Nonlinear Schrödinger equations with amplitude-dependent Wadati potentials

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(Received 12 July 2022; accepted 27 October 2022; published 14 November 2022)

Complex Wadati-type potentials of the form $V(x) = -w^2(x) + iw_x(x)$, where $w(x)$ is a real-valued function, are known to possess a number of intriguing features, unusual for generic non-Hermitian potentials. In the present paper, we introduce a class of nonlinear Schrödinger-type problems which generalize the Wadati potentials by assuming that the base function $w(x)$ depends not only on the transverse spatial coordinate, but also on the amplitude of the field. Several examples of prospective physical relevance are discussed, including models with the nonlinear dispersion or with the derivative nonlinearity. The numerical study indicates that the generalized model inherits the remarkable features of standard Wadati potentials, such as the existence of continuous soliton families, the possibility of symmetry-breaking bifurcations when the model obeys the parity-time symmetry, the existence of constant-amplitude waves, and the eigenvalue quartets in the linear-instability spectra. Our results deepen the current understanding of the interplay between nonlinearity and non-Hermiticity and expand the class of systems which enjoy the exceptional combination of properties unusual for generic dissipative nonlinear models.

DOI: 10.1103/PhysRevE.106.054209

I. INTRODUCTION

The combination of nonlinearity and non-Hermiticity can have a dramatic impact on the properties of a waveguiding system. In particular, the presence of a non-Hermitian complex potential (which in physical terms takes into account the energy exchange with the environment) can heavily alter the structure of stationary localized nonlinear modes or solitons propagating in the system. Whereas in conservative waveguides the solitons exist as continuous families parametrized by an "internal" parameter, such as the soliton frequency or amplitude, the dissipative solitons most usually exist as isolated points [1,2], which from the dynamical point of view, behave as attractors (provided that the soliton is stable). A prototypical model that illustrates this dissimilarity between conservative and dissipative systems is the generalized nonlinear Schrödinger equation (GNLSE),

$$i\Psi_t = -\Psi_{xx} + V(x)\Psi + W(x)F(|\Psi|^2)\Psi, \tag{1}$$

where complex-valued functions $V(x)$ and $W(x)$ can be referred to as a linear and a nonlinear potential, respectively, and a real-valued function $F(\cdot)$, such that $F(0) = 0$, specifies the nonlinearity. Equation (1) is a canonical model describing evolution of nonlinear waves in various physical settings. In particular, in optical applications, Ψ corresponds to the dimensionless amplitude of the electric field. Its dependence on coordinate t describes evolution of the pulse along the propagation direction, and x is the transverse coordinate. Effective complex optical potentials correspond to the presence of spatial regions with gain and loss [3,4], which, in particular, can be implemented using coherent multilevel atoms driven by external laser fields [5,6].

When the imaginary parts of both potentials vanish, the model becomes conservative, and its stationary nonlinear modes $\Psi(x,t) = e^{-i\mu t}\psi(x)$ can be parametrized by the continuous change of the real frequency μ. However, if at least one of the potentials is complex, then the continuous families of solitons typically disappear. At the same time, there exist at least two types of non-Hermitian potentials that do support continuous families of nonlinear modes (see a more detailed discussion in Ref. [7]). Systems of the first type obey the parity-time ($\mathcal{PT}$) symmetry [3,8]: In this case real and imaginary parts of functions $V(x)$ and $W(x)$ are even and odd functions of x, respectively [9,10]. The second type corresponds to the so-called Wadati-type potentials [11] for which

$$V(x) = -w^2(x) + iw_x(x), \quad W(x) \equiv 1, \tag{2}$$

where $w(x)$ is a real-valued differentiable function which is not required to have any particular symmetry [at the same time, if $w(x)$ is even, then the resulting Wadati potential $V(x)$ becomes $\mathcal{PT}$ symmetric, i.e., the two types of potentials are partially overlapping]. In what follows, it will be convenient to say that the GNLSE (1) with potentials (2) corresponds to *linear* Wadati potentials. Apart from the existence of continuous families of nonlinear modes [12], linear Wadati potentials are known to support several other remarkable features. When the nonlinearity is absent, i.e., $F \equiv 0$, linear Wadati potentials can have an all-real spectrum of eigenvalues [13] and undergo distinctive phase transition from all-real to the complex spectrum through an exceptional point or a self-dual spectral singularity [14,15]. Returning to the nonlinear setup, Wadati-type potentials support the existence of constant-amplitude nonlinear waves [16], feature unusual dynamical behavior near the phase-transition threshold [17], and eigenvalue quartets in the linear-stability spectrum [18]. Additionally, when a Wadati potential is $\mathcal{PT}$ symmetric, it allows for bifurcations of families of non-$\mathcal{PT}$-symmetric modes [19], which is

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impossible for $\mathcal{PT}$-symmetric potentials of general form. Whereas the theory of Wadati potentials is far from being complete, on the qualitative level some of their peculiar properties can be explained by the existence of a "conserved" (i.e., x-independent) quantity which constrains the shape of stationary modes [20].

Unusual and not yet fully understood properties of Wadati potentials encourage to look for their possible generalizations. In the meantime, most of the presently available studies of Wadati potentials are basically limited by the GNLSE with spatially uniform power-law and saturating nonlinearities [21,22]. In the present paper, we introduce a more profound generalization of Wadati potentials by considering the situation where the base function $w(x)$ depends not only on the spatial coordinate x, but also on the amplitude of the field Ψ. We argue that the extended system preserves some of the properties of linear Wadati potentials. Particular realizations of the introduced generalization lead to nonlinear systems of potential physical relevance. Those include a GNLSE equation with the additional higher-order nonlinearity that emerges due to the stimulated response in optical fibers and a GNLSE with the derivative nonlinearity. Combining our analytical understandings with simple demonstrative computations, we construct continuous families of nonlinear modes and discuss their linearization spectra. Additionally, we show that when the generalized system is $\mathcal{PT}$ symmetric, it undergoes a symmetry-breaking bifurcation which results in continuous families of non-$\mathcal{PT}$-symmetric solitons.

The rest of the paper is organized as follows. Section II outlines the derivation of the generalized model and presents several particular realizations of the extended system. Section III contains a case study of a model with the derivative nonlinearity. Section IV concludes the paper.

II. CONSTRUCTION OF GENERALIZED POTENTIALS

For stationary modes $\Psi(x,t) = e^{-i\mu t}\psi(x)$, Eq. (1) with linear Wadati potential (2) becomes

$$\mu\psi = -\psi_{xx} + (-w^2 + iw_x)\psi + F(|\psi|^2)\psi. \quad (3)$$

The latter stationary equation has been studied in the previous literature [12,18–24]. In particular, it is known that if Eq. (3) is considered as a dynamical system with x playing the role of the evolution variable, then the respective "dynamics" is constrained by a conserved (i.e., x-independent) quantity [20]. Our construction of the generalized model relies on a "gauge transformation" which converts the stationary equation (3) to another ordinary differential equation, where the x-independent "integral of motion" has an especially simple form [25]. Indeed, using the substitution $\psi(x) = \phi(x)\exp[i\int w(x)dx]$, where $\phi(x)$ is a new stationary field, from (3) we obtain

$$\mu\phi = -\phi_{xx} - 2iw(x)\phi_x + F(|\phi|^2)\phi. \quad (4)$$

Multiplying the latter equation by ϕ_x^* and adding it with its complex conjugate, we obtain the integral of motion,

$$\mu|\phi|^2 = -|\phi_x|^2 + \int_0^{|\phi|^2} F(\xi)d\xi + \text{const}, \quad (5)$$

where "const" is an arbitrary x-independent quantity. Clearly, for localized modes with $\lim_{x\to\pm\infty}\phi(x) = \lim_{x\to\pm\infty}\phi_x(x) = 0$ this constant must be zero. Using the obtained integral of motion as a qualitative argument, continuous families of nonlinear localized modes have been constructed in linear Wadati potentials [20]. In the meantime, since the integral (5) does not contain function $w(x)$ explicitly, Eq. (4) can be generalized naturally by assuming that the base function $w(x)$ depends not only on the spatial coordinate x, but also on the amplitude of the field. This idea suggests replacing Eq. (4) with the following more general one:

$$\mu\phi = -\phi_{xx} - 2iA(x, |\phi|^2)\phi_x + F(|\phi|^2)\phi, \quad (6)$$

where $A(\cdot,\cdot)$ is a real-valued function of two variables. It is easy to check that identity (5) remains valid for the newly introduced Eq. (6). We further make the "inverse" gauge transformation $\phi(x) = \psi(x)\exp[i\int z(x)dx]$, where $z(x) := B(x, |\psi|^2)$ is another real-valued function, which converts (6) into the following equation:

$$\mu\psi = -\psi_{xx} + \left(2AB + B^2 - i\frac{\partial B(x,|\psi|^2)}{\partial x} - i\frac{\partial B(x,|\psi|^2)}{\partial|\psi|^2}(|\psi|^2)_x\right)\psi - 2i(A+B)\psi_x + F(|\psi|^2)\psi. \quad (7)$$

The obtained stationary equation corresponds to the following temporal evolution problem:

$$i\Psi_t = -\Psi_{xx} + \left(2AB + B^2 - i\frac{\partial B(x,|\Psi|^2)}{\partial x} - i\frac{\partial B(x,|\Psi|^2)}{\partial|\Psi|^2}(|\Psi|^2)_x\right)\Psi - 2i(A+B)\Psi_x + F(|\Psi|^2)\Psi. \quad (8)$$

Equation (8) is the central point of the present paper. It represents the generalization of the previously studied GNLSE with linear Wadati potentials, which can be recovered from Eq. (8) by setting $A = -B = w(x)$. To distinguish this model from the known case of linear Wadati potentials, we will say that Eq. (8) corresponds to *nonlinear* Wadati potentials. By construction, the stationary version of this model [i.e., Eq. (7)] has the x-independent conserved quantity and is, therefore, expected to support continuous families of nonlinear localized modes and share other properties of linear Wadati potentials.

For different choices of A and B the obtained equation (8) acquires specific shapes which can be of potential relevance for physical applications.

Example 1. Choosing $A = -B = w(x) + \sigma|\Psi|^2$, where σ is a real coefficient, from (8) we obtain the following equation:

$$i\Psi_t = -\Psi_{xx} + (-w^2 + iw_x)\Psi - 2\sigma w(x)|\Psi|^2\Psi - \sigma^2|\Psi|^4\Psi + i\sigma(|\Psi|^2)_x\Psi + F(|\Psi|^2)\Psi. \quad (9)$$

The case of $\sigma = 0$ recovers the previously considered GNLSE with linear Wadati potentials. Let us comment on the newly appeared terms that contain the coefficient σ. The term $2\sigma w(x)|\Psi|^2\Psi$ can be considered as cubic nonlinearity with a real-valued nonlinear potential whose spatial shape is given by the base function $w(x)$. The term $\sigma^2|\Psi|^4\Psi$ corresponds to the spatially uniform quintic focusing nonlinearity. Assuming that σ is small, this term can be neglected in the leading order. In fact, this term can be "eliminated" by a simple redefinition of function F. The most unconventional term in Eq. (9) corresponds to $i\sigma(|\Psi|^2)_x\Psi$. In fiber optics, the higher-order nonlinear terms of this form have been used to take into account nonlinear dispersion resulting from the stimulated Raman scattering of ultrashort pulses, see, e.g., Refs. [26–29].

Example 2. Choosing $A = -B = \sigma w(x)|\Psi|^2$, we arrive at the following equation, where the nonlinear gain-and-loss term and nonlinear dispersion are spatially modulated by function $w(x)$ and its derivative $w_x(x)$:

$$i\Psi_t = -\Psi_{xx} + (-\sigma^2 w^2|\Psi|^4 + i\sigma w_x|\Psi|^2)\Psi + i\sigma w(x)(|\Psi|^2)_x\Psi + F(|\Psi|^2)\Psi. \quad (10)$$

Example 3. The choices of $A = w(x) - \sigma|\Psi|^2/2$ and $B = -w(x) + \sigma|\Psi|^2$ transform Eq. (8) to the following model:

$$i\Psi_t = -\Psi_{xx} + (-w^2 + iw_x)\Psi + \sigma w(x)|\Psi|^2\Psi + F(|\Psi|^2)\Psi - i\sigma(|\Psi|^2\Psi)_x. \quad (11)$$

In the latter equation, the spatially inhomogeneous cubic nonlinearity $\sigma w(x)|\Psi|^2\Psi$ is present together with the *derivative* nonlinearity $i\sigma(|\Psi|^2\Psi)_x$. The derivative NLSE $i\Psi_t = -\Psi_{xx} - i\sigma(|\Psi|^2\Psi)_x$ is integrable by the inverse scattering method [30,31] and has been widely used to describe nonlinear waves in plasma [32,33] and optical fibers [34]. Note that the "additional" terms in Eq. (11) (i.e., those with coefficient σ) are conservative, i.e., do not contribute to the power balance equation: indeed, introducing the power $P(t) = \int_{-\infty}^{\infty}|\Psi|^2dx$ from Eq. (11) we compute

$$\frac{dP(t)}{dt} = 2\int_{-\infty}^{\infty} w_x|\Psi|^2dx. \quad (12)$$

Several recent studies have addressed the derivative NLSE with $\mathcal{PT}$-symmetric potentials [35–38]. Being motivated, in part, by this newly emerged interest in non-Hermitian extensions of the derivative NLSE, in the next section, we perform a more detailed examination of the model (11).

III. WADATI POTENTIALS WITH THE DERIVATIVE NONLINEARITY

A. Constant-amplitude waves

Despite their spatially inhomogeneous structure, the linear Wadati potentials are known to support constant-amplitude solutions [16]. Similar solutions can be found in the introduced generalized models. They can be constructed easily by noting that a pair $\phi(x) \equiv \rho_0$ and $\mu = F(\rho_0^2)$, where ρ_0 is an arbitrary real amplitude, solves Eq. (4). In particular, for Eq. (11) the constant-amplitude solution reads

$$\Psi(x,t) = \rho_0 \exp\left\{i\int w(x)dx - i\sigma\rho_0^2 x - iF\left(\rho_0^2\right)t\right\}. \quad (13)$$

In linear Wadati potentials, constant-amplitude waves of the analogous form have been used to examine the development of modulational instability in non-Hermitian optical media [16] and to implement coherent perfect absorption of nonlinear waves [15].

B. Computing the continuous families of nonlinear modes using the numerical shooting approach

Let us now use the demonstrative computation to argue that Eq. (11) supports continuous families of stationary localized modes $\Psi = e^{-i\mu t}\psi(x)$, where μ is a real parameter, and $\lim_{x\to\pm\infty}\psi(x) = 0$. The corresponding procedure is essentially a generalization of the approach from Ref. [20]. It is more convenient to describe it in terms of the equivalent Eq. (6). We reduce the dimensionality of the underlying phase space by employing the representation $\phi(x) = \rho(x)\exp[i\int v(x)dx]$, where $\rho(x)$ and $v(x)$ are real-valued functions. Separating Eq. (6) into real and imaginary parts, we arrive at the system,

$$\mu\rho = -\rho_{xx} + v^2\rho + 2A(x,\rho^2)v\rho + F(\rho^2)\rho, \quad (14)$$

$$-v_x\rho - 2v\rho_x - 2A(x,\rho^2)\rho_x = 0. \quad (15)$$

We fix some $\mu < 0$ and assume that $A(x,\rho^2(x))$ decays fast as $x \to \pm\infty$. Then solutions $\phi_\pm(x)$ that decay, respectively, at $+\infty$ and $-\infty$ have simple asymptotic behavior $\phi_\pm(x) = \exp(-\sqrt{|\mu|}|x|)(C_\pm + o(1)_{x\to\pm\infty})$, where $C_\pm$ are real x-independent constants. The asymptotic behavior of derivatives $\phi_{x,\pm}(x)$ can be obtained by differentiating the asymptotic equations for $\phi_\pm(x)$. Choosing some sufficiently large $x_\infty \gg 1$, for any C_+ and C_- we can use the found asymptotic behavior to approximate $[\rho_\pm(\pm x_\infty), \rho_{x,\pm}(\pm x_\infty), v_\pm(\pm x_\infty)]$ and then use those as initial values for the numerical solution of system (14) and (15), computing in this way solutions $[\rho_+(x), \rho_{x,+}(x), v_+(x)]$ on the interval $[0, x_\infty)$ and $[\rho_-(x), \rho_{x,-}(x), v_-(x)]$ on the interval $(-x_\infty, 0]$. Looking for a continuously differentiable localized mode $\phi(x)$ which decays at both infinities, we need to find a solution to the following system of three equations:

$$\rho_+(0) = \rho_-(0), \quad \rho_{+,x}(0) = \rho_{-,x}(0), \quad v_+(0) = v_-(0), \quad (16)$$

with respect to two unknowns (i.e., two "shooting parameters") C_+ and C_-. At first glance, system (16) seems overdetermined and, therefore, is not expected to have a solution. However, the integral (5) imposes an additional relation between the functions: $\mu\rho_\pm^2(x) = -\rho_{\pm,x}^2(x) - v_\pm^2(x)\rho_\pm^2(x) + \int_0^{\rho_\pm^2(x)}F(\xi)d\xi$. These constraints imply that if any two equations (say the first two) of system (16) are satisfied, then the third equation is satisfied automatically [speaking more precisely, the third equation will be satisfied in the form $v_+^2(0) = v_-^2(0)$; however, the spurious solutions with $v_+(0) = -v_-(0)$ can be easily filtered out in the practical realization of the procedure]. Therefore, system (16) is not overdetermined and can be expected to have one or several solutions each of which corresponds to a localized mode. Since the described procedure can be fulfilled for any $\mu < 0$, the continuous in μ families of localized modes are indeed expected to exist.

Proceeding now to a numerical illustration, we look for stationary modes of Eq. (11) with the derivative nonlinearity and, therefore, consider $A(x,\rho^2(x)) = w(x) - \sigma\rho^2(x)/2$ with

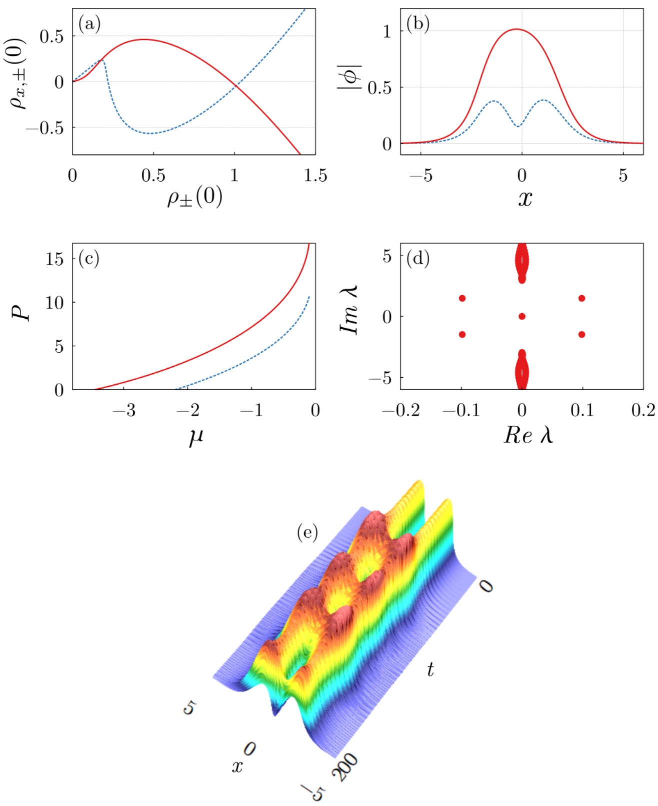

FIG. 1. (a) Dependencies $\rho_+(0)$ vs $\rho_{+,x}(0)$ (solid red curve) and $\rho_-(0)$ vs $\rho_{-,x}(0)$ (dotted blue curve) obtained by the numerical shooting procedure for $\mu = -2$. (b) Amplitudes of nonlinear modes $|\psi(x)| = |\phi(x)|$ corresponding to the two intersections in (a). Solid red and dotted blue profiles correspond to the fundamental and first excited states, respectively. (c) Continuous curves corresponding to the families of fundamental (solid red curve) and first excited (dotted blue curve) nonlinear modes. In (a)–(c) we use $F(|\psi|^2) = 0$ and $\sigma = 1$. (d) Linear stability eigenvalues λ (shown with red circles) for the mode from the excited family at $\mu = -3$ for $F(|\psi|^2) = -|\psi|^2$ and $\sigma = 0.25$. (e) Temporal evolution $|\Psi(x,t)|$ of an initial condition taken as a slightly perturbed stationary mode whose linear-instability spectrum is shown in (d).

an asymmetric base function $w(x)$ in the form

$$w(x) = \tanh 2(x - 2.5) - \tanh(x - 2.5). \tag{17}$$

For linear Wadati potentials, this base function has been considered in Ref. [18]. To keep the model as simple as possible but still nontrivial, in this example we remove the conventional nonlinearity by setting $F \equiv 0$. In Fig. 1(a) we illustrate the numerical shooting procedure by presenting the dependencies $\rho_+(0)$ vs $\rho_{+,x}(0)$ and $\rho_-(0)$ vs $\rho_{-,x}(0)$ obtained from the numerical solution of systems (14) and (15) under the gradual increase of shooting parameters C_+ and C_- departing from zero, and for the fixed value of μ. Apart from the origin (which corresponds to the trivial zero solution with $C_+ = C_- = 0$), the shown curves feature two intersections, which correspond to the particular values of the shooting parameters (C_+, C_-) for which the first two equations of system (16) are satisfied. It has been checked that the third equation of this system is also satisfied for both intersections and, hence, they indeed correspond to two localized modes. Spatial profiles $|\psi(x)| = |\phi(x)|$ of these modes are shown in Fig. 1(b). Gradually changing μ and tracing the found intersections, we construct two continuous families which are visualized in Fig. 1(c) in the form of continuous dependencies $P(\mu)$, where $P = \int_{-\infty}^{\infty} |\psi|^2 dx$ is the squared L^2 norm of the solution (that corresponds to the optical beam power). In the limit $P \to 0$ the solutions become small amplitude, and the corresponding values of μ approach the eigenvalues of the underlying linear eigenvalue problem which can be formally obtained from (3) by setting $F(|\psi|^2) = 0$.

Whereas the discussion in this subsection has been limited to the spatially localized modes whose amplitude $|\rho(x)|$ vanishes rapidly enough as $x \to \infty$ and $x \to -\infty$, the consideration can be extended toward inclusion of stationary modes with nonvanishing asymptotic behavior. In particular, similar arguments can be elaborated for kinklike solutions with $\lim_{x\to\pm\infty} \rho(x) = r_\pm$, where r_+ and r_- are nonzero constants, provided that the corresponding asymptotic behavior can be exhaustively described using several shooting parameters.

C. Eigenvalue quartets in the linear-instability spectrum

For linear Wadati potentials, it has been found numerically that the complex eigenvalues of the linear-stability operator always appear in quartets $(\lambda, \lambda^*, -\lambda, -\lambda^*)$ [18]. This numerical observation (which, to the best of our knowledge, has not yet received any analytical confirmation) is rather intriguing, because the eigenvalue quartets are more typical to Hamiltonian systems or to setups constrained by some explicit symmetry (such as $\mathcal{PT}$-symmetric systems). In the meantime, the GNLSE with Wadati potentials does not apparently feature a Hamiltonian structure and has no readily evident symmetry. In order to examine if the eigenvalue quartets persist in our generalized model, we perform the standard linearization procedure using the perturbed stationary-mode substitution in the form $\Psi(x,t) = e^{-i\mu t}[\psi(x) + a(x)e^{\lambda t} + b^*(x)e^{\lambda^* t}]$, where $a(x)$ and $b(x)$ are small perturbations, and complex λ is the linear stability eigenvalue. The positive real part of λ defines the increment of an eventual instability. Using this substitution in Eq. (11) and keeping only the linear stability in $a(x)$ and $b(x)$ terms, we obtain a linear-stability eigenproblem for eigenvalue λ and eigenvector $(a,b)^T$ (since the linearization procedure is straightforward, we do not present the corresponding linear-stability equations herein). For a numerical illustration we consider Eq. (11) with the base function given by (17) and cubic nonlinearity $F(|\psi|^2) = -|\psi|^2$. Then for $\sigma = 0$ our model recovers that from Ref. [18], where the existence of quartets has been observed numerically. Increasing σ and evaluating numerically the linear-stability spectra, we observe that the eigenvalue quartets persist for $\sigma \neq 0$. For example, for $\mu = -3$ and $\sigma = 0.25$ we obtain eigenvalues $\lambda_{1,2} \approx 0.098\,3725 \pm i1.484\,634$ and $\lambda_{3,4} \approx -0.098\,3725 \pm i1.484\,634$, i.e., the numerical results allow to conjecture the found eigenvalues indeed form a quartet, at least, with the accuracy 10^{-6}. A rigorous proof of this fact remains an interesting issue for future studies.

To corroborate the presence of the instability, we have simulated nonlinear dynamics governed by Eq. (11) using a modification of the finite-difference Crank-Nicolson scheme adapted for the derivative nonlinearity [39,40]. Choosing the

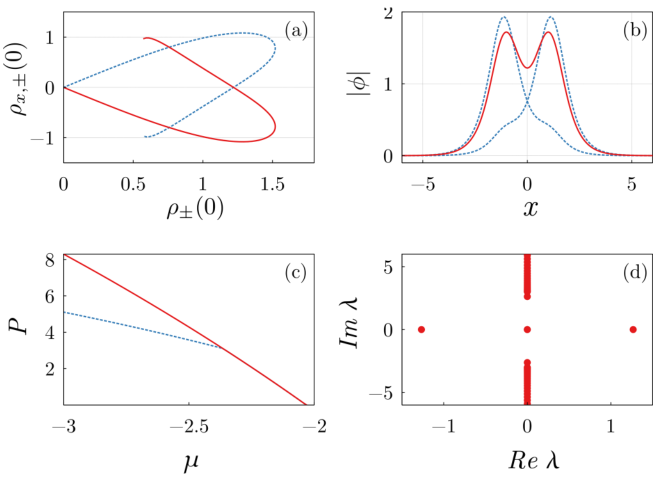

FIG. 2. (a) Dependencies $\rho_+(0)$ vs $\rho_{+,x}(0)$ (solid red curve) and $\rho_-(0)$ vs $\rho_{-,x}(0)$ (dotted blue curve) obtained by the numerical shooting procedure for $\mathcal{PT}$-symmetric system at $\mu = -3$ constructed by increasing the shooting parameters from $C_\pm = 0$ to $C_\pm = 34$. (b) Amplitudes of nonlinear modes corresponding to the intersections in (a). Solid red curve corresponds to the $\mathcal{PT}$-symmetric soliton and two dotted blue profiles correspond to a pair of symmetry-broken solitons. (c) Continuous families of $\mathcal{PT}$-symmetric (solid red curve) and asymmetric (dotted blue curve) solitons. (d) Linear-stability eigenvalues for the $\mathcal{PT}$-symmetric soliton at $\mu = -3$. In this figure $F(|\psi|^2) = -|\psi|^2$ and $\sigma = 0.4$.

slightly perturbed unstable stationary mode as an initial condition, from its evolution shown in Fig. 1(e) we observe that the stationary mode preserves its shape for $t \lesssim 50$, but for longer times the shape of numerical solution oscillates aperiodically.

D. Symmetry-breaking bifurcation and non-$\mathcal{PT}$-symmetric modes in the $\mathcal{PT}$-symmetric case

Next, we illustrate that in the case when the introduced generalized model becomes $\mathcal{PT}$ symmetric, it can support asymmetric (more precisely, not $\mathcal{PT}$ symmetric) nonlinear modes. As in the case of linear Wadati potentials, Eq. (11) becomes $\mathcal{PT}$ symmetric [8], i.e., invariant under the transformation $x \to -x$, $t \to -t$, and $i \to -i$ when the base function $w(x)$ is even. Symmetric and asymmetric modes can be searched using the numerical shooting approach outlined above. For a numerical illustration, we use a bimodal base function in the form $w(x) = 2e^{-(x-1.2)^2} + 2e^{-(x+1.2)^2}$ which results in an effectively double-well potential. In Fig. 2(a) we plot the corresponding dependencies $\rho_+(0)$ vs $\rho_{+,x}(0)$ and $\rho_-(0)$ vs $\rho_{-,x}(0)$. Three intersections in Fig. 2(a) correspond to a pair of asymmetric modes [the two intersections with $\rho_x(0) \neq 0$] and to a $\mathcal{PT}$-symmetric mode [the intersection with $\rho_x(0) = 0$]. Spatial profiles of symmetric and asymmetric modes are plotted in Fig. 2(b). Changing the value of μ, we construct continuous families of $\mathcal{PT}$-symmetric and asymmetric modes. Two asymmetric families bifurcate from the symmetric one after a supercritical pitchfork bifurcation: in other words, the symmetric family is stable for small P but becomes unstable right after the bifurcation. A representative plot of linear stability eigenvalues for the symmetric family past the symmetry-breaking bifurcation is shown in Fig. 2(d). In Fig. 3 we present several representative examples of numerically computed nonlinear dynamics in the $\mathcal{PT}$-symmetric system. In agreement with the predictions of the linear-stability analysis, we observe that above the symmetry-breaking bifurcation the symmetric family is unstable [Fig. 3(a)]: the initially perturbed symmetric stationary mode shortly evolves to a breatherlike oscillating entity. The asymmetric families [Fig. 3(b)] and the symmetric family below the symmetry-breaking bifurcation are stable, and the corresponding numerical solutions preserve the stationary shape for indefinitely long time.

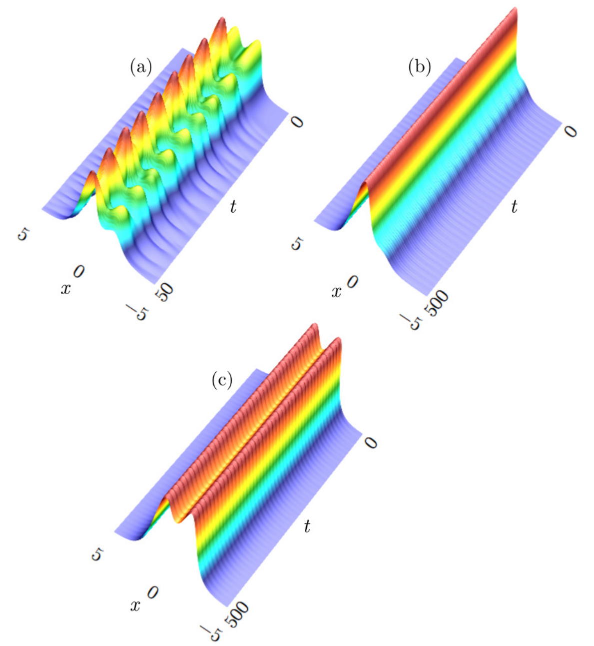

FIG. 3. Temporal evolutions of initial conditions taken as slightly perturbed stationary modes from Fig. 2: unstable symmetric (a) and stable asymmetric (b) modes above the symmetry-breaking bifurcation (specifically, at $\mu = -3$) and the stable symmetric mode (c) below the symmetry-breaking bifurcation (specifically, at $\mu = -2.2$). The plots show $|\Psi(x, t)|$.

E. Absence of the dynamical conservation law

Results of the previous subsections do not imply that nonlinear Wadati potentials simply inherit any property of linear Wadati potentials without even a minor modification. As an example of a dissimilarity between the two types of potentials, we recall that linear Wadati potentials admit a dynamical (i.e., t-independent) conserved quantity given as [17]

$$J = \int_{-\infty}^{\infty} [i\Psi\Psi_x^* - w(x)|\Psi|^2]dx. \tag{18}$$

Indeed, for linear Wadati potentials from Eqs. (1) and (2) we compute $dJ/dt = 0$. In the meantime, for nonlinear Wadati potentials from Eq. (11) we evaluate

$$\frac{dJ}{dt} = \sigma \int_{-\infty}^{\infty} [2w(|\Psi|^4)_x dx + i(|\Psi|^2)_x(\Psi_x\Psi^* - \Psi_x^*\Psi)]dx$$
$$= \sigma \int_{-\infty}^{\infty} (|\Psi|^4)_x[2w - (\arg\Psi)_x]dx.$$

Hence, the dynamical conserved quantity (18) does not carry over to nonlinear Wadati potentials. Of course, this does not imply that the extended model (11) does not have any dynamical conservation law at all. However, so far we have not been able to find a generalization of the dynamical integral (18) for nonlinear Wadati potentials.

IV. CONCLUSION

The nonlinear Schrödinger equation with complex Wadati-type potentials is known to feature a variety of unusual and intriguing properties. In the meantime, the generalizations of Wadati potentials are rather scarce, and most of the activity in this direction is presently limited by spatially homogeneous power-law and saturating nonlinearities. In this paper, we have proposed a significant extension of Wadati potentials. The main idea of our approach is to consider the base function of a Wadati potential as depending not only on the spatial coordinate, but also on the amplitude of the field. The resulting extended model admits a conserved (i.e., independent of the transverse coordinate) quantity. Using this conserved quantity, we have employed a demonstrative computation approach to argue that the generalized model supports continuous families of bright solitons. The numerical study of the linear-instability spectra indicates that unstable solitons feature eigenvalue quartets, which is another remarkable peculiarity of Wadati potentials. We have demonstrated that the generalized parity-time-symmetric model supports a supercritical symmetry-breaking bifurcation that gives birth to continuous families of non-$\mathcal{PT}$-symmetric solitons. Numerical simulations of nonlinear dynamics indicate that unstable stationary modes can dynamically transform to nearly periodic breatherlike solutions. The introduced model also admits a straightforward generalization of constant-amplitude nonlinear waves known to exist in conventional Wadati potentials. In contrast to the previously known models with Wadati potentials, our generalization incorporates spatially modulated nonlinearity where the shape of the modulation is determined by the base function of the Wadati potential. Additional nonlinear terms can include spatially uniform nonlinear dispersion or derivative nonlinearity. Our findings essentially broaden the class of systems which enjoy the unique combination of peculiar features of nonlinear and non-Hermitian systems typical to Wadati-type and $\mathcal{PT}$-symmetric potentials.

ACKNOWLEDGMENT

The work was supported by the Priority 2030 Federal Academic Leadership Program.

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