


## Thermodynamics and criticality of $su(m)$ spin chains of Haldane-Shastry type

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We study the thermodynamics and critical behavior of  $su(m)$  spin chains of Haldane-Shastry type at zero chemical potential, both in the  $A_{N-1}$  and  $BC_N$  cases. We evaluate in closed form the free energy per spin for arbitrary values of  $m$ , from which we derive explicit formulas for the energy, entropy, and specific heat per spin. In particular, we find that the specific heat features a single Schottky peak, whose temperature is well approximated for  $m \lesssim 10$  by the corresponding temperature for an  $m$ -level system with uniformly spaced levels. We show that at low temperatures the free energy per spin of the models under study behaves as that of a one-dimensional conformal field theory with central charge  $c = m - 1$  (with the only exception of the Frahm-Inozemtsev chain at zero value of its parameter). However, from a detailed study of the ground-state degeneracy and the low-energy excitations, we conclude that these models are only critical in the antiferromagnetic case, with a few exceptions that we fully specify.

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### I. INTRODUCTION

In this paper we shall consider a broad class of spin chains with long-range interactions modeled on the Haldane-Shastry chain [1,2], whose interactions can be expressed in terms of the generators of the  $su(m)$  algebra in the fundamental representation. More precisely, if  $N$  denotes the number of sites and  $m$  is the number of internal degrees of freedom the canonical basis of the system's Hilbert space  $\otimes_{i=1}^N \mathbb{C}^m$  is spanned by the vectors

$$|s_1, \dots, s_N\rangle := |s_1\rangle \otimes \dots \otimes |s_N\rangle, \quad (1)$$

where  $s_i \in \{1, \dots, m\}$ . We define the permutation and spin flip operators  $S_{ij}$  and  $S_i$  ( $1 \leq i < j \leq N$ ) by the usual formulas

$$S_{ij}|\dots s_i \dots s_j \dots\rangle = |\dots s_j \dots s_i \dots\rangle, \quad (2)$$

$$S_i|\dots s_i \dots\rangle = |\dots m - s_i + 1 \dots\rangle. \quad (3)$$

The latter operators can be easily expressed in terms of the (Hermitian)  $su(m)$  generators  $T_k^\alpha$  in the fundamental representation, where  $1 \leq \alpha \leq m^2 - 1$  and the subindex labels the chain sites. Indeed, using the normalization  $\text{tr}(T_k^\alpha T_k^\beta) = 2\delta_{\alpha\beta}$  we have

$$S_{ij} = \frac{1}{m} + \frac{1}{2} \sum_{\alpha=1}^{m^2-1} T_i^\alpha T_j^\alpha, \quad S_i = T_i^\gamma + \frac{1}{2m} [1 - (-1)^m],$$

where the index  $\gamma$  is fixed but arbitrary. In particular, for  $m = 2$  we have  $T_k^\alpha = \sigma_k^\alpha$ , and hence [3]  $S_i = \sigma_i^1$ ,  $S_{ij} = \frac{1}{2}(1 + \sigma_i \cdot \sigma_j)$ , where  $\sigma^\alpha$  ( $\alpha = 1, 2, 3$ ) are the Pauli matrices. Note that

the operators  $S_{ij}$  obey the standard permutation algebra

$$S_{ij}O_j = O_iS_{ij}, \quad S_{ij}O_i = O_jS_{ij}, \quad S_{ij}O_k = O_kS_{ij},$$

where  $k \neq i, j$  and  $O_k$  is any operator acting on the  $k$ th site.

The first class of spin chains we shall be interested in are the Haldane-Shastry (HS), Polychronakos-Frahm (PF) [4–6] and Frahm–Inozemtsev (FI) [7] chains. They can be collectively defined through the formula [8]

$$H_\pm = \sum_{i < j} J_{ij} (1 \mp S_{ij}), \quad (4)$$

where

$$J_{ij} = \frac{J}{2N^2 \sin^2(\xi_i - \xi_j)}, \quad \xi_k = \frac{k\pi}{N} \quad (\text{HS chain}), \quad (5)$$

$$J_{ij} = \frac{J}{N(\xi_i - \xi_j)^2}, \quad H_N(\xi_k) = 0 \quad (\text{PF chain}), \quad (6)$$

$$J_{ij} = \frac{J}{2N^2 \sinh^2(\xi_i - \xi_j)}, \quad L_N^{c-1}(e^{2\xi_k}) = 0 \quad (\text{FI chain}). \quad (7)$$

Here  $J > 0$  is a real constant fixing the energy scale,  $H_N$  denotes the Hermite polynomial of degree  $N$  and  $L_N^{c-1}$  is a generalized Laguerre polynomial of degree  $N$  with positive parameter  $c$ . For reasons that will become clear in the sequel, we shall sometimes refer to the ferromagnetic models  $H_+$  as *bosonic* and to the antiferromagnetic ones  $H_-$  as *fermionic*. From the previous expressions we see that the HS chain can be naturally considered as a circular chain with equally spaced sites and spin-spin interactions inversely proportional to the square of the chord distance, while the PF and FI chains are better regarded as linear chains with sites  $\xi_k$  defined in Eqs. (6) and (7) and, respectively, rational (inverse square) or hyperbolic interactions.

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The spin chains discussed above are all naturally related to the  $A_{N-1}$  classical root system. The second type of chain we shall deal with is the variant of the HS chain related to the  $BC_N$  root system (HS-B chain), whose Hamiltonian shall be taken as [9,10]

$$H_{\pm} = \frac{J}{4N^2} \sum_{i < j} \left[ \frac{1 \mp S_{ij}}{\sin^2(\xi_i - \xi_j)} + \frac{1 \mp \tilde{S}_{ij}}{\sin^2(\xi_i + \xi_j)} \right] + \frac{J}{8N^2} \sum_i \left( \frac{\beta_1}{\sin^2 \xi_i} + \frac{\beta_2}{\cos^2 \xi_i} \right) (1 + \varepsilon S_i), \quad (8a)$$

with

$$P_N^{\beta_1-1, \beta_2-1}(\cos 2\xi_k) = 0. \quad (8b)$$

Here  $J > 0$ ,  $P_N^{\beta_1-1, \beta_2-1}$  is a Jacobi polynomial of degree  $N$  and parameters  $\beta_{1,2} > 0$ ,  $\varepsilon = \pm 1$ , and we have used the abbreviation  $\tilde{S}_{ij} := S_{ij}S_iS_j = S_iS_jS_{ij}$ . This model can be regarded as the *open* version of the HS chain, with chain sites  $z_j := e^{2i\xi_j}$  lying on the upper unit circle (although in general not uniformly spaced). The spin at  $z_j$  interacts both with the remaining spins at  $z_k$  (with  $k \neq j$ ) and their reflections  $\bar{z}_k$  with respect to the real axis, the interaction strength being inversely proportional to the square of the (chord) distances  $|z_j - z_k|$  and  $|z_j - \bar{z}_k|$ .

The HS spin chain was originally introduced as a parent Hamiltonian for the Gutzwiller variational ground state for the one-dimensional Hubbard model in the limit of infinite on-site energy. In fact, the (exact) ground state of the HS chain contains a Jastrow factor reminiscent of the Laughlin wave function in the fractional quantum Hall effect [11–13]. The remarkable properties of the HS chain became apparent shortly after its introduction. For instance, it was soon found that its spinon excitations behave as effective particles obeying Haldane's fractional statistics [12,14,15]. This can indeed be regarded as the simplest realization of anyons in one dimension. The HS chain is completely integrable [16–18], and is actually invariant under the Yangian quantum algebra  $Y[\mathfrak{sl}(m)]$  even for a finite number of sites, which in part explains the high degeneracies of its spectrum [19]. It is closely related to the Wess-Zumino-Novikov-Witten (WZNW) model at level 1 [14], and can also be embedded into a larger class of models constructed from chiral vertex operators of an appropriate conformal field theory [20,21].

A characteristic property of the HS chain that distinguishes it from short-range chains like the Heisenberg model is the fact that it can be obtained as the strong interaction (large coupling) limit of an integrable one-dimensional system, namely the (spin) Sutherland model [22–24]. In fact, the PF and FI chains can be analogously derived from the integrable spin Calogero [25,26] and Frahm-Inozemtsev [27] dynamical models. Similarly, the HS-B chain is the large coupling limit of the spin Sutherland model of  $BC_N$  type [9,10,28]. As first pointed out by Polychronakos [6], the connection between the spin chains (4)–(8) and the dynamical spin models of Calogero-Sutherland type mentioned above can be exploited to derive the chains' partition functions in closed form [10,29–31]. From the common structure of these partition functions and their relation to the representation theory of the Yangian algebra in terms of border strips and their associated

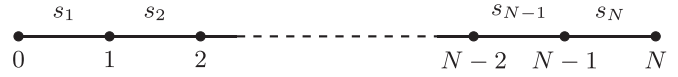


FIG. 1. Linear vertex model associated to  $su(m)$  spin chain of HS type.

motifs [32], a remarkable equivalence between the  $A_{N-1}$  chains (4)–(7) and certain (inhomogeneous) vertex models was established in Ref. [33]. Indeed, the spectrum of the latter chains (with the correct degeneracy for each energy) is the same as that of a suitable linear vertex model with  $N + 1$  vertices that we shall next describe. Each vertex is labeled by an integer in the range  $0, 1, \dots, N$ , and is connected to its nearest neighbors by a bond which can take the values  $1, \dots, m$ . Thus, a configuration of the model is determined by a *bond vector*

$$\mathbf{s} := (s_1, \dots, s_N) \in \{1, 2, \dots, m\}^N,$$

where  $s_i$  denotes the value of the bond joining the  $(i - 1)$ th and the  $i$ th vertices (see Fig. 1). The energy of the configuration represented by the bond vector  $\mathbf{s}$  can be computed through the formula

$$E(\mathbf{s}) = J \sum_{i=1}^{N-1} \delta_{\pm}(s_i, s_{i+1}) \mathcal{E}(x_i), \quad x_i := i/N. \quad (9)$$

Here the *dispersion function*  $\mathcal{E}(x)$ , which depends on the chain considered, is given by

$$\mathcal{E}(x) = \begin{cases} x(1-x), & \text{(HS chain),} \\ x, & \text{(PF chain),} \\ x(x + \gamma_N), & \text{(FI chain),} \end{cases} \quad (10)$$

with  $\gamma_N := (c - 1)/N$ , while the function  $\delta_{\pm}$  (where the  $\pm$  sign corresponds to the double sign in  $H_{\pm}$ ) is defined by

$$\delta_{+}(i, j) = \begin{cases} 0, & i \leq j, \\ 1, & i > j, \end{cases} \quad \delta_{-}(i, j) = \begin{cases} 0, & i < j, \\ 1, & i \geq j. \end{cases} \quad (11)$$

We thus see that the spin degrees of freedom behave as bosons (respectively, fermions) in the ferromagnetic (respectively, antiferromagnetic) case. Note also that the vectors with components  $\delta_{\pm}(s_i, s_{i+1})$  are closely connected to the *motifs* introduced by Haldane [14]. An analogous description of the spectrum of the HS-B chain was recently found in Ref. [34]. More precisely, in this case the vertex model has an additional vertex and a last bond  $s_{N+1}$  assuming the *fixed* (half-integer) value  $m_{\varepsilon} + \frac{1}{2}$ , where

$$m_{\varepsilon} := \frac{1}{2} [m + \varepsilon \pi(m)],$$

and  $\pi(m) = \frac{1}{2} [1 - (-1)^m]$  is the parity of  $m$ . Thus, in this case

$$E(\mathbf{s}) = J \sum_{i=1}^N \delta_{\pm}(s_i, s_{i+1}) \mathcal{E}(x_i), \quad (12)$$

with  $\delta_{\pm}$  as above and dispersion function

$$\mathcal{E}(x) = x \left( \gamma_N + 1 - \frac{x}{2} \right), \quad \gamma_N := \frac{1}{2N} (\beta_1 + \beta_2 - 1). \quad (13)$$

The thermodynamics of spin chains of HS type has been studied ever since the early work of Haldane, who derived an

expression for the entropy of the  $su(2)$  HS chain by means of the spinon description of its spectrum [35]. Shortly afterwards, Sutherland and Shastry [36] addressed the general  $su(m)$  case, outlining a complicated procedure for computing the free energy which involves two successive integrations. This procedure, however, only yields an explicit expression for  $m = 2$ . Around the same time, a heuristic formula for the free energy per spin of the PF and FI chains (with no magnetic field or chemical potential term) was presented in Refs. [5,7], again only for the  $su(2)$  case. In fact, a systematic study of the thermodynamics of the  $A_{N-1}$  chains (4)–(7) (with an additional chemical potential or magnetic field term) using the transfer matrix method was undertaken for the first time in Ref. [37], and extended later to the supersymmetric case in Ref. [38]. The key idea in this respect is the fact that using Eq. (9) (or, actually, its generalization to allow for a chemical potential term) it is straightforward to express the partition function as the trace of a product of  $N$  site-dependent  $m \times m$  transfer matrices. In our case, it follows from Eq. (9) that the partition function of the three  $A_{N-1}$  chains can be collectively written as

$$Z = \text{tr}[A(x_0)A(x_1) \cdots A(x_{N-1})],$$

where the  $m \times m$  transfer matrix  $A(x)$  has entries

$$A_{\mu\nu} = e^{-\beta J \mathcal{E}(x) \delta_{\pm}(\mu, \nu)}, \quad 1 \leq \mu, \nu \leq m. \quad (14)$$

Here  $\beta := 1/T$  is the inverse temperature (taking Boltzmann's constant  $k_B$  as unity), and the dispersion relation  $\mathcal{E}(x)$  is given by Eq. (10). Since the matrix  $A(x)$  has positive entries for all  $x \in [0, 1]$ , the classical Perron theorem [39,40] implies that  $A(x)$  has a positive simple eigenvalue  $\lambda_1(x)$  which exceeds the modulus of any other eigenvalue. From this fact it readily follows that in the thermodynamic limit  $N \rightarrow \infty$  the free energy per spin  $f(T)$  of the chains (4)–(7) can be expressed as

$$f(T) = -T \int_0^1 \ln \lambda_1(x) dx. \quad (15)$$

As shown in Ref. [41], similar expressions are valid for the HS-B chain (8). Indeed, from Eq. (12) we obtain

$$Z = \text{tr}[A(x_1) \cdots A(x_{N-1})B],$$

where  $A_{\mu\nu}(x)$  is defined as above but using Eq. (13) for the dispersion relation, and the  $m \times m$  matrix  $B$  has entries

$$B_{\mu\nu} = e^{-\beta J(\gamma_N + 1/2) \delta_{\pm}(\mu, m_\varepsilon + \frac{1}{2})}, \quad 1 \leq \mu, \nu \leq m.$$

Since all the matrices in the expression for the partition function have again positive entries, it follows from Perron's theorem that the thermodynamic free energy per spin is given by Eq. (15) also in this case (see Ref. [41] for details). Note that when computing the thermodynamic free energy from Eq. (15) we must replace the parameter  $\gamma_N$  in Eqs. (10) and (13) by

$$\gamma := \lim_{N \rightarrow \infty} \gamma_N \geq 0.$$

In fact, Eq. (15) has been shown to hold for the  $su(m|n)$  supersymmetric version of the chains (4)–(8) studied in this paper, even with the addition of a general chemical potential term [38,41]. Thus, the thermodynamic functions of all of

these models can be computed in closed form provided that the Perron eigenvalue  $\lambda_1(x)$  of the transfer matrix  $A(x)$  in Eq. (14) can be explicitly found. So far, this has only been done in the  $su(2)$  case (bosonic or fermionic) [37,41] and in the supersymmetric case [42] with  $1 \leq m, n \leq 2$  [38,41].

The main aim of this paper is to derive a remarkably simple expression for  $\lambda_1(x)$  for all the HS-type  $su(m)$  chains (4)–(8), valid for arbitrary values of  $m$ . Thus, the thermodynamic functions of these models can be evaluated in closed form. We stress that such closed-form expressions had been obtained so far only for the  $su(2)$  case, even at zero chemical potential.

Another problem we shall address in this work is the study of the critical behavior of the chains (4)–(8). As is well known, a strong indication of the critical character of a model is the low-temperature behavior of its free energy. The reason for this is that at low temperatures the free energy per unit length of a  $(1+1)$ -dimensional conformal field theory (CFT) with central charge  $c$  behaves as [43,44]

$$f(T) \simeq f(0) - \frac{\pi c T^2}{6v}, \quad (16)$$

where  $v$  is the Fermi velocity and we are using natural units  $\hbar = 1$ . It is thus expected that the free energy of a critical system obey the latter asymptotic formula at sufficiently low temperatures, with  $c$  equal to the central charge of the effective CFT governing the model's low-energy behavior. Using the explicit expression (15) for the free energy per spin of the chains (4)–(8), we shall show that Eq. (16) is satisfied for these models with central charge  $c = m - 1$  (with the only exception of the FI chain with  $\gamma = 0$ ). This result agrees with the calculation in Ref. [45] for the supersymmetric PF chain (using a different method), and is consistent with the fact that the low-energy excitations of the (original) HS chain (5) are governed by the  $su(m)_1$  WZNW model [14,46,47]. It should be emphasized, however, that (16) is only a necessary condition for criticality. Indeed, a CFT—and thus a truly critical model—must have low-energy excitations with a linear energy-momentum relation and the degeneracy of its ground state should remain finite in the thermodynamic limit. Using Eqs. (9)–(10) and (12)–(13), we shall prove that both of these conditions hold in our case. In this way we shall show that the chains (4)–(7) (with  $\gamma > 0$  for the FI chain) are critical only in the fermionic (antiferromagnetic) case, which is again in agreement with the results for the HS chain in Ref. [48]. On the other hand, we shall see that the HS-B chain is critical not only in the fermionic case, but also in the bosonic (ferromagnetic) one when  $m = 2$ , or  $m = 3$  and  $\varepsilon = -1$ .

We shall close this section with a few words on the paper's organization. In Sec. II we recall the formula for the free energy per spin of  $su(m)$  chains of HS-type in terms of the Perron eigenvalue of a suitable transfer matrix derived in Refs. [37,41]. We then evaluate this eigenvalue, thus obtaining a simple closed-form expression for the free energy of the latter models valid for arbitrary values of  $m$ . This expression is used in Sec. III to derive explicit formulas for the energy, entropy, and specific heat per spin. We also study the approximation of these functions by those of an  $m$ -level system with uniformly spaced levels, and establish the existence of a single Schottky peak in the specific heat for all values of  $m$ . In Sec. IV we analyze the critical behavior of the models

under study by first determining the low-temperature behavior of their free energy, and then examining in detail the ground-state degeneracy and the existence of low-energy excitations with linear energy-momentum relation. The paper ends with a technical Appendix in which we deduce the full asymptotic expansion of the free energy per spin at low temperatures used in Sec. IV.

## II. FREE ENERGY

In this section we shall evaluate the Perron (“dominant”) eigenvalue  $\lambda_1(x)$  of the transfer matrix  $A(x)$  given by Eq. (14), which as we have seen determines the free energy per spin of the HS-type chains (4)–(8) through the integral (15). To this end, we first recall that according to Perron’s theorem the dominant eigenvalue of a positive matrix [49] possesses a positive eigenvector. In fact, a corollary of Perron’s theorem posits that the dominant eigenvalue of a positive matrix is the only eigenvalue possessing a positive eigenvector. Since this property shall be essential in what follows, we shall briefly summarize its proof. Indeed, suppose that  $\mathbf{v}$  is a positive eigenvector of a positive matrix  $P$  with eigenvalue  $\lambda$ , and denote by  $\lambda_1$  the dominant eigenvalue of  $P$ . Since the transpose matrix  $P^T$  is also positive and has the same spectrum as  $P$ , its dominant eigenvalue is also  $\lambda_1$ , and therefore there exists a positive vector  $\mathbf{w}$  such that  $P^T \mathbf{w} = \lambda_1 \mathbf{w}$ . We then have

$$\lambda_1(\mathbf{w}, \mathbf{v}) = (P^T \mathbf{w}, \mathbf{v}) = (\mathbf{w}, P\mathbf{v}) = \lambda(\mathbf{w}, \mathbf{v}),$$

where we have used the fact that  $P^T$  and  $\lambda_1$  are real. Since both  $\mathbf{v}$  and  $\mathbf{w}$  are positive vectors, from the latter equality it follows that  $\lambda = \lambda_1$ , as claimed.

In fact, if an eigenvalue  $\lambda$  of a positive matrix  $P$  is known, Perron’s theorem provides a simple test to ascertain whether  $\lambda$  is the dominant eigenvalue of  $P$ . Indeed, if we denote by  $C_{ij}^\lambda$  the  $(i, j)$  cofactor of the matrix  $\lambda - P$ , we then have the elementary identity

$$\sum_{j=1}^m (\lambda \delta_{ij} - P_{ij}) C_{kj}^\lambda = \delta_{ik} \det(\lambda - P) = 0,$$

where  $m$  is the order of  $P$ . From the previous identity it follows that *any* row of the cofactor matrix  $(C_{ij}^\lambda)_{i,j=1}^m$  is an eigenvector of  $P$  with eigenvalue  $\lambda$ . However, as part of the proof of Perron’s theorem, it is shown that *all* the cofactors  $C_{ij}^\lambda$  corresponding to the dominant eigenvalue  $\lambda_1$  are *positive* [40]. From the discussion in the previous paragraph we then obtain the following elementary test: an eigenvalue  $\lambda$  of a positive matrix  $P$  is its dominant eigenvalue if and only if *any* row of the cofactor matrix of  $\lambda - P$  is a positive vector.

Let us now turn to the computation of the dominant eigenvalue of the positive matrix  $A(x)$  in Eq. (14). To begin with, from the definition (11) of  $\delta_+$  it follows that in the *bosonic* case the matrix  $A(x)$  has the following structure

$$A(x) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a^m & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^m & a^m & \cdots & 1 & 1 \\ a^m & a^m & \cdots & a^m & 1 \end{pmatrix},$$

where we have set [50]

$$a := e^{-\beta J \mathcal{E}(x)/m}.$$

An elementary calculation shows that

$$\lambda(x) = \sum_{k=0}^{m-1} a^k = \frac{1 - a^m}{1 - a}$$

is an eigenvalue of  $A(x)$ . We could now apply the previous test to check whether  $\lambda(x)$  is the dominant eigenvalue of  $A(x)$ , but in this case it is easier to observe that the vector

$$\mathbf{v} = (1, a, \dots, a^{m-1})$$

is a positive eigenvector of  $A(x)$  with eigenvalue  $\lambda(x)$ . Indeed,

$$\begin{aligned} [A(x)\mathbf{v}]_k &= \sum_{i=1}^{k-1} a^m a^{i-1} + \sum_{i=k}^m a^{i-1} = \sum_{i=k-1}^{m+k-2} a^i \\ &= a^{k-1} \lambda(x) = \lambda(x) v_k. \end{aligned}$$

From the previous discussion it follows that  $\lambda(x) = \lambda_1(x)$ . We thus obtain the following remarkable formula for the free energy per spin  $f_+(T)$  of the *bosonic* HS-type chains (4)–(8):

$$\begin{aligned} f_+(T) &= -T \int_0^1 \ln \left( \sum_{k=0}^{m-1} e^{-\beta J \mathcal{E}(x)k/m} \right) dx \\ &= -T \int_0^1 \ln \left( \frac{1 - e^{-\beta J \mathcal{E}(x)}}{1 - e^{-\beta J \mathcal{E}(x)/m}} \right) dx \\ &= f_0 - T \int_0^1 \ln \left[ \frac{\sinh(\beta J \mathcal{E}(x)/2)}{\sinh(\beta J \mathcal{E}(x)/(2m))} \right] dx, \end{aligned} \quad (17)$$

where

$$f_0 := \frac{J}{2} \left( 1 - \frac{1}{m} \right) \mathcal{E}_0, \quad \mathcal{E}_0 := \int_0^1 \mathcal{E}(x) dx.$$

The constant  $\mathcal{E}_0$ , which can be easily found from Eqs. (10) and (13), takes the values 1/6 (HS chain), 1/2 (PF chain), and  $J/2 + 1/3$  (FI and HS-B chains).

An analogous result for the fermionic case can be easily derived noting that

$$A_-(x) = A_+(x) + (a^m - 1)\mathbb{1},$$

where for clarity’s sake we have denoted by  $A_-(x)$  [respectively,  $A_+(x)$ ] the matrix  $A(x)$  in the fermionic (respectively, bosonic) case. Hence, the dominant eigenvalue of the matrix  $A_-(x)$  is given by

$$\lambda_1(x) = \sum_{k=0}^{m-1} a^k + a^m - 1 = a \sum_{k=0}^{m-1} a^k.$$

Substituting into Eq. (15) we obtain the following simple formula for the free energy per spin  $f_-(T)$  in the *fermionic* case:

$$f_-(T) = f_+(T) + \frac{J \mathcal{E}_0}{m}. \quad (18)$$

For  $m = 2$ , the previous formulas for  $f_\pm$  coincide with those in Refs. [37,41]. As an additional consistency check, note that as  $T \rightarrow \infty$  from Eqs. (17) and (18) we easily

obtain

$$f(T) \simeq -T \ln m,$$

in agreement with the elementary identity

$$Z(T) \underset{T \rightarrow \infty}{\simeq} m^N.$$

*Remark 1.* The thermodynamic free energy of the PF chain can be computed in closed form in terms of the dilogarithm function [51]

$$\text{Li}_2(z) := - \int_0^z \frac{\ln(1-t)}{t} dt,$$

analytic in the cut complex plane  $\mathbb{C} \setminus [1, \infty)$ , where the integral is taken along any path in the latter set joining the origin

$$u_{\pm} = \frac{\partial}{\partial \beta} (\beta f_{\pm}) = \frac{1}{2} \left( 1 \mp \frac{1}{m} \right) \mathcal{E}_0 - \frac{1}{2} \int_0^1 \mathcal{E}(x) \left[ \coth \left( \frac{\beta \mathcal{E}(x)}{2} \right) - \frac{1}{m} \coth \left( \frac{\beta \mathcal{E}(x)}{2m} \right) \right] dx, \quad (19)$$

$$s = \beta(u_{\pm} - f_{\pm}) = \int_0^1 \left\{ \ln \left[ \frac{\sinh \left( \frac{\beta \mathcal{E}(x)}{2} \right)}{\sinh \left( \frac{\beta \mathcal{E}(x)}{2m} \right)} \right] - \frac{\beta \mathcal{E}(x)}{2} \left[ \coth \left( \frac{\beta \mathcal{E}(x)}{2} \right) - \frac{1}{m} \coth \left( \frac{\beta \mathcal{E}(x)}{2m} \right) \right] \right\} dx, \quad (20)$$

$$c_V = -\beta^2 \frac{\partial u_{\pm}}{\partial \beta} = \frac{\beta^2}{4} \int_0^1 \mathcal{E}(x)^2 \left[ \frac{1}{m^2} \text{csch}^2 \left( \frac{\beta \mathcal{E}(x)}{2m} \right) - \text{csch}^2 \left( \frac{\beta \mathcal{E}(x)}{2} \right) \right] dx. \quad (21)$$

Using the explicit formula for the free energy per spin of the PF chain from the previous section we readily find the following simple expressions for its thermodynamic functions:

$$\begin{aligned} u_+ &= u_- - \frac{1}{2m} = -f_+ - T \ln \left( \frac{1 - e^{-\beta}}{1 - e^{-\beta/m}} \right), \\ s &= -2\beta f_+ - \ln \left( \frac{1 - e^{-\beta}}{1 - e^{-\beta/m}} \right), \\ c_V &= -2\beta f_+ - 2 \ln \left( \frac{1 - e^{-\beta}}{1 - e^{-\beta/m}} \right) + \beta \left( \frac{1}{e^{\beta} - 1} - \frac{1/m}{e^{\beta/m} - 1} \right). \end{aligned}$$

From the first equality in Eq. (17) it follows that for large  $T$  (i.e., for  $T \gg \mathcal{E}(1)$  for the PF, FI and HS-B chains or  $T \gg \mathcal{E}(1/2) = 1/4$  for the HS chain) the ferromagnetic free energy per spin  $f_+(T)$  can be well approximated replacing  $\mathcal{E}(x)$  by its mean value  $\mathcal{E}_0$  over the interval  $[0,1]$ . In other words, we have

$$f_+(T) \underset{T \rightarrow \infty}{\simeq} -T \ln \left( \sum_{k=0}^{m-1} e^{-k\beta \frac{\mathcal{E}_0}{m}} \right) =: f_m(T).$$

The right-hand side is the partition function of an  $m$ -level system with uniformly spaced levels  $E_k = k\mathcal{E}_0/m$ ,  $k = 0, \dots, m-1$ . In fact, at sufficiently high temperatures the thermodynamic functions of the HS-type chains (4)–(8) behave qualitatively as those of the corresponding  $m$  level system, as can be seen, for instance, from Fig. 2 for the HS chain with  $m = 2, \dots, 5$  (we omit the corresponding plots for the PF, FI, and HS-B chains, which are very similar). It should be noted, however, that at low temperatures the thermodynamic functions of the HS-type chains behave quite differently than those of their corresponding  $m$ -level system. Indeed, at low temperatures the free energy of the  $m$ -level

to the point  $z$ . Indeed, we have

$$f_+(T) = \frac{T^2}{J} \left[ m \text{Li}_2(e^{-\beta J/m}) - \text{Li}_2(e^{-\beta J}) - \frac{\pi^2}{6} (m-1) \right].$$

### III. THERMODYNAMICS

The energy density  $u$ , entropy per spin  $s$  and specific heat per spin  $c_V$  of the HS-type chains (4)–(8) can be readily computed using Eqs. (17) and (18). Before doing so, to simplify our formulas we shall set without loss of generality  $J = 1$ , so that energy and temperature become dimensionless. With this proviso, we find

system,

$$\begin{aligned} f_m(T) &= -T \ln \left( \frac{1 - e^{-\beta \mathcal{E}_0}}{1 - e^{-\beta \mathcal{E}_0/m}} \right) \\ &= -T e^{-\beta \mathcal{E}_0/m} + O(T e^{-2\beta \mathcal{E}_0/m}), \end{aligned}$$

is exponentially small, and so are its remaining thermodynamic functions. On the other hand, from the discussion in the next section [cf. Eq. (26)] it follows that for the HS-type chains  $f_{\pm}(T) - f_{\pm}(0) \sim -T^2$  as  $T \rightarrow 0+$  [or  $\sim -T^{3/2}$  for the FI chain with  $\gamma = 0$ ; cf. Eq. (A2)]. Thus,  $u - u(0) \sim T^2$ ,  $s \sim T$ , and  $c_V \sim T$  as  $T \rightarrow 0+$  [or  $u - u(0) \sim T^{3/2}$ ,  $s \sim T^{1/2}$ , and  $c_V \sim T^{1/2}$  for the FI chain with  $\gamma = 0$ ].

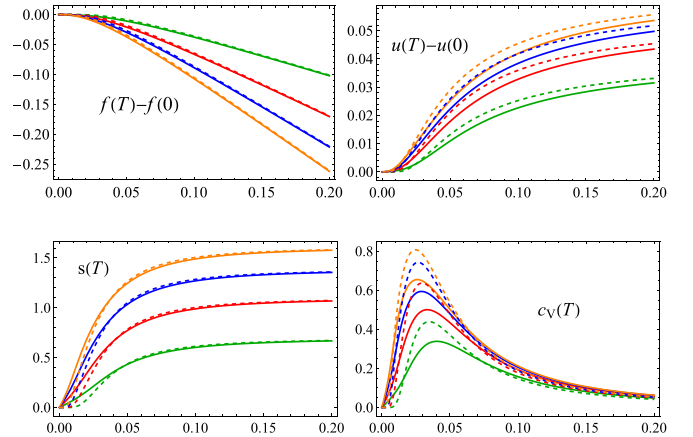


FIG. 2. Thermodynamic free energy, energy, entropy, and specific heat per spin of the  $su(m)$  HS chain (4)–(5) with  $m = 2, 3, 4, 5$  (green, red, blue, and orange solid lines, respectively) vs temperature  $T$ , compared to their counterparts for an  $m$ -level system (dashed lines).

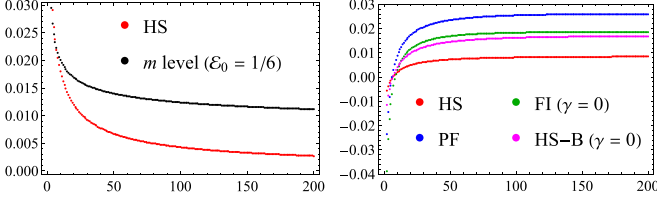


FIG. 3. Left: Temperature  $T_m$  of the Schottky peak of the  $su(m)$  HS chain with  $m = 2, \dots, 200$  compared to its rough  $m$ -level approximation  $\mathcal{E}_0/(2t_m) = 1/(12t_m)$ . Right: Difference  $\mathcal{E}_0/(2t_m) - T_m$  for the HS, PF, FI HS-B chains (with  $\gamma = 0$  for the last two chains) for  $m = 2, \dots, 200$ .

From Fig. 2 (and similar plots for the PF, FI, and HS-B chains that we are not displaying) it is apparent that, at least for low values of  $m$ , the specific heat per spin of the HS-type chains features a single absolute maximum (i.e., a Schottky peak). This behavior can be qualitatively understood by analyzing the specific heat of the corresponding  $m$ -level system, given by

$$c_{V,m} = t^2 \left[ \frac{1}{m^2} \operatorname{csch}^2(t/m) - \operatorname{csch}^2 t \right], \quad t := \frac{\beta \mathcal{E}_0}{2}.$$

Indeed, although we have not found an analytic proof, we have compelling numerical evidence that  $c_{V,m}(t)$  has a single critical point  $t_m$  in the half-line  $(0, \infty)$  which is an absolute maximum. It is therefore to be expected that the specific heat of the  $su(m)$  HS-type chains feature a single Schottky peak at a temperature  $T_m$  of the order of  $\mathcal{E}_0/(2t_m)$ . In fact, for low values of  $m \lesssim 10$  the Schottky temperature  $T_m$  is reasonably well approximated by the  $m$ -level value  $\mathcal{E}_0/(2t_m)$ , as can be seen from Fig. 3 (left) for the HS chain (and similarly for the other HS-type chains). On the other hand, for large  $m$  the difference  $\mathcal{E}_0/(2t_m) - T_m$  appears to tend to a constant positive value, while  $T_m \rightarrow 0$  (see Fig. 3, right).

#### IV. CRITICAL BEHAVIOR

As mentioned in the Introduction, one of the hallmarks of criticality is the low-temperature behavior of the free energy, given by Eq. (16). In our case, using Eqs. (17) and (18) with  $J = 1$  we obtain

$$f_{\pm}(T) - f_{\pm}(0) = -T \int_0^1 \ln(1 - e^{-\beta \mathcal{E}(x)}) dx + T \int_0^1 \ln(1 - e^{-\beta \mathcal{E}(x)/m}) dx. \quad (22)$$

As  $T \rightarrow 0+$ , the main contribution to both of these integrals comes from a small neighborhood of the points where the dispersion relation  $\mathcal{E}(x)$  vanishes, i.e.,  $x = 0, 1$  for the HS chain and  $x = 0$  for the PF, FI, and HS-B chains. For the latter three chains  $\mathcal{E}(x)$  is monotonically increasing over the interval  $[0, 1]$ , so that performing the changes of variables  $y = \beta \mathcal{E}(x)$

and  $y = \beta \mathcal{E}(x)/m$  in the integrals in Eq. (22) we have

$$f_{\pm}(T) - f_{\pm}(0) = -T^2 \int_0^{\beta \mathcal{E}(1)} \ln(1 - e^{-y})(\mathcal{E}^{-1})'(Ty) dy + mT^2 \int_0^{\frac{\beta \mathcal{E}(1)}{m}} \ln(1 - e^{-y})(\mathcal{E}^{-1})'(mTy) dy, \quad (23)$$

where  $\mathcal{E}^{-1} : 0 \rightarrow \mathcal{E}(1)$  denotes the inverse function of  $\mathcal{E} : 0 \rightarrow 1$ . Since the main contribution to both integrals comes from the point  $y = 0$ , we can approximate  $(\mathcal{E}^{-1})'(Ty)$  and  $(\mathcal{E}^{-1})'(mTy)$  by  $(\mathcal{E}^{-1})'(0) = 1/\mathcal{E}'(0)$  (assuming that  $\gamma > 0$  for the FI chain) and push the upper limit in each integral to  $+\infty$ , thus obtaining

$$\begin{aligned} f_{\pm}(T) - f_{\pm}(0) &= \frac{(m-1)T^2}{\mathcal{E}'(0)} \int_0^{\infty} \ln(1 - e^{-y}) dy + O(T^3) \\ &= -\frac{(m-1)T^2}{\mathcal{E}'(0)} \operatorname{Li}_2(1) + O(T^3) \\ &= -\frac{(m-1)\pi^2}{6\mathcal{E}'(0)} T^2 + O(T^3); \end{aligned} \quad (24)$$

see the Appendix for more details [52]. It follows from Eq. (24) that the free energy per spin of the PF, FI (with  $\gamma > 0$ ) and HS-B chains behaves as that of a CFT with central charge  $c = m - 1$ . To see this, note first of all that the variable  $x$  can be regarded as  $p/\pi$ , where  $p$  is the momentum (defined modulo  $2\pi$ ). Indeed, since the dispersion relation is monotonic the interval  $0 \leq x \leq 1$  corresponds to the positive momentum range  $0 \leq p \leq \pi$  and not to the full range  $-\pi \leq p \leq \pi$ . As the relation between energy  $\mathcal{E}(x)$  and momentum  $p = \pi x$  is linear near  $p = 0$ , the Fermi velocity is given by

$$v = \left. \frac{d\mathcal{E}}{dp} \right|_{p=0} = \frac{\mathcal{E}'(0)}{\pi}, \quad (25)$$

so that Eq. (24) can indeed be written as

$$f_{\pm}(T) - f_{\pm}(0) = -\frac{(m-1)\pi}{6v} T^2 + O(T^3). \quad (26)$$

For the HS chain (4)–(5), the dispersion relation  $\mathcal{E}(x)$  is symmetric about  $x = 1/2$  and increasing in the interval  $[0, 1/2]$ . Hence, we can write

$$\begin{aligned} f_{\pm}(T) - f_{\pm}(0) &= -2T \int_0^{1/2} \ln(1 - e^{-\beta \mathcal{E}(x)}) dx \\ &\quad + 2T \int_0^{1/2} \ln(1 - e^{-\beta \mathcal{E}(x)/m}) dx, \end{aligned} \quad (27)$$

and proceeding as above we arrive at

$$f_{\pm}(T) - f_{\pm}(0) = -\frac{(m-1)\pi^2}{3\mathcal{E}'(0)} T^2 + O(T^3) \quad (28)$$

(see again the Appendix for details on the error term). However, in this case the symmetry of the dispersion relation about  $x = 1/2$  implies that the relation between  $x$  and  $p$  is  $p = 2\pi x$  (the interval  $0 \leq x \leq 1$  now corresponds to the full momentum range  $0 \leq p \leq 2\pi$ ). Hence,

$$v = \left. \frac{d\mathcal{E}}{dp} \right|_{p=0} = \frac{\mathcal{E}'(0)}{2\pi}, \quad (29)$$

and Eq. (26) also holds in this case. Thus, Eq. (26) is valid for all the HS-type chains, except for the FI chain with  $\gamma = 0$ . (In fact, as shown in the Appendix, for the latter chain  $f_{\pm}(T) - f_{\pm}(0) \sim -T^{3/2}$  as  $T \rightarrow 0+$ .)

As mentioned in the Introduction, to ascertain the criticality of a quantum system we must also examine the degeneracy of its ground state and study its low-energy excitations. Both of these problems can be addressed with the help of Eqs. (9), (10) and (12), (13) for the energy spectrum. We shall start by constructing the bond vectors for the ground state of each of the chains (4)–(8). In this way we shall easily determine the ground-state energy and its degeneracy.

Consider, first, the PF, FI, and HS-B chains, whose dispersion relation  $\mathcal{E}(x)$  is monotonically increasing over the interval  $[0,1]$ . In the fermionic case  $\delta(s_i, s_{i+1}) = 0$  if and only if  $s_i < s_{i+1}$ , and hence the bond vectors  $\mathbf{s}_g$  yielding the ground state of the PF and FI chains are obtained by placing [53]  $r := \lfloor N/m \rfloor$  sequences  $(1, \dots, m)$  starting from the right end and filling the remaining  $N - rm$  components with an increasing sequence  $(s_1, \dots, s_{N-rm}) \in \{1, \dots, m\}^{N-rm}$ :

$$\mathbf{s}_g = (s_1, \dots, s_{N-rm}, 1, \dots, m, \dots, 1, \dots, m), \quad (30)$$

$$1 < s_1 < \dots < s_{N-rm}.$$

The ground-state degeneracy is therefore  $\binom{m}{N-rm} \ll N$  as  $N \rightarrow \infty$ . In the case of the HS-B chain, since  $\delta(s_N, s_{N+1}) = \delta(s_N, m_\varepsilon + \frac{1}{2})$ , to obtain the ground state we must take  $s_N = m_\varepsilon$ . Hence, the ground-state bond vectors are in this case

$$\mathbf{s}_g = (s_1, \dots, s_{N-rm}, 1, \dots, m, \dots, 1, \dots, m, 1, \dots, m_\varepsilon),$$

$$1 < s_1 < \dots < s_{N-rm},$$

where now  $r = \lfloor (N - m_\varepsilon)/m \rfloor$ . The ground-state degeneracy is again  $\binom{m}{N-rm} \ll N$ . Thus, the degeneracy of the fermionic PF, FI, and HS-B chains remains finite in the thermodynamic limit. The situation is completely different in the bosonic case, since now  $\delta(s, s) = 0$ . Thus, the ground-state bond vectors of the PF and FI chains are of the form

$$\mathbf{s}_g = (\underbrace{s_1, \dots, s_1}_{k_1}, \dots, \underbrace{s_r, \dots, s_r}_{k_r}), \quad (31)$$

with  $k_1 + \dots + k_r = N$  and  $1 \leq s_1 < \dots < s_r \leq m$ . Thus, in this case the ground-state degeneracy is given by

$$d_g = \sum_{r=1}^m \mathcal{P}(N; r) \binom{m}{r},$$

where  $\mathcal{P}(N; r)$  denotes the number of partitions of the integer  $N$  in  $r$  parts. In particular, the ground-state degeneracy clearly tends to infinity in the thermodynamic limit. On the other hand, for the bosonic HS-B chain we must impose the additional restriction  $s_r \leq m_\varepsilon$ , so that in this case we have

$$d_g = \sum_{r=1}^{m_\varepsilon} \mathcal{P}(N; r) \binom{m_\varepsilon}{r}.$$

It follows that in this case the ground state is nondegenerate if and only if  $m_\varepsilon = 1$ , i.e., for  $m = 2$  or  $m = 3$  and  $\varepsilon = -1$ . The ground-state degeneracy is otherwise infinite (at least  $N + 1$ ) in the thermodynamic limit.

The above analysis must be slightly modified in the case of the HS chain, whose dispersion relation is increasing over

the interval  $[0, 1/2]$  and symmetric about  $1/2$ . For this reason, to the bond vectors (30) we should add their “reflected” counterparts

$$(1, \dots, m, \dots, 1, \dots, m, s_1, \dots, s_{N-rm}).$$

Thus, in this case the ground-state degeneracy is again finite in the thermodynamic limit  $[2\binom{m}{N-rm}]$  when  $N$  is not a multiple of  $m$ , or 1 when it is]. Finally, in the bosonic case the ground-state bond vector is still of the form (31), and hence the ground-state degeneracy is infinite in the thermodynamic limit. In summary, from the analysis of the ground-state degeneracy we conclude that only the fermionic chains (4)–(8), and the bosonic HS-B chain with  $m = 2$  or  $m = 3$ ,  $\varepsilon = -1$ , can be truly critical. It is interesting to observe in this respect that the bosonic  $su(3)$  HS-B chain with  $\varepsilon = 1$  has exactly the same thermodynamic functions as its counterpart with  $\varepsilon = -1$ , but only the latter can be critical.

Let us now turn to the study of the low-energy excitations over the ground state in the possible critical cases identified in the previous paragraph. To begin with, consider the fermionic  $su(m)$  HS chain. A low-energy excitation over a ground state with bond vector  $\mathbf{s}_g$  is obtained, for instance, replacing a component  $s_i$  with  $i \ll N$  (or  $N - i \ll N$ ) and  $s_i < s_{i+1}$ , by  $s'_i \geq s_{i+1}$ . In the first case we add an energy

$$\Delta E = \mathcal{E}(x_i) \simeq \mathcal{E}'(0)x_i = O(1/N),$$

while in the second one

$$\Delta E = \mathcal{E}(x_i) = \mathcal{E}(1 - x_i) \simeq \mathcal{E}'(0)(1 - x_i) = O(1/N).$$

On the other hand, it is well known [14] that the state described by a bond vector  $\mathbf{s}$  has momentum

$$P(\mathbf{s}) = 2\pi \sum_{i=1}^{N-1} x_i \delta(s_i, s_{i+1}) \quad \text{mod } 2\pi.$$

Hence, by replacing  $s_i$  by  $s'_i$  we add a momentum

$$\Delta P = \pm 2\pi x_i, \quad \text{mod } 2\pi,$$

where the “+” sign corresponds to the case  $i \ll N$ . Thus, in this case the model is critical, and its Fermi velocity is given by

$$v = \lim_{N \rightarrow \infty} \left| \frac{\Delta E}{\Delta P} \right| = \frac{\mathcal{E}'(0)}{2\pi},$$

in agreement with Eq. (29). Note that this conclusion is consistent with the discussion in Ref. [48] for the  $su(m|n)$ -supersymmetric HS chains.

For the fermionic PF, FI, and HS-B chains the situation is quite different, as the Hamiltonian of these models is not invariant under translations along the lattice and thus momentum is not conserved. Of course, these models still possess low-energy excitations obtained by exciting a component of the ground-state bond vector with index  $i \ll N$  (both in the fermionic case and for the bosonic HS-B chain with  $m = 2$  or  $m = 3$ ,  $\varepsilon = -1$ ), with energy

$$\Delta E = \mathcal{E}(x_i) \simeq \mathcal{E}'(0)x_i = O(1/N).$$

By analogy with the HS chain, we assign to an energy eigenstate with bond vector  $\mathbf{s}$  an effective momentum

$$P(\mathbf{s}) = \pi \sum_{i=1}^{N-\eta} x_i \delta(s_i, s_{i+1}), \quad \text{mod } 2\pi, \quad (32)$$

where  $\eta = 0$  for the HS-B chain and  $\eta = 1$  otherwise. Note that, as explained above, in this case the factor multiplying the sum is  $\pi$  instead of  $2\pi$ , since the dispersion relation of the PF, FI, and HS-B chains is monotonically increasing over the interval  $[0,1]$ , and thus this interval represents only the positive momentum range. With this definition the change in effective momentum of the low-energy excitations described above is  $\Delta P = \pi x_i$ , and hence their Fermi velocity is given by Eq. (25). In summary, the HS-type chains (4)–(8) are all critical in the fermionic case, while in the bosonic case only the HS-B chain is critical when  $m = 2$  or  $m = 3$ ,  $\varepsilon = -1$ .

*Remark 2.* The PF chain is known to possess Yangian invariance [54], and it is conjectured that the same is true for the FI chain in view of the structure of its partition function and the high degeneracy of its spectrum [31]. Likewise, the HS-B chain is also known to have twisted Yangian symmetry [9]. It is thus conceivable that the effective momentum (32) could be related to the eigenvalues of one of the conserved Yangian generators for these models.

### V. CONCLUSIONS

In this paper we have completely determined in closed form the thermodynamics of the  $su(m)$  spin chains of Haldane-Shastry type (with zero chemical potential) (4)–(8) for all  $m$ . Our method relies on the fact that the energy spectrum of these models coincides with that of an appropriate vertex model, which makes it possible to express their thermodynamic free energy as an integral involving the Perron eigenvalue of a position-dependent  $m \times m$  transfer matrix. We have been able to compute this eigenvalue in closed form for arbitrary values of  $m$  by applying the classical Perron theorem on positive matrices and some of its consequences. This yields an explicit expression for the free energy per spin of these models, which fully determines their thermodynamics. We have found that at sufficiently high temperatures the thermodynamic functions of the  $su(m)$  HS-type chains behave qualitatively as those of an  $m$ -level system with uniformly spaced levels. In particular, for all values of  $m$  the specific heat features a single Schottky peak, whose temperature is close to that of the corresponding  $m$ -level system for  $m \lesssim 10$ .

Using our explicit formula for the free energy per spin, we have also examined the critical behavior of the  $su(m)$  HS-type chains (4)–(8). We have first shown that the low-temperature behavior of the free energy coincides with that of a CFT with central charge  $c = m - 1$ , both in the ferromagnetic (bosonic) and the antiferromagnetic (fermionic) regimes (with  $\gamma > 0$  for the FI chain). This low temperature behavior of the free energy is, however, only a necessary condition for criticality. To ascertain whether the models under study are critical or not, we have used the motif-based description of their spectrum to study the degeneracy of their ground state and the existence of low-energy excitations with a linear energy-momentum relation. In this way we have shown that the antiferromagnetic

chains are all critical (again, with  $\gamma > 0$  for the FI chain), whereas in the ferromagnetic case only the  $su(2)$  and  $su(3)$  HS-B chain (with  $\varepsilon = -1$  in the latter case) are critical.

Our main result does not appear to be easily generalizable to the case of nonzero chemical potential, or to  $su(m|n)$  supersymmetric chains, and in fact the known expressions of the thermodynamic functions of these models for low values of  $m$  and  $n$  have a more complicated structure than those found in this paper. Note, however, that our closed-form expression for the Perron eigenvalue of  $su(m)$  chains in the zero chemical potential case could be used to identify and approximate this eigenvalue for sufficiently small values of the chemical potentials, which would in turn yield a corresponding approximation for the thermodynamic functions.

### ACKNOWLEDGMENTS

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### APPENDIX: ASYMPTOTIC EXPANSION OF THE FREE ENERGY

In this Appendix we shall provide the details of the computation of the asymptotic approximation to the free energy per spin of the chains (4)–(8) at low temperatures [see, e.g., Eq. (24)]. In fact, we shall derive a complete asymptotic expansion of the latter function for  $T \rightarrow 0+$ .

Consider, to begin with, the FI chain with  $\gamma = 0$ , for which  $\mathcal{E}(x) = x^2$  and hence  $\mathcal{E}^{-1}(y) = \sqrt{y}$ ,  $0 \leq y \leq 1$ . From Eq. (23) we have

$$f_{\pm}(T) - f_{\pm}(0) = -T^2[g(T) - mg(mT)], \quad (A1)$$

with

$$g(T) = \frac{1}{2} T^{-1/2} \int_0^{\beta} \ln(1 - e^{-y}) \frac{dy}{\sqrt{y}}.$$

Since

$$\begin{aligned} & \left| \int_0^{\infty} \ln(1 - e^{-y}) \frac{dy}{\sqrt{y}} - \int_0^{\beta} \ln(1 - e^{-y}) \frac{dy}{\sqrt{y}} \right| \\ &= - \int_{\beta}^{\infty} \ln(1 - e^{-y}) \frac{dy}{\sqrt{y}} = O(T^{1/2} e^{-\beta}), \end{aligned}$$

we can write

$$\begin{aligned} g(T) &= \frac{1}{2} T^{-1/2} \int_0^{\infty} \ln(1 - e^{-y}) \frac{dy}{\sqrt{y}} + O(e^{-\beta}) \\ &= -\frac{\sqrt{\pi}}{2} \zeta(3/2) T^{-1/2} + O(e^{-\beta}), \end{aligned}$$

where  $\zeta(z) := \sum_{n \geq 1} n^{-z}$  is Riemann’s zeta function. From Eq. (A1) we finally obtain

$$\begin{aligned} & f_{\pm}(T) - f_{\pm}(0) \\ &= -(\sqrt{m} - 1) \frac{\sqrt{\pi}}{2} \zeta(3/2) T^{3/2} + O(T^2 e^{-\beta/m}), \quad (A2) \end{aligned}$$

which shows that the FI chain is not critical for  $\gamma = 0$ .



Consider next the PF chain. In this case we simply have

$$\int_0^1 \ln(1 - e^{-\beta x}) dx = T \int_0^\beta \ln(1 - e^{-y}) dy = -\frac{\pi^2 T}{6} + O(Te^{-\beta}),$$

and from Eq. (22) we find

$$f_{\pm}(T) - f_{\pm}(0) = -(m - 1) \frac{\pi^2 T^2}{6} + O(T^2 e^{-\beta/m}).$$

This is in agreement with Eq. (24), since in this case  $\mathcal{E}(x) = x$ .

Let us next deal with the FI and HS-B chains with  $\gamma > 0$ , for which we respectively have

$$(\mathcal{E}^{-1})'(z) = \begin{cases} (\gamma^2 + 4z)^{-1/2}, & 0 \leq z \leq \gamma + 1, \\ [(\gamma + 1)^2 - 2z]^{-1/2}, & 0 \leq z \leq \gamma + \frac{1}{2}. \end{cases}$$

Since in both cases all the derivatives of  $\mathcal{E}(z)$  are bounded in their respective domains we have

$$(\mathcal{E}^{-1})'(z) = \sum_{k=0}^n c_k \frac{z^k}{k!} + O(z^{n+1}), \quad c_k := \left. \frac{d^{k+1}(\mathcal{E}^{-1})}{dz^{k+1}} \right|_{z=0}.$$

From these formulas we obtain the estimates

$$g(T) = \int_0^{\beta \mathcal{E}(1)} \ln(1 - e^{-y}) (\mathcal{E}^{-1})'(Ty) dy = \sum_{k=0}^n c_k T^k \int_0^{\beta \mathcal{E}(1)} \frac{y^k}{k!} \ln(1 - e^{-y}) dy + O(T^{n+1}).$$

Moreover, since

$$\int_{\beta \mathcal{E}(1)}^\infty \frac{y^k}{k!} \ln(1 - e^{-y}) dy = O(\beta^k e^{-\beta \mathcal{E}(1)}),$$

the upper limit in all of the integrals above can be pushed to infinity at the cost of an exponentially small term. We thus have

$$g(T) = \sum_{k=0}^n c_k T^k \int_0^\infty \frac{y^k}{k!} \ln(1 - e^{-y}) dy + O(T^{n+1}) = -\sum_{k=0}^n c_k \zeta(k + 2) T^k + O(T^{n+1}),$$

which is equivalent to the infinite asymptotic expansion

$$g(T) \sim -\sum_{k=0}^\infty c_k \zeta(k + 2) T^k. \tag{A3}$$

By Eq. (A1), the latter formula yields the following asymptotic series for the free energy per spin of the FI and HS-B chains with  $\gamma > 0$ :

$$f_{\pm}(T) - f_{\pm}(0) \sim -\sum_{k=0}^\infty (m^{k+1} - 1) c_k \zeta(k + 2) T^{k+2}. \tag{A4}$$

The coefficients  $c_k$  can be easily computed in both cases, with the result

$$c_k = \begin{cases} (-2)^k (2k - 1)!! \gamma^{-(2k+1)} & \text{(FI),} \\ (-1)^k (2k - 1)!! (\gamma + 1)^{-(2k+1)} & \text{(HS-B),} \end{cases}$$

where  $(-1)!! := 1$ . In particular, from Eqs. (10)–(13) it easily follows that the first term in the asymptotic series (A4) coincides with Eq. (24).

The above argument must be slightly modified to deal with the HS chain and the HS-B chain with  $\gamma = 0$ , since in both cases  $(\mathcal{E}^{-1})'$  becomes infinite at the right endpoint  $\mathcal{E}(1)$ . For instance, for the HS chain we have

$$(\mathcal{E}^{-1})'(z) = (1 - 4z)^{-1/2}, \quad 0 \leq z \leq 1/4;$$

note that in this case  $\mathcal{E}^{-1}$  is the inverse of  $\mathcal{E} : [0, 1/2] \rightarrow [0, 1/4]$ , since  $\mathcal{E}(x)$  is increasing over  $[0, 1/2]$  and symmetric about the half-point  $x = 1/2$ . Using Eq. (27) and performing the change of variable  $\beta \mathcal{E}x = y$  we again arrive at Eq. (A1), where  $g(T)$  is now given by

$$g(T) = 2 \int_0^{\beta/4} \ln(1 - e^{-y}) (1 - 4Ty)^{-1/2} dy. \tag{A5}$$

To deal with the divergence of the last term at the upper limit of the integral, we first note that

$$(1 - 4z)^{-1/2} = \frac{(-1)^{n+1}}{2^{n+1} (2n + 1)!!} \frac{d^{n+1}}{dz^{n+1}} (1 - 4z)^{\frac{2n+1}{2}}.$$

We next define

$$h(z) = \frac{(-1)^{n+1}}{2^{n+1} (2n + 1)!!} \left[ (1 - 4z)^{\frac{2n+1}{2}} - \sum_{k=0}^{2n+1} c_k \frac{z^k}{k!} \right],$$

with

$$c_k = \left. \frac{d^k}{dz^k} \right|_{z=0} (1 - 4z)^{\frac{2n+1}{2}},$$

so that  $h(z) = O(z^{2n+2})$ . Differentiating  $n + 1$  times the expression for  $h(z)$  we arrive at the identity

$$(1 - 4z)^{-1/2} = h^{(n+1)}(z) + \frac{(-1)^{n+1}}{2^{n+1} (2n + 1)!!} \frac{d^{n+1}}{dz^{n+1}} \times \sum_{k=n+1}^{2n+1} c_k \frac{z^k}{k!} = h^{(n+1)}(z) + \frac{(-1)^{n+1}}{2^{n+1} (2n + 1)!!} \sum_{j=0}^n c_{n+j+1} \frac{z^j}{j!}.$$

Taking into account that

$$c_{n+j+1} = (2n + 1)!! (2j - 1)!! (-1)^{n+1} 2^{n+j+1},$$

we finally obtain

$$(1 - 4z)^{-1/2} = h^{(n+1)}(z) + \sum_{j=0}^n 2^j (2j - 1)!! \frac{z^j}{j!}.$$

Substituting this expression into Eq. (A5) yields

$$g(T) = \sum_{j=0}^n 2^{j+1} (2j - 1)!! T^j \int_0^{\beta/4} \frac{y^j}{j!} \ln(1 - e^{-y}) dy + 2\beta^{n+1} \int_0^{\beta/4} \ln(1 - e^{-y}) \frac{d^{n+1}}{dy^{n+1}} h(Ty) dy.$$

The asymptotic expansion of the first term is straightforward:

$$\begin{aligned} & \sum_{j=0}^n 2^{j+1} (2j-1)!! T^j \int_0^{\beta/4} \frac{y^j}{j!} \ln(1 - e^{-y}) dy \\ &= \sum_{j=0}^n 2^{j+1} (2j-1)!! T^j \int_0^\infty \frac{y^j}{j!} \ln(1 - e^{-y}) dy + O(\beta^n e^{-\beta/4}) \\ &= - \sum_{j=0}^n 2^{j+1} (2j-1)!! \zeta(j+2) T^j + O(\beta^n e^{-\beta/4}). \quad (\text{A6}) \end{aligned}$$

We claim that the second term in the previous equation for  $g(T)$  is  $O(T^{n+1})$ . This is easily proved by integrating by parts  $n+1$  times:

$$\begin{aligned} & \beta^{n+1} \int_0^{\beta/4} \ln(1 - e^{-y}) \frac{d^{n+1}}{dy^{n+1}} h(Ty) dy \\ &= \sum_{k=0}^n (-1)^{n-k} \beta^{n+1-k} \varphi^{(n-k)}(y) h^{(k)}(Ty) \Big|_0^{\beta/4} \\ & \quad + (-\beta)^{n+1} \int_0^{\beta/4} \varphi^{(n+1)}(y) h(Ty) dy, \end{aligned}$$

with  $\varphi(y) := \ln(1 - e^{-y})$ . The boundary terms vanish at  $y = 0$ , since  $h(z) = O(z^{2n+2})$ . On the other hand,

$$\beta^{n-k+1} \varphi^{(n-k)}(\beta/4) h^{(k)}(1/4) = O(\beta^{n-k+1} e^{-\beta/4}),$$

for  $k = 0, \dots, n$ , as the first  $n$  derivatives of  $h$  are bounded. Thus, the boundary terms are  $O(\beta^{n+1} e^{-\beta/4})$ , while

$$\beta^{n+1} \int_0^{\beta/4} \varphi^{(n+1)}(y) h(Ty) dy = O(T^{n+1}),$$

since  $h(Ty) = O[(Ty)^{2n+2}]$  and the integral  $\int_0^\infty y^{2n+2} \varphi^{(n+1)}(y) dy$  is convergent. Putting all of the above together we obtain the asymptotic series

$$g(T) \sim - \sum_{k=0}^\infty 2^{k+1} (2k-1)!! \zeta(k+2) T^k,$$

from which it follows that

$$\begin{aligned} f_\pm(T) - f_\pm(0) &\sim - \sum_{k=0}^\infty (m^{k+1} - 1) 2^{k+1} \\ &\quad \times (2k-1)!! \zeta(k+2) T^{k+2}. \quad (\text{A7}) \end{aligned}$$

In particular, truncating the series after the first term and using Eq. (10) we obtain Eq. (28).

Consider, finally, the HS-B chain with  $\gamma = 0$ , for which  $\mathcal{E}(x) = x(1 - x/2)$ . From Eqs. (22) and (27) it easily follows that

$$f_\pm(T) - f_\pm(0) = 2[f_{\text{HS},\pm}(T/2) - f_{\text{HS},\pm}(0)],$$

whence we obtain the asymptotic series

$$f_\pm(T) - f_\pm(0) \sim - \sum_{k=0}^\infty (m^{k+1} - 1) (2k-1)!! \zeta(k+2) T^{k+2}.$$

Again, the first term in this series is easily seen to yield Eq. (24).

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