajectories were used. Solid lines represent the EB parameters of the ML distributions.

IV. DISCUSSION

In the previous studies [54,55], it has been found that the PDF of the first passage time can be expressed by the sum of two power-law distributions with exponents $1 - H$ and $2H$. In particular, it suggests that the PDF follows a power-law distribution with exponent $1 - H$ for short time and a power-law distribution with exponent $2H$ for large time when $H < 1/3$. Thus, the power-law exponent of the FPT distribution in the asymptotic behavior becomes different from Eq. (28). Our results of the power-law exponent of the FPT distribution in Fig. 2 are estimated for a finite time, which suggests that the exponent is in a good agreement with $1 - H$. However, in the asymptotic behavior, it will change. This will affect our statement that the normalized occupation time distribution converges to the Mittag-Leffler distribution with order $1 - H$ for $H < 1/3$.

As an application of the occupation time statistics of the fBm in a finite domain, we consider a Langevin equation with fluctuating diffusivity [56]. The dynamic equation is described by $\dot{x}(t) = \sqrt{2D(t)}\xi(t)$, where $\xi(t)$ is a white Gaussian noise. We define the fluctuating diffusivity $D(t)$ by the fBm:

$$D(t) = \begin{cases} 1, & B_H(t) \in A \\ 0, & B_H(t) \notin A \end{cases}, \tag{32}$$

where A is a finite domain. The PDF of durations of a freezing state, i.e., $D(t) = 0$ follows a power-law distribution. In particular, durations are IID random variables following a power-law distribution with exponent $\alpha = 1/2$ when $H = 1/2$. The motion of this model is almost the same as that of the continuous-time random walk (CTRW). Since the durations

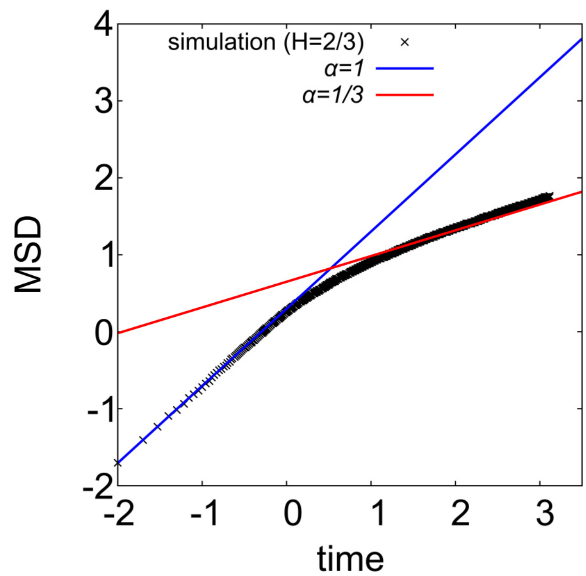

FIG. 6. Ensemble-averaged mean square displacement for the fractional CTRW for $H = 2/3$ in the log-log scale, where $A = [-0.01, 0.1]$. The blue line represents a linear scaling, i.e., $\langle x(t)^2\rangle = 2t$ and the red line represents a sublinear scaling, i.e., subdiffusion: $\langle x(t)^2\rangle \propto t^\alpha$. We use 10^4 trajectories and 2^{17} steps.

are not independent random variables when $H \neq 1/2$, this model is a kind of extension of the CTRW to the CTRW with correlated waiting times. Thus, we call this model the fractional CTRW. The mean square displacement (MSD) of $x(t)$ can be obtained as

$$\langle x(t)^2\rangle = 2\langle D(t)\rangle t. \quad (33)$$

The ensemble average of $D(t)$ is calculated by $\langle D(t)\rangle \propto \langle T_A(t)\rangle/t \propto t^{\alpha-1}$. Therefore, the MSD exhibits anomalous diffusion

$$\langle x(t)^2\rangle \propto t^\alpha, \quad (34)$$

in the long-time limit. For short-time behavior, the MSD exhibits normal diffusion because $D(t) = 1$ when $B_H(t) \in A$. In Fig. 6, we calculated the MSD and confirmed normal diffusion in the short time and the subdiffusion in the long-time behavior, which is a similar behavior to a recently proposed model of the generalized Langevin equation with fluctuating diffusivity [57].

V. CONCLUSION

We found trajectory-to-trajectory fluctuations of occupation times in a finite domain for the fBm. In particular, the EB parameter converges to a nonzero constant for $t \to \infty$. Thus, the normalized occupation time converges in distribution for any H. When $|\varepsilon|$ is small enough, the EB parameter converges to that of the ML distribution with order $1 - H$. On the other hand, the EB parameter slightly deviates from that of the ML distribution when $|\varepsilon|$ is not small enough. This is a non-Markov extension of the infinite ergodic theory. The Hurst exponent H in the fBm characterizes the occupation time statistics.

ACKNOWLEDGMENT

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APPENDIX A: NUMERICAL METHOD OF THE FBM

We conduct numerical simulations of the fBm by two different methods [58].

1. Method 1

Mandelbrot and Van Ness defined the fractional Brownian motion. An approximate expression of the fBm defined by Eq. (2) is

$$B_H(t) \sim \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \sum_{i=-\infty}^{nt} \left(t - \frac{i}{n}\right)^{H-\frac{1}{2}} n^{-\frac{1}{2}} \xi_i, \quad (A1)$$

where ξ_i is the white Gaussian noise. The difference equation of the fBm is approximately given by

$$\Delta B_H(t) = \frac{n^{-\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)} \left(\sum_{i=n(t-1)}^{nt-1} \left(t - \frac{i}{n}\right)^{H-\frac{1}{2}} \xi_i + \sum_{i=-M+nt}^{n(t-1)-1} \left(\left(t - \frac{i}{n}\right)^{H-\frac{1}{2}} - \left(t - 1 - \frac{i}{n}\right)^{H-\frac{1}{2}} \right) \xi_i \right), \quad (A2)$$

where $\Delta B_H(t) = B_H(t+1) - B_H(t)$.

2. Method 2: Davis and Harte method

The mean and the covariance of the increments $\Delta B_H(t)$ are given by $\langle \Delta B_H(t)\rangle = 0$ and $\langle \Delta B_H(t)\Delta B_H(s)\rangle = \gamma(t - s)$, respectively, where

$$\gamma(t) = (t+1)^{2H} + (t-1)^{2H} - 2t^{2H}, \quad (A3)$$

for positive integers t. First, we define the linear array $\{W_n\}$ as follows:

$$W_0 = V_0, \quad (A4)$$

$$W_n = \frac{1}{\sqrt{2}}(V_n + iV_{2N-n}), \text{ for } n = 1, \ldots, N-1, \quad (A5)$$

$$W_N = V_N, \quad (A6)$$

$$W_n = -\frac{i}{\sqrt{2}}(V_n + iV_{2N-n}), \text{ for } n = N+1, \ldots, 2N-1, \quad (A7)$$

where $V_0, V_1, \cdots, V_{2N-1}$ are independent Gaussian random numbers, with $\langle V_n\rangle = 0$ and $\langle V_m V_n\rangle = \delta_{m,n}$. This mean is $\langle W_n\rangle = 0$ and covariance is

$$\langle W_n W_{n'}\rangle = \delta_{n,0}\delta_{n',0} + \delta_{n+n',2N}. \quad (A8)$$

Second, we define the linear array $\{\lambda_n\}$ as follows:

$$\lambda_n = \sum_{k=0}^{2N-1} \Gamma_k e^{i\pi \frac{nk}{N}}, \tag{A9}$$

where

$$\Gamma_k = \begin{cases} \gamma(k), & (0 \leqslant k \leqslant N) \\ \gamma(2N-k), & (N+1 \leqslant k \leqslant 2N-1) \end{cases}.$$

The set of increments of the fBm is obtained as:

$$\Delta B_H(n) = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} W_k \sqrt{\lambda_k} e^{i\pi \frac{nk}{N}}. \tag{A10}$$

APPENDIX B: DETAILED FORM FOR THE PROPAGATOR IN A SEMI-INFINITE DOMAIN

The propagator $Z_+(x_0, x, t)$ is calculated by using the perturbation theory [53]

$$\begin{aligned} Z_+(x_0, x, t) &= Z_+^{(0)}(x_0, x, t) + \varepsilon Z_+^{(1)}(x_0, x, t) \\ &= \frac{x}{\sqrt{4\pi t^3}} e^{-\frac{x^2}{4t}} (1 + \varepsilon(A(x,t) + B_0(x_0) \\ &\quad + B_1(x,t) + B_2(x,t) - a_1 \ln x_0)), \end{aligned} \tag{B1}$$

where $A(x,t) = (1 + \omega(\frac{x^2}{2t} - 3))$, $B_0(x_0) = 3 - 2\gamma_E + \ln(\frac{\omega}{2}) - 4\ln x_0$, $B_1(x,t) = (\frac{x^2}{2t} - 2)(\gamma_E - 1 + 2\ln\frac{x}{\sqrt{2t}} - \ln 2) - 2$, $B_2(x,t) = (\frac{x^2}{2t} - 1)\ln(\frac{4t}{\omega}) + \mathcal{I}(x,t)$, and

$$\mathcal{I}(x,t) = \frac{x^2}{12t} {}_2F_2\left(1, 1; \frac{5}{2}, 3; \frac{x^2}{4t}\right) + \pi\left(1 - \frac{x^2}{2t}\right) \mathrm{erfi}\left(\frac{x}{2\sqrt{t}}\right). \tag{B2}$$

APPENDIX C: DERIVATION OF THE PROPAGATOR IN A FINITE DOMAIN

Here, we derive the propagator in a finite domain. The propagator can be obtained in the same way as in Eq. (25). We calculate the propagator $Z_{\rm in}(x, x_0, t)$. The propagator in a finite domain can be written as

$$\begin{aligned} Z_{\rm in}(x_0, x, t) &= \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] e^{-\mathcal{S}_0} \Theta_A[x] + \varepsilon \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \left(\frac{1}{2}\mathcal{S}_1 + 2\mathcal{S}_0(1 + \ln\omega)\right) e^{-\mathcal{S}_0} \Theta_A[x] + o(\varepsilon^2) \\ &= Z_{\rm in}^{(0)}(x_0, x, t) + \varepsilon Z_{\rm in}^{(1)}(x_0, x, t), \end{aligned} \tag{C1}$$

where $\Theta_A[x]$ is 1 if $B_H(t) \in A \equiv [-L, L]$ and 0 otherwise, and $Z_{\rm in}^{(0)}$ is the propagator the Brownian for the finite domain. By the separation of variables, we have

$$Z_{\rm in}^{(0)}(x_0, x, t) = \sum_{n=1}^{\infty} \frac{1}{L} \cos\left(\frac{2n-1}{2L}\pi x_0\right) \cos\left(\frac{2n-1}{2L}\pi x\right) e^{-(\frac{2n-1}{2L}\pi)^2 t}. \tag{C2}$$

Using Eq. (23), $Z_{\rm in}^{(1)}(x_0, x, t)$ can be calculated as follows:

$$\begin{aligned} Z_{\rm in}^{(1)}(x_0, x, t) &= -\int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \left(-\frac{1}{4}\int_0^t dt_1 \int_0^t dt_2 \frac{\partial_{t_1} x(t_1) \partial_{t_2} x(t_2)}{|t_1 - t_2|} - 2\mathcal{S}^{(0)}[x](1 + \log\omega)\right) e^{-\mathcal{S}^{(0)}[x]} \Theta[x] \\ &= 2(1 + \log\omega) \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \mathcal{S}^{(0)}[x] e^{-\mathcal{S}^{(0)}[x]} \Theta[x] + \frac{1}{4}\int_0^t dt_1 \int_0^t dt_2 \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \frac{\partial_{t_1} x(t_1) \partial_{t_2} x(t_2)}{|t_1 - t_2|} e^{-\mathcal{S}^{(0)}[x]} \Theta[x] \\ &=: Z_{\rm in}^{\alpha}(x_0, x, t) + Z_{\rm in}^{\beta}(x_0, x, t). \end{aligned}$$

The first term becomes

$$\begin{aligned} Z_{\rm in}^{\alpha}(x_0, x, t) &= 2(1 + \log\omega) \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] \mathcal{S}^{(0)}[x] e^{-\mathcal{S}^{(0)}[x]} \Theta[x] \\ &= -2(1 + \log\omega) \frac{\partial}{\partial a}\bigg|_{a=1} \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}[x] e^{-a\mathcal{S}^{(0)}} \Theta[x] \\ &= -2(1 + \log\omega) \frac{\partial}{\partial a}\bigg|_{a=1} \sum_n \frac{a}{L} \cos\left(\frac{2n-1}{2L}\pi x_0\right) \cos\left(\frac{2n-1}{2L}\pi x\right) e^{-\frac{1}{a}(\frac{2n-1}{2L}\pi)^2 t} \\ &= -2(1 + \log\omega) \sum_n \left(1 + \left(\frac{2n-1}{2L}\pi\right)^2 t\right) \cos\left(\frac{2n-1}{2L}\pi x_0\right) \cos\left(\frac{2n-1}{2L}\pi x\right) e^{-(\frac{2n-1}{2L}\pi)^2 t}, \end{aligned}$$

and the second term becomes

$$Z_{\rm in}^{\beta}(x_0,x,t) = \frac{1}{8L^5}\int_0^t dt_1 \int_{t_1+\omega}^t dt_2 \int_{-L}^{L} dx_1 \int_{-L}^{L} dx_2 \sum_{j,k,l,m,n} \frac{1}{t_2-t_1} \cos\left(\frac{2j-1}{2L}\pi x_1\right) e^{-\left(\frac{2j-1}{2L}\pi\right)^2 t_1}\left(1+\cos\left(\frac{2k-1}{L}\pi x_1\right)\right)$$
$$\times \cos\left(\frac{2l-1}{2L}\pi x_1\right)\cos\left(\frac{2l-1}{2L}\pi x_2\right) e^{-\left(\frac{2l-1}{2L}\pi\right)^2 (t_2-t_1)}\left(1+\cos\left(\frac{2m-1}{L}\pi x_2\right)\right)$$
$$\times \cos\left(\frac{2n-1}{2L}\pi x_2\right)\cos\left(\frac{2n-1}{2L}\pi x\right) e^{-\left(\frac{2n-1}{2L}\pi\right)^2 (t-t_2)}. \tag{C3}$$

Calculating the above equations by using MATHEMATICA, we obtain the leading term of $Z_{\rm in}(x_0, x, t)$ in the long-time limit:

$$Z_{\rm in}(x_0,x,t) \sim e^{-t}(1-\varepsilon t \,{\rm Ei}(t)e^{-t}) f(x), \tag{C4}$$

where $f(x)$ is a function of x.

APPENDIX D: CORRELATION OF THE SUM OF THE RANDOM VARIABLES

Here, we estimate the correlation of a sum of dependent positive random variables, $X_n + X_{n+1}$. X_n is independent of X_k for $k = 1, \cdots, n-2$. In other words, X_n depends on X_{n-1}. We quantify the strength of the correlation as follows:

$$C_X = \frac{\langle X_1X_2\rangle - \langle X_1\rangle\langle X_2\rangle}{\langle X\rangle^2}, \tag{D1}$$

where $\langle X_n\rangle$ is the expected value of X_n. Furthermore, the strength of the correlation for the sum of consecutive random variables, i.e., $Y_n = X_{2n-1} + X_{2n}$, becomes

$$C_Y = \frac{\langle Y_1Y_2\rangle - \langle Y_1\rangle\langle Y_2\rangle}{\langle Y\rangle^2} = \frac{\langle X_2X_3\rangle - \langle X_2\rangle\langle X_3\rangle}{\langle Y\rangle^2} = C_X\frac{\langle X\rangle^2}{\langle Y\rangle^2} < C_X, \tag{D2}$$

where we used $\langle Y\rangle > \langle X\rangle$ because random variable X is positive. Therefore, the correlation of Y_n gets weaker than that of X_n.

[1] D. A. Darling and M. Kac, On occupation times for Markov processes, Trans. Amer. Math. Soc. **84**, 444 (1957).

[2] J. Lamperti, An occupation time theorem for a class of stochastic processes, Trans. Amer. Math. Soc. **88**, 380 (1958).

[3] E. Dynkin, Some limit theorems for sums of independent random variables with infinite mathematical expectations, *Selected Translations in Mathematical Statistics and Probability*, Vol. 1 (American Mathematical Society, Providence, RI, 1961), p. 171.

[4] W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley & Sons, London-New York-Sydney-Toronto, 1968), Vol. 1.

[5] Y. Kasahara, Limit theorems of occupation times for Markov processes, Publ. RIMS, Kyoto Univ. **12**, 801 (1977).

[6] C. Godrèche and J. M. Luck, Statistics of the occupation time of renewal processes, J. Stat. Phys. **104**, 489 (2001).

[7] S. N. Majumdar and A. Comtet, Local and Occupation Time of a Particle Diffusing in a Random Medium, Phys. Rev. Lett. **89**, 060601 (2002).

[8] S. Burov and E. Barkai, Occupation Time Statistics in the Quenched Trap Model, Phys. Rev. Lett. **98**, 250601 (2007).

[9] A. Rebenshtok and E. Barkai, Distribution of Time-Averaged Observables for Weak Ergodicity Breaking, Phys. Rev. Lett. **99**, 210601 (2007).

[10] A. Rebenshtok and E. Barkai, Weakly non-ergodic statistical physics, J. Stat. Phys. **133**, 565 (2008).

[11] X. Brokmann, J. P. Hermier, G. Messin, P. Desbiolles, J. P. Bouchaud, and M. Dahan, Statistical Aging and Nonergodicity in the Fluorescence of Single Nanocrystals, Phys. Rev. Lett. **90**, 120601 (2003).

[12] A. C. Barato, E. Roldán, I. A. Martínez, and S. Pigolotti, Arcsine Laws in Stochastic Thermodynamics, Phys. Rev. Lett. **121**, 090601 (2018).

[13] T. Akimoto, Distributional Response to Biases in Deterministic Superdiffusion, Phys. Rev. Lett. **108**, 164101 (2012).

[14] R. Miura, A note on look-back options based on order statistics, Hitotsubashi J. Commerce Manage. **27**, 15 (1992).

[15] J. Akahori, Some formulae for a new type of path-dependent option, Ann. Appl. Probab. **5**, 383 (1995).

[16] A. Clauset, M. Kogan, and S. Redner, Safe leads and lead changes in competitive team sports, Phys. Rev. E **91**, 062815 (2015).

[17] J. Aaronson, *An Introduction to Infinite Ergodic Theory* (American Mathematical Society, Providence, 1997).

[18] Y. He, S. Burov, R. Metzler, and E. Barkai, Random Time-Scale Invariant Diffusion and Transport Coefficients, Phys. Rev. Lett. **101**, 058101 (2008).

[19] T. Neusius, I. M. Sokolov, and J. C. Smith, Subdiffusion in time-averaged, confined random walks, Phys. Rev. E **80**, 011109 (2009).

[20] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, Anomalous diffusion models and their properties: Non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking, Phys. Chem. Chem. Phys. **16**, 24128 (2014).

[21] T. Miyaguchi and T. Akimoto, Intrinsic randomness of transport coefficient in subdiffusion with static disorder, Phys. Rev. E **83**, 031926 (2011).

[22] T. Miyaguchi and T. Akimoto, Ergodic properties of continuous-time random walks: Finite-size effects and ensemble dependences, Phys. Rev. E **87**, 032130 (2013).

[23] T. Akimoto and E. Yamamoto, Distributional behavior of diffusion coefficients obtained by single trajectories in annealed transit time model, J. Stat. Mech. (2016) 123201.

[24] T. Akimoto and E. Yamamoto, Distributional behaviors of time-averaged observables in the Langevin equation with fluctuating diffusivity: Normal diffusion but anomalous fluctuations, Phys. Rev. E **93**, 062109 (2016).

[25] T. Albers and G. Radons, Exact Results for the Nonergodicity of d-Dimensional Generalized Lévy Walks, Phys. Rev. Lett. **120**, 104501 (2018).

[26] T. Albers and G. Radons, Nonergodicity of d-dimensional generalized Lévy walks and their relation to other space-time coupled models, Phys. Rev. E **105**, 014113 (2022).

[27] T. Akimoto, E. Barkai, and G. Radons, Infinite invariant density in a semi-Markov process with continuous state variables, Phys. Rev. E **101**, 052112 (2020).

[28] E. Barkai, G. Radons, and T. Akimoto, Transitions in the Ergodicity of Subrecoil-Laser-Cooled Gases, Phys. Rev. Lett. **127**, 140605 (2021).

[29] E. Barkai, G. Radons, and T. Akimoto, Gas of sub-recoiled laser cooled atoms described by infinite ergodic theory, J. Chem. Phys. **156**, 044118 (2022).

[30] T. Akimoto, E. Barkai, and G. Radons, Infinite ergodic theory for three heterogeneous stochastic models with application to subrecoil laser cooling, Phys. Rev. E **105**, 064126 (2022).

[31] B. B. Mandelbrot and J. W. V. Ness, Fractional Brownian motions, fractional noises and applications, SIAM Rev. **10**, 422 (1968).

[32] J.-H. Jeon, V. Tejedor, S. Burov, E. Barkai, C. Selhuber-Unkel, K. Berg-Sørensen, L. Oddershede, and R. Metzler, *In Vivo* Anomalous Diffusion and Weak Ergodicity Breaking of Lipid Granules, Phys. Rev. Lett. **106**, 048103 (2011).

[33] J. Szymanski and M. Weiss, Elucidating the Origin of Anomalous Diffusion in Crowded Fluids, Phys. Rev. Lett. **103**, 038102 (2009).

[34] I. Bronstein, Y. Israel, E. Kepten, S. Mai, Y. Shav-Tal, E. Barkai, and Y. Garini, Transient Anomalous Diffusion of Telomeres in the Nucleus of Mammalian Cells, Phys. Rev. Lett. **103**, 018102 (2009).

[35] S. C. Weber, A. J. Spakowitz, and J. A. Theriot, Bacterial Chromosomal Loci Move Subdiffusively through a Viscoelastic Cytoplasm, Phys. Rev. Lett. **104**, 238102 (2010).

[36] E. Kepten, I. Bronshtein, and Y. Garini, Ergodicity convergence test suggests telomere motion obeys fractional dynamics, Phys. Rev. E **83**, 041919 (2011).

[37] K. Burnecki, E. Kepten, J. Janczura, I. Bronshtein, Y. Garini, and A. Weron, Universal algorithm for identification of fractional Brownian motion. A case of telomere subdiffusion, Biophys. J. **103**, 1839 (2012).

[38] A. Sabri, X. Xu, D. Krapf, and M. Weiss, Elucidating the Origin of Heterogeneous Anomalous Diffusion in the Cytoplasm of Mammalian Cells, Phys. Rev. Lett. **125**, 058101 (2020).

[39] T. Akimoto, E. Yamamoto, K. Yasuoka, Y. Hirano, and M. Yasui, Non-Gaussian Fluctuations Resulting from Power-Law Trapping in a Lipid Bilayer, Phys. Rev. Lett. **107**, 178103 (2011).

[40] A. Zoia, A. Rosso, and S. N. Majumdar, Asymptotic Behavior of Self-Affine Processes in Semi-Infinite Domains, Phys. Rev. Lett. **102**, 120602 (2009).

[41] J. L. A. Dubbeldam, V. G. Rostiashvili, A. Milchev, and T. A. Vilgis, Fractional Brownian motion approach to polymer translocation: The governing equation of motion, Phys. Rev. E **83**, 011802 (2011).

[42] V. V. Palyulin, T. Ala-Nissila, and R. Metzler, Polymer translocation: the first two decades and the recent diversification, Soft Matter **10**, 9016 (2014).

[43] J. H. P. Schulz, E. Barkai, and R. Metzler, Aging Effects and Population Splitting in Single-Particle Trajectory Averages, Phys. Rev. Lett. **110**, 020602 (2013).

[44] T. Akimoto and E. Barkai, Aging generates regular motions in weakly chaotic systems, Phys. Rev. E **87**, 032915 (2013).

[45] J. H. P. Schulz, E. Barkai, and R. Metzler, Aging Renewal Theory and Application to Random Walks, Phys. Rev. X **4**, 011028 (2014).

[46] T. Akimoto, T. Sera, K. Yamato, and K. Yano, Aging arcsine law in Brownian motion and its generalization, Phys. Rev. E **102**, 032103 (2020).

[47] T. Sadhu, M. Delorme, and K. J. Wiese, Generalized Arcsine Laws for Fractional Brownian Motion, Phys. Rev. Lett. **120**, 040603 (2018).

[48] T. Sadhu and K. J. Wiese, Functionals of fractional Brownian motion and the three arcsine laws, Phys. Rev. E **104**, 054112 (2021).

[49] H. E. Hurst, Long term storage, An experimental study (1965).

[50] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus* (Springer Science & Business Media, 2012), Vol. 113.

[51] D. R. Cox, *Renewal Theory* (Methuen, London, 1962).

[52] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed. (Wiley, New York, 1971), Vol. 2.

[53] K. J. Wiese, S. N. Majumdar, and A. Rosso, Perturbation theory for fractional Brownian motion in presence of absorbing boundaries, Phys. Rev. E **83**, 061141 (2011).

[54] L. P. Sanders and T. Ambjörnsson, First passage times for a tracer particle in single file diffusion and fractional Brownian motion, J. Chem. Phys. **136**, 175103 (2012).

[55] M. Bologna, F. Vanni, A. Krokhin, and P. Grigolini, Memory effects in fractional Brownian motion with Hurst exponent $h < 1/3$, Phys. Rev. E **82**, 020102(R) (2010).

[56] T. Uneyama, T. Miyaguchi, and T. Akimoto, Fluctuation analysis of time-averaged mean-square displacement for the Langevin equation with time-dependent and fluctuating diffusivity, Phys. Rev. E **92**, 032140 (2015).

[57] T. Miyaguchi, Generalized Langevin equation with fluctuating diffusivity, Phys. Rev. Res. **4**, 043062 (2022).

[58] T. Dieker, Simulation of fractional Brownian motion, Ph.D. thesis, Masters Thesis, Department of Mathematical Sciences, University of Twente, 2004.