Nonequilibrium dynamics in a three-state opinion-formation model with stochastic extreme switches

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We investigate the nonequilibrium dynamics of a three-state kinetic exchange model of opinion formation, where switches between extreme states are possible, depending on the value of a parameter q. The mean field dynamical equations are derived and analyzed for any q. The fate of the system under the evolutionary rules used in S. Biswas *et al.* [Physica A **391**, 3257 (2012)] shows that it is dependent on the value of q and the initial state in general. For $q = 1$, which allows the extreme switches maximally, a quasiconservation in the dynamics is obtained which renders it equivalent to the voter model. For general q values, a "frozen" disordered fixed point is obtained which acts as an attractor for all initially disordered states. For other initial states, the order parameter grows with time t as $\exp[\alpha(q)t]$ where $\alpha = \frac{1-q}{3-q}$ for $q \neq 1$ and follows a power law behavior for $q = 1$. Numerical simulations using a fully connected agent-based model provide additional results like the system size dependence of the exit probability and consensus times that further accentuate the different behavior of the model for $q = 1$ and $q \neq 1$. The results are compared with the nonequilibrium phenomena in other well-known dynamical systems.

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I. INTRODUCTION

One of the main motivations in studying nonequilibrium phenomena is to check what kind of steady states can be reached using different initial conditions. In the well-known Ising-Glauber model at zero temperature, on lattices or networks, several studies have been made to show that the steady states may not be the equilibrium steady states [1–14]. Exit probability, a quantity related to the type of final state reached from an initially biased state, has also been studied extensively in recent times in spin and opinion-formation models [15–26]. In systems with more than two states, several other interesting features, like the two-stage ordering process, have been noted [26]. In addition, how a system evolves to a stable state starting from an unstable fixed point is also a matter of interest [27].

Opinion dynamics models relevant to social phenomena have received extensive attention recently [28–31]. These models typically show a rich nonequilibrium behavior. Usually, the opinion of an agent is updated following the interaction with other individuals; sometimes the influence of media is also incorporated. In the numerous models studied so far, the interaction and the choice of the interacting agent(s) play crucial roles. The simplest models involve binary opinions typically represented by 0,1 or ± 1. The voter model [32,33], in which an agent just copies the opinion of another randomly picked up agent, is one of the simplest and earliest opinion dynamics models. Later, models involving more complexities have been constructed [29,30]. The binary models obviously cannot capture all the intricacies of the real world. Hence, models with three or more opinion states as well as continuous values of opinions have been considered in the recent past. The voter model can be generalized with a larger number of states easily [34] while other multistate models which involve the effect of more neighbors have also been considered [35,36]. In comparison to the simple binary-state models, here the opinions are not merely flipped but can change in more than one possible way. We focus our attention on the so-called kinetic exchange models where pairwise interactions are considered at each step [37]. However, these models generally have some restrictions. In particular, in the kinetic exchange models most recently studied with three discrete opinion states quantified by -1, 0, and 1 (assumed to represent e.g., left, central, and right ideologies), a transition from 1 to -1 or vice versa (i.e., an extreme switch of opinion) is not allowed to the best of our knowledge [38–42]. Also, in many other similar three-state models such a restriction is imposed [43–49]. However, human behavior being complex and unpredictable, such switches cannot be completely ruled out. In fact, there are real-world examples where even political cadres or leaders shift their allegiance to parties with totally opposite principles [50,51]. The reasons may be associated with immediate gains and selfish interests, lack of strong ideological beliefs, etc. We consider a model for opinion dynamics where extreme switches are allowed to happen and see how the dynamics are affected by this. It may be added here that for the multistate voter model or Potts-type models, such extreme switches can take place; however, in the relevant studies, the effect of such switches has not been the issue of interest specifically [34–36].

In this article, we have considered a kinetic exchange model of opinion dynamics with three states, with the possibility of switching between extreme opinions. In the mean field approach, the equations for the time derivatives are set up for the three population densities of different opinions and solved numerically. We have introduced a parameter q which governs the probability with which switches between extreme opinions can occur and studied its effect on the time evolution. q varies between zero and unity; the zero case is already considered where no such switch is allowed [38]. In parallel, numerical simulations have been conducted using a fully connected agent-based model. The model and quantities of interest are discussed in the next section, followed by the results presented in Sec. III, and, finally in the concluding section, the results are discussed and compared to existing results in similar models.

II. MEAN FIELD KINETIC EXCHANGE MODEL

We have considered a kinetic exchange model (KEM) for opinion formation which incorporates three opinion values that are quantified by $0, \pm 1$. The possible correspondence with left, central, and right ideologies has already been mentioned. The three opinion values may even mimic a two-party voting system, where the ± 1 opinions represent support for the two parties while people with zero opinion (the neutral population) are those who refrain from voting for either of them. The opinion of an individual is updated by taking into account her present opinion and an interaction with a randomly chosen individual in the fully connected model. The opinion of the ith individual is denoted by $o_i(t)$. The time evolution of o_i, after an interaction with the kth individual, chosen randomly, is given by

$$o_i(t+1) = o_i(t) + \mu o_k(t), \tag{1}$$

where μ can be interpreted as an interaction parameter. The opinions are bounded in the sense $|o_i| \leqslant 1$ at all times and therefore o_i is taken as 1 (-1) if it is more (less) than 1 (-1). There is no self-interaction so $i \neq k$ in general. This evolutionary rule was introduced in [38]. Here time is assumed to be discrete but one can easily use a continuous-time model as will be done in this paper.

In several previous works [26,38–42], μ, the interaction parameter, has been chosen randomly, allowing also negative values albeit being bounded: $|\mu| \leqslant 1$. Such a bound allows a transition between opinion values with a difference of maximum ± 1 only. In the present work, the interaction parameter μ is allowed to take two discrete values. The values are $\mu = 1$ and $\mu = 2$ which occur with probabilities $1 - q$ and q, respectively. Hence, for example, if an agent with opinion $+1$ interacts with another with opinion -1 and $\mu = 2$, her opinion can change to -1, the other extreme value. The possibilities of all the interactions and resulting opinions are shown in Fig. 1 for the extreme values $q = 0$ and $q = 1$. Note that in the present work only positive values of μ are allowed.

The densities of the three populations with opinion $0, \pm 1$ are denoted by $f_0, f_{\pm 1}$ with $f_0 + f_{+1} + f_{-1} = 1$. The ensemble averaged order parameter obtained from the time dependent equations for the densities is given as $\langle O(t) \rangle = f_{+1} - f_{-1}$ with $-1 \leqslant \langle O(t) \rangle \leqslant 1$.

Usually, to study opinion dynamics models, one starts with a random disordered configuration such that the average opinion is 0. Given that there are three states, one can choose this

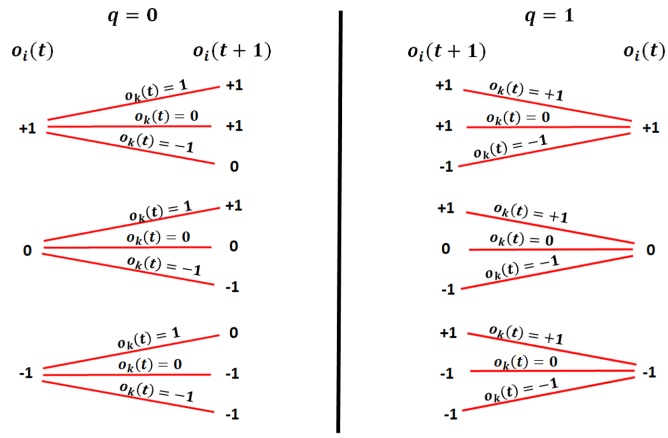

FIG. 1. The updated opinions of the ith individual following an interaction with another individual (denoted by k) for all possible opinion values at time t are shown for $q = 0$ (left panel), which implies $\mu = 1$ and $q = 1$ (right panel) for which $\mu = 2$.

state with different combinations of f_i's, keeping $f_{+1} = f_{-1}$. A conventional choice is $f_0 = f_{\pm 1} = 1/3$.

One can also study the effect of an initial bias in the distribution of opinions in the starting configuration of the system. The homogeneous configuration being one with all the densities equal to 1/3, one can consider a deviation from this such that the net opinion is nonzero by choosing $f_0 = 1/3$, $f_{+1} = 1/3 + \Delta/2$, and $f_{-1} = 1/3 - \Delta/2$. Here $-2/3 \leqslant \Delta \leqslant 2/3$. Apart from this case, one can take other initial configurations which have a net nonzero opinion. We have discussed such cases as well to show the initial configuration dependence.

We present in this paper the rate equations derived analytically using mean field theory for the three densities, and study their behavior as functions of time. The fixed point analysis of the equations present some interesting and nonintuitive results. We also obtain the exit probability. Here the exit probability E is considered as a function of Δ; i.e., $E(\Delta)$ is the probability that the final configuration has $f_{+1} = 1$ starting from $f_{+1} = 1/3 + \Delta/2$ and $f_{-1} = 1/3 - \Delta/2$. The saturation value of $\langle O \rangle$ is related to $E(\Delta)$ by

$$\langle O \rangle_{\text{sat}} = 2E(\Delta) - 1, \tag{2}$$

from which the exit probability can be estimated.

We have also conducted numerical simulations by considering an agent-based model where each agent can interact with any other agent. Here, the order parameter for a given configuration is defined as $\bar{O}(t) = \frac{|\sum o_i(t)|}{N}$ where N is the system size with $\langle \bar{O} \rangle$ denoting the configuration average. To calculate the exit probability $E(\Delta)$, we directly estimate the fraction of configurations which reach the consensus state with all opinions equal to 1.

To solve the coupled differential equations, the Euler method has been used and in the Monte Carlo method, system sizes ranging from 100 to 2^{16} have been simulated with the number of configurations ranging from 10^4 to 10^5.

From the simulations, it is also possible to estimate the average consensus times for different system sizes. All the results are presented in the next section.

III. RESULTS

We present in this section the mean field analytical solution in detail and also the results obtained from numerical simulations.

A. Mean field rate equations

To set up the rate equations for the f_i's, we need to treat the time variable as continuous. Assume that the opinion changes from i to j $(i, j = 0, \pm 1)$ in time Δt with the transition rate given by $w_{i\to j}$. Then we have the following set of w_{ij}'s:

$$\begin{aligned} w_{+1\to+1} &= f_{+1}^2 + f_0 f_{+1}, \\ w_{0\to+1} &= f_0 f_{+1}, \\ w_{-1\to+1} &= q f_{-1} f_{+1}, \\ w_{+1\to 0} &= (1-q) f_{+1} f_{-1}, \\ w_{0\to 0} &= f_0^2, \\ w_{-1\to 0} &= (1-q) f_{-1} f_{+1}, \\ w_{+1\to-1} &= q f_{+1} f_{-1}, \\ w_{0\to-1} &= f_0 f_{-1}, \\ w_{-1\to-1} &= f_{-1}^2 + f_0 f_{-1}. \end{aligned}$$

Hence, in general, we have $f_i(t+\Delta t) = f_i(t) + \sum_j w_{j\to i}\Delta t - \sum_j w_{i\to j}\Delta t$ such that, taking $\Delta t \to 0$, we get

$$\frac{df_{+1}}{dt} = f_0 f_{+1} - (1-q) f_{+1} f_{-1} \tag{3}$$

and

$$\frac{df_{-1}}{dt} = f_0 f_{-1} - (1-q) f_{-1} f_{+1}. \tag{4}$$

The time evolution of the ensemble averaged order parameter $\langle O(t)\rangle$ satisfies

$$\frac{d\langle O(t)\rangle}{dt} = f_0 \langle O(t)\rangle. \tag{5}$$

B. Fixed points and steady states

There will be some trivial fixed points corresponding to the initial conditions that have any of the three densities equal to 1. Here, obviously, there will be no evolution of the system at all. We consider more general cases in the following.

Equation (5) shows that a steady state for $\langle O\rangle$ is obtained when $\langle O\rangle = 0$ and/or $f_0 = 0$. Consider first the case when we have a disordered steady state, i.e., $\langle O(t\to\infty)\rangle = 0$. If the initial state is disordered, Eq. (5) indicates that O will remain zero, i.e., will not evolve although the individual densities may change in time. We show in the following that for all values of q, there exists a nontrivial disordered fixed point at which there is no evolution of not only the order parameter but also of the individual densities. This special fixed point may be termed the frozen fixed point (FFP) as the system does not undergo any change at all right from the beginning, although none of the densities have value unity.

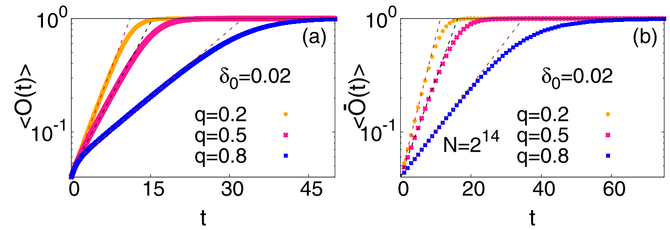

FIG. 2. The order parameter versus time variation near the frozen fixed point shows an exponential growth given by $\exp[\alpha(q)t]$ for any $q \neq 1$ as is evident from the (a) analytical as well as (b) simulation results. Also shown are the best fitted curves of the form given in Eq. (8).

At the FFP, $f_{+1} = f_{-1} = x > 0$ is a constant in time. Using this in Eq. (3) or Eq. (4), one gets

$$\frac{dx}{dt} = x - (3-q)x^2 = 0. \tag{6}$$

Ignoring the solution $x = 0$, we get $x = \dfrac{1}{3-q}$; i.e., the fixed point is given by

$$f_{+1} = f_{-1} = \frac{1}{3-q}, \quad f_0 = \frac{1-q}{3-q}. \tag{7}$$

The stability of the FFP can be checked by introducing small deviations about these values. These deviations can be introduced in different ways. We first consider a deviation such that the initial state has a nonzero order. Taking $f_{+1}(t) = x^* + \delta$ and $f_{-1}(t) = x^* - \delta$ where $x^* = \frac{1}{3-q}$ is the FFP value, we get from Eq. (3)

$$\frac{d(x^*+\delta)}{dt} = (x^*+\delta) - (2-q)(x^*+\delta)(x^*-\delta) - (x^*+\delta)^2.$$

Linearizing the above, one finally gets

$$\delta(t) = \delta_0 \exp[\alpha(q)t], \tag{8}$$

where δ_0 is the initial value of δ and

$$\alpha(q) = \frac{1-q}{3-q}. \tag{9}$$

The order parameter which is equal to $2\delta(t)$ should show the same behavior and indeed both the analytical solution and simulations show the expected initial exponential growth with the value of the exponent very close to α given by Eq. (9) (see Fig. 2). The simulation results have some finite size effects which is not unexpected [Fig. 2(b)]. The fact that $\alpha(q) > 0$ (for $q \neq 1$) implies the FFP is an unstable one for all values of q (except $q = 1$) when the deviation favors a finite order.

On the other hand, if we start from any disordered state, it can be shown that the system will flow towards the FFP. Here, with $f_+ = f_{-1}$ initially, they will remain the same in time as indicated by the rate equations. Hence the state can be characterized by $f_\pm = x^* + \rho$ and $f_0 = 1 - 2(x^* + \rho)$. In this case, we obtain

$$\rho(t) = \rho_0 \exp[-t], \tag{10}$$

i.e., the state flows to the FFP with a rate independent of q. Here ρ_0 is the initial value of ρ. Hence the FFP acts as an

TABLE I. Fraction of neutral opinion, positive opinion, and negative opinion considered for the initial configuration.

Initial configuration	f_0	f_{+1}	f_{-1}
Set I	1/3	1/3	1/3
Set II	1/2	1/4	1/4
Set III	5/10	3/10	2/10
Set IV	1/2	$1/4 + 0.01$	$1/4 - 0.01$

attractor for all initially disordered states. We have checked that the above form is indeed obeyed for any value of q (not shown).

C. Time evolution of the densities and the order parameter

Having obtained the fixed point and the behavior of the system close to it for any value of q, we proceed to study the time evolution of the relevant variables in more detail in this section. We will first discuss this for $q = 1$, which is obviously a special point in the parameter space. For other values of q also, we present the results which show consistency with the theoretical analysis. The data for the time evolution of the three densities and the order parameters have been obtained by numerically solving the analytical equations and also using Monte Carlo simulations. Only two of these four quantities are independent; however, it is more informative to present the results for all of them.

We have used four different sets of initial conditions stated in Table I. Of these, sets I and II are both disordered with set I corresponding to the homogeneous case. Set III represents an arbitrary initial condition that favors order. Set IV is also ordered initially and can be regarded as a small deviation from set II. For all these cases, for $q = 1$, f_0 falls rapidly within a few steps as shown in Figs. 3(a) and 4(a) obtained using both the methods. This behavior of f_0 may be easily understood from the transition possibilities, as we note (see Fig. 1) that, for opinion zero, there is no flux to this state from opinion values ± 1 while there is an outgoing flux when the zero opinion changes to other values. This leads to the behavior of $\langle O(t)\rangle$ in Eq. (5) as $\frac{d\langle O\rangle}{dt} \approx 0$; i.e., a quasiconservative system is obtained. Note that $f_0 \to 0$ implies $f_{\pm 1}$ are independent of time for $q = 1$, but not necessarily equal to 1 or 0. Hence, the consensus state (i.e., either f_{+1} or f_{-1} equal to 1) is not reached in general such that the value of the ensemble averaged opinion is less than 1. We exclude here the trivial cases where $f_{+1} = 1$ or $f_{-1} = 1$ initially. The results using the mean field equations are shown in Fig. 3 for different initial states given in Table I. It is seen that, as expected, for sets I and II, which are disordered initially, the system evolves to the FFP 0, 1/2, 1/2. For the other sets we see that the system reaches a steady state (which is not a consensus state) within a few steps with the final value of the order parameter close to the initial one and $f_0 = 0$. For sets III and IV, we have used initial states with a bias to the $+1$ opinion and $O(t)$ is therefore positive in all the cases.

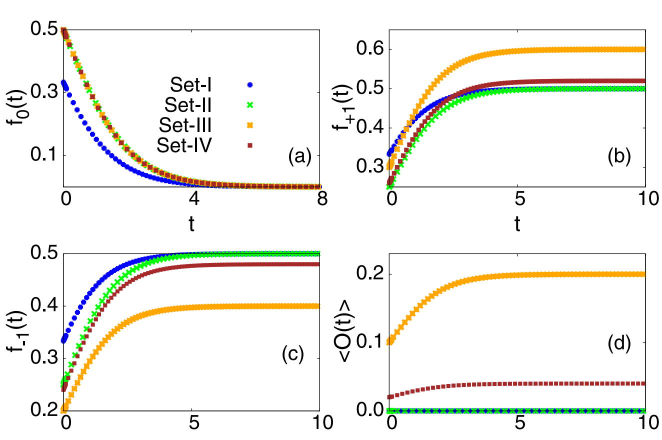

FIG. 3. Results for $q = 1$ obtained from the analytical solution using the initial configurations given in Table I. The three densities and the ensemble averaged order parameter are shown as functions of time in (a), (b), (c), and (d), respectively. The time evolution of sets I and II merge within a few steps as expected.

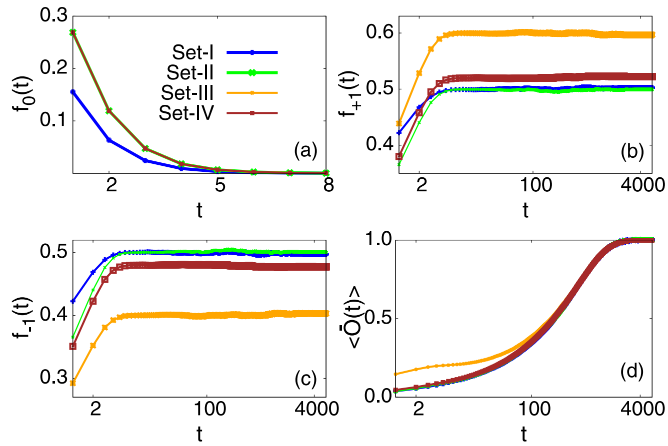

FIG. 4. Results for $q = 1$ obtained from Monte Carlo simulation (for system size $N = 2^{10}$) using the initial configurations given in Table I. The three densities and the ensemble averaged order parameter are shown as functions of time in (a), (b), (c), and (d), respectively.

The corresponding simulation results are shown in Fig. 4. Here the consensus states are reached for all the sets of initial states including the disordered ones (sets I and II). This is because, in simulations, since we have a finite system size, a random fluctuation can drive the system to a consensus state in an individual configuration. Therefore, the data which are shown for the ensemble average of the *absolute value* of the order parameter show $\langle \bar{O} \rangle \to 1$ at large times for all initial states. This is analogous to the kinetics in the Ising-Glauber model at zero temperature in one dimension, where we have a conservation such that the ensemble averaged magnetization is zero. In simulations, however, an individual configuration indeed reaches the all-spin-up or -down state so that the absolute value of magnetization reaches unity even after configuration averaging.

Let us next discuss the case for $q = 0$, the other extreme limit. This is the case when extreme switches are not allowed and it is identical to the model considered in [38] when all interactions are positive and equal to 1. In that case, an ordered state is expected at long times. However, as already discussed in the last section, the $\Delta = 0$ point is the FFP here leading to

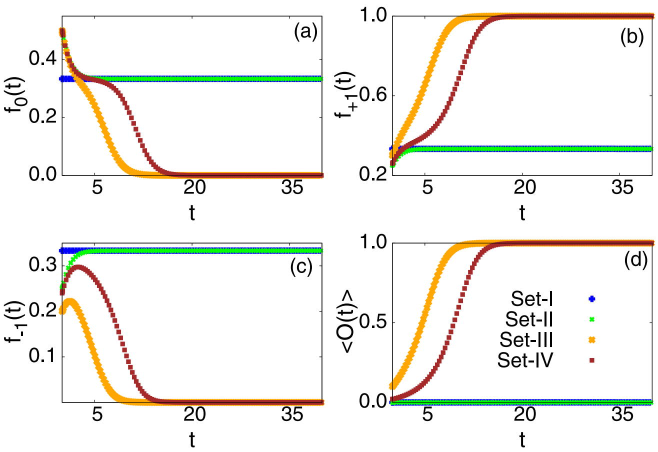

FIG. 5. Results for $q = 0$ obtained from analytical solution using the initial configurations given in Table I. The three densities and the ensemble averaged order parameter are shown as functions of time in (a), (b), (c), and (d), respectively. The time evolutions of sets I and II merge within a few steps as expected.

$\langle O(t)\rangle = 0$ for all t when the time evolution is studied using the analytical equations. The results are presented in Fig. 5. We note that set I does not evolve at all and for set II, the densities evolve before terminating at the FFP, consistent with the analysis presented in the previous section. Initial states with nonzero order show that the system reaches a consensus state. We also observe that if the initial state is close to a disordered state (set IV), the system spends a longer time to reach consensus. Once again, in the numerical simulations $\langle \bar{O}(t \to \infty)\rangle = 1$ for all initial configurations as shown in Fig. 6. In the completely disordered case, again random fluctuations are responsible for driving an individual system to a consensus state.

For other values of $q \neq 1$, the qualitative behavior of the time evolution is similar to that of $q = 0$. One gets consensus states starting from initially biased states. The disordered

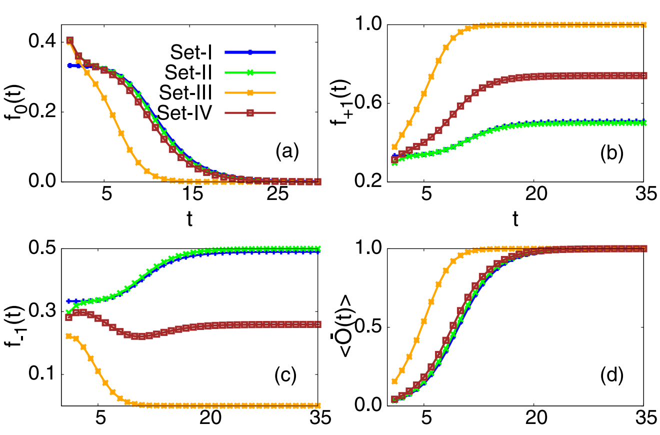

FIG. 6. Results for $q = 0$ obtained from Monte Carlo simulations (for system size $N = 2^{10}$) using the initial configurations given in Table I. The three densities and the ensemble averaged order parameter are shown as functions of time in (a), (b), (c), and (d), respectively.

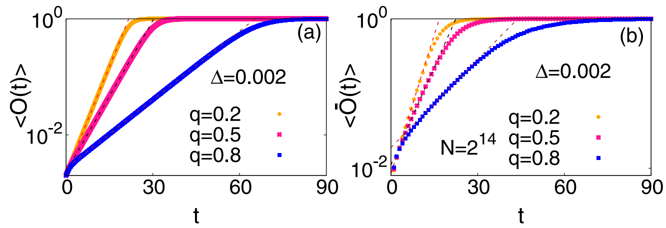

FIG. 7. The order parameter versus time variations for the initial condition $f_\pm = 1/3 \pm \Delta/2$ shown for different values of q using (a) the analytical method and (b) numerical simulation. The best fitting curves for the growth show an exponential form $\exp[\beta(q)t]$.

states flow to the FFP when the time evolution is studied by solving the differential equations numerically, as expected. Consensus is reached in individual configurations starting from any initial state in the numerical simulations.

We have already discussed the growth of the order parameter for initially ordered states with small deviations from the FFP taken in a particular manner. In this section we discussed the time evolution using various other initial configurations. It is found that any initially ordered state finally attains consensus and the growth of the order parameter is found to be exponential in all cases given by $\exp[\beta(q)t]$. In particular we show in Fig. 7 the case when the initial condition is $f_\pm = 1/3 \pm \Delta/2$ with $\Delta = 0.002$.

The values of $\alpha(q)$ and $\beta(q)$ obtained from the numerical solution of the rate equations as well as using Monte Carlo simulations are very close to each other as shown in Fig. 8. So we conclude that when the system orders for $q \neq 1$, the initial growth of the order parameter is given by a unique exponential form, independent of the initial condition, with the exponent given by Eq. (9).

Since the magnitude of the order parameter increases, a steady state must imply $f_0 = 0$. With $f_0 = 0$, we have from Eqs. (3) and (4) that in the steady state, either f_{+1} or f_{-1} must be zero (or unity). Hence the consensus state will be reached for all $q \neq 1$.

For $q = 1$, although the analytical results show that the consensus states are not reached in the thermodynamic limit, for finite systems, we find a unique behavior for the growth of the order parameter from the numerical simulation. Instead of

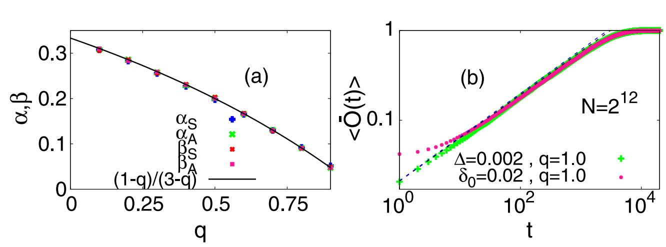

FIG. 8. (a) The values of α and β (subscript A and S denoting analytical and simulation results, respectively) shown against q agree very well with the analytical form given by Eq. (9). (b) The order parameter against time for $q = 1$ obtained using the numerical simulations is shown to follow a power law behavior with the exponent $= 0.49 \pm 0.01$. The initial conditions are the same as in Figs. 2(b) and 7(b).

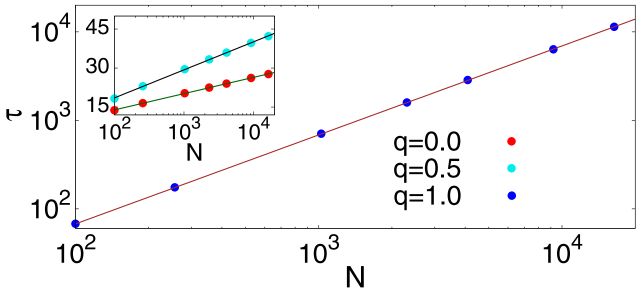

FIG. 9. Variation of average consensus time for different values of q. The main plot shows the $q = 1$ results obtained using system sizes up to $N = 2^{14}$. Inset shows results for two values of $q \neq 1$ obtained in system sizes $\leqslant 2^{16}$ agents.

exponential, it displays a slower power law variation with the exponent very close to 0.5. The data are presented in Fig. 8(b).

D. Consensus times

From the simulations, τ, the average time to reach the consensus state, has been estimated for different system sizes. Once again, we find different behavior for $q \neq 1$ and $q = 1$. For $q = 1$, $\tau \propto N$, while for other values of q, the consensus time τ depends logarithmically on the system size, with τ increasing with q. Figure 9 shows the data.

E. Biased initial conditions and exit probability

To calculate the exit probability, we consider the biased initial condition $f_0 = 1/3$ and $f_{\pm 1} = 1/3 \pm \Delta/2$. As already mentioned, we calculate the exit probability as a function of Δ.

The exit probability for $q = 1$ has a completely different behavior compared to other values of q. We find that it has linear variation given by $E(\Delta) = 1/2 + 3\Delta/4$ shown in Fig. 10(a). A linear behavior is expected in conserved systems but it is intriguing that even here, there is a linear behavior although the system is not exactly conserved. In this case, the simulations also agree as we take into account whether the consensus reached is for all $+1$ or all -1 states. Figure 10(b) shows the results are consistent with a linear variation of $E(\Delta)$ when $q = 1$ and shows that it is also independent of the system size.

Analytical results for the exit probability, for $q \neq 1$, shows a step-function-like behavior: $E(\Delta) = 1$ for $\Delta > 0$ and is

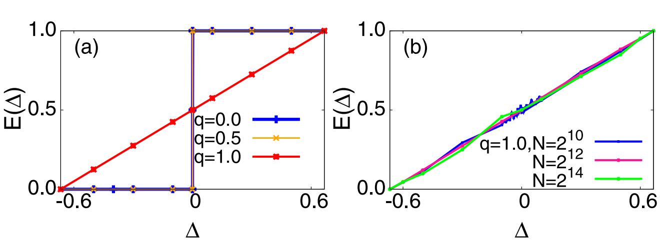

FIG. 10. (a) $E(\Delta)$ against Δ obtained in the analytical method using Eq. (2) for different values of q. (b) $E(\Delta)$ for $q = 1$ using the numerical simulations for different system sizes shows a system-size-independent linear behavior.

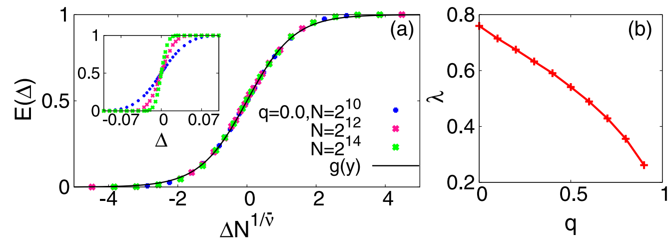

FIG. 11. (a) Data collapse of $E(\Delta)$ for different system sizes is shown against scaled Δ values for $q = 0$. Inset shows the raw data. (b) The variation of the parameter λ against q.

equal to zero for $\Delta < 0$. The fixed point analysis also indicates $E(\Delta = 0) = 1/2$. Hence any biased state with a majority opinion equal to 1 (-1) will end up with all opinions equal to 1 (-1).

The simulation results for $E(\Delta)$ for $q \neq 1$ show strong finite size dependence. When plotted against $N^{1/\bar{\nu}}\Delta$, the data collapse in a single curve indicating

$$E(\Delta) = g(N^{1/\bar{\nu}}\Delta), \tag{11}$$

where g is a scaling function. This is true for any value of $q \neq 1$ with a universal value of $\bar{\nu} \approx 2$. Figure 11(a) shows the data for $q = 0$.

The scaling function g in Eq. (11) can be approximated by

$$g(y) = [1 + \tanh(\lambda y)]/2, \tag{12}$$

as obtained earlier in a few other models [18,24–27]. We find that λ decreases as the value of q increases from zero [Fig. 11(b)].

In the thermodynamic limit, the exit probability shows the step function behavior. However, for finite systems, it is S shaped and from the finite size analysis we can conclude that the range of Δ over which it is neither zero nor unity, to a large extent, is inversely proportional to $\lambda N^{1/\bar{\nu}}$.

We end this section commenting that the two different behaviors of the exit probability shown in Fig. 10(a) are analogous to the Ising-Glauber model in dimensions greater than unity and the voter model (in any dimension) respectively for $q \neq 1$ and $q = 1$.

IV. SUMMARY AND DISCUSSIONS

In this paper, the evolution of the opinions in a kinetic exchange model has been studied using both analytical and numerical methods. The three discrete opinion values used here are quantized by $0, \pm 1$. The mean field differential equations for the rate of change of the population densities having the three opinions have been derived and analyzed. Here, the parameter q determines the value of the interaction μ, a random variable, which can have binary values 1 and 2. When $\mu = 2$, which occurs with a probability q, there is a possibility that the opinion value switches from one extreme value to the other. The $q = 0$ case, where μ can have a single value equal to unity, has been considered earlier in several studies in different contexts.

Let us first summarize the main results obtained:

(a) Any initially ordered state will reach a consensus state for $q \neq 1$.

(b) A frozen disordered fixed point exists; all initially disordered states flow there.

(c) The growth of the order parameter is exponential for $q \neq 1$.

(d) A quasiconservation exists for $q = 1$ leading to different saturation behaviors of the order parameter and exit probability.

The results are qualitatively different for $q = 1$ and $q \neq 1$. The analytical solution, which is valid in the thermodynamic limit, shows that for $q = 1$ the dynamics are quasiconservative as the order parameter remains constant after a very short transient time. This indicates that the system does not order fully for any initial configuration with initial order parameter less than 1. The linear behavior of the exit probability is similar to what is seen for a conservative dynamics as for example in the voter model in all dimensions and the Ising-Glauber model in one dimension. This is actually quite interesting, as the present model does not strictly conserve the order parameter; the saturation value is not exactly equal to the initial one. But the linear behavior of the exit probability can still occur if the saturation value of the order parameter varies linearly with the initial value, which we have checked to be true here.

The $q = 1$ case is in fact very similar to the voter model: as f_0 goes to zero very fast, it effectively renders the system to a binary opinion model within a short time scale with the transition rates identical to those in the voter model [52]. Like the voter model, here the agent adopts the opinion of the other agent with whom she interacts irrespective of her own opinion. We also obtain the result that the average consensus time is proportional to N for $q = 1$, a result valid for the mean field voter model.

In the analytical approach, one essentially obtains the ensemble averages in the thermodynamic limit. Initial configurations with nonzero order will eventually reach the consensus state for $q \neq 1$. We also find that this growth behavior is unique, i.e., does not depend on the initial state but only on q. This is not surprising as it is expected that there will be a single time scale in the system. Such exponential growths have been recently observed in the mean field Ising model with finite coordination number also [27].

The analytical approach also leads to the interesting result that an initially disordered state, that can be realized in many ways, will flow towards the so-called frozen fixed point at a rate independent of q. In comparison, in binary models like the Ising model, the disordered state is unique, characterized by exactly half of the relevant degrees of freedom belonging to one state. Hence no such flow can be observed there.

In the numerical simulations, one can keep track of the individual configurations. For all q values we get a consensus state finally for states starting from partially disordered states. For $q \neq 1$, this is the same result obtained from analytical treatment. However, it is not expected that consensus will be obtained for $q = 1$ for any initial state and for $q \neq 1$, for initially fully disordered configurations. This contradictory result obtained in the simulations is argued to be due to finite size effects. In finite systems, random fluctuations can drive the system to a consensus state (which implies that the absolute value of the order parameter is unity) even if the initial configuration is fully disordered. This has been observed in spin models also, e.g., in the one-dimensional Ising-Glauber model for which the ensemble averaged order parameter is conserved but still consensus states can be reached in numerical simulations starting from disordered states. Numerical simulations also show that for $q = 1$, the growth follows a power law behavior, which is much slower than exponential. As a result, the consensus time is linear in N for $q = 1$, compared to the weak logarithmic dependence on the system size when $q \neq 1$.

The exit probability for $q \neq 1$ indicates a step function behavior in the thermodynamic limit. It shows strong finite size effects as indicated from the numerical simulations. As observed in some other models, a scaling behavior is obtained, dictated by two parameters $\bar{\nu}$ and λ. The value of $\bar{\nu}$ is independent of q, a result similar to that in several other models where also $\bar{\nu}$ does not depend on the model parameter. However, the value of $\bar{\nu} \approx 2$ is clearly different from the ones found earlier for Ising-like and other opinion dynamics models [18,24–27]. λ, on the other hand, is dependent on the parameter q, which was also found to be true in the other models. The linear variation of the exit probability in the $q = 1$ case, independent of system size, also indicates that one will get minority spreading here [53].

In conclusion, the present results indicate that a society attains stability when people have less influence on others, i.e., q is small, with the consensus state attained very fast. Essentially, the $q \neq 1$ model is qualitatively similar to the $q = 0$ model, with a q dependent time scale to reach the consensus state which diverges as $1/\alpha \propto (1 - q)^{-1}$. So the extreme switches cause a delay in reaching the consensus as they increase in number. $q = 1$, which allows the maximum possible switches between extreme opinion values, essentially leads to a fragmented society. That this does not happen usually signifies that real systems may be mimicked by a $q \neq 1$ value in this model. It also shows that an initially disordered society will remain so when one considers the ensemble average; however, individual configurations do reach consensus. From the perspective of statistical physics, we have presented a model with a rich behavior as q is changed; at $q = 1$ a voter-model-like behavior is seen that changes to a finite dimensional spin-model-like behavior for any $q < 1$. As future studies, it will be interesting to consider negative interactions between the agents which will introduce a noise that can drive an order-disorder transitions. This will also make it closer to reality and would open up the possibility to compare with time dependent real data. Another interesting possibility is to consider general opinion values instead of $\pm 1, 0$ [54–56] and introduce transition between any two states and see how it compares with the present case.

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