Kinetic theory of one-dimensional inhomogeneous long-range interacting N-body systems at order $1/N^2$ without collective effects

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Long-range interacting systems irreversibly relax as a result of their finite number of particles, N. At order $1/N$, this process is described by the inhomogeneous Balescu-Lenard equation. Yet, this equation exactly vanishes in one-dimensional inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances. In the limit where collective effects can be neglected, we derive a closed and explicit $1/N^2$ collision operator for such systems. We detail its properties, highlighting in particular how it satisfies an H theorem for Boltzmann entropy. We also compare its predictions with direct N-body simulations. Finally, we exhibit a generic class of long-range interaction potentials for which this $1/N^2$ collision operator exactly vanishes.

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I. INTRODUCTION

Because they are composed of a finite number of particles, long-range interacting N-body systems unavoidably relax towards their thermodynamical equilibrium [1–4], should it exist [5,6]. Such a dynamics is sourced by Poisson shot noise: for a fixed total mass, the larger the number of particles, N, the slower the diffusion. Kinetic theory aims at describing this long-term relaxation. Here, we are interested in inhomogeneous systems, i.e., systems with a nontrivial mean-field orbital structure [2]. When limited to $1/N$ effects, i.e., two-body correlations, the system's evolution is generically described by the inhomogeneous Balescu-Lenard (BL) equation [7,8]. When collective effects are neglected, i.e., when one neglects the system's ability to amplify its own self-generated fluctuations [9], this intricate kinetic equation becomes the inhomogeneous Landau equation [10]. Since both equations are valid at order $1/N$, they describe evolutions on timescales of order $NT_{\rm d}$, with $T_{\rm d}$ being the dynamical time.

For one-dimensional (1D) inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances, the BL equation collision term vanishes exactly (see, e.g., Refs. [11–19]). This is a kinetic blocking and we refer to Ref. [20] for a detailed review of the literature on that regard. Kinetically blocked systems can only evolve under the effect of three-body correlations. Their relaxation occurs therefore on a timescale of order $N^2T_{\rm d}$, or even larger. In this paper, we derive an appropriate kinetic equation for this (very) slow process.

Steps in that direction were successively performed as follows: (i) in Ref. [21] a first attempt was made at deriving a $1/N^2$ equation for the one-dimensional (homogeneous) Hamiltonian mean field model (HMF) model [22] starting from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy and neglecting collective effects; (ii) this was further clarified in Ref. [23], which emphasized the main properties of this collision operator and compared it with numerical simulations; and (iii) finally, these results were generalized in Ref. [20] to homogeneous systems with an arbitrary interaction potential.

The kinetic equation derived in Ref. [20] was restricted to homogeneous 1D systems. Here, we go beyond this limitation and focus on inhomogeneous systems, while still neglecting collective effects. In the limit where collective amplification can be neglected, we present a closed, explicit, and well-posed kinetic equation that describes these systems' relaxation on $N^2T_{\rm d}$ timescales. In addition to reviewing the key properties of this collision operator, we also quantitatively compare its predictions with direct N-body simulations. Remarkably, we present a class of interaction potentials for which this $1/N^2$ collision term exactly vanishes whatever the system's distribution function: we call this a second-order kinetic blocking.

The paper is organized as follows. In Sec. II, we spell out the inhomogeneous $1/N^2$ kinetic equation, as given by Eq. (4). In Sec. III, we present the main properties of this equation, while in Sec. IV, we investigate its steady states. In Sec. V, we quantitatively compare the prediction of this theory with direct numerical simulations of particles interacting on the unit sphere. Finally, we conclude in Sec. VI. In all these sections, technical details are deferred either to the Appendices or to appropriate references.

II. KINETIC EQUATION

We consider a population of N particles of individual mass $\mu = M_{\rm tot}/N$, with $M_{\rm tot}$ being the system's total mass. The 1D canonical (specific) phase coordinates are denoted by $\mathbf{w} = (\theta, J)$, with θ being the 2π-periodic angle and J the action [2]. The system's total specific Hamiltonian reads

$$H = \sum_{i=1}^{N} U_{\rm ext}(\mathbf{w}_i) + \sum_{i<j}^{N} \mu\, U(\mathbf{w}_i, \mathbf{w}_j), \tag{1}$$

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with $U_{\rm ext}(\mathbf{w})$ being a given external potential and $U(\mathbf{w},\mathbf{w}')$ the pairwise interaction potential between the particles. We assume that the pairwise interaction satisfies the symmetries $U(\mathbf{w},\mathbf{w}')=U(|\theta-\theta'|,\{J,J'\})$. When Fourier expanded with respect to the angles, it becomes

$$U(\mathbf{w},\mathbf{w}')=\sum_{k=-\infty}^{+\infty}U_k(J,J')e^{ik(\theta-\theta')}, \tag{2}$$

where the coefficients $U_k(J,J')\in\mathbb{R}$ satisfy the symmetries (i) $U_{-k}(J,J')=U_k(J,J')$ and (ii) $U_k(J',J)=U_k(J,J')$.

We describe the system with its distribution function (DF), $F=F(\mathbf{w})$, normalized to $\int d\mathbf{w}F=M_{\rm tot}$. The system is in a quasistationary equilibrium, so that both the mean DF, $F=F(J)$, and the mean Hamiltonian, $H_0(\mathbf{w})=U_{\rm ext}(\mathbf{w})+\int d\mathbf{w}'\,U(\mathbf{w},\mathbf{w}')F(\mathbf{w}')=H_0(J)$, depend only on the action. The present system is said to be inhomogeneous because (i) the coupling coefficients $U_k(J,J')$ depend on the particles' actions and (ii) to every action is associated an orbital frequency, $\Omega(J)=dH_0/dJ$. Characterizing relaxation amounts to characterizing $\partial F(J,t)/\partial t$.

In the limit where only $1/N$ effects are accounted for,[1] the dynamics of $F(J,t)$ is described by the inhomogeneous BL equation [7,8]. It reads

$$\frac{\partial F(J)}{\partial t}=2\pi^2\mu\frac{\partial}{\partial J}\Bigg[\int dJ_1\,\big|\psi^{\rm d}_{\rm tot}(J,J_1)\big|^2\,\delta_{\rm D}\big(\Omega[J]-\Omega[J_1]\big)\times\left(\frac{\partial}{\partial J}-\frac{\partial}{\partial J_1}\right)F(J)\,F(J_1)\Bigg], \tag{3}$$

where we dropped the time dependence of the DFs for clarity. Here, the total coupling coefficients follow from $|\psi^{\rm d}_{\rm tot}(J,J_1)|^2=\sum_k|k||\psi^{\rm d}_{kk}(J,J_1,k\Omega[J])|^2$; the detailed expression of the dressed coupling coefficients $\psi^{\rm d}_{kk'}(J,J_1,\omega)$ can be found in Appendix G of Ref. [24]. Importantly, we emphasize that the symmetry of Eq. (2) imposes $\psi^{\rm d}_{kk'}\propto\delta_{kk'}$; i.e., only 1:1 resonances are permitted.

For a system with a monotonic frequency profile,[2] $J\mapsto\Omega(J)$, the diffusion flux from Eq. (3) exactly vanishes. Indeed, the resonance condition $\delta_{\rm D}(\Omega[J]-\Omega[J_1])$ imposes that only local two-body resonances of the form $J_1=J$ are permitted. This leads to $\partial F/\partial t=0$ in Eq. (3). As a consequence, one-dimensional inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances generically have a vanishing BL flux. This is a kinetic blocking (see, e.g., Refs. [12,14,16,17]); i.e., these systems cannot relax via two-body correlations of order $1/N$. Relaxation is then significantly delayed as it can only occur through the weaker $1/N^2$ three-body correlations. This is our focus here.

In order to derive an appropriate $1/N^2$ kinetic equation for the present system, we generalize the result from Ref. [20], which focused on homogeneous systems with an arbitrary interaction potential. Yet, accounting for inhomogeneity makes it so that calculations become rapidly difficult to handle given the large number of terms appearing. All these aspects are dealt with within a MATHEMATICA code that is distributed in the Supplemental Material [25].

Building upon Ref. [20], the key steps of the derivation are highlighted in Appendix A. In short, we proceed by (i) writing the usual BBGKY coupled evolution equations for the one-, two- and three-body DFs [26]; (ii) rewriting these equations as evolution equations for the one-body DF, $F(J,t)$, and the two- and three-body correlation functions using the cluster expansion [27]; (iii) truncating these equations at order $1/N^2$ and splitting the two-body correlation function in its $1/N$ and $1/N^2$ components [21]; (iv) neglecting collective effects by assuming that the system is dynamically hot enough so as to not strongly amplify its own self-generated perturbations;[3] and (v) solving explicitly a sequence of four differential equations. Once these steps have been implemented, it remains to perform a large number of integration by parts, symmetrizations, and relabelings to reach a simple expression for the final $1/N^2$ collision operator. This is, by far, the most challenging part of the calculation where the use of a computer algebra system appears mandatory. All the details are given in Ref. [25].

Ultimately, the kinetic equation reads

$$\frac{\partial F(J)}{\partial t}=2\pi^3\mu^2\frac{\partial}{\partial J}\Bigg[\sum_{k_1,k_2}\frac{1}{k_1^2(k_1+k_2)}\mathcal{P}\int\frac{dJ_1}{(\Omega[J]-\Omega[J_1])^4}\times\int dJ_2\,\mathcal{U}_{k_1k_2}(\mathbf{J})\,\delta_{\rm D}[\mathbf{k}\cdot\mathbf{\Omega}]\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{J}}\right)F_3(\mathbf{J})\Bigg], \tag{4}$$

where the sum over k_1 and k_2 is restricted to the integers such that k_1, k_2, and (k_1+k_2) are all nonzero, and $\mathcal{P}$ stands for Cauchy principal value. In that expression, we also shortened the notations by introducing the vectors $\mathbf{J}=(J,J_1,J_2)$, $\mathbf{\Omega}=(\Omega[J],\Omega[J_1],\Omega[J_2])$, $\mathbf{k}=(k_1+k_2,-k_1,-k_2)$, and $F_3(\mathbf{J})=F(J)F(J_1)F(J_2)$. Equation (4) finally involves the (positive) coupling coefficients $\mathcal{U}_{k_1k_2}(\mathbf{J})$, whose detailed expression is given in Appendix B. We note that Eq. (4) is sourced by three-body correlations, i.e., it involves the DF three times. Therein, relaxation occurs only when the three-body resonance condition $\delta_{\rm D}[\mathbf{k}\cdot\mathbf{\Omega}]$ is satisfied. Finally, Eq. (4) also differs from Eq. (3) because it does not involve collective effects.

In the limit of a homogeneous system, one may replace (i) the action J by the velocity v, (ii) the orbital frequency $\Omega(J)$ by v, and (iii) the action-dependent coefficients $U_k(J,J')$ by the velocity-independent ones, U_k. Following such replacements, one exactly recovers the homogeneous $1/N^2$ kinetic equation already derived in Ref. [20].

Equation (4) is the main result of this work. It describes the (very) long-term relaxation of dynamically hot one-dimensional inhomogeneous systems, as driven by $1/N^2$

[1] We must also assume that the mean DF is linearly stable, i.e., the linearized Vlasov equation, $\partial\delta F/\partial t+[\delta F,H_0]+[F,\delta H(\delta F)]=0$ (see Appendix A), supports no unstable modes (see, e.g., Chap. 5.3 in Ref. [2]).

[2] For example, this is the case in 1D homogeneous systems where, up to prefactors, one has $(J,\Omega[J])\to(v,v)$, with v being the velocity.

[3] It is the same assumption that allows one to derive the (simpler) Landau equation from the BL one, at order $1/N$ [10].

effects. Equation (4) is rather general since it applies to any arbitrary long-range interaction potential that follows Eq. (2). In the coming section, we explore some of the main properties of this kinetic equation.

III. PROPERTIES

A. Conservation laws

Equation (4) satisfies a couple of important conservation laws. Up to prefactors, those are

$$\begin{aligned} M(t) &= \int dJ\, F(J,t) && \text{(total mass)};\\ P(t) &= \int dJ\, J\, F(J,t) && \text{(total momentum)};\\ E(t) &= \int dJ\, H_0(J)\, F(J,t) && \text{(total energy)}. \end{aligned} \tag{5}$$

We refer to Appendix C 1 for the proof of these conservations.

B. Dimensionless rescaling

It is enlightening to estimate the typical relaxation time predicted by Eq. (4) using a dimensionless rewriting. We denote the system's typical frequency with Ω_0 and set the dynamical time to $T_\mathrm{d} = 1/\Omega_0$. We define the typical DF's action dispersion $J_0 = [\langle (J - \langle J\rangle)^2\rangle]^{1/2}$ with $\langle X\rangle = \int d\mathbf{w} X F/M_\mathrm{tot}$, and we introduce the dimensionless DF $\overline{F} = J_0 F/M_\mathrm{tot}$. Finally, we assume that the interaction potential satisfies $U \propto G$. Injecting these various elements into Eq. (4), we find $\partial\overline{F}/\partial\bar{t} = 1/(N^2Q^4)\ldots$, with $\bar{t} = t/T_\mathrm{d}$ and all the remaining terms being dimensionless. Here, we introduced the system's dimensionless stability parameter (see, e.g., Ref. [28])

$$Q = \frac{J_0\,\Omega_0}{G M_\mathrm{tot}}. \tag{6}$$

The larger Q is, the hotter the system is, i.e., the weaker the collective effects are.[4] The system's relaxation time T_r, when driven by Eq. (4), therefore scales like

$$T_\mathrm{r} \simeq Q^4 N^2 T_\mathrm{d}. \tag{7}$$

As expected, the hotter the system is, the slower the long-term relaxation is. Given that Eq. (4) has been derived by neglecting collective effects, it can only be applied to systems with $Q \gg 1$.

C. Well-posedness

Because it involves a high-order resonance denominator, it is not obvious that Eq. (4) is well-posed, i.e., that there are no divergences when $J_1 \to J$. Following the same approach as in Ref. [20], we show in Appendix C 2 that one can rewrite Eq. (4) under an alternative form for which the principal value can be computed.

IV. STEADY STATES

A. *H* theorem

Up to prefactors, the system's entropy is defined as

$$S(t) = -\int dJ\, s[F(J,t)], \tag{8}$$

with $s[F] = F\ln(F)$ being Boltzmann's entropy.[5] As detailed in Appendix D 1, one can show that Eq. (4) drives an evolution of the entropy according to

$$\begin{aligned} \frac{dS}{dt} = \frac{2\pi^3\mu^2}{3}\sum_{k_1,k_2}\int d\mathbf{J}\,\frac{\mathcal{U}_{k_1k_2}(\mathbf{J})}{k_1^2(k_1+k_2)^2}\mathcal{P}\left(\frac{1}{(\Omega[J]-\Omega[J_1])^4}\right)\\ \times\frac{\delta_\mathrm{D}[\mathbf{k}\cdot\boldsymbol{\Omega}]}{F_3(\mathbf{J})}\left((k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)}\right)^2. \end{aligned} \tag{9}$$

All the terms in this integral are positive. Hence, Eq. (4) satisfies an H theorem, i.e.,

$$\frac{dS}{dt} \geqslant 0. \tag{10}$$

Equation (4) therefore drives an irreversible relaxation.

B. Boltzmann distribution

For the present case, the thermodynamical equilibria, i.e., the Boltzmann DFs, are generically of the form

$$F_\mathrm{B}(J) \propto e^{-\beta H_0(J) + \gamma J}, \tag{11}$$

with β and γ being two Lagrange multipliers associated with the conservation of $E(t)$ and $P(t)$ in Eq. (5). When injected in Eq. (4), the DF from Eq. (11) gives

$$\begin{aligned} \frac{\partial F_\mathrm{B}(J)}{\partial t} &\propto \delta_\mathrm{D}[\mathbf{k}\cdot\boldsymbol{\Omega}]\,\{-\beta\,[\mathbf{k}\cdot\boldsymbol{\Omega}] + \gamma\,[(k_1+k_2) - k_1 - k_2]\}\\ &= 0. \end{aligned} \tag{12}$$

As expected, Boltzmann DFs are equilibria of Eq. (4).

C. Constraint from the *H* theorem

Following the calculation of dS/dt in Eq. (9), let us now investigate what are the most generic steady states of Eq. (4). For simplicity, we assume that there exists (k_1, k_2) such that $\mathcal{U}_{k_1k_2}(\mathbf{J}) \neq 0$ when $\mathbf{k}\cdot\boldsymbol{\Omega} = 0$, i.e., at resonance. An obvious way of ensuring $dS/dt = 0$ is for the last term in Eq. (9) to systematically vanish. Since $J \mapsto \Omega(J)$ is monotonic, we can define the function $G(\Omega) = F'(J[\Omega])/F(J[\Omega])$ and we find the constraint

$$\forall\Omega_1,\Omega_2:\; G\left(\frac{k_1\Omega_1 + k_2\Omega_2}{k_1+k_2}\right) = \frac{k_1 G(\Omega_1) + k_2 G(\Omega_2)}{k_1+k_2}, \tag{13}$$

namely, a weighted average. For this constraint to be satisfied for all Ω_1 and Ω_2, the function $\Omega \mapsto G(\Omega)$ must necessarily be affine; i.e., one has

$$G(\Omega) = -\beta\,\Omega + \gamma. \tag{14}$$

[4]The typical scaling of Q can readily be found from the system's inhomogeneous response matrix (see Chap. 5.3 in Ref. [2]).

[5]Boltzmann's entropy is expected, *a priori*, to be the system's entropy in the present weak-coupling limit [27].

Integrating once with respect to J, Eq. (14) recovers the Boltzmann DF from Eq. (11). Provided that there exists one (k_1, k_2) for which $\mathcal{U}_{k_1k_2}(\mathbf{J}) \neq 0$ at resonance, the only equilibrium DFs of Eq. (4) are the Boltzmann DFs.

D. Second-order kinetic blocking

It is interesting to determine whether or not one can design an interaction potential and a frequency profile so that for all (k_1, k_2) one has $\mathcal{U}_{k_1k_2}(\mathbf{J}) = 0$ at resonance. Following Eq. (9), this would then impose $dS/dt = 0$ whatever the considered DF. For simplicity, in this section, we assume the simple frequency profile $\Omega(J) \propto J$.

In Appendix D 2, we show that potentials of the form

$$U(\mathbf{w}, \mathbf{w}') \propto |J - J'|^{\alpha} \sum_{\substack{k=1 \\ k \equiv 0 \,[\mathrm{mod}\, d]}}^{+\infty} \frac{1}{|k|^{\alpha}} \cos[k\,(\theta - \theta')], \tag{15}$$

with α being an arbitrary power law index and $d \geqslant 1$ an arbitrary integer, are generically blocked for the $1/N^2$ dynamics driven by Eq. (4). More precisely, such potentials ensure that the coupling function $\mathcal{U}_{k_1k_2}(\mathbf{J})$ (Appendix B) satisfies

$$\forall k_1, k_2, J, J_1\text{:}\ \mathcal{U}_{k_1k_2}(J, J_1, J_2^{\mathrm{res}}) = 0, \tag{16}$$

with $J_2^{\mathrm{res}} = J_2^{\mathrm{res}}[\mathbf{k}, J, J_1]$, the resonant action complying with the constraint $\mathbf{k} \cdot \boldsymbol{\Omega} = 0$. For the frequency profile $\Omega[J] \propto J$, it simply reads $J_2^{\mathrm{res}} = [(k_1 + k_2)J - k_1J_1]/k_2$. In Appendix D 2, we show that (i) Eq. (15) allows one to recover the local interaction potentials already unveiled in Ref. [20], and (ii) for particular values of α, the harmonic sum from Eq. (15) can be explicitly performed.

Following Eq. (15), *any* (linearly stable) DF, $F = F(J)$, when embedded within the mean potential $H_0 = \frac{1}{2}J^2$ and pairwise interaction from Eq. (15), satisfies $\partial F/\partial t = 0$ when plugged into Eq. (4). We call such a situation a second-order kinetic blocking; i.e., both the $1/N$ BL flux and the $1/N^2$ flux from Eq. (4) exactly vanish. In the limit where collective effects can effectively be neglected, we expect that such systems will relax on the much longer timescale $O(N^3T_{\mathrm{d}})$. Given the difficulty of (i) deriving a kinetic equation at order $1/N^3$ and (ii) numerically integrating such dynamics on extremely long timescales, we postpone their investigation to future works.

V. APPLICATION

In order to test Eq. (4), we perform numerical simulations of classical Heisenberg spins embedded within an external potential (see, e.g., Refs. [15,16]). We refer to Appendix E for a detailed presentation of the setup.

In Fig. 1, we illustrate the initial diffusion flux $\partial F/\partial t = \partial \mathcal{F}/\partial J$. This figure shows a good quantitative agreement between the predicted and measured fluxes. The remaining slight mismatch likely stems from contributions associated with collective effects.

VI. CONCLUSIONS

We presented a $1/N^2$ closed and explicit kinetic equation for long-range interacting one-dimensional inhomogeneous systems. The collision operator from Eq. (4) generalizes the classical inhomogeneous Landau kinetic equation to regimes where the $1/N$ relaxation exactly vanishes by symmetry. Equation (4) conserves the total mass, momentum, and energy, and it satisfies an H theorem. We exhibited a class of long-range interaction potentials for which this $1/N^2$ collision term exactly vanishes. Finally, we showed how Eq. (4) quantitatively matches with measurements from direct N-body simulations.

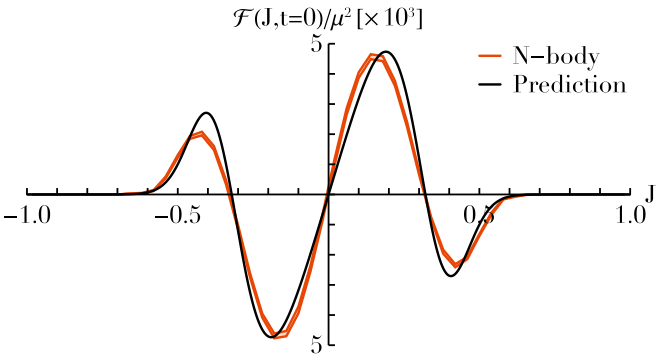

FIG. 1. Initial diffusion flux $\partial F/\partial t = \partial \mathcal{F}/\partial J$ in action space as measured in direct N-body simulations and predicted by Eq. (4). Detailed parameters are given in Appendix E.

Naturally, the present work is only one step toward ever more detailed characterizations of the long-term relaxation of finite-N systems. As such it would be worthwhile to (i) generalize Eq. (4) to also account for collective effects—a significantly challenging endeavor, and (ii) investigate, both analytically and numerically, systems driven by the interaction potential from Eq. (15), for which we expect a second-order kinetic blocking, i.e., a relaxation on N^3T_{d} timescales. Finally, effective applications of Eq. (4) to interpret spinor condensate and easy-plane ferromagnet experiments could be investigated.

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APPENDIX A: DERIVATION

Most of the derivation of Eq. (4) follows the same lines as in Ref. [20] (see Appendix A therein). Here, we only highlight the key changes stemming from inhomogeneity.

1. BBGKY hierarchy

The dynamics of the system is exactly described by the BBGKY equations for the n-body DFs, $F_n(\mathbf{w}_1, \ldots, \mathbf{w}_n, t)$ (see Appendix A 1 in Ref. [20]). It reads

$$\frac{\partial F_n}{\partial t} + [F_n, H_n]_n + \int d\mathbf{w}_{n+1}\,[F_{n+1}, \delta H_{n+1}]_n = 0, \tag{A1}$$

with the Poisson bracket

$$[f,h]_n = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial\theta_i}\frac{\partial h}{\partial J_i} - \frac{\partial f}{\partial J_i}\frac{\partial h}{\partial\theta_i}\right). \tag{A2}$$

In Eq. (A1), the n-body Hamiltonian H_n follows from Eq. (1) with the replacement $N \to n$. In Eq. (A1), the specific interaction energy δH_{n+1} simply reads

$$\delta H_{n+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{n+1}) = \sum_{i=1}^{N} U(\mathbf{w}_i,\mathbf{w}_{n+1}). \tag{A3}$$

Importantly, Eq. (A1) provides us with the evolution equations for the one-, two-, and three-body DFs (F_1, F_2, and F_3). This is the starting point of the derivation.

2. Cluster expansion

To perform perturbative expansions with respect to the small parameter $1/N$, the n-body DFs are developed using the cluster expansion [27]. Doing so, one introduces the two- and three-body correlation functions $G_2(\mathbf{w}_1,\mathbf{w}_2)$ and $G_3(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)$. As an example, G_2 is defined via

$$F_2(\mathbf{w}_1,\mathbf{w}_2) = F(\mathbf{w}_1)F(\mathbf{w}_2) + G_2(\mathbf{w}_1,\mathbf{w}_2), \tag{A4}$$

where, from now on, we write the one-body DF as $F = F_1$. Once this expansion has been performed, the quantities at our disposal scale with respect to N like $F \sim 1$, $G_2 \sim 1/N$, and $G_3 \sim 1/N^2$.

These expansions are then injected into Eq. (A1) to obtain evolution equations for $\partial F/\partial t$, $\partial G_2/\partial t$, and $\partial G_3/\partial t$. All these calculations are explicitly performed in Ref. [25].

3. Truncation at order $1/N^2$

The next step is to truncate these three evolution equations at order $1/N^2$. We perform the following operations.

(i) We introduce the small parameter $\epsilon = 1/N$ and decompose the two-body correlation function as

$$G_2 = \epsilon\, G_2^{(1)} + \epsilon^2\, G_2^{(2)}. \tag{A5}$$

Similarly, the other parameters at our disposal are rescaled as $\mu \to \epsilon\,\mu$ and $G_3 \to \epsilon^2\, G_3$.

(ii) We truncate the BBGKY evolution equations up to order ϵ^2. We split the evolution equation for $\partial G_2/\partial t$ into two, respectively, for $\partial G_2^{(1)}/\partial t$ and $\partial G_2^{(2)}/\partial t$.

(iii) We leverage our assumption of a quasistationarity, i.e., $F = F(J)$ and $H_0 = H_0(J)$, hence introducing the orbital frequencies $\Omega(J)$.

(iv) We neglect the contributions from collective effects; i.e., we perform replacements of the form

$$\int d\mathbf{w}_3\, G_2^{(1)}(\mathbf{w}_2,\mathbf{w}_3)\,\partial_{\theta_1} U(\mathbf{w}_1,\mathbf{w}_3) \to 0, \tag{A6}$$

and similarly for $G_2^{(2)}$ and G_3.

(v) In the hot limit, we neglect the source term in $G_2^{(1)}\times G_2^{(1)}$ in $\partial G_3/\partial t$, as its contribution is a factor $1/Q$ smaller than the source term in $G_2^{(2)}$.

4. Solving the equations

Following all these steps, we have at our disposal a set of four coupled partial differential equations. These equations can be solved sequentially, starting with $G_2^{(1)}$, then G_3, then $G_2^{(2)}$, and finally F. This is done as follows.

(i) We perform Fourier expansions with respect to the angles θ_i.

(ii) We rely on the assumption of timescale separation, hence fixing $F(J,t) = \text{const.}$ when solving for fluctuations.

(iii) We impose the initial conditions $G_2^{(1)}(t=0) = 0$ and similarly for $G_2^{(2)}$ and G_3.

Ultimately, we obtain an explicit time-dependent expression for $G_2^{(2)}(t)$. Leveraging once again timescale separation, we consider the limit $t \to +\infty$ of $G_2^{(2)}(t)$ in the evolution equation for $\partial F/\partial t$. A typical time integral is then replaced asymptotically by

$$\lim_{t\to+\infty}\int_0^t dt_1\, e^{i(t-t_1)\omega_{\mathrm{R}}} = \pi\,\delta_{\mathrm{D}}(\omega_{\mathrm{R}}) + i\,\mathcal{P}\left(\frac{1}{\omega_{\mathrm{R}}}\right), \tag{A7}$$

where $\omega_{\mathrm{R}} \in \mathbb{R}$ is a linear combination of $\mathbf{\Omega}$.

5. Simplifying the expressions

At this stage, we are left with a kinetic equation involving thousands of terms. The computer algebra system allows for efficient manipulations of these expressions. The key steps are as follows.

(i) We implement systematic relabelings of the actions (J_1,J_2) and resonance numbers (k_1,k_2), so that the resonant frequencies are all of the form $\omega_{\mathrm{R}} = \mathbf{k}\cdot\mathbf{\Omega}$.

(ii) We integrate by parts with respect to the actions so that no derivatives act on δ_{D}, and only first-order derivatives of the DF and coupling coefficients are present.

(iii) We use the scaling relations of δ_{D} and $\mathcal{P}$, e.g., $\delta_{\mathrm{D}}(\alpha x) = \delta_{\mathrm{D}}(x)/|\alpha|$, to take out resonance numbers.

(iv) The frequency profile being monotonic, we use

$$\int dJ_2\, f(J_1,J_2)\,\delta_{\mathrm{D}}\big(\Omega[J_1]-\Omega[J_2]\big) = \frac{f(J_1,J_1)}{|\Omega'(J_1)|}. \tag{A8}$$

(v) We use the resonance condition $\delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]$ to make the replacements $(\Omega-\Omega_2)\to -(k_1/k_2)(\Omega-\Omega_1)$ and $(\Omega_1-\Omega_2)\to -([k_1+k_2]/k_2)(\Omega-\Omega_1)$, with the shortened notation $(\Omega,\Omega_1,\Omega_2) = \mathbf{\Omega}$. Principal values are therefore only expressed as functions of $(\Omega-\Omega_1)$.

All these manipulations are automated using tailored rules in MATHEMATICA, which can all be found in Ref. [25]. Ultimately, one obtains the closed result from Eq. (4).

APPENDIX B: COUPLING COEFFICIENTS

The coupling coefficients $\mathcal{U}_{k_1k_2}(\mathbf{J})$ appearing in Eq. (4) are generically given by

$$\mathcal{U}_{k_1k_2}(\mathbf{J}) = \big[(\Omega[J]-\Omega[J_1])\,\mathcal{U}^{(1)}_{k_1k_2}(\mathbf{J}) + k_2\,\mathcal{U}^{(2)}_{k_1k_2}(\mathbf{J})\big]^2. \tag{B1}$$

The coupling functions appearing in this expression read

$$\begin{aligned}\mathcal{U}^{(1)}_{k_1k_2}(\mathbf{J}) = {} & k_2(k_1+k_2)\{U_{k_1+k_2}(J,J_2)\,\partial_{J_2}U_{k_1}(J_1,J_2) \\ & - U_{k_2}(J,J_2)\,\partial_J U_{k_1}(J,J_1)\} \\ & + k_1(k_1+k_2)\{U_{k_1}(J,J_1)\,\partial_J U_{k_2}(J,J_2) \\ & - U_{k_1+k_2}(J,J_1)\,\partial_{J_1}U_{k_2}(J_1,J_2)\} \\ & - k_1k_2\{U_{k_2}(J_1,J_2)\,\partial_{J_1}U_{k_1+k_2}(J,J_1) \\ & - U_{k_1}(J_1,J_2)\,\partial_{J_2}U_{k_1+k_2}(J,J_2)\} \end{aligned} \tag{B2}$$

and

$$\begin{aligned}\mathcal{U}^{(2)}_{k_1k_2}(\mathbf{J}) = {} & (k_1+k_2)\,\frac{d\Omega}{dJ}\,U_{k_1}(J,J_1)\,U_{k_2}(J,J_2) \\ & - k_1\,\frac{d\Omega}{dJ_1}\,U_{k_1+k_2}(J,J_1)\,U_{k_2}(J_1,J_2) \\ & - k_2\,\frac{d\Omega}{dJ_2}\,U_{k_1}(J_1,J_2)\,U_{k_1+k_2}(J,J_2). \end{aligned} \tag{B3}$$

We refer the reader to Ref. [25] for the associated derivation.

APPENDIX C: PROPERTIES

1. Conservation laws

We can generically rewrite Eq. (4) as

$$\frac{\partial F(J)}{\partial t} = \frac{\partial \mathcal{F}(J)}{\partial J}, \tag{C1}$$

with $\mathcal{F}(J)$ being the diffusion flux. The time derivatives of Eq. (5) then read

$$\begin{aligned}\frac{dM}{dt} &= \int dJ\,\frac{\partial \mathcal{F}}{\partial J}, \\ \frac{dP}{dt} &= -\int dJ\,\mathcal{F}(J), \\ \frac{dE}{dt} &= -\int dJ\,\Omega(J)\,\mathcal{F}(J). \end{aligned} \tag{C2}$$

The conservation of the total mass, $M(t)$, follows from the absence of any boundary contributions.

For the conservation of $P(t)$ and $E(t)$, we investigate

$$\int dJ\,\mathcal{F}(J) = \sum_{k_1,k_2}(k_1+k_2)\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J}), \tag{C3}$$

with $A_{k_1k_2}(\mathbf{J})$ given by Eq. (4). With the relabelings $J \leftrightarrow J_1$ and $(k_1,k_2) \to (-k_1-k_2,k_2)$, Eq. (C3) becomes [25]

$$\int dJ\,\mathcal{F}(J) = -\sum_{k_1,k_2}k_1\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J}). \tag{C4}$$

Similarly, with the relabelings $J \leftrightarrow J_2$ and $(k_1,k_2) \to (-k_1,k_1+k_2)$, Eq. (C3) becomes [25]

$$\int dJ\,\mathcal{F}(\mathbf{J}) = -\sum_{k_1,k_2}k_2\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J}). \tag{C5}$$

We can now go back to the computation of the conserved quantities in Eq. (C2). By adding $\frac{1}{3}$ of Eqs. (C3)–(C5), we finally obtain

$$\begin{aligned}\frac{dP}{dt} &= -\tfrac{1}{3}\sum_{k_1,k_2}\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J})\,\{(k_1+k_2)-k_1-k_2\} = 0, \\ \frac{dE}{dt} &= -\tfrac{1}{3}\sum_{k_1,k_2}\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J})\,\{\mathbf{k}\cdot\boldsymbol{\Omega}\} = 0, \end{aligned} \tag{C6}$$

where the final equality stems from the presence of the resonance condition $\delta_{\mathrm{D}}[\mathbf{k}\cdot\boldsymbol{\Omega}]$ in Eq. (4).

2. Well-posedness

Following the same approach as in Ref. [20], we define the set of fundamental resonances as

$$\{(k,k')\,|\,0<k,k'\}. \tag{C7}$$

For a given fundamental resonance, (k,k'), we define the associated set of resonance pairs (k_1,k_2) with

$$\begin{aligned}\mathcal{R}(k,k') = {} & \{(k,k'),(k+k',-k),(k,-k-k'), \\ & \times\ (k',k),(k+k',-k'),(k',-k-k')\}. \end{aligned} \tag{C8}$$

We note that (i) all (k_1,k_2) in $\mathcal{R}(k,k')$ satisfy $k_1 \gtrless 0$, and (ii) even for $k=k'$, this set still contains six elements.

Following these definitions, we rewrite Eq. (4) as [25]

$$\frac{\partial F(J)}{\partial t} = 2\pi^3\mu^2\,\frac{\partial}{\partial J}\left[\sum_{k,k'>0}\mathcal{F}_{kk'}(J)\right], \tag{C9}$$

where $\mathcal{F}_{kk'}(J)$ stands for the flux generated by the fundamental resonance (k,k'). It reads

$$\mathcal{F}_{kk'}(J) = \mathcal{P}\int\frac{dJ_1}{(\Omega[J]-\Omega[J_1])^4}\sum_{(k_1,k_2)\in\mathcal{R}(k,k')}\mathcal{C}_{k_1k_2}(J,J_1). \tag{C10}$$

Here, $\mathcal{C}_{k_1k_2}(J,J_1)$ follows from Eq. (4) and reads

$$\mathcal{C}_{k_1k_2}(J,J_1) = \int dJ_2\,\frac{\mathcal{U}_{k_1k_2}(\mathbf{J})}{k_1^2(k_1+k_2)}\,\delta_{\mathrm{D}}[\mathbf{k}\cdot\boldsymbol{\Omega}]\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{J}}\right)F_3(\mathbf{J}). \tag{C11}$$

Combining Eqs. (C10) and (C11), we must ultimately perform an integration with respect to dJ_1dJ_2. As in Ref. [20], we can propose an alternative writing for $\mathcal{C}_{k_1k_2}(J,J_1)$ by performing the relabeling $J_1 \leftrightarrow J_2$. We obtain [25]

$$\overline{\mathcal{C}}_{k_1k_2}(J,J_1) = \int dJ_2\,\frac{\overline{\mathcal{U}}_{k_1k_2}(\mathbf{J})}{k_2^2(k_1+k_2)}\,\delta_{\mathrm{D}}[\overline{\mathbf{k}}\cdot\boldsymbol{\Omega}]\left(\overline{\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{J}}\right)F_3(\mathbf{J}). \tag{C12}$$

As in Eq. (4), we introduced $\overline{\mathbf{k}} = (k_1+k_2,-k_2,-k_1)$ along with the coupling function

$$\overline{\mathcal{U}}_{k_1k_2}(\mathbf{J}) = \left[(\Omega[J]-\Omega[J_1])\,\overline{\mathcal{U}}^{(1)}_{k_1k_2}(\mathbf{J}) - k_1\overline{\mathcal{U}}^{(2)}_{k_1k_2}(\mathbf{J})\right]^4. \tag{C13}$$

In that expression, $\overline{\mathcal{U}}^{(1)}_{k_1k_2}(\mathbf{J})$ [respectively, $\overline{\mathcal{U}}^{(2)}_{k_1k_2}(\mathbf{J})$] are directly obtained from Eq. (B2) [respectively, Eq. (B3)] by performing the relabeling $J_1 \leftrightarrow J_2$.

To obtain a well-posed expression for the flux $\mathcal{F}_{kk'}(J)$, it remains to appropriately combine the two possible writings, $\mathcal{C}_{k_1k_2}(J,J_1)$ and $\overline{\mathcal{C}}_{k_1k_2}(J,J_1)$. Once again, the computer algebra system proves undoubtedly useful [25]. Starting from

Eq. (C10), we consider the writing

$$\begin{aligned}\mathcal{F}_{kk'}(J) = \mathcal{P}\int \frac{dJ_1}{(\Omega[J]-\Omega[J_1])^4} \\ \times \{(\mathcal{C}_{kk'}+\mathcal{C}_{k'k}) \\ + (\mathcal{C}_{k+k',-k}+\mathcal{C}_{k+k',-k'}) \\ + (\gamma_{kk'}\,\mathcal{C}_{k,-k-k'}+(1-\gamma_{kk'})\,\overline{\mathcal{C}}_{k,-k-k'}) \\ + (\gamma_{kk'}\mathcal{C}_{k',-k-k'}+(1-\gamma_{kk'})\,\overline{\mathcal{C}}_{k',-k-k'})\},\end{aligned} \tag{C14}$$

with the weight

$$\gamma_{kk'} = \frac{(k-k')^2}{k^2+k'^2}. \tag{C15}$$

Such a choice is always well-defined since $k, k' > 0$ [see Eq. (C7)]. This symmetrization is the same as in Ref. [20].

We can then rewrite Eq. (C14) as

$$\mathcal{F}_{kk'}(\Omega) = \mathcal{P}\int d\Omega_1\,\frac{K_{kk'}(\Omega,\Omega_1)}{(\Omega-\Omega_1)^4}, \tag{C16}$$

where we used the fact that $J \mapsto \Omega(J)$ is monotonic, introducing therefore $\Omega = \Omega[J]$ and $\Omega_1 = \Omega[J_1]$. After calculation [25], for $\delta\Omega \to 0$, one finds

$$K(\Omega, \Omega+\delta\Omega) \simeq O[(\delta\Omega)^3], \tag{C17}$$

making the principal value in Eq. (C16) well-posed.

APPENDIX D: STEADY STATES

1. *H* theorem

Starting from Eq. (8), the time-derivative of Boltzmann's entropy is given by

$$\frac{dS}{dt} = \int dJ\,\frac{F'(J)}{F(J)}\,\mathcal{F}(J). \tag{D1}$$

Using the same symmetrizations as in Eq. (C6), we find

$$\begin{aligned}\frac{dS}{dt} = \tfrac{1}{3}\sum_{k_1,k_2}\int d\mathbf{J}\,A_{k_1k_2}(\mathbf{J}) \\ \times\left\{(k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)}\right\}.\end{aligned} \tag{D2}$$

It then only remains to inject the expression of $A_{k_1k_2}(\mathbf{J})$ [Eq. (C3)] to obtain dS/dt as given in Eq. (9).

2. Second-order kinetic blocking

Let us consider interaction potentials of the form

$$U_k(J,J') \propto \frac{\delta_{k|d}}{|k|^\alpha}\,|J-J'|^\alpha, \tag{D3}$$

with α an arbitrary power-law index and $d \geqslant 1$ being an arbitrary integer. We also introduced the function $\delta_{k|d} = 1$ if $k \equiv 0\,[\mathrm{mod}\,d]$ and 0 otherwise.

As detailed in Ref. [25], for such a potential, when embedded within the frequency profile $\Omega(J) \propto J$, one finds

$$\begin{aligned}\mathcal{U}_{k_1k_2}(J,J_1,J_2^{\mathrm{res}}) \propto [(k_1+k_2)\,\delta_{k_1|d}\,\delta_{k_2|d} \\ - k_1\,\delta_{k_1+k_2|d}\,\delta_{k_2|d} - k_2\,\delta_{k_1+k_2|d}\,\delta_{k_1|d}]^2 \\ = 0,\end{aligned} \tag{D4}$$

where we used $\delta_{k|d}\delta_{k'|d} = \delta_{k|d}\delta_{k+k'|d}$. We have therefore devised a generic second-order kinetic blocking.

For $\alpha = 0$, Eq. (D3) becomes the simple interaction potential $U_k(J,J') \propto \delta_{k|d}$. This makes the pairwise interaction independent of the particles' actions. This class of potential was already unveiled in Ref. [20], when investigating homogeneous 1D systems at order $1/N^2$. In particular, Ref. [20] showed that this interaction potential becomes

$$U(\mathbf{w},\mathbf{w}') \propto 1 - \frac{1}{d}\sum_{k=0}^{d-1}\delta_{\mathrm{D}}\big[(\theta-\theta') - k\tfrac{\pi}{d}\big]. \tag{D5}$$

As discussed in Ref. [20], such potentials amount to exactly local interactions: they do not drive any relaxation.

For $d = 1$ and $\alpha > 0$ being an even integer, one can explicitly compute $U(\mathbf{w},\mathbf{w}')$ from $U_k(J,J')$ as given by Eq. (D3). Up to prefactors, one obtains (see 9.622.3 in Ref. [29])

$$U(\mathbf{w},\mathbf{w}') \propto \big(J-J'\big)^{2n} B_{2n}\big[\tfrac{1}{2\pi}w_{2\pi}(\theta-\theta')\big], \tag{D6}$$

with $n \geqslant 1$ being an integer. In that expression, $B_{2n}(x)$ is a Bernoulli polynomial, e.g., $B_2(x) = x^2 - x + \frac{1}{6}$. We also introduced the "wrapping" function

$$w_{2\pi}(\theta) \equiv \theta\,[\mathrm{mod}\,2\pi]; \quad 0 \leqslant w_{2\pi}(\theta) < 2\pi. \tag{D7}$$

Investigating systems driven by interaction potentials as in Eq. (D6) will be the subject of future works.

APPENDIX E: NUMERICAL SIMULATIONS

We consider the same system as in Ref. [18], i.e., a set of N particles evolving on the unit sphere. This setup mimics "vector resonant relaxation" in galactic nuclei [30]. Given an axis $\widehat{\mathbf{z}}$, the spherical coordinates are denoted with (ϑ,ϕ), so that $\mathbf{w} = [\phi = \theta, \cos(\vartheta) = J]$ are canonical coordinates with the associated unit vector $\widehat{\mathbf{L}} = \widehat{\mathbf{L}}(\mathbf{w})$.

Following Ref. [16], we fix the external potential U_{ext} and the pairwise interaction potential U to be

$$U_{\mathrm{ext}}(\widehat{\mathbf{L}}) = D\,\big(\widehat{\mathbf{L}}\cdot\widehat{\mathbf{z}}\big)^2; \quad U(\widehat{\mathbf{L}},\widehat{\mathbf{L}}') = G\,\widehat{\mathbf{L}}\cdot\widehat{\mathbf{L}}'. \tag{E1}$$

Following Ref. [30], the individual equations of motion read $d\widehat{\mathbf{L}}_i/dt = \partial H/\partial\widehat{\mathbf{L}}_i \times \widehat{\mathbf{L}}_i$. For the particular choice from Eq. (E1), we readily find

$$\frac{\partial H}{\partial\widehat{\mathbf{L}}_i} = 2D\,\big(\widehat{\mathbf{L}}_i\cdot\widehat{\mathbf{z}}\big)\,\widehat{\mathbf{z}} + G\,\mathbf{S}, \tag{E2}$$

with $\mathbf{S} = \sum_{i=1}^N \mu\,\widehat{\mathbf{L}}_i$ being the system's magnetization. Since $\mathbf{S}$ is "shared" by all the particles, the rates of changes, $\{d\widehat{\mathbf{L}}_i/dt\}_i$, can be computed in $O(N)$ operations.

The invariants of the present system are

$$\forall i:\ |\widehat{\mathbf{L}}_i| = 1; \quad S_z = \mathbf{S}\cdot\widehat{\mathbf{z}}; \quad E_{\rm tot} = H. \tag{E3}$$

Because of the gauge invariance associated with the constraints, $|\widehat{\mathbf{L}}_i| = 1$, $d\widehat{\mathbf{L}}_i/dt$ is uniquely defined, while the gradient $\partial H/\partial\widehat{\mathbf{L}}_i$ is not. As such, we define, unambiguously, the precession vector $\boldsymbol{\Omega}_i = \widehat{\mathbf{L}}_i \times d\widehat{\mathbf{L}}_i/dt$, so that

$$\frac{d\widehat{\mathbf{L}}_i}{dt} = \boldsymbol{\Omega}_i \times \widehat{\mathbf{L}}_i; \quad \boldsymbol{\Omega}_i \cdot \widehat{\mathbf{L}}_i = 0. \tag{E4}$$

In order to exactly conserve all the $|\widehat{\mathbf{L}}_i|$, the system's dynamics is integrated using the structure-preserving classical fourth-order Munthe-Kaas scheme [31]. We refer to Sec. 5.1 in Ref. [32] for a presentation of this explicit scheme.

For the results presented in Fig. 1, we considered $N = 1\,024$, $G = -1$, $D = 15$, and $M_{\rm tot} = 1$. The integration was performed with a fixed time step $h = 5\times10^{-4}$, with a dump every $\Delta t = 100$ and integrated up to $t = 2\times10^{5}$. Running one realization requires about 32 h of computation on a single core. At the final times, the relative errors in S_z and $E_{\rm tot}$ were typically 10^{-7}.

For the initial conditions, as in Ref. [18], we consider

$$F(\mathbf{w}) = Ce^{-(J/\sigma)^4}, \tag{E5}$$

with C a normalization constant. We fixed the DF's initial dispersion to $\sigma = 0.45$. This ensures that the system is linearly stable and hot enough (see Fig. 6 in Ref. [18]).

To obtain Fig. 1, we performed a total of 1024 independent realizations. The action, $-1 \leqslant J \leqslant 1$, is binned in 50 equal size bins. For each realization, each action bin and each time dump, we determine the number of particles left to that action. This is then averaged over the available realizations. Finally, we fit the associated time series with an affine time dependence: its slope is proportional to the local diffusion flux $\mathcal{F}(J, t = 0)$. To estimate the measurement errors, we follow the same bootstrap approach as in Appendix F of Ref. [23]. In Fig. 1, we report the 16% and 84% level lines over 100 bootstrap resamplings.

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