Noisy voter model: Explicit expressions for finite system size

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Urn models are classic stochastic models that have been used to describe a diverse kind of complex systems. Voter and Ehrenfest's models are very well-known urn models. An opinion model that combines these two models is presented in this work and it is used to study a noisy voter model. In particular, at each temporal step, an Ehrenfest's model step is done with probability α or a voter step is done with probability $1-\alpha$. The parameter α plays the role of noise. By performing a spectral analysis, it is possible to obtain explicit expressions for the order parameter, susceptibility, and Binder's fourth-order cumulant. Recursive expressions in terms of the dual Hahn polynomials are given for first passage and return distributions to consensus and the equal coexistence of opinions. In the cases where they follow power-law distributions, their exponents are computed. This model has a pseudocritical noise value that depends on the system size; a discussion about thermodynamic limits is given.

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I. INTRODUCTION

Voter and Ehrenfest's models are very well-known urn models. The classic versions of these models consider two urns and N labeled balls. Ehrenfest's model considers noninteracting balls and it was originally proposed to explain the second law of thermodynamics [1]. The voter model involves interacting particles and it is used to describe the evolution of opinions in a population [2]. Although these are simple mathematical models, they have been inspiring research for a long time.

There is an increasing interest in understanding some opinion models on complex networks [3], considering the inclusion of strong opinions [4,5], external perturbations [6–8], contrarians [9], active links [10], and persuasion in small groups [11], among others. In general, most of the results are compared with those from fully connected networks. However, there are only a few explicit solutions and sometimes they are only known for N very large. Thus, in most cases, comparisons are done by using numerical simulations and they may be not accurate enough. Since urn models are equivalent to the description of fully connected networks, explicit expressions for finite system sizes could be very useful.

In this work we consider an opinion model that combines voter and Ehrenfest's models. The proposed model has an order-disorder transition and it is a particular case of more general models, [8,12–14]. In particular, our model is the noisy voter model that was proposed by Kirman with $\delta = \varepsilon$ instead of $\delta = 2\varepsilon$ [12]. We are interested in the stationary distribution, magnetization, susceptibility, and Binder's fourth-order cumulant for finite system sizes as well as in the behavior of the system after taking thermodynamic limit. To find some relevant properties of opinion models, like the time it takes for the system to reach consensus, we relate our urn model to a random walk on a one-dimensional finite grid and compute first passage and return distributions.

This work is organized in the following manner. In the next section, the model is described, a spectral analysis of the transition matrix is performed, and the stationary distribution is given in terms of the Beta function. In Sec. III magnetization, susceptibility, and Binder's fourth-order cumulant are computed. In Sec. IV first passage and return distributions are analyzed. In the cases where they follow power-law distributions, their exponents are computed. Finally, we provide a discussion of the main results.

II. MODEL

We consider two urns, A and B, and N labeled balls. Each ball is located in one of the urns. Since the number of balls is constant, the state of the system is described by the number

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of balls in one of the urns. We consider that balls in urn A have opinion $+1$ and balls in urn B have opinion -1. The number of balls or agents with opinion $+1$ is the number of balls inside the urn A. We use this quantity to describe the system dynamics.

In our model, at each time step, one of the following two different actions can take place.

(1) A label is chosen randomly and the ball with that label is removed from the urn where it is located and added to the other urn.

(2) Two labels are chosen randomly, the ball with the first chosen label stays in its urn. The ball with the second chosen label goes to the same urn where the first one belongs.

Action 1 has a probability α and action 2 has a probability $1-\alpha$, where $\alpha \in [0,1]$. In other words, at each time step, an Ehrenfest's step is done with probability α or a voter step is done with probability $1-\alpha$. If $\alpha = 0$, the model is reduced to the voter model and if $\alpha = 1$ it becomes the Ehrenfest's model.

In terms of opinion, when the update is given by a voter step, agents can change their opinions due to the interaction with other agents. On the other hand, if the update is due to an Ehrenfest's step, agents change their opinions without interaction. Since each ball has the same probability to be chosen during a Ehrenfest's step $1/N$ the probability to change opinion spontaneously is larger from the current majority to the current minority opinion than the other way around. The parameter α plays the role of a noise and it is also referred to as social temperature. In the next sections system behavior is analyzed for different values of α.

A. Spectral analysis

As it has been expressed, the number of total balls N remains constant; thus, the state of the system can be described by the number of balls in the urn A, which will be denoted by $i = 0, 1, \ldots, N$. With the rules described above, the transition probability matrix is a tridiagonal matrix given by

$$M = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{N-1} & b_{N-1} & a_{N-1} \\ & & & c_N & b_N \end{pmatrix},$$

where

$$\begin{aligned} a_i &= (1-\alpha)\frac{i}{N}\frac{N-i}{N-1} + \alpha\frac{N-i}{N}, \\ b_i &= (1-\alpha)\left(1 - 2\frac{i}{N}\frac{N-i}{N-1}\right), \\ c_i &= (1-\alpha)\frac{i}{N}\frac{N-i}{N-1} + \alpha\frac{i}{N}. \end{aligned} \tag{1}$$

The contributions due to the voter model are represented by the first terms on the right-hand side of Eq. (1), while the second terms correspond to the Ehrenfest's model contributions. This model is the Kirman's model with $\varepsilon = \delta = \alpha$ [12], it also corresponds to those of Refs. [8,14] (with parameters $\{1, \alpha, 1, \alpha, 0, 0\}$ and $N_0 = N_1$, respectively).

To find the eigenvalues and eigenvectors of the transition matrix M, we proceed as in the work of Karlin and McGregor [15]. We define recursively a sequence of polynomials $p_0, p_1, \ldots, p_N$ by means of the three-term recurrence relation

$$\begin{aligned} x p_i(x) &= a_i p_{i+1}(x) + b_i p_i(x) + c_i p_{i-1}(x), \\ p_0 &= 1, \quad p_{-1} = 0, \end{aligned} \tag{2}$$

for $i = 0, \ldots, N-1$. Note that these are N equations which involve the coefficients of the first N rows of M. If we denote $v = [p_0(x), \ldots, p_N(x)]'$, where $'$ denotes transposed, then we have

$$Mv = xv - [0, \ldots, 0, x p_N(x) - b_N p_N(x) - c_N p_{N-1}(x)]'.$$

The first N entries of the equation above are given by the recurrence relation (2). The $(N+1)$th entry, however, gives an extra condition. The eigenvalues of M are precisely the zeros of the polynomial of degree $N+1$ given by $x p_N(x) - b_N p_N(x) - c_N p_{N-1}(x)$ and the eigenvectors by evaluating $[p_0(x), \ldots, p_N(x)]'$ at those zeros. This procedure provides the spectral decomposition of M explicitly, as long as we can relate the polynomials $\{p_i\}$ to a known family of orthogonal polynomials.

To determine the polynomials $\{p_i\}$, we identify our model with that from Ref. [13] whose parameters a, b, and ν are related to α and N by

$$a = b = \frac{\alpha N - 1}{1-\alpha}, \quad \nu = \frac{\alpha N + N - 2\alpha}{N-1}, \tag{3}$$

so the recurrence coefficients become

$$\begin{aligned} a_i &= \frac{\nu(N-i)(a+1+i)}{N(N+2a+2)}, \qquad b_i = 1 - a_i - c_i, \\ c_i &= \frac{\nu i(N+a+1-i)}{N(N+2a+2)}. \end{aligned}$$

It follows from Eqs. (6) and (7) of Ref. [13] that the polynomials

$$R_i(x, a, a, N) = p_i\left(-\frac{\nu x}{N(N+2a+2)} + 1\right),$$

are the dual Hahn polynomials, which can be written in terms of a ${}_3F_2$ series as

$$\begin{aligned} R_n(\lambda(x), a, a, N) &= {}_3F_2\left[\begin{matrix} -x, x+2a+1, -n \\ a+1, -N \end{matrix}; 1\right] \\ &= \sum_{j=0}^{n} \frac{(-n)(x+2a+1)_j(-x)_j}{j!(a+1)_j(-N)_j}, \end{aligned}$$

where $\lambda(x) = x(x+2a+1)$ and $(a)_j = a(a+1)\ldots(a+j-1)$ is the standard Pochhammer symbol. The dual Hahn polynomials $R_i(x, a, a, N)$ satisfy the following orthogonality relations:

$$\sum_{x=0}^{N} \omega_x R_\ell[\lambda(x), a, a, N] R_k[\lambda(x), a, a, N] = \frac{\delta_{k,\ell}}{\pi(\ell)},$$

where

$$\begin{aligned} \omega_x &= \binom{N}{x}\frac{(2a+1)_x}{(N+2a+2)_x}\frac{2x+2a+1}{2a+1}, \\ \pi(\ell) &= \binom{N}{\ell}\frac{\beta(a+1+\ell, N+a+1-\ell)}{\beta(a+1, a+1)}, \end{aligned} \tag{4}$$

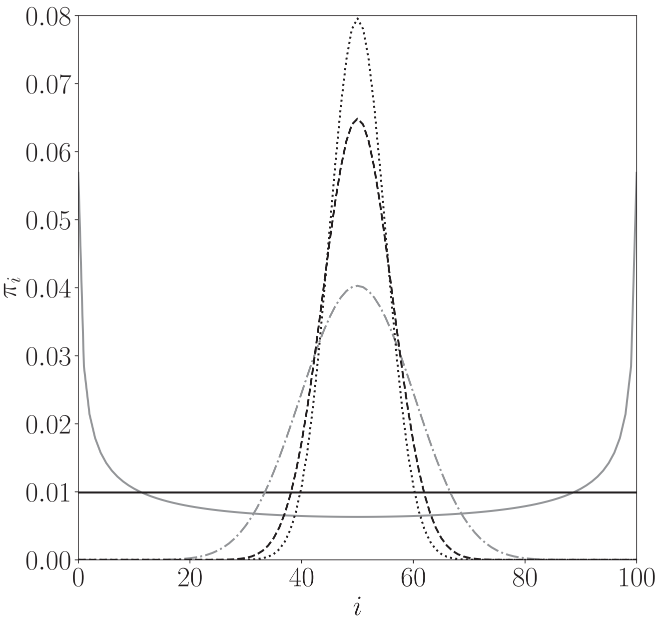

FIG. 1. Stationary probability distributions considering $N = 100$ for the following values of α: 0.005 (solid grey line); $0.01 = 1/N$ (solid black line); 0.15, (dashed grey line); 0.5 (dashed black line); and 1 (dotted black line).

with $x,\ \ell = 0, 1, \ldots, N,\ \beta(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, the beta function, and $\Gamma(x)$ the gamma function. The eigenvalues of the transition matrix M are given by

$$1 - \frac{\nu\lambda(x)}{N(N + a + b + 2)}, \qquad x = 0, 1, \ldots, N.$$

These solutions are valid for $\alpha \in (0, 1)$. For $\alpha = 0$ hypotheses of Ref. [15] are not fulfilled, while for $\alpha = 1$ a proper limit has to be taken to obtain the Krawtchouk polynomials [13].

B. Stationary distribution

The model presented in this work can be seen as a discrete random walk on a one-dimension finite grid. In this case the position of the walker can be associated with the number of balls in urn A. The boundary conditions depends on α being absorbing for $\alpha = 0$ and reflecting otherwise. According to Karling and McGregor [15], the explicit expression for the stationary distribution, which is related to the norm of the orthogonal polynomials, is given by Eq. (4), $\pi_i = \pi(i)$, $i = 0, 1, \ldots, N$.

In Fig. 1 we show the stationary distribution for different values of α considering $N = 100$. Results are similar for different values of N. For the particular case $\alpha = 1/N$, i.e., $a = 0$, the stationary distribution is a uniform distribution for all N. If $\alpha > 1/N$ the stationary distribution has a unique maximum at $N/2$, but if $\alpha < 1/N$ it has two maxima, one at zero, and the second one at N. In other words, for $\alpha = 1/N$ there is a balance between the voter and Ehrenfest's models, while for $\alpha > 1/N$ ($\alpha < 1/N$), Ehrenfest's (voter) model weighs more than the voter (Ehrenfest's) model. States 0 and N are absorbing states if $\alpha = 0$, i.e., when the model is reduced to the voter model. If $\alpha \in (0, 1/N)$, states zero and N are no longer absorbing states although they can be considered sticky states. In the following section we define an order parameter and study its behavior as a function of a.

III. FINITE SIZE, ANALYTICAL RESULTS

Although for $\alpha \in (0, 1/N)$ there is no absorbing state, the states 0 and N are sticky and it is more probable to find the system near those states. However, if $\alpha \in (1/N, 1)$ the maximum of the stationary probability distribution is at $N/2$, thus the likelihood to find the system around this state is larger. Then, if $\alpha \in (0, 1/N)$ the system tends to be neat with most of the balls in one urn, which represents most of the agent with the same opinion. However, if $\alpha \in (1/N, 1)$, the system tends to be disordered, with around half of the balls in each urn, which represents a polarized society. The pseudocritical noise value is given by $\alpha_c(N) \equiv 1/N$. When N goes to infinity, the system has an order-disorder transition at $\alpha_c = 0$. To study this transition, we define the opinion of the system by $m = (2i - N)/N$. It is clear that $m \in [-1, 1]$. Due to the symmetry of the problem $\langle m \rangle = 0$, for all α, then we define the *average opinion* of the system by $\langle |m| \rangle$, which can be taken as an order parameter. Other magnitudes of interest are the *susceptibility*, which is defined by $\chi = N(\langle m^2 \rangle - \langle |m| \rangle^2)$ and the Binder's fourth-order cumulant, which is given by $U = 1 - \langle m^4 \rangle/(3\langle m^2 \rangle^2)$. To give more elegant explicit expressions, we consider an even number of agents, $N = 2M$.

We recall from (4) that the stationary distribution π_i is the multiplicative inverse of the norm of the dual Hahn polynomials. Therefore it is precisely the orthogonality measure of the Hahn polynomials, see, for instance, Sec. 9.5 and 9.6 of Ref. [16]. We can compute explicitly the first few moments of the Hahn polynomials and obtain the explicit expressions for $\langle |m| \rangle$, χ, and U. See Refs. [17,18] for a discussion on higher moments. Thus,

$$\begin{aligned}
\langle |m| \rangle &= \frac{(2M)!(a + M + 1)\beta(a + M + 1, a + M + 1)}{(M!)^2(a + 1)\beta(a + 1, a + 1)}, \\
\chi &= 2M\left(\frac{(a + M + 1)}{(2a + 3)M} - \langle |m| \rangle^2\right), \\
U &= 1 - \frac{(2a + 3)[(3M - 1)a + 3M^2 + 3M - 1]}{3M(2a + 5)(a + M + 1)}, \qquad (5)
\end{aligned}$$

where a is given in Eq. (3). When $\alpha = \alpha_c(N)$, the parameter a becomes zero for all N. Thus we define $a_c = 0$.

The average opinion, susceptibility, and Binder's fourth-order cumulant are shown in Figs. 2–4 as a function of α for different values of N. As we expected, the larger the value of N, the smaller the values of $\langle |m| \rangle$ for $\alpha > 0$. The susceptibility shows a maximum for a value of α less than $\alpha_c(N)$ and its amplitude is proportional to N. The maximum of χ is around $\alpha_m(N) \sim 1/(2N)$ or, analogously, around $a_m \sim -1/2$ for all N. Binder's fourth-order cumulant is independent of N at $\alpha_c = 0$, in particular, $U(\alpha_c) = 2/3$, as it was expected, since, at this point, a phase transition takes place. Even more, the larger

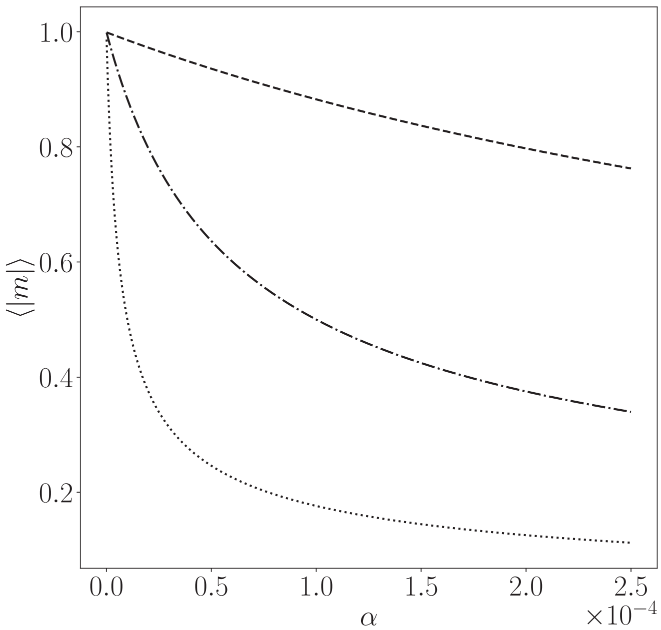

FIG. 2. Average order parameter $\langle|m|\rangle$ vs α for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line).

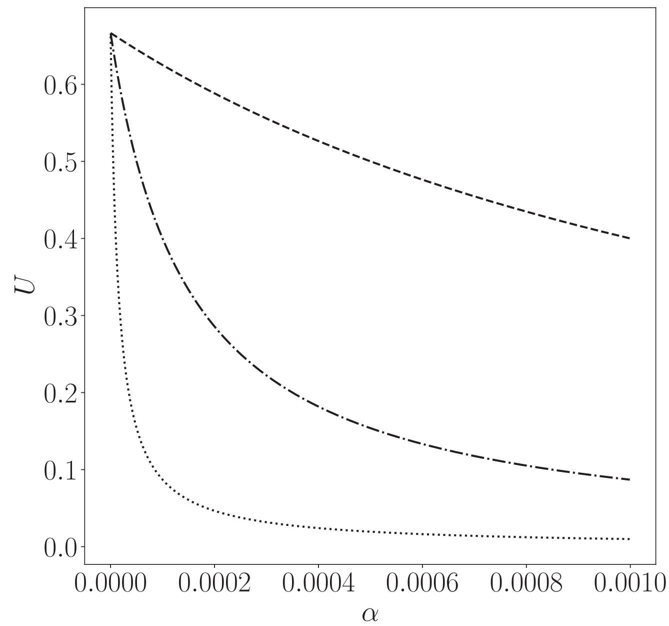

FIG. 4. Binder's fourth-order cumulant, U vs α for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line).

N, the larger the absolute value of the Binder's fourth-order cumulant derivative at α_c.

Considering the finite-size scaling relations

$$\langle|m|\rangle = N^{-\beta_{\mathrm{exp}}/\nu_{\mathrm{exp}}} f_m[N^{1/\nu_{\mathrm{exp}}}(\alpha - \alpha_c)],$$
$$\chi = N^{\gamma_{\mathrm{exp}}/\nu_{\mathrm{exp}}} f_\chi[N^{1/\nu_{\mathrm{exp}}}(\alpha - \alpha_c)],$$
$$\frac{dU}{d\alpha} = N^{1/\nu_{\mathrm{exp}}} f_{U'}[N^{1/\nu_{\mathrm{exp}}}(\alpha - \alpha_c)],$$

the exponents ν_{exp}, β_{exp}, and γ_{exp}, can be determined by evaluating Eq. (5) and the derivative of U at a_c. It is easy to verify that $\nu_{\mathrm{exp}} = 1$, $\beta_{\mathrm{exp}} = 0$, and $\gamma_{\mathrm{exp}} = 1$. These scaling laws are verified in Figs. 5–7 where a universal curve is obtained. We have to notice that if we consider a d-dimensional fully

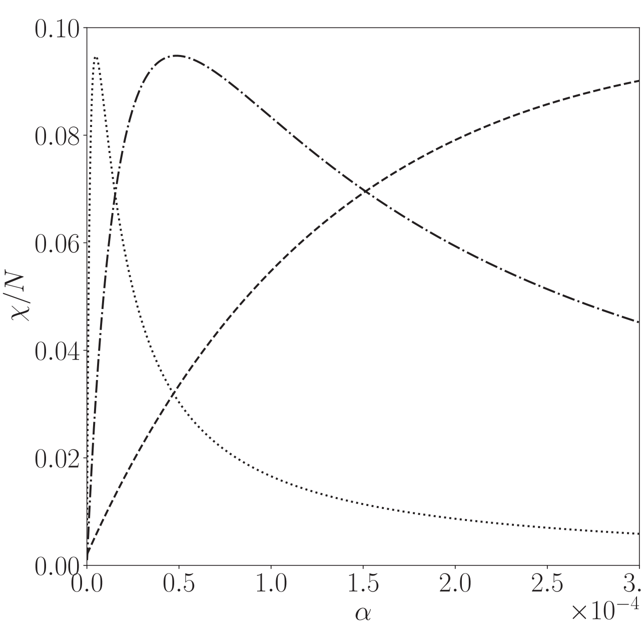

FIG. 3. Ratio of susceptibility to system size χ/N vs α for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line).

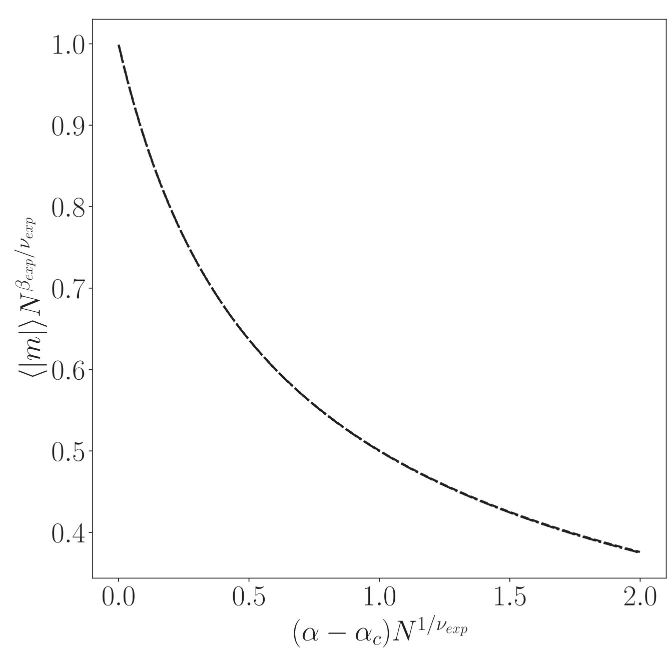

FIG. 5. Average order parameter $\langle|m|\rangle N^{\beta_{\mathrm{exp}}/\nu_{\mathrm{exp}}}$ vs $(\alpha - \alpha_c)N^{1/\nu_{\mathrm{exp}}}$ for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line). The values $\beta_{\mathrm{exp}} = 0$ and $\nu_{\mathrm{exp}} = 1$ give the collapse.

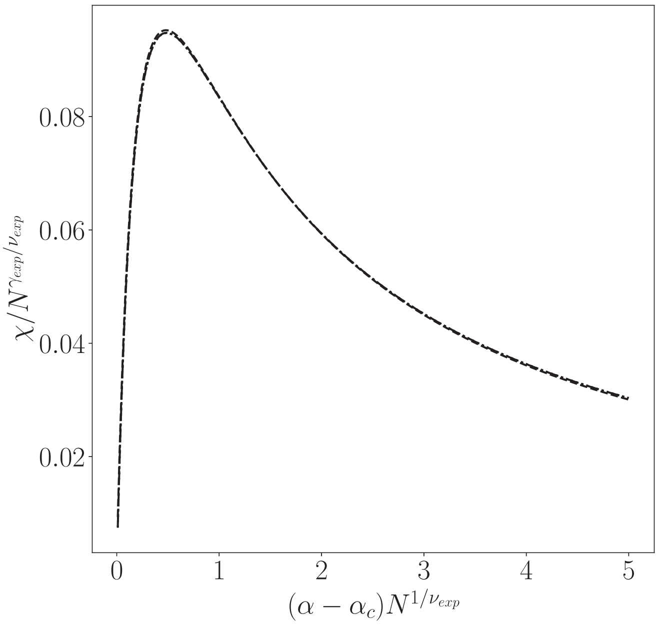

FIG. 6. Ratio of susceptibility to system size $\chi/N^{\gamma_{\rm exp}/\nu_{\rm exp}}$ vs $(\alpha - \alpha_c)N^{1/\nu_{\rm exp}}$ for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line). The values $\gamma_{\rm exp} = 1$ and $\nu_{\rm exp} = 1$ give the collapse.

connected lattice with linear dimension L, N would be equivalent to L^d. Since the critical social temperature is zero, the corresponding Rushbrooke relationship has to be used [19]. In this case, $\alpha_{\rm exp} + 2\beta_{\rm exp} + \gamma_{\rm exp} \geqslant 1$, where $\alpha_{\rm exp}$ is the heat capacity critical exponent.

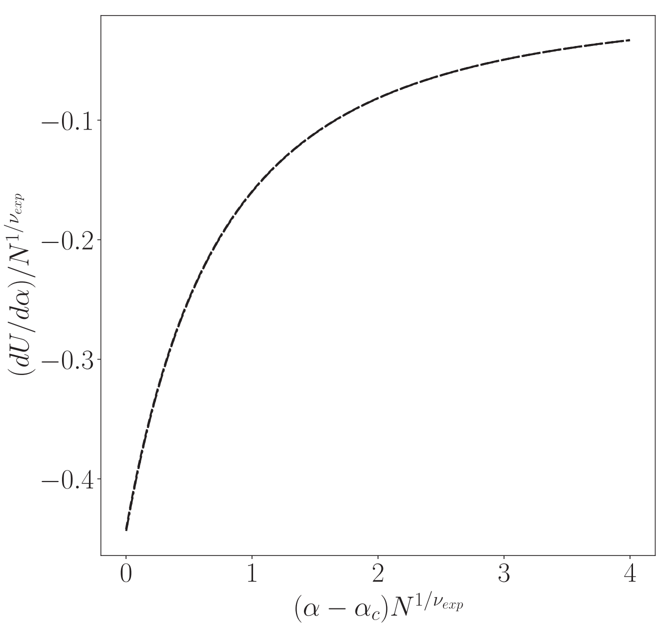

FIG. 7. Ratio of Binder's fourth-order cumulant derivative to system size $\frac{dU}{d\alpha}/N^{1/\nu_{\rm exp}}$ vs $(\alpha - \alpha_c)N^{1/\nu_{\rm exp}}$ for the following values of N: 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line). The value $\nu_{\rm exp} = 1$ gives the collapse.

IV. CONSENSUS TIME AND FIRST RETURN PROBABILITY DISTRIBUTIONS

Some properties of this opinion model can be computed by using tools from the random walk theory. For instance, it is possible to give an analytical expression for the expected consensus time, i.e., the average time to reach consensus for the first time. Of course, states zero and N represent consensus states and state $N/2$ represents the state with equal coexistence of opinions. From the point of view of random walk theory, states zero, N, and $N/2$ are also particular states since the first two are the boundary of the domain and the last is a state of attraction for $\alpha \neq 0$, i.e., the mean first passage time from state i to state $N/2$ is less than that from $N/2$ to i, for all $i \neq N/2$. This can be proved straightforwardly by identifying the mean first passage time from state i to state i as $1/\pi_i$ and using the theorem of Ref. [20]. Thus, state $N/2$ is a particular state even for $a < 0$.

Since states zero and N are equivalent states, in this section we compute the expected first passage time from state $N/2$ to state N and the first return probability distribution for states $N/2$ and 0.

There are two very interesting regions, one characterized by $-1 < a < 0$ and the other one for a large enough. The region for negative values of a is associated with an ordered system. On the other hand, if a is large enough, α is near one, i.e., our model is near to Ehrenfest's model. In this case, the diagonal elements of the transition matrix are near zero, see Eq. (1). In other words, in the limit $\alpha = 1$ the first return probability distribution only has positive values for even numbers of steps, i.e., it is not allowed to go back at the initial state after an odd number of steps. Thus, for $\alpha \approx 1$ or, equivalently a large enough, a difference between the values of the first return probability distribution for an even and odd numbers of steps should be appreciated with a coalescent point as it happens in the random walk of Ref. [21].

To compute numerically some of these quantities, we first simplify the notation. We will denote $R_\ell(x) = R_\ell[\lambda(x), a, a, N]$, so that the n-step transition probabilities are given in terms of the dual Hahn polynomials by

$$P^{(n)}_{k,\ell} = \rho(\ell)\sum_{x=0}^{N}\omega_x R_\ell(x)R_k(x)\left(1 - \frac{v\lambda(x)}{N(N+2a+2)}\right)^n.$$

We denote by $f^{(n)}_{i,j}$ the probability of reaching the state j for the first time in n steps given that the process started at state i:

$$f^{(n)}_{i,j} = P(X_n = j,\ X_m \neq j \text{ for } 1 \leqslant m \leqslant n-1 | X_0 = i).$$

In particular, for each integer $n \in \mathbb{N}$, we let $f^{(n)}_i$ be the probability that, starting from state i, the first return to state i occurs in the nth step. This can be computed recursively by the well-known formula [15]

$$f^{(n)}_i = P^{(n)}_{i,i} - \sum_{k=0}^{n-1} f^{(k)}_i P^{(n-k)}_{i,i}. \quad (6)$$

Here we define $f^{(0)}_i = 0$ for all i.

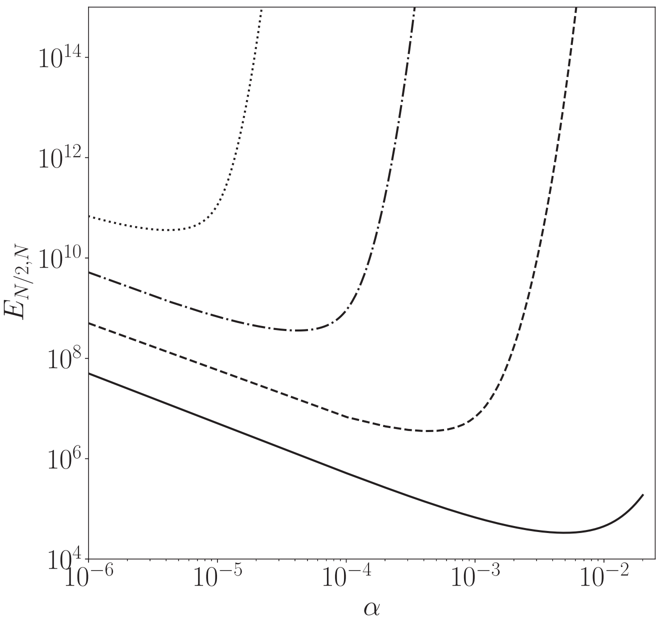

FIG. 8. Expected time to from polarization to consensus $E_{N/2,N}$ in function of α for the following values of N: 100 (solid line); 1000 (dashed line); 10 000, (dashed-dotted line); 100 000 (dotted line).

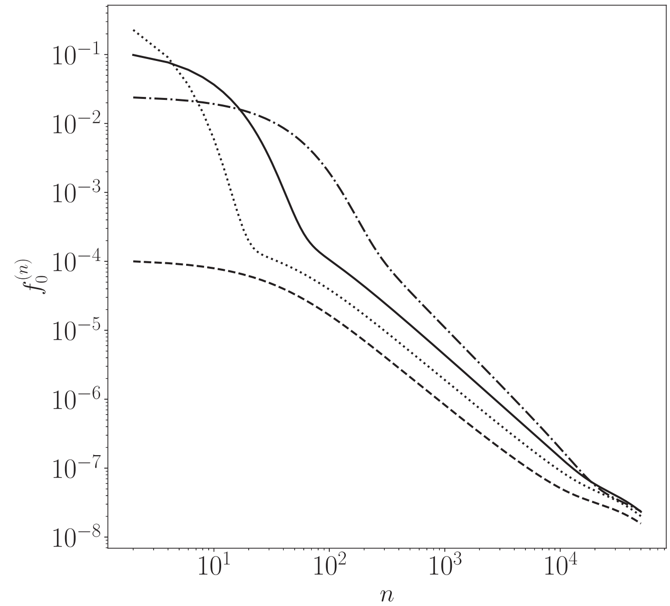

FIG. 9. First return distribution $f_0^{(n)}$ for the following values of a, 0, (dashes line); $-1/4$, (dotted line); $-1/2$, (solid line), and $-3/4$, (dashed-dotted line).

We also introduce the expected first passage time $E_{i,j}$ from state i to state j:

$$E_{i,j} = \sum_{n=1}^{\infty} n f_{i,j}^{(n)}.$$

In the following subsections we analyze $E_{N/2,N}$, $f_{N/2,N}^{(n)}$, $f_0^{(n)}$, and $f_{N/2}^{(n)}$ for different values of α.

A. Expected first passage time from $N/2$ to N

It follows from Refs. [13, (16)] that the expected first passage time E_{ij} is given by

$$E_{ij} = \frac{N(N+2a+2)}{\nu(2a+1)} \sum_{k=1}^{j} \frac{(2a+1)_k[(-j)_k - (-i)_k]}{k(-N)_k(a+1)_k}.$$

This sum can be computed numerically for a given i and j. In Fig. 8 we show these expected times with $i = N/2$ and $j = N$ as a function of α for different values of N. All these curves have a minimum value around $\alpha \sim 1/(2N)$ or $a \sim -1/2$. One may have expected a strictly increasing function since, for $a \in (-1, 0)$, the smaller a, the more sticky the boundaries. However, the boundaries are not fully absorbent, thus, there is a probability to stay near the opposite boundary and it becomes more difficult to leave this if $a \in (-1, -1/2)$. It means, the expected time to go from $N/2$ to N is affected by those trajectories that pass by some close neighborhood of state 0. Those trajectories tend to stay longer near state 0 if $a \in (-1, -1/2)$. The minimum expected time from a completely polarized state to consensus is located at the same value of a that gives the maximum susceptibility value, as it is expected since, at this point, the fluctuations are maximum. In this model consensus is a nonabsorbing state and since $N/2$ is a state of attraction the time to reach the state $N/2$ after visiting a consensus state is even less on average.

B. First return distribution for a consensus sate

We now compute the first return distribution for a consensus sate. In this subsection we consider $a < 0$. If $a > 0$, the stationary distribution has a maximum in $N/2$ and consensus states are no longer sticky states. Thus, once the walker leaves a consensus state, the larger a, the longer it takes to go back at the initial state (on average). We can compute numerically the first return distribution for a consensus sate by using the recursive form that was shown in Eq. (6). In Fig. 9 we show a log-log plot for these distributions for $N = 100$ and consider the following values of a, 0 (dashes line): $-1/4$, (dotted line); $-1/2$, (solid line), and $-3/4$, (dashed-dotted line). For an intermediate number of steps, all these curves also behave as power-law distributions. For a number of steps large enough, finite system size effects are present.

Although they behave as a power law, they do not share the exponent $\lambda_{\exp}$. We numerically compute these distributions for many values of a and estimate their exponent by fitting the power-law interval. In Fig. 10 these exponents are shown as a function of a. For the particular value $a = -1/2$, we can find analytically the exponent for large N.

When the total number of balls N is large enough, the density $x_\rho = i/N$ can be regarded as continuous. Thus the probability density $p(x_\rho, t)$ satisfies a Fokker-Planck equation given by

$$\frac{\partial p}{\partial t} = -\frac{[A(x_\rho)p]}{\partial x_\rho} + \frac{1}{2}\frac{\partial^2[B(x_\rho)p]}{\partial x_\rho^2},$$

where $A(x_\rho)$ and $B(x_\rho)$ are time-independent drift and diffusion coefficients. By writing the master equation of our model and taking the proper approximations for large N [22], we

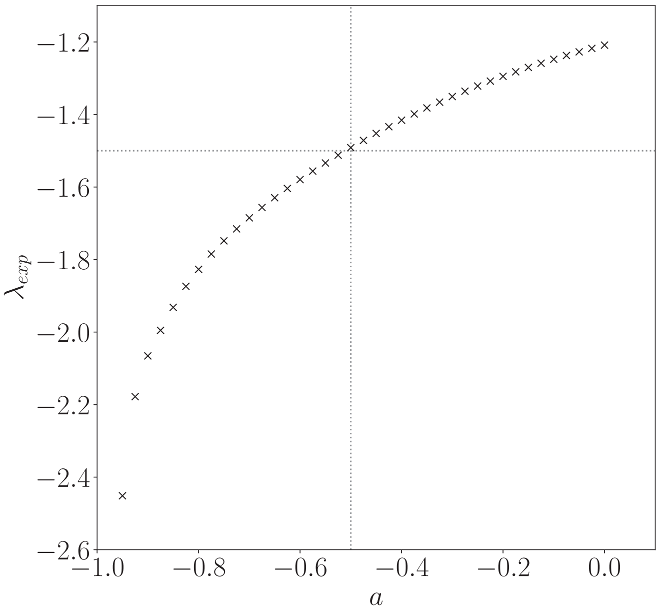

FIG. 10. Exponents of power-law distributions associated with $f_0^{(n)}$ vs a. Vertical and horizontal lines indicate the values $a = -1/2$ and $\lambda = -3/2$, respectively.

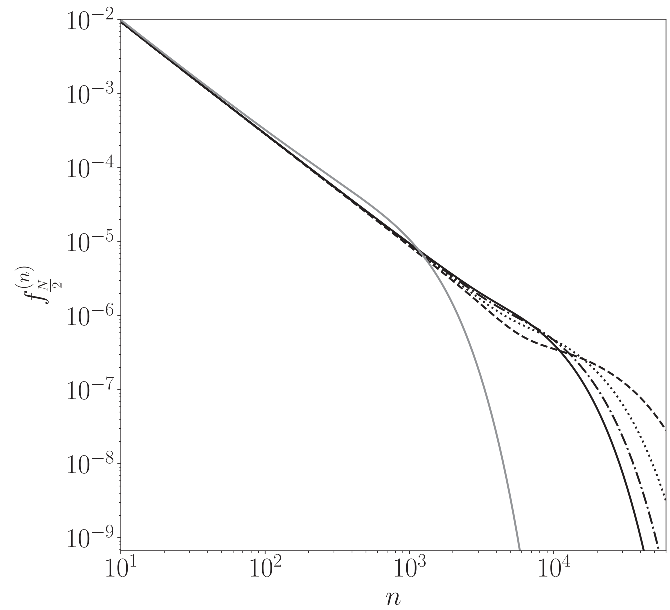

FIG. 11. First return distribution $f^{(n)}_{N/2}$ for the following values of a: 10, (grey solid line); 0, (black solid line); $-3/4$, (dashed-dotted line); $-1/2$, (dotted line); and $-1/4$, (dashed line).

obtain

$$A(x_\rho) = \frac{a+1}{N+a}(1-2x_\rho),$$
$$B(x_\rho) = \frac{1}{N+a}\left(\frac{a+1}{N} + 2x_\rho(1-x_\rho)\right).$$

These expressions are different to those in Ref. [23] because Kirman's model with $\delta = 2\varepsilon$ is used in that article. In the same work, Artime and coworkers showed that first passage distribution $f(x_f, t|x_0)$, i.e., the probability density for the stochastic variable to take a value in a small environment of x_f for the first time at time t, provided that it was x_0 at time zero, can be obtained by solving the eigenvalue problem of the Fokker-Planck equation. In particular, by introducing the so-called Liouville-Green transformation, the eigenvalue problem

$$\frac{[A(x_\rho)X_n]}{\partial x_\rho} - \frac{1}{2}\frac{\partial^2[B(x_\rho)X_n]}{\partial x_\rho^2} = \lambda_n X_n,$$

becomes

$$\frac{d^2Y_n(y)}{dy} + [\lambda_n - \Delta(y)]Y_n(y) = 0,$$

where

$$y(x_\rho) = \left|\int_{x_f}^{x_\rho}\sqrt{\frac{2}{B(x')}}dx'\right|,$$
$$Y_n = B^{1/4}(x_\rho)w^{1/2}(x_\rho)X_n(x_\rho),$$
$$w(x_\rho) = B(x_\rho)\exp\left[-2\int^{x_\rho}\frac{A(x')}{B(x')}dx'\right],$$
$$\Delta = \frac{16(A^2 + A'B - AB') + 3B'^2 - 4BB''}{32B},$$

and $n = 0, 1, 2, \ldots$. The first passage distribution is related to the eigenfunctions X_n by

$$f(x_f, t|x_0) = \frac{1}{2}B(x_f)w(x_0)\left|\sum_{n=0}^{\infty}X_n(x_0)X_n'(x_f)e^{-\lambda_n t}\right|,$$

see Ref. [23]. For our model,

$$\Delta = \frac{(2a+1)}{8(N+a)} \times \left(\frac{4N(1+2a)x_\rho(x_\rho-1) + N(2a-1) - 4(a+1)}{2Nx_\rho(1-x_\rho)+(a+1)}\right),$$

and it vanishes for all x_ρ, only for $a = -1/2$. For this particular case, it is very easy to solve the eigenvalue problem and $f(x_f, t|x_0)$ follows a power-law distribution with exponent $-3/2$ as in the case of the so-called Wentzel-Kramers-Brillouin (WKB) approximation (see Ref. [23]). This is indicated in Fig. 10 by the intersection between the vertical and horizontal lines. This value is slightly different to our numerical value. We attribute this difference to the fact that the numerical computation was performed with $N = 100$ and the value $\lambda_{\rm exp} = -3/2$ is valid for very large N.

C. First return distribution for $N/2$

We can study the first return time distribution for state $N/2$, i.e., we can compute numerically the values of $f^{(n)}_{N/2}$ by using the recursive relation of Eq. (6). Figure 11 shows that the first return distribution follows a power-law distribution with exponent $-3/2$ for a_c. For $a > a_c$ the effects of finite size are present for a smaller number of steps n. For $a < a_c$ there is a supercritical behavior for n large enough. For $a > 0$ there is a subcritical regime. However, we know that if the parameter α goes to 1, the Ehrenfest model should be obtained. In this case,

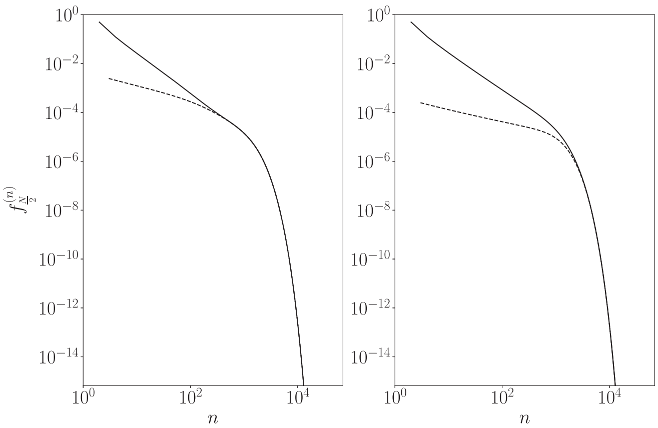

FIG. 12. First return distribution $f^{(n)}_{N/2}$ for even (solid line) an odd (dashed line) number of steps. On the left $a = 100\,000$, and on the right $a = 1\,000\,000$. In both cases $N = 1000$.

the probability to stay in the same state at the next temporal step b_i, vanishes. Thus, the probability to return for the first time to the site $N/2$ should be zero for an odd number of steps and positive for an even number of steps. If $\alpha \to 1$, $a \to \infty$, thus for a large enough a difference between the first return probability for an even or odd numbers of steps has to be appreciated. Figure 12 shows this difference and it is possible to appreciate that the coalescent point moves to the right when a becomes larger. Although it could be possible to study this behavior with a similar approach of the authors of Ref. [21], to the best of our knowledge, there is no explicit expression of the associated dual Hahn polynomials.

V. CONCLUSION

In this work we present a simple combination of two classical urn models to study a noisy voter model. In general, the expression noisy voter model refers to the voter model whose probability transitions are modified by noise. In this work, the noise is present by Ehrenfest's model steps. The aims of this work were to study some properties of the transition order-disorder as well as the first passage and return distributions. For the first aim we use the explicit expression of the stationary distribution and the second aim is tackled by knowing all the eigenvectors of the transition matrix in terms of the dual Hahn orthogonal polynomials. Although the spectral analysis was already performed in Ref. [13], we determined the critical exponents, as well as the maximum of the susceptibility, around $\alpha = 1/(2N)$ or $a = -1/2$. We also studied in detail the first return distributions $f^{(n)}_0$ and analyzed the behavior of their power-law exponents.

One of the first approaches to study a noisy voter model was introduced by Kirman. Although his model had two parameters, ε and δ, his results were given for $\delta = 2\varepsilon$ and N large [12]. If Kirman's model, with the given relationship between the parameters, is written in terms of Dette's model, the parameter a becomes

$$a = \frac{\varepsilon(N+1) - 1}{1 - 2\varepsilon}.$$

Thus, the critical point is at $a = 0$, and thus, $\varepsilon = 1/(N+1)$ and the stationary distribution is given by Eq. (4) that is equivalent to that presented in Ref. [8] for $N_0 = N_1$ and whose expression for N large is as that in Ref. [12]. Other examples of noisy voter models could be those models presented in Refs. [3,24], where the interactions are with some neighbors on the network and the noise is represented by constant parameters.

In general, most of the works studied the behavior of the system for N large by using the mean-field approximation or Fokker-Planck formalism losing, in this way, some characteristics that were present for finite system sizes. Even more, the thermodynamic limit is always taken with α constant. Our approach allows to study the thermodynamic limit $N \to \infty$ and $\alpha \to 0$ while their product αN remains constant. In particular, we can write $\alpha N = a(1-\alpha) + 1$; thus, if α goes to zero at the same time N goes to infinity the product αN goes to $a + 1$. This implies that the region $a < 0$ is that associated with the thermodynamic limit $N \to \infty$ and $\alpha \to 0$ faster than $1/N$.

In this work, magnetization, susceptibility, Binder's fourth-order cumulant, and critical exponents could be determined by using discrete random walks tools. Even more, this approach is very useful to study the system behavior when $a < 0$, or equivalently, where the system is at the voter regime or the thermodynamic limit was taken as explained above. In that case, a simulation approach would required a long computational time to obtain accurate curves. In particular, it is very difficult to obtain Fig. 9 by numerical simulations and, even worse, Fig. 10.

On the other hand, at the Ehrenfest regime, for a very large, a difference for return probability behaviors is appreciated for an odd and an even number of steps. This can be a very important property to consider for some systems. For instance, if it is considered a random walker that moves according the transition probabilities given by Eq. (1) and that has a light that is on after even steps or off otherwise, it could be interesting to know the probability that the walker arrives to the border with the light on.

It is worth mentioning that the maximum at susceptibility is given when $E_{N/2,N}$ is minimum, as it was expected since the fluctuations are maximum. This happens for $a \approx -1/2$. For $a < -1/2$ both boundaries become very sticky and it becomes very difficult for the walker to leave them. Thus, $a = 0$ indicates a change between the Ehrenfest and voter regimes, while $a = -1/2$ indicates that the boundaries become very sticky and difficult to leave. For $a < -1/2$, or equivalently $\alpha < 1/(2N)$, boundaries have a relevant role in the system dynamics.

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[1] P. Ehrenfest and T. Ehrenfest, Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem, Phys. Z. **8**, 311 (1907).
[2] S. Redner, Reality-inspired voter models: A mini-review, C. R. Phys. **20**, 275 (2019).
[3] A. Carro, R. Toral, and M. San-Miguel, The noisy voter model on complex networks, Sci. Rep. **6**, 24775 (2016).
[4] A. L. M. Vilela and H. Eugene Stanley, Effect of strong opinions on the dynamics of the majority-vote model, Sci. Rep. **8**, 8709 (2018).
[5] D. Braha and M. A. M. de Aguiar, Voting contagion: modeling and analysis of a century of U.S. presidential elections, PLoS One **12**, e0177970 (2017).
[6] D. D. Chinellato, I. R. Epstein, D. Braha, Y. Bar-Yam, and M. A. M. de Aguiar, Dynamical response of networks under external perturbations: exact results, J. Stat. Phys. **159**, 221 (2015).
[7] D. Harmon, M. Lagi, M. A. M. de Aguiar, D. D. Chinellato, D. Braha, I. R. Epstein, and Y. Bar-Yam, Anticipating economic market crises using measures of collective panic, PLoS One **10**, e0131871 (2015).
[8] M. Ramos, M. A. M. de Aguiar, and D. Braha, Opinion dynamics on networks under correlated disordered external perturbations, J. Stat. Phys. **173**, 54 (2018).
[9] N. Khalil and R. Toral, The noisy voter model under the influence of contrarians, Physica A **515**, 81 (2019).
[10] I. Caridi, S. Manterola, V. Semeshenko, and P. Balenzuela, Topological study of the convergence in the voter model, Appl. Network Sci. **4**, 119 (2019).
[11] A. Czaplicka, C. Charalambous, R. Toral, and M. San-Miguel, Biased-voter model: How persuasive a small group can be? Chaos Solitons Fractals **161**, 112363 (2022).
[12] A. Kirman, Ants, rationality, and recruitment, Quarterly J. Econ. **108**, 137 (1993).
[13] H. Dette, On a generalization of the Ehrenfest urn model, J. Appl. Probab. **31**, 930 (1994).
[14] W. Pickering and C. Lim, Solution to urn models of pairwise interaction with application to social, physical, and biological sciences, Phys. Rev. E **96**, 012311 (2017).
[15] S. Karlin and J. McGregor, Random walks, Illinois J. Math. **3**, 66 (1959).
[16] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their Q-Analogues* (Springer, Berlin, 2010).
[17] P. Njionou-Sadjang, W. Koepf, and M. Foupouagnigni, On moments of classical orthogonal polynomials, J. Math. Anal. Appl. **424**, 122 (2015).
[18] D. Dominici, Polynomial sequences associated with the moments of hypergeometric weights, SIGMA **12**, 044 (2016).
[19] V. Udodov, Violating of the Essam-Fisher and Rushbrooke relationships at low temperatures, WJCMP **05**, 55 (2015).
[20] O. Krafft and M. Schaefer, Mean passage times for tridiagonal transition matrices and a two-parameter Ehrenfest urn model, J. Appl. Probab. **30**, 964 (1993).
[21] S. A. Menchón and P. Román, Emergence of power law distributions for odd and even lifetimes, Phys. Rev. E **102**, 062143 (2020).
[22] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (North-Holland, Amsterdam, 2007).
[23] O. Artime, N. Khalil, R. Toral, and M. S. Miguel, First-passage distributions for the one-dimensional Fokker-Planck equation, Phys. Rev. E **98**, 042143 (2018).
[24] B. L. Granovsky and N. Madras, The noisy voter model, Stoch. Process. Their Appl. **55**, 23 (1995).