Size-independent scaling analysis for explosive percolation

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The Achlioptas process, a percolation algorithm on random network, shows a rapid second-order phase transition referred to as explosive percolation. To obtain the transition point and critical exponent β for percolations on a random network, especially for bond percolations, we propose a new scaling analysis that is independent of the system size. The transition point and critical exponent β are estimated for the product-rule (PR) and da Costa-Dorogovtsev-Goltsev-Mendes (dCDGM) ($m = 2$) models of the Achlioptas process, as well as for the Erdős-Rényi (ER) model, which is a classical model in which the analytic values are known. The validity of the scaling analysis is confirmed, especially for the transition point. The estimations of β are also consistent with previously reported values for the ER and dCDGM(2) models, whereas the β estimation for the PR model deviates somewhat. By introducing a parameter representing the maximum cluster size, we develop an extrapolation scheme for the critical exponent β from the simulation just at the transition point in order to obtain a more accurate value. The estimated value of β is improved compared with that obtained by the scaling analysis for the ER model and is also consistent with the β value obtained for the dCDGM(2) model, whereas its deviation from the previously reported value is larger for the PR model. We discuss the accuracy of the present estimations and draw conclusions about their reliability.

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I. INTRODUCTION

Percolation phenomena have been an active topic of study in statistical physics because of their applicability to models of conductive materials and the spread of infectious diseases and because they are related to phase transitions and critical phenomena. In 2009, Achlioptas *et al.* introduced explosive percolation [1] in the field of random networks. They slightly modified the Erdős-Rényi (ER) model, in which a random network is produced by adding a new edge randomly, to a rule by adding one of two randomly chosen edges to avoid increasing the cluster size [2]. The model in which edges are added by such rules is generally called the Achlioptas process, of which there are several variations. When a network is developed by this process, the phase transition becomes more abrupt than that in the ER model, hence the name "explosive." Whether the transition in the Achlioptas process is first or second order was initially a subject of controversy, but it has been confirmed to be a second-order transition [3]. This abrupt but second-order transition behavior has also attracted attention as a critical phenomenon and has been studied by numerical methods. Notably, difficulties have been observed in the analysis of finite size scaling [4]. In abrupt phase transitions in which the exponent β is very small, the order parameter becomes very steep near the transition and it is somehow troublesome to estimate the transition point and exponent precisely through the finite-size scaling of the order parameter.

In the present study, we investigated a numerical analysis method for explosive percolation to estimate the transition point and the critical exponent β. For this purpose, we propose a new system-size-independent scaling analysis for percolations on a random network, especially for bond percolations. The proposed method enables us to analyze numerical data without finite-size effects, which means that the results are equivalent to those in an infinite system. Such a size-independent scaling analysis has been presented as dynamical scaling analysis in the nonequilibrium relaxation method [5] for phase transitions and critical phenomena, which has been attracting attention as an alternative to finite-size scaling. We also propose an extrapolation scheme, by introducing a parameter representing the maximum cluster size, for the critical exponent β from the simulation just at the transition point in order to obtain a more accurate value. The estimated value of β is expected to be improved compared with that obtained by the scaling analysis.

This paper is organized as follows. The percolation phenomenon and models of percolation on random network are described together with the classical percolation model in Sec. II. In Sec. III, we propose the new scaling analysis and present the corresponding numerical results. In Sec. IV, the critical exponent β is precisely estimated by introducing an extrapolation scheme. Section V summarizes and discusses the results.

II. MODELS

A. Percolation on a lattice

In classical site percolations, each vertex on a lattice point is assumed to be occupied with probability p or to be empty with probability $1 - p$, and the cluster formed by occupied vertices connected to each other as nearest-neighboring pairs

is considered. The size of clusters does not increase linearly with p; large clusters suddenly appear at a certain value p_c [6]. When the occupancy probability p is less than p_c, only small clusters exist. However, when p is greater than p_c, one large cluster is dominant. When the number of vertices N is sufficiently large, $N \to \infty$, this large cluster is called an infinite cluster.

The probability of a vertex belonging to the infinite cluster, which is denoted by $P(p)$, is recognized as the order parameter, and p is the control parameter; thus, the system undergoes a phase transition at $p = p_c$. The function $P(p)$ can be written as [6]

$$P(p) = 1 - \sum_s s n_s, \tag{2.1}$$

where n_s is the number of clusters with size s divided by N. Some physical quantities are known to exhibit anomalies near the transition point, and $P(p)$ is expected to behave algebraically as [6]

$$P(p) = a|p - p_c|^\beta, \tag{2.2}$$

where β is referred to as a critical exponent and is a quantity that characterizes the critical phenomenon.

B. Percolations on random network

In percolations on random network, a number of vertices are prepared first instead of a lattice and each edge between two of these vertices is considered to be connected or not. The explosive percolation was introduced by Achlioptas *et al.* [1] and has continued to be widely investigated [7]. In the Erdős-Rényi (ER) model, which is one of the models of classical percolation on a random network, a new edge is connected to a network at random. By contrast, in the Achlioptas process, a slight modification is made to the edge-connection rule.

For the product-rule (PR) model, which is a typical explosive percolation model, let us randomly select two edges as a candidate for a new connection. Each edge connects two vertices, each of which is included by a cluster; we refer to this cluster as a connected cluster for a vertex. The size of the connected cluster is considered for each vertex. Each edge is associated with a product of the sizes of two connected clusters. We adopt one of the selected edges so that the product is smaller; it is called the PR. Figure 1 shows an image for the PR model. The rule used to choose a connecting edge in the PR model is provided to make it difficult for the cluster size to be larger.

Note that we can consider other rules for explosive percolation in choosing edges. For example, in the da Costa-Dorogovtsev-Goltsev-Mendes (dCDGM) model[8–10], choosing m vertices at random, the one belonging to the smallest cluster is selected. Two vertices are selected independently in this way and a new edge connecting them is added. The integer value m is fixed and distinguishes the model as the dCDGM(m) model. These models are generally called the Achlioptas process [11]. Because the transition is confirmed to be second order even though it is abrupt, the critical exponents can be defined as in Sec. II A.

The Achlioptas processes such as the PR model can be applied to lattice percolation because the rules are defined by the manner in which edges are chosen [4]. However, in the present paper, we investigate the process applied to random networks and introduce a new method to analyze the phase transition for the explosive percolation. We investigated the PR and dCDGM($m = 2$) models as well as the ER model, for which the solution is known analytically. In Table I, we summarize the previously obtained transition points and critical exponents β for these models [8,12].

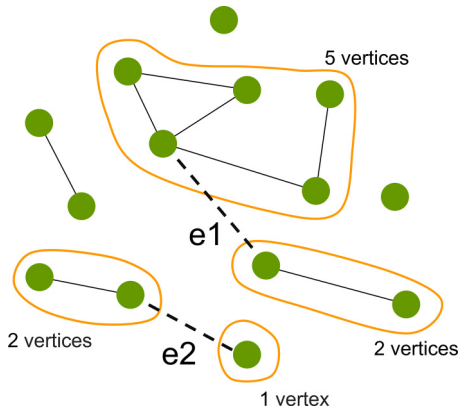

FIG. 1. Process by which an edge is added for the PR model: e1 and e2 are randomly selected candidates for the added edge. The product of the cluster sizes at both ends of e1 is 5 2 = 10, and that of e2 is 2 1 = 2. Therefore, e2 is chosen.

III. SCALING ANALYSIS

A. Method

We propose a scaling analysis for models of percolation on random networks, in which the system size does not appear. Extending the probability of an infinite cluster, $P(t)$, defined in Eq. (2.1), where t is the average number of edges per vertex corresponding to parameter p in lattice models, we define a new function $P(x, t)$ and introduce a new parameter x representing the maximum cluster size:

$$P(x, t) = 1 - \sum_{s<x} s n_s, \tag{3.1}$$

where n_s is the number of clusters with size s divided by N. This function $P(x, t)$ represents the probability that a randomly selected vertex belongs to a cluster greater than or equal to x.

Equation (3.1) becomes equivalent to Eq. (2.1) as $x \to \infty$, and we expect the asymptotic behavior near the transition point to be

$$P(\infty, t) = P(t) = a|t - t_c|^\beta. \tag{3.2}$$

TABLE I. Previously obtained transition point and critical exponent.

Model	t_c	β	Reference
ER	0.5	1	
PR	0.888449(2)	0.0861(5)	[12]
dCDGM(2)	0.923207509297(2)	0.05557108(1)	[8]

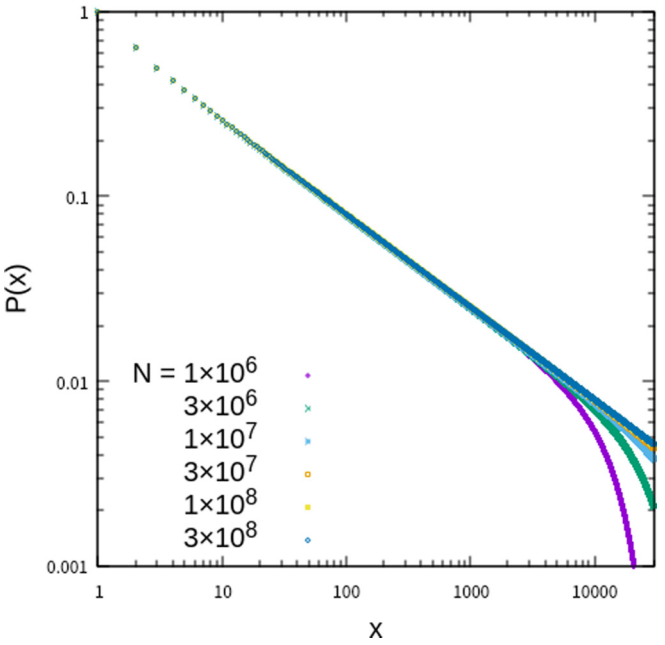

FIG. 2. Size dependence for the ER model on a double-logarithmic scale.

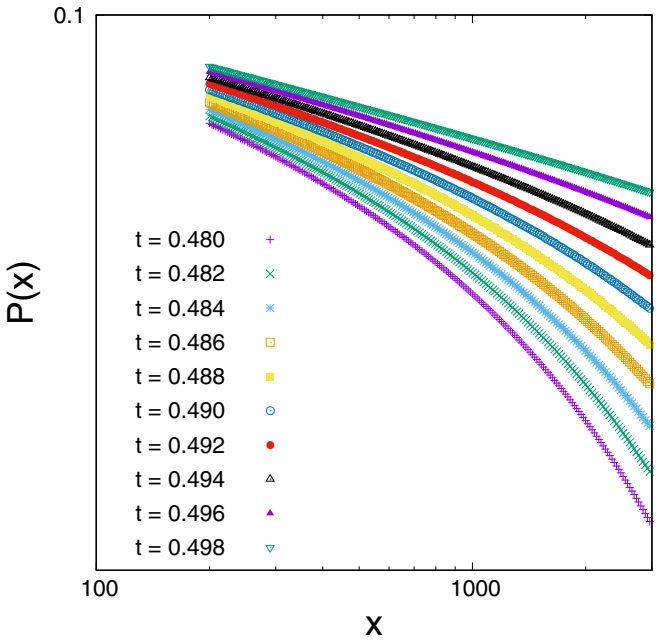

FIG. 3. Results of $P(x,t)$ for the ER model on a double-logarithmic scale.

Because no infinite clusters exist at $t < t_c$, we have $P(\infty, t) = 0$. However, if $t > t_c$, the existence of an infinite cluster means that $P(\infty, t)$ has a positive finite value. Under this consideration, we expect the following asymptotic ($x \to \infty$) behaviors for every t case:

$$P(x,t) \sim \begin{cases} A\exp(-x/\xi) & (t < t_c), \\ x^{-\lambda_P} & (t = t_c), \\ p_\infty + A'\exp(-x/\xi) & (t > t_c). \end{cases} \tag{3.3}$$

where we introduce another function ξ that depends on t as $\xi = \xi(t)$ and exponent λ_P.

If a second-order phase transition is assumed, the asymptotic behavior of ξ is expected to be

$$\xi(t) = b|t - t_c|^{-\nu}, \tag{3.4}$$

where we define a new exponent ν satisfying the relation

$$\lambda_P = \beta/\nu. \tag{3.5}$$

Note we use the symbols ξ and ν here because Eq. (3.3) shows a behavior similar to the correlation function for classical percolation on a lattice, where $P(x,t)$ corresponds to the correlation function with x as the distance between two vertices, and we use ξ as the correlation length and ν as its critical exponent. Of course, no concept of length exists in the present models on random networks. We therefore will not further consider the relation among these quantities.

The transition point t_c and critical exponent β can be estimated from the behavior of $P(x,t)$. Noting Eqs. (3.2)–(3.5) and referring to the asymptotic behavior of the correlation function in percolation on a lattice where A and A' would include an algebraic dependence of $x^{-\lambda_P}$, $P(x,t)$ is expected to satisfy the following scaling form in the asymptotic regime in $t \sim t_c$:

$$P(x,t) = \xi^{-\lambda_P}\Psi(x/\xi). \tag{3.6}$$

Appropriate values of t_c and β as well as a, b, ν in Eqs. (3.2) and (3.4) should be chosen such that $\xi^{\lambda_P}P(x,t)$ is fitted as a function of $x/\xi(t)$ on a scaling function Ψ. To efficiently carry out such a scaling plot, we apply the method introduced for the dynamical scaling, in which Bayesian inference and the kernel method are used [13].

B. Numerical results

We applied the previously discussed scaling analysis to the ER, PR, and dCDGM(2) models. First, we present the results for the ER model, for which the analytic solution is known. To perform the scaling analysis precisely, we need to confirm that the size N is sufficiently large for the numerical data of $P(x,t)$ to show no size dependence up to the maximum value of x used in the simulation. We simulated at $t = 1/2$ for several sizes and calculated $P(x,t)$, as plotted in Fig. 2. We found that, for x values $x \leqslant 3000$, no size dependence appears in the data for the system with $N = 1 \times 10^8$ or larger. The simulation was performed using these parameters, and 2240 independent samples were used for averaging. The value of t was varied from 0.480 to 0.498 with an increment of 0.002. The results for $P(x,t)$ are plotted in Fig. 3, and the resultant scaling plot is shown in Fig. 4. The transition points t_c and the critical exponent β were evaluated and are shown in Table II.

TABLE II. Estimated t_c and β values obtained using the present scaling analysis.

Model	t_c	β
ER	0.49978(7)	1.010(9)
PR	0.88843(1)	0.0889(7)
dCDGM(2)	0.923211(5)	0.056(1)

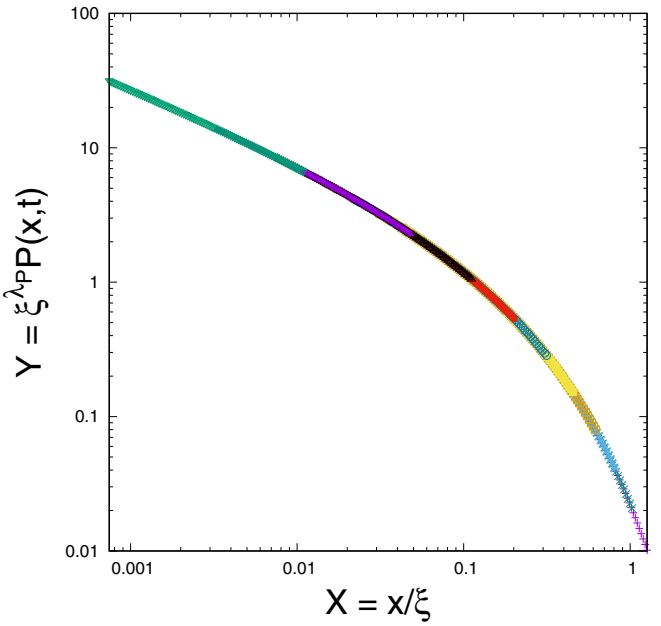

FIG. 4. Scaling plot of $P(x,t)$ for the ER model on a double-logarithmic scale.

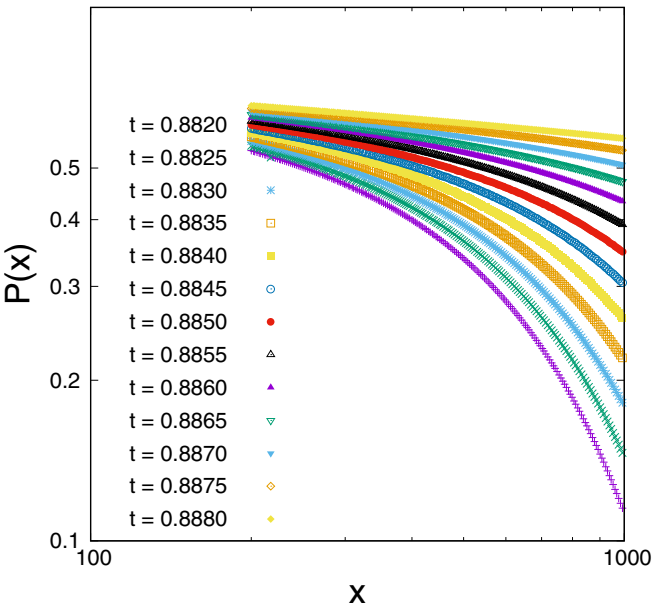

FIG. 6. Results of $P(x,t)$ for the PR model.

The error bar was estimated by the bootstrap method [13]. The deviations from the analytic values $t_c = 1/2$, $\beta = 1$ are approximately 0.044% and 1.0%, respectively, demonstrating that the present analysis is reliable.

The same analysis was performed for the PR model. The size dependence at $t = 0.888\,45$ is shown in Fig. 5. The results show that the simulation for the system with $N = 1 \times 10^8$ up to $x \leqslant 1000$ was sufficiently large. The simulation was performed using these parameters, and 2240 independent samples were used for averaging. The value of t was varied from 0.8820 to 0.8880 with an increment of 0.005. The results are plotted in Fig. 6, and the scaling plot is shown in Fig. 7. The transition points t_c and the critical exponent β were estimated and are shown in Table II. A comparison of the results with those of Grassberger (Table I) reveals that the transition point agrees very well and that the β value is consistent but deviates slightly from the previously reported values.

For the dCDGM(2) model, only the results for t_c and β are listed in Table II. It was confirmed that the simulation for the system with $N = 1 \times 10^8$ to $x \leqslant 3000$ was sufficiently large. The simulation was performed using these parameters, and

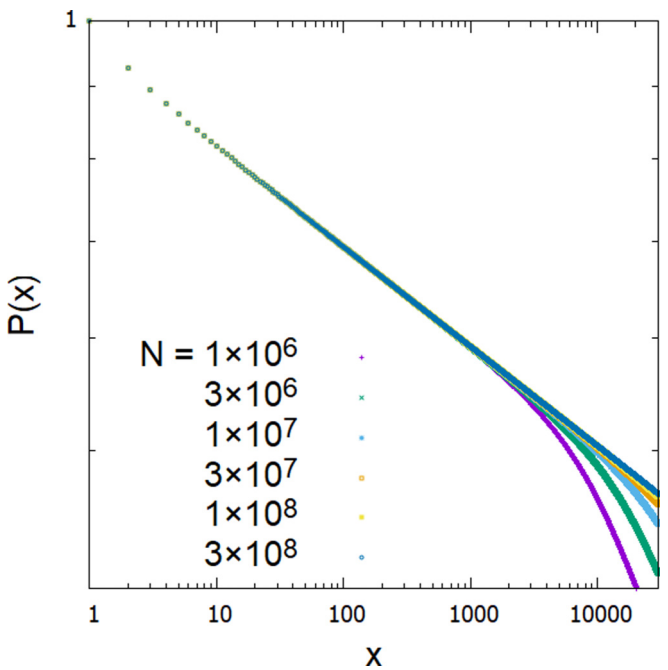

FIG. 5. Size dependence for the PR model.

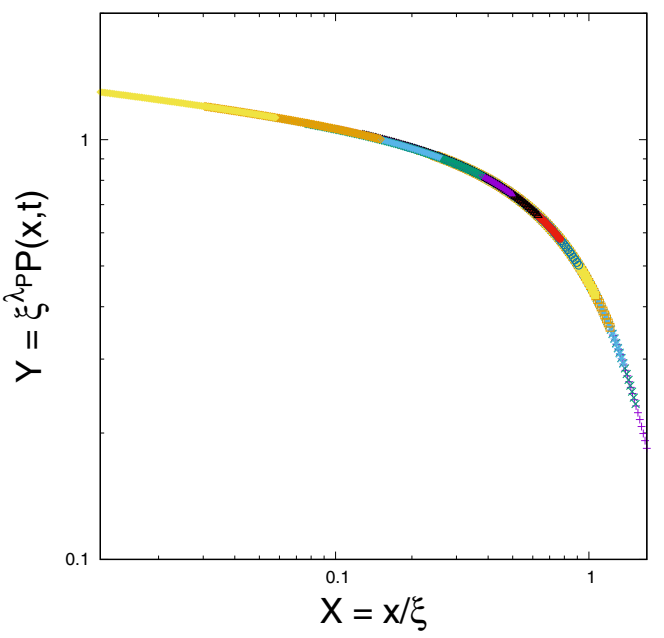

FIG. 7. Scaling plot of $P(x,t)$ for the PR model.

4480 independent samples were used for averaging, where the value of t was varied from 0.9197 to 0.9232 with an increment of 0.0005. The t_c and β values are consistent with those of da Costa (Table I).

IV. ESTIMATION OF β BY EXTRAPOLATION

Although the value of β estimated by the scaling analysis in Sec. III results in a good scaling plot of $P(x,t)$ for the range of x to a maximum observed value, β may deviate from the true value corresponding to that for $x \to \infty$. In this section, we investigate an extrapolation scheme for β estimated in finite x values to that at $x \to \infty$.

A. Method of extrapolation

We define another function $Q(x,t)$ as the partial derivative of $P(x,t)$ with t:

$$Q(x,t) \equiv \frac{\partial P(x,t)}{\partial t}. \tag{4.1}$$

From Eqs. (3.2) and (3.3), $Q(x,t)$ is expected to diverge algebraically as $x \to \infty$ at the transition point:

$$Q(x,t_c) \sim x^{\lambda_Q}, \tag{4.2}$$

where the exponent λ_Q is obtained through β defined in Eq. (3.2) and ν defined in Eq. (3.4) as

$$\lambda_Q = (1-\beta)/\nu. \tag{4.3}$$

For convenience, the logarithmic derivatives of $P(x,t_c)$ and $Q(x,t_c)$ are defined for a finite value of x:

$$\lambda_P(x) = -\frac{d\ln P(x,t_c)}{d\ln x}, \tag{4.4}$$

$$\lambda_Q(x) = \frac{d\ln Q(x,t_c)}{d\ln x}, \tag{4.5}$$

which can asymptotically approach the exponents λ_P and λ_Q, respectively, as $x \to \infty$ and are referred to as the "local exponents" of λ_P and λ_Q. Because the exponents λ_P, λ_Q, and β are related through Eqs. (3.5) and (4.3), we define the local exponent $\beta(x)$ as

$$\beta(x) = \frac{\lambda_P(x)}{\lambda_P(x)+\lambda_Q(x)}, \tag{4.6}$$

which can approach β as $x \to \infty$.

Although the function $Q(x,t)$ is defined by the partial derivative of $P(x,t)$ with t, as in Eq. (4.1), we did not use a numerical derivative of $P(x,t)$ to obtain it; we instead evaluated it by simulation directly. To obtain $Q(x,t)$, we substituted the operation ∂t by an operation that adds a few edges to the network and then evaluated $\partial P(x,t)$ by the difference of $P(x,t)$. In the case of the ER model, for each sample, we simulated the following procedure. Edges were added until $t = t_c$. Then, for all remaining unconnected edges, the value of $\partial P(x,t)$ was averaged for the case where the edge was chosen in the next step, which provided $Q(x,t_c)$. In the case of the PR model or the dCDGM model, for each sample, we simulated the following procedure. Edges were added until $t = t_c$. The value of $\partial P(x,t)$ was then averaged for N independent additions of a single edge, which provided $Q(x,t_c)$.

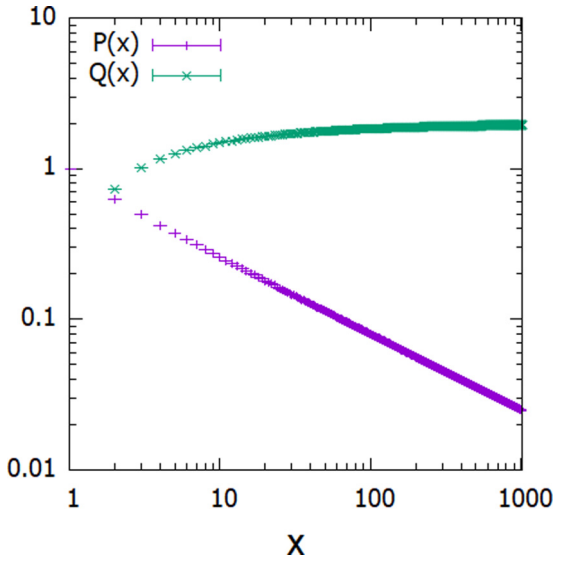

FIG. 8. Results of $P(x,t_c)$ and $Q(x,t_c)$ for the ER model on a double-logarithmic scale

B. Numerical results

First, we present the analysis for the ER model. The simulation was performed at $t_c = 1/2$, the analytic value of the transition point, with the size $N = 1 \times 10^9$; 22 400 independent samples were used for averaging. The results of $P(x,t_c)$ and $Q(x,t_c)$ are plotted in Fig. 8. Note that, because the β value for the ER model is close to unity, the exponent λ_Q gives a small value from Eq. (4.3), providing the gently increasing behavior observed in Fig. 8. From the data of these functions, we calculate $\lambda_P(x)$ and $\lambda_Q(x)$ defined in Eqs. (4.4) and (4.5) by the numerical derivative of a linearly fitted line in an appropriate interval of x. Then, the local exponent $\beta(x)$ is calculated by the use of Eq. (4.6) and is plotted as a function of $1/x^{0.713}$ in Fig. 9; the exponent 0.713 in the horizontal axis was chosen so that the extrapolation shows the best linearity [14]. Note that the calculated local exponent $\beta(x)$ clearly depends on x, which indicates that this extrapolation is important for obtaining a precise value corresponding to $x \to \infty$. The estimated β is listed in Table III, where the error bar was evaluated from the guidelines plotted in the figure; the estimation is obtained by the solid line and the error bar is determined by the broken lines. The analysis was found to be valid, with an error of approximately 0.73% compared to the analytic value. Note that the obtained value is closer to the analytic value than that estimated by the scaling method and reported in Table II. Thus, the value of β estimated by the extrapolation method is expected to be precise.

TABLE III. Estimations of β by extrapolation.

Model	β
ER	0.993(1)
PR	0.0933(9)
dCDGM(2)	0.0548(3)

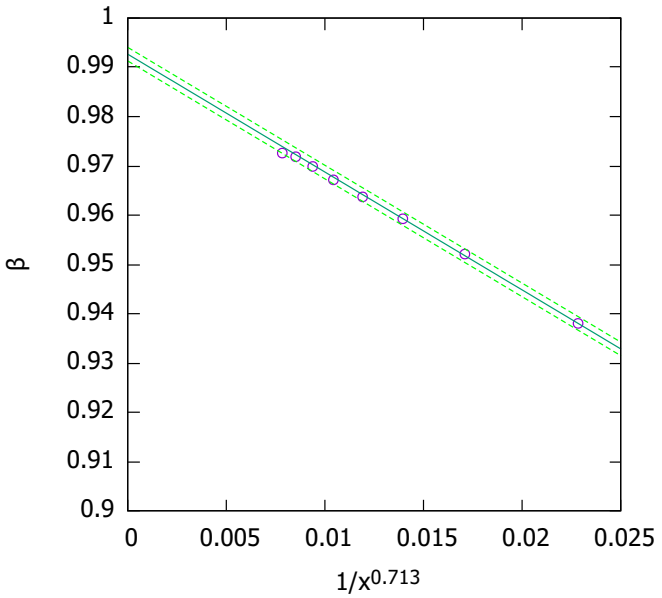

FIG. 9. Extrapolation of β for the ER model. The best fitting is indicated by a solid line, and broken lines are a guide to the eye, highlighting the errors that reveal the asymptotic behavior.

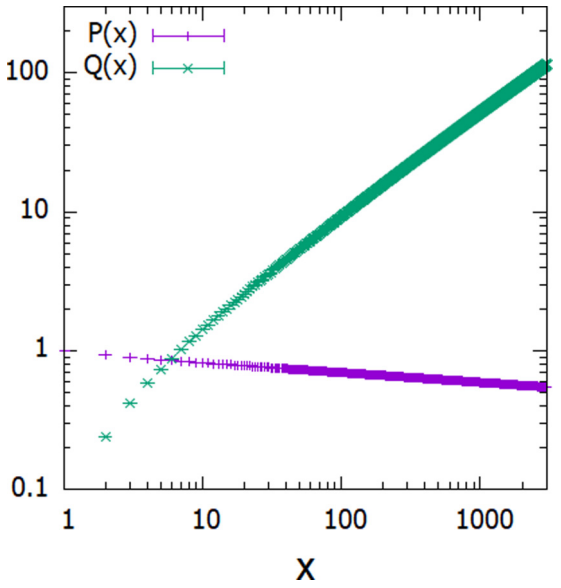

FIG. 10. Results of $P(x, t_c)$ and $Q(x, t_c)$ for the PR model.

Let us discuss the importance and reliability of the extrapolation scheme for the estimation of exponent β. Since the two functions $P(x, t)$ and $Q(x, t)$ are related analytically as in Eq. (4.1), the resulting value of β by the scaling analysis of $P(x, t)$ in Sec. III and that by the present extrapolation scheme by the use of both $P(x, t)$ and $Q(x, t)$ have the same origin. As seen in Fig. 9, the estimated value of β depends on the observed region of x, the precise value can be obtained by taking the limit of $x \to \infty$. Similarly, the resulting value of β by the scaling analysis would also be deviated from a true value depending on the maximum value of observed x; this deviation is expected to disappear asymptotically when $x \to \infty$. In the simulation, it is necessary to increase the number of vertices N for calculating these functions with larger x region. Thus, the extrapolation scheme is easier for taking the limit, and we introduce it to estimate a precise β. Note that, for the ER model, the improvement of estimated β between Tables II and III is small, and further the error bar is not appropriate in the latter. However, because of the asymptotic behavior on x stated above, we expect that the estimated value can be improved in the extrapolation scheme. We consider that one should not compare the error bars between in the scaling analysis and in the extrapolation scheme seriously. The former is recognized as a statistical one, while the latter is not statistical but a measure for the stability of data. It is obtained by the guide lines in Fig. 9, which are plotted to guide the eye for the extrapolation, in which the plotted data are included.

The same analysis was performed for the PR model. The simulation was performed at $t_c = 0.888\,434$, which was estimated as described in Sec. III, with the size $N = 5 \times 10^8$; 4480 independent samples were used for averaging. The results of $P(x, t_c)$ and $Q(x, t_c)$ are plotted in Fig. 10. The local exponent $\beta(x)$ is plotted as a function of $1/x^{0.699}$ in Fig. 11; the exponent 0.699 in the horizontal axis was chosen so that the extrapolation shows the best linearity. The estimated β value is listed in Table III, where the error bar was evaluated from the guidelines plotted in the figure. The value of β deviates from that reported by Grassberger [12] (Table I) to a greater extent than the value estimated from the scaling in Table II. We will discuss this point later.

The same analysis was performed for the dCDGM(2) model. The simulation was performed at $t_c = 0.923212$, as estimated in Sec. III, with the size $N = 5 \times 10^8$; 5600 independent samples were used for averaging. The local exponent $\beta(x)$ is plotted as a function of $1/x^{0.716}$ in Fig. 13; the results of $P(x, t_c)$ and $Q(x, t_c)$ are shown in Fig. 12. The β value,

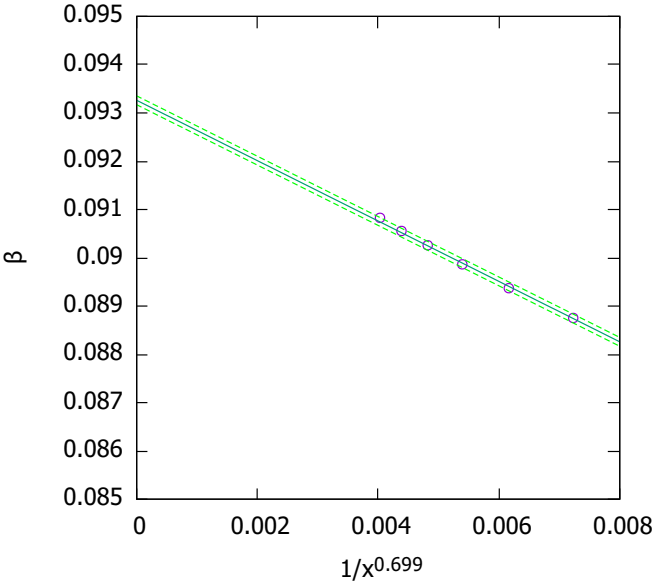

FIG. 11. Extrapolation of β for the PR model.

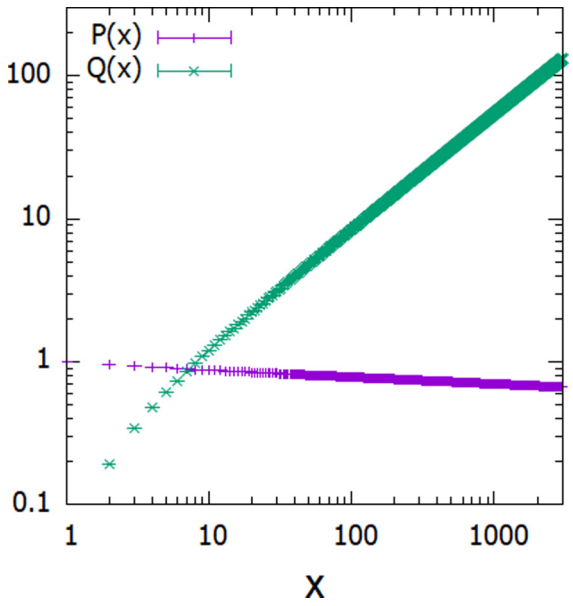

FIG. 12. Results of $P(x, t_c)$ and $Q(x, t_c)$ for the dCDGM(2) model.

which is shown in Table III, does not substantially deviate from the scaling result.

V. SUMMARY AND DISCUSSION

The explosive percolation that appears in the Achlioptas process was investigated by introducing a size-independent scaling method for percolations on a random network. The transition point and critical exponent β were estimated for two models of the Achlioptas process: the product-rule (PR) and dCDGM(2) models. We further applied the scaling method to the Erdős-Rényi (ER) model, which is a classical model in which the analytic values are known, and confirmed the validity of the analysis. The estimations of the transition point

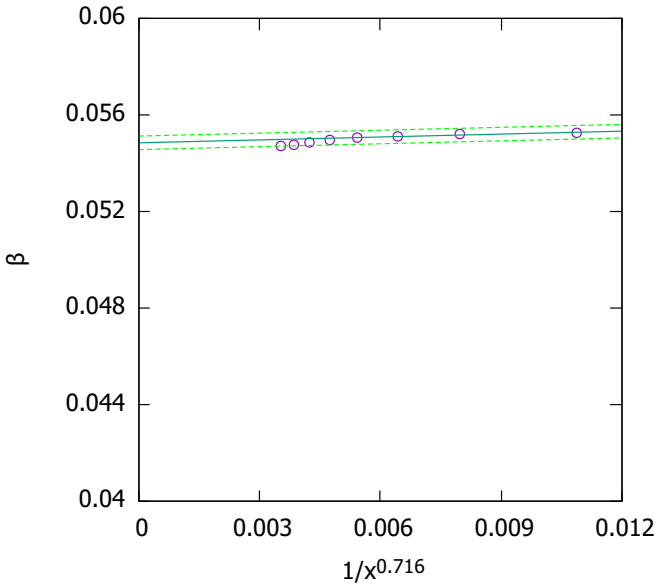

FIG. 13. Extrapolation of β for the dCDGM(2) model.

for these models were in good agreement with those reported previously. The estimations of β were also consistent with previously reported values, whereas the β value estimated using the PR model deviated slightly. To obtain a more accurate value of β, we proposed a scheme of extrapolating the simulation just at the transition point by introducing a parameter representing the maximum cluster size. The estimated value of β was improved compared with that estimated by the scaling analysis for the ER model and was also consistent with the value estimated using the dCDGM(2) model.

By contrast, the deviation of the β value from the previously reported value became larger for the PR model. The value of β for the PR model was previously estimated as 0.0861(5) by finite-size scaling analysis [12]; let us refer to this value as β_0. The β value was obtained as 0.0889(7) by the present size-independent scaling analysis, which was confirmed to be reliable by the results for the ER and dCDGM(2) models. The extrapolation scheme for β was also confirmed to be reliable for the ER and dCDGM(2) models, whereas that for the PR model, 0.0933(9), showed a greater deviation; let us call this value β_E. We consider that $\beta(x)$ in Eq. (4.6) should approach the true value of β in the limit of $x \to \infty$. However, our data for $P(x, t)$ and $Q(x, t)$ only extend to a finite value of x; thus, the extrapolation scheme becomes necessary. Increasing x further in the present analysis is difficult because the system size needs to be larger to avoid size dependence. Notably, the extrapolation scheme is provided for the data at the transition point and the quality of the result is influenced by the accuracy of the transition point in the simulation.

We examined the dependence of the estimated β_E on the value of t_c used in the extrapolation scheme for the PR model. For both the lower and upper bounds of t_c of the PR model listed in Table II, $t_{c-} \equiv t_c - \Delta t_c = 0.888\,421$ and $t_{c+} \equiv t_c + \Delta t_c = 0.888\,447$, respectively, the same analysis as in Sec. IV was performed, where 3360 samples were used for averaging, with a size of $N = 3 \times 10^8$. We obtained $\beta_- \equiv 0.096(2)$ for $t_c = t_{c-}$ and $\beta_+ \equiv 0.0903(7)$ for $t_c = t_{c+}$. A small but finite monotonic t_c dependence of β was observed. We found that the values of β estimated by extrapolation in $t_{c-} \leqslant t_c \leqslant t_{c+}$ are all larger than β_0 for $t_c = 0.888\,449$, the previous estimation reported in Table I. In addition, β_+ for $t_c = t_{c+}$ still deviates from and is larger than the previously reported value β_0 even though the values of t_c are similar. Furthermore, it is seen in Fig. 11 that the $\beta(x)$ is monotonic increasing with x, and the smallest value of observed $\beta(x)$ is larger than β_0. Since the true value is given by $\beta(x)$ with $x \to \infty$ if our argument for the asymptotic behavior in the extrapolation scheme is correct, the present estimation β_E is expected to be closer to the true value. These results suggest that the previous estimation β_0 contains an error stemming from a finite-size effect.

Improving the accuracy of the scaling analysis or extrapolation scheme requires conducting a simulation for larger values of x, which requires a larger system size N and more samples. In addition, for the extrapolation scheme, the accuracy of the transition point t_c used in the simulation is important.

The present size-independent scaling analysis and the extrapolation scheme can be applied to a wide range of

percolation phenomena, including lattice percolations. As an example, we applied the extrapolation scheme to the classical bond percolation on a square lattice in two dimensions and obtained $\beta = 0.1369(4)$, where $\beta = 5/36$ was pointed out analytically [6].

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[14] It is not clear what the value of this exponent is related to and which value is appropriate. Our aim is to estimate the asymptotic limit of $\beta(x)$ for $x \to \infty$, while we have the data up to a finite value of x. Therefore, for a practical reason, we determine the value of exponent so that the linearity is best in the observed x regime.