


Environment-mediated entropic uncertainty in charging quantum batteries

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We studied the dynamics of entropic uncertainty in Markovian and non-Markovian systems during the charging of open quantum batteries (QBs) mediated by a common dissipation environment. In the non-Markovian regime, the battery is almost fully charged efficiently, and the strong non-Markovian property is beneficial for improving the charging power. In addition, the results show that the energy storage is closely related to the couplings of the charger-reservoir and battery-reservoir; that is, the stronger coupling of a charger reservoir improves energy storage. In particular, entanglement is required to obtain the most stored energy and is accompanied by the least tight entropic bound. Interestingly, it was found that the tightness of the entropic bound can be considered as a good indicator of the energy transfer in different charging processes, and the complete energy transfer always corresponds to the tightest entropic bound. Our results provide insight into the optimal charging efficiency of QBs during practical charging.

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I. INTRODUCTION

The concept of quantum battery (QB) was proposed by Alicki and Fannes [1], who investigated the extractability of the amount of work done by a quantum system during unitary evolution. It is a device used to temporarily store energy. In recent years, a series of studies have been conducted on energy changes in QBs [1–6]. In practice, both quantum and classical batteries focus on the efficiency with which energy (in the case of the battery) is transferred during charging and discharging to extract as much energy as possible from the battery. QB is considered an open system because of its interaction with the surrounding environment. This leads to the leakage of energy from the QB into the environment. Thus far, many scholars have proposed various schemes for studying energy transfer efficiency by quantum effects [7–9]. In principle, the characterizations for improving the performance of the QB during charging are reflected by higher energy storage and the smaller optimal charging time (i.e., the shortest time λ_s to reach the peak energy storage) [5,6,10–15]. Therefore, it is necessary to pursue a better charging process by tuning the control parameters in the transfer process.

In addition, the uncertainty relation was originally proposed by Heisenberg where we cannot accurately predict the measurement outcomes with respect to two arbitrary incompatible observables [16]. Heisenberg's uncertainty principle is regarded as a cornerstone of quantum mechanics. Subsequently, Kennard [17] and Robertson [18] proposed an expression of the uncertainty principle in the form of standard deviation: $\Delta\hat{P}_1\Delta\hat{P}_2 \geq \frac{1}{2}|\langle[\hat{P}_1, \hat{P}_2]\rangle|$, where $\Delta(\bullet)$ represents the standard deviation, and $\langle\circ\rangle$ in the lower bound represents the corresponding expected value. Note that this relation leads to a trivial result, that is, the lower

bound will become zero when the system's state is prepared in one of the two observables' eigenstates. On this premise, Deutsch introduced information entropy to describe the uncertainty principle and present the entropic uncertainty relation (EUR) [19]: $H(\hat{P}_1) + H(\hat{P}_2) \geq \log_2(\frac{2}{1+\sqrt{c}})^2$, where $c = \max_{ij}|(P_{1i} | P_{2j})|^2$, $|P_{1i}\rangle$ and $|P_{2j}\rangle$ being with the eigenstates of the measurement operator \hat{P}_1 , \hat{P}_2 , and the Shannon entropy $H(\hat{P}_k) = -\sum_i p_{ki} \log_2 p_{ki}$ ($k = 1, 2$) with $p_{ki} = \langle P_{ki} | \hat{\rho} | P_{ki} \rangle$ being the probability of obtaining the i th measurement result. Later, Kraus [20] and Maassen and Uffink [21] improved Deutsch's results. Notably, this relationship is suitable for single-particle systems. Thus, one may ask how to express the uncertainty relation. If the measured particle is correlated with another. Berta *et al.*, Ming *et al.*, and Xie *et al.* proposed a new form of uncertainty relation called quantum memory-assisted entropic uncertainty relation (QMA-EUR) [22–24]. Their investigation focuses on a two-measurement case. Later on, Liu *et al.* expanded Berta *et al.*'s version to the case of multiple measurements, and put forward the general expression of a QMA-EUR related to multiple measurements as [25]

$$\sum_{x=1}^N S(\hat{P}_x|B) \geq -\log_2(b) + (N-1)S(A|B), \quad (1)$$

where $S(A|B) = S(\hat{\rho}_{AB}) - S(\hat{\rho}_B)$ is the von Neumann conditional entropy [26], and the von Neumann entropy $S(\hat{\rho}) = -\text{tr}(\hat{\rho} \log_2 \hat{\rho})$. Specifically, $b = \max_{i_N} \{\sum_{i_2 \sim i_{N-1}} \max_{i_1} [c(u_{i_1}^1, u_{i_2}^2)] \prod_{x=2}^{N-1} [c(u_{i_x}^x, u_{i_{x+1}}^{x+1})]\}$ and $u_{i_x}^x$ is the i th eigenvector of the operator \hat{P}_x .

Recently, some scholars have studied the change in entanglement in the QB [27], and the relationship between energy density and entanglement [28] during quantum battery charging, which indicates that entanglement plays an important role in energy transfer during the charging process. In addition, the connection between entanglement and uncertainty has

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been disclosed [29]. With this in mind, the tightness of the lower bound on the entropic uncertainty relation plays a role during the charging of quantum batteries. Motivated by this, we first investigate the optimization of the charging process by different coupling regimes: Markovian and non-Markovian regimes. When the battery is stable, we look for more stored energy by investigating the qubit-environment coupling and the initial entanglement of the charger and battery. We found that a stronger charger-environment coupling is beneficial for obtaining more stored energy. Moreover, we study the variation in the entropic uncertainty relation (the tightness of the entropic bound) and the relation with energy storage in charging QBs. Remarkably, it was found that the tightness of the entropic bound can be considered an indicator of the energy transfer, and the complete energy transfer is always accompanied by the tightest entropic bound. That is, the tightness of the uncertainty bound can reflect the charging efficiency in the QB.

The structure of this paper is as follows. In Sec. II, charging model of an open quantum battery system is reviewed. In Sec. III, the time evolution of stored energy and power in different coupling regimes is studied in the model. In Sec. IV, we investigate the steady-state energy of the QB and its relation to the tightness of the lower bounds on entropic uncertainty. Furthermore, we investigate the tightness at different energy transfer rates (the proportion of energy transferred from the charger to the battery) in the charging process. Finally, we conclude our paper with a concise summary in Sec. V.

II. CHARGING MODEL

We consider a quantum battery charging model consisting of a quantum charger A and a quantum battery B , which is coupled to a common zero-temperature bosonic reservoir, without coupling between A and B . Each cell of A and B is a two-level system with the same transition frequency $\omega_A = \omega_B = \omega_0$, and the ground and excitation states are $|g\rangle$ and $|e\rangle$, respectively. The Hamiltonian H of the entire system is composed of H_0 and H_{int} [30–32],

$$H = H_0 + H_{\text{int}}, \quad (2)$$

where

$$H_0 = \sum_{i=A,B} \omega_0 \sigma_i^+ \sigma_i^- + \sum_k \omega_k c_k^\dagger c_k, \quad (3)$$

and

$$H_{\text{int}} = (\beta_1 \sigma_A^+ + \beta_2 \sigma_B^+) \sum_k g_k c_k + (\beta_1 \sigma_A^- + \beta_2 \sigma_B^-) \sum_k g_k c_k^\dagger. \quad (4)$$

H_0 represents the free Hamiltonian of the qubit and the reservoir, where σ_i^+ and σ_i^- are the Pauli rising and lowering operators of the qubit, ω_k and $c_k^\dagger(c_k)$ are the frequency and creation (annihilation) operators of the k th mode of the field, respectively. H_{int} denotes the interaction between the two-qubit system and the reservoir. β_1 and β_2 are defined as the interaction strengths between qubits A and B and the reservoir, respectively, and are dimensionless real parameters. And $g_k \beta_1$ and $g_k \beta_2$ are the coupling constants between the qubits and the reservoir. Then, the collective coupling constant

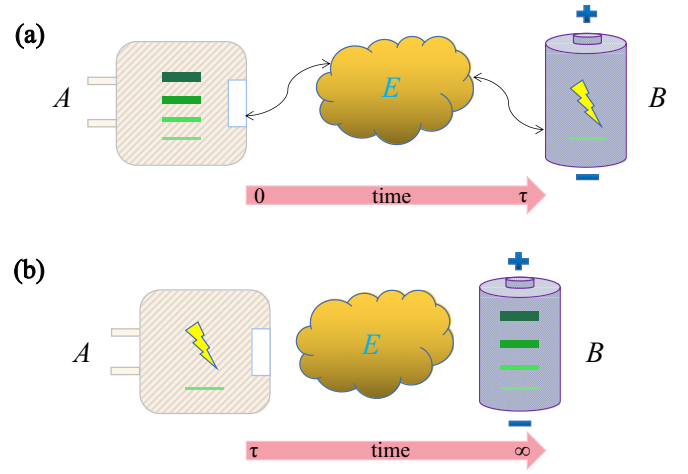


FIG. 1. (a) Within time $t \in [0, \tau]$, the charger (A) and battery (B) interact with the environment (E). (b) When $t > \tau$, at the end of the charging process, the battery is in the steady state and its energy is conserved.

$\beta_T = \sqrt{\beta_1^2 + \beta_2^2}$ and the relative interaction strength $\zeta_i = \beta_i / \beta_T$ ($i = 1, 2$). It should be noted that different effective couplings of A and B with a reservoir can be obtained under different conditions for β_1 and β_2 . In addition, H_{int} exists within the time interval: $t \in [0, \tau]$ of the charging process (see Fig. 1). Outside this interval, there is no interaction between A (B) and the environment, furthermore, there is no interaction between A and B . Therefore, this is the wireless charging process of QB in an environmental medium [30]. Note that, this charging model corresponds to physically feasible experimental settings by the all-optics platform [30,33,34].

Under the limitation that the two-qubit system and reservoir are separable states at the beginning, we consider the case in which there is only one excitation state in the two-qubit system and the reservoir is in a vacuum state. The initial state of the entire system is

$$|\varphi(0)\rangle = \eta_{01} |e\rangle_A |g\rangle_B |0\rangle_X + \eta_{02} |g\rangle_A |e\rangle_B |0\rangle_X, \quad (5)$$

where η_{01} and η_{02} are the probabilities amplitudes and $|0\rangle_X$ is the vacuum state of the reservoir. The time evolution of a single excited state can be obtained as

$$|\varphi(t)\rangle = \eta_1(t) |e\rangle_A |g\rangle_B |0\rangle_X + \eta_2(t) |g\rangle_A |e\rangle_B |0\rangle_X + \sum_k \eta_k(t) |g\rangle_A |g\rangle_B |1_k\rangle_X, \quad (6)$$

where $\eta_1(t)$ and $\eta_2(t)$ are probability amplitudes. We consider the case in which the environment is regarded as an electromagnetic field in a lossless cavity, and the Lorentzian form of the spectral density of the cavity field is [35]

$$J(\omega) = \frac{v^2 \lambda}{\pi(\omega - \omega_0)^2 + \pi \lambda^2}, \quad (7)$$

where λ is the spectrum width, $1/\lambda$ is the correlation time of the reservoir, and v is the effective coupling strength satisfying $L = v \beta_T$ where L is the vacuum Rabi frequency. Then we define a dimensionless real number $R = L/\lambda$ where strong coupling has $R \gg 1$ and weak coupling with $R \ll 1$ [36–42].

In this case, Markovian evolution occurs under weak coupling, and non-Markovian dynamics and memory effects occur under strong coupling. And then the analytic solution for $\eta_i(t)$ ($i = 1, 2$) comes in the following form [30–32]:

$$\eta_1(t) = [\zeta_2^2 + \zeta_1^2 \mu(t)]\eta_{01} - \zeta_1 \zeta_2 [1 - \mu(t)]\eta_{02}, \quad (8)$$

and

$$\eta_2(t) = [\zeta_1^2 + \zeta_2^2 \mu(t)]\eta_{02} - \zeta_1 \zeta_2 [1 - \mu(t)]\eta_{01}, \quad (9)$$

where the Lorentzian form is $\mu(t) = e^{-\frac{\lambda t}{2}} [\cosh(\frac{\kappa t}{2}) + \frac{\lambda}{\kappa} \sinh(\frac{\kappa t}{2})]$ and $\kappa = \sqrt{\lambda^2 - 4L^2}$ [30–32]. Finally, we have

$$\mu(t) = \begin{cases} e^{-\frac{\lambda t}{2}} [\cosh(x) + \sinh(x)/\chi_1], & 0.5 \geq R \geq 0, \\ e^{-\frac{\lambda t}{2}} [\cos(y) + \sin(y)/\chi_2], & R \geq 0.5, \end{cases} \quad (10)$$

where $x = \chi_1 \sqrt{\lambda^2 t^2}/2$, $y = \chi_2 \sqrt{\lambda^2 t^2}/2$ and $\chi_1 = \sqrt{1 - 4R^2}$, $\chi_2 = \sqrt{4R^2 - 1}$.

III. PERFORMANCE OF QUANTUM BATTERIES WITH DIFFERENT COUPLING REGIMES

During the charging process of the quantum batteries, the coupling strength between the qubits (charger and battery) and the reservoir is considered to be an important element in determining excellent charging performance. From Eqs. (5) and (6), the reduced density matrices can be expressed as

$$\rho_{AB}(0) = |\eta_{01}|^2 |eg\rangle_{AB} \langle eg| + |\eta_{02}|^2 |ge\rangle_{AB} \langle ge| + \eta_{01} \eta_{02}^* |eg\rangle_{AB} \langle ge| + \eta_{01}^* \eta_{02} |ge\rangle_{AB} \langle eg|, \quad (11)$$

and

$$\rho_{AB}(t) = |\eta_1(t)|^2 |eg\rangle_{AB} \langle eg| + |\eta_2(t)|^2 |ge\rangle_{AB} \langle ge| + \eta_1(t) \eta_2(t)^* |eg\rangle_{AB} \langle ge| + \eta_1(t)^* \eta_2(t) |ge\rangle_{AB} \langle eg| + [1 - |\eta_1(t)|^2 - |\eta_2(t)|^2] |gg\rangle_{AB} \langle gg| \quad (12)$$

of the two-qubit system. Note that $\eta_i(t) = \eta_i(t)^*$, ($i = 1, 2$). The reduced-density matrices $\rho_A(t)$ and $\rho_B(t)$ of A and B can also be obtained. Then we take $\eta_{01} = \sin \theta e^{i\phi}$ and $\eta_{02} = \cos \theta$. According to Eq. (12) and the reduced density matrix $\rho_B(t)$, the energy of the battery at time t is represented by $E_B(t) = \text{tr}[H_B \rho_B(t)]$. Hence, the change in battery energy during charging can be quantified by

$$\Delta E_B(t) = E_B(t) - E_B(0) = \omega_0 [|\eta_2(t)|^2 - \cos^2 \theta]. \quad (13)$$

Another effective parameter in the charging process is the charging power of the QB. In this case, the instantaneous power of the charging process is as follows:

$$P = \frac{E_B(t)}{t}. \quad (14)$$

A. Markovian dynamics

In the weakly coupled regime, the general behavior of energy variation can be obtained. We consider the maximum energy storage and the faster charging speed (i.e., greater instantaneous power) as indicators to optimize the charging process and find the smallest optimal charging time λt_s when the maximum energy obtained by the battery is fixed.

Figure 2(a) shows the effect of coupling strength on stored energy with $\zeta_1 = \zeta_2 = \frac{1}{\sqrt{2}}$. First, the stored energy increases to its maximum over time. Second, in the long-time limit, the change in the coupling between the qubit and the reservoir cannot increase the peak energy storage, however, it can significantly optimize the shortest time λt_s to reach the peak energy storage. Figure 2(b) represents the variation in the energy stored over a larger range of coupling. The energy did not change monotonically with time. Different from Fig. 2(a), the enhanced coupling strength enables the QB to store more energy with smaller λt_s . Figure 2(c) reveals the contribution of coupling strength to instantaneous power during charging. A stronger interaction between the two-qubit system and the reservoir was found to speed up the charging. Compared with Figs. 2(c) and 2(d) also reveals that the improvement of the coupling strength is beneficial to the charging power, which is increased by about four times. Moreover, the change trend of power does not change like that of energy, that is, it reaches the maximum in a relatively short period of time and then decreases slowly. In short, in Markovian systems, a higher coupling strength reduces the optimal charging time and attains more stored energy, therefore, the charging performance of the battery is improved.

B. Non-Markovian dynamics

Figure 3(a) clearly shows the help of a memory effect for maintaining high energy storage. The stored energy oscillates efficiently, and the first wave packet represents the maximum stored energy, which corresponds to the optimal charging time. The peaks of the other wave packets decreased with respect to λt . It can be seen that changing the coupling strength does not significantly improve the maximum stored energy, and a decrease in λt_s can be realized by an increase in the coupling. Figure 3(b) illustrates that coupling enhancement can significantly increase the charging speed, which is the same as Figs. 2(c) and 2(d).

In the Markovian regime, one can obtain that the maximum energy of the weak coupling regime is about $0.34\omega_0$ and the optimal charging time is about 4. However, the non-Markovian system has advantages both in terms of maximum energy storage (the battery is almost full) and optimal charging time $\lambda t_s \approx 0.006$, owing to the unique memory effect of the non-Markovian regime, which provides an important way to overcome energy leakage into the environment.

IV. THE TIGHTNESS OF AN ENTROPIC BOUND AND STORED ENERGY

In the open quantum battery system, the decoherence effect of the environment leads to an energy leakage. Although greater non-Markovian properties allow the battery to charge satisfactorily for short periods of time, there still exists an additional improvement in energy storage for a longer charging time, which is regarded as another consideration for an excellent charging protocol for quantum batteries. Therefore, in this section, we pursue higher stored energy in the steady state of a quantum battery under the influence of initial en-

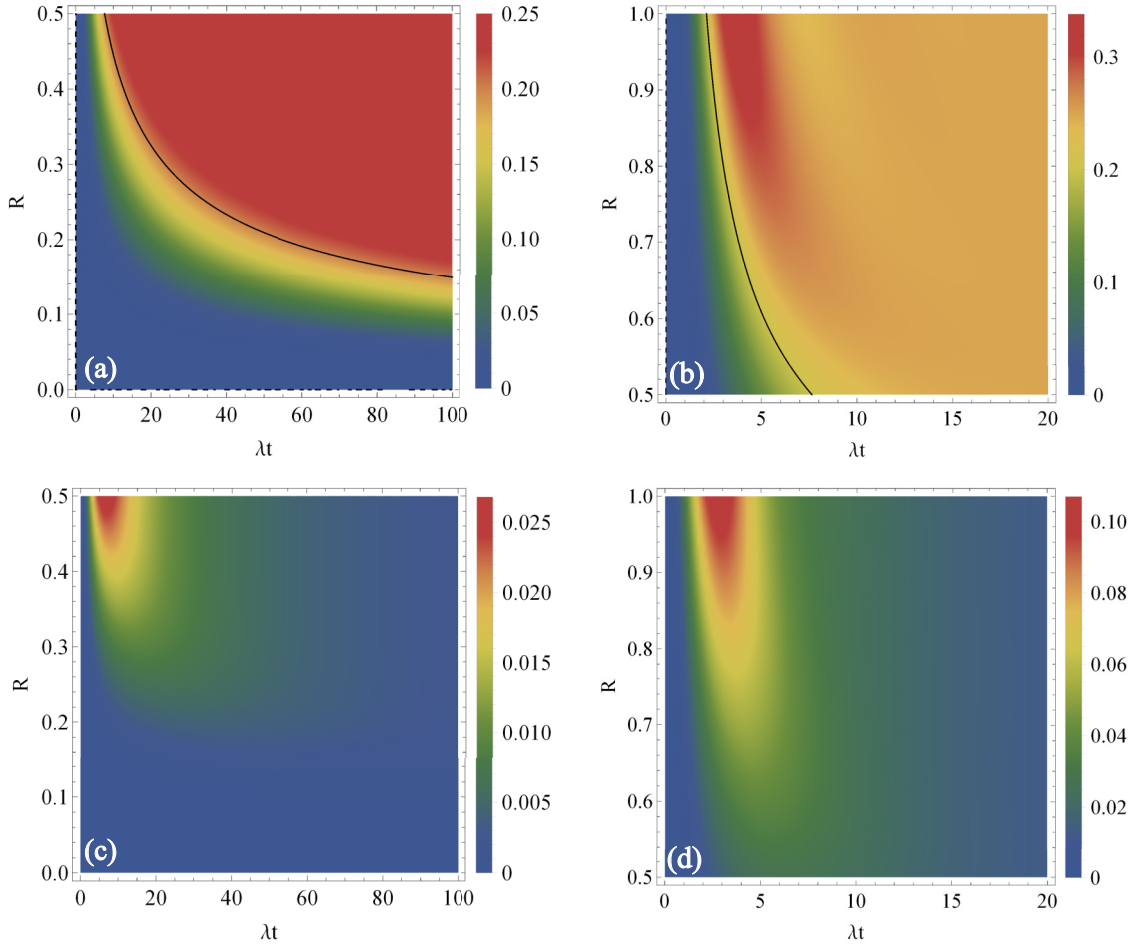


FIG. 2. Markovian charging process. The stored energy of battery $\Delta E_B(t)/\omega_0$ and instantaneous power P/ω_0 as function of the dimensionless quantity λt and coupling strength R are plotted in (a), (b), (c), and (d), respectively. We choose the initial separable states with $\theta = 0.5\pi$, and the same interactions of the charger reservoir and the battery reservoir ($\zeta_1 = \zeta_2 = \frac{1}{\sqrt{2}}$). Graphs (a) and (c): $0 \leq R \leq 0.5$; Graphs (b) and (d): $0.5 \leq R \leq 1$.

tanglement and relative interaction strength (characterized by comparing the charger-reservoir ζ_1 and battery-reservoir coupling strength ζ_2). Note that ζ_1 and ζ_2 were normalized).

Furthermore, we investigate the role of the entanglement and tightness of the entropic bound in the process of boosting stored energy.

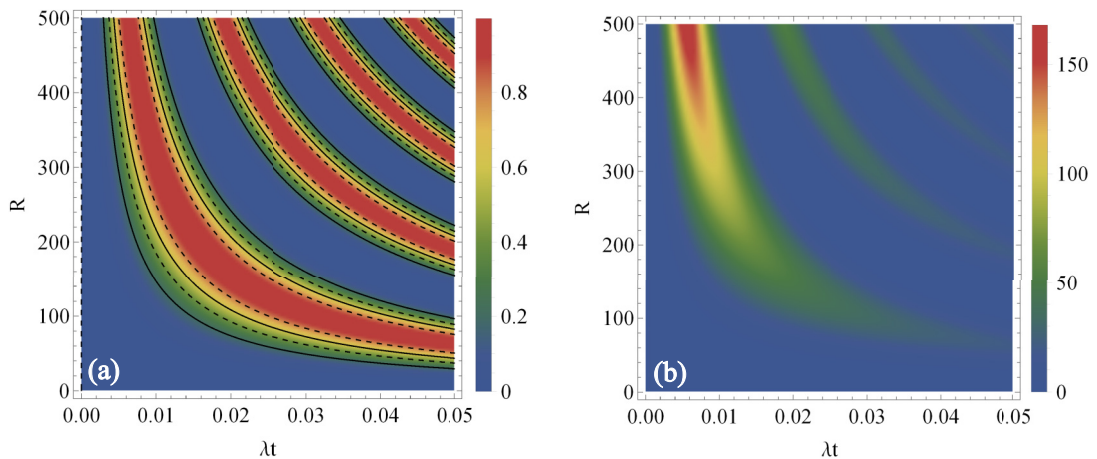


FIG. 3. In the Non-Markovian charging process the stored energy of the battery $\Delta E_B(t)/\omega_0$ and instantaneous power P/ω_0 as a function of the dimensionless quantity λt and coupling strength R are plotted in Graphs (a) and (b), respectively. $1 \leq R \leq 500$ is set, and all other parameters are the same as those in Fig. 2.

For a system with an X -type density matrix, the concurrence of the system [43] is $C_{\rho(t)} = 2 \max\{0, |\rho_{14}| - \sqrt{|\rho_{22}\rho_{33}|}, |\rho_{23}| - \sqrt{|\rho_{11}\rho_{44}|}\}$, where ρ_{ij} corresponds to the entries in row i and column j of matrix $\rho(t)$. Consequently, the concurrence of $\rho_{AB}(t)$ is calculated as $C_{\rho_{AB}(t)} = 2|\eta_1(t)\eta_2^*(t)|$.

To probe the dynamics of entropic uncertainty, we resort to a pair of Pauli operators σ_x and σ_z as the incompatibility, which leads to postmeasurement states,

$$\rho(\sigma_x|B) = \frac{1}{2}[\rho(t)_{AB} + (\sigma_x \otimes I)\rho(t)_{AB}(\sigma_x \otimes I)], \quad (15)$$

and

$$\rho(\sigma_z|B) = \frac{1}{2}[\rho(t)_{AB} + (\sigma_z \otimes I)\rho(t)_{AB}(\sigma_z \otimes I)]. \quad (16)$$

where I denotes an identity matrix. With respect to $\rho(\sigma_x|B)$, eigenvalues can be calculated as

$$\gamma_1^x = \gamma_2^x = (1 - D)/4, \quad \gamma_3^x = \gamma_4^x = (1 + D)/4, \quad (17)$$

where $D = \sqrt{1 - 4|\eta_2(t)|^2 + 4|\eta_1(t)|^2|\eta_2(t)|^2 + 4|\eta_2(t)|^4}$ for $\rho(\sigma_z|B)$, we have eigenvalues

$$\gamma_1^z = 0, \quad \gamma_2^z = |\eta_1(t)|^2, \quad \gamma_3^z = |\eta_2(t)|^2, \quad (18)$$

$$\gamma_4^z = 1 - |\eta_1(t)|^2 - |\eta_2(t)|^2. \quad (19)$$

Finally, we obtain the magnitude of the entropic uncertainty as $U_i^{xz}(t) = 2 \sum_{i=1}^2 \varepsilon_i^B \log_2 \varepsilon_i^B - \sum_{j=1}^4 \gamma_j^x \log_2 \gamma_j^x - \sum_{j=1}^4 \gamma_j^z \log_2 \gamma_j^z$ [left-hand side of Eq. (1)]. When the system is stable, we have

$$\begin{aligned} U_i^{xz}(\infty) &= |\eta_2(\infty)|^2 \log_2 |\eta_2(\infty)|^2 - M \log_2 M \\ &\quad - 2[|\eta_2(\infty)|^2 - 1] \log_2 [1 - |\eta_2(\infty)|^2] \\ &\quad - |\eta_1(\infty)|^2 \log_2 |\eta_1(\infty)|^2 \\ &\quad + (F - 1) \log_2 [(1 - F)/4]/2 \\ &\quad - (F + 1) \log_2 [(F + 1)/4]/2, \end{aligned} \quad (20)$$

and lower bound $U_r^z(t) = 1 + \sum_{i=1}^2 \varepsilon_i^B \log_2 \varepsilon_i^B - \sum_{j=1}^4 \varepsilon_j^{AB} \log_2 \varepsilon_j^{AB}$ [right-hand side of Eq. (1)]. The corresponding lower bound in the steady state is in the form of

$$\begin{aligned} U_r^z(\infty) &= 1 + |\eta_2(\infty)|^2 \log_2 |\eta_2(\infty)|^2 \\ &\quad - [|\eta_2(\infty)|^2 - 1] \log_2 [1 - |\eta_2(\infty)|^2] \\ &\quad - M \log_2 M - (1 - M) \log_2 (1 - M), \end{aligned} \quad (21)$$

where ε_i^B and ε_j^{AB} are the eigenvalues of $\rho_B(t)$ and $\rho_{AB}(t)$, $F = \sqrt{1 + 4|\eta_1(\infty)|^2|\eta_2(\infty)|^2 - 4|\eta_2(\infty)|^2 + 4|\eta_2(\infty)|^4}$, and $M = 1 - |\eta_1(\infty)|^2 - |\eta_2(\infty)|^2$. The superscripts of entropic uncertainty and lower bounds indicate the Pauli operator chosen for the measurement and the number of measurement operators, respectively.

After a sufficient charging period, i.e., $t \rightarrow \infty$, the internal energy of the battery tends to be stable, and at this time we get $\mu(\infty) = 0$, $\eta_1(\infty) = \zeta_2^2 \eta_{01} - \zeta_1 \zeta_2 \eta_{02}$, and $\eta_2(\infty) = \zeta_1^2 \eta_{02} - \zeta_1 \zeta_2 \eta_{01}$. Thus, the concurrence can be expressed as follows:

$$C_{\rho_{AB}(\infty)} = 2|\eta_1(\infty)\eta_2^*(\infty)|, \quad (22)$$

and the stored energy of the battery in the steady state is obtained by

$$\Delta E_B(\infty) = \omega_0[|\eta_2(\infty)|^2 - \cos^2 \theta]. \quad (23)$$

Similarly, entropic uncertainty $U_i^{xz}(\infty)$ and its lower bound $U_r^z(\infty)$ in the steady state can be calculated. The tightness of the lower bound is given by $\Delta U^{xz}(\infty) = U_i^{xz}(\infty) - U_r^z(\infty)$, which reflects the general characteristics of the entropic uncertainty relation. The tighter lower bound is quantified by a smaller $\Delta U^{xz}(\infty)$.

In addition, as a parameter in the probability amplitude of the initial state, θ is reflected in the initial charger-battery entanglement in the charging scheme. The concurrent initial state can be expressed as $C_{\rho_{AB}(0)} = 2|\eta_{01}\eta_{02}^*|$. After the calculation, it was found that the entanglement of the initial state is only related to θ and is independent of phase ϕ , which reaches a maximum value of 1 when $\theta = 0.25\pi$ and monotonically decreases to 0 when θ approaches the boundary value 0.5π . Therefore, in the charging process, we consider the influence of the initial entanglement on battery performance according to the change in θ .

In Figs. 4(a)–4(c), entanglement and stored energy have a similar evolution process whereas they are opposite to the tightness of entropic bound. All three quantities reach an extreme value when the initial entanglement is at a minimum. Moreover, when the initial state is separable ($\theta = 0.5\pi$), the stored energy is effective [$\Delta E_B(\infty) \geq 0$] for all values of the relative interaction strength, and the stored energy reaches a maximum [$\Delta E_B(\infty)_{\max} = 0.25\omega_0$] when the coupling strengths of the charger reservoir and the battery reservoir are symmetrical ($\zeta_1 = \zeta_2 = 1/\sqrt{2}$). The results also show that entanglement is beneficial for energy storage when $\phi = 0$. Although a tighter entropic bound is not required for the growth of stored energy, there is a close relationship between them, that is, the highest stored energy strictly corresponds to the least tight entropic bound, which never occurs with respect to entanglement.

In Figs. 4(d)–4(f) with the reversal of the relative phase to $\phi = \pi$, it can be seen that entanglement reaches the peak value when the system is initially the maximum entangled state, which is completely opposite to that of the case of $\phi = 0$. Simultaneously, the entropic bound became less tight. Furthermore, there is effective stored energy [$\Delta E_B(\infty) \geq 0$], regardless of the initial charger-battery entanglement. Compared to the case of $\phi = 0$, the peak value of the stored energy increases to approximately $0.33\omega_0$ when $\phi = \pi$, accompanied by an asymmetric charging process ($\zeta_1 \approx 0.8$) and a certain initial entanglement ($\theta \approx 1.3$). Comparing Figs. 4(d) and 4(e), it is to find that charger-battery entanglement is necessary on the parameter plane $\theta - \zeta_1$ to obtain the most stored energy. Interestingly, the flip of the relative phase does not change the close relationship between the tightness of the entropic bound and the stored energy, that is, the most stored energy when $\phi = \pi$ strictly corresponds to the least tight entropic bound.

We find that the stored energy of the steady state is always negatively correlated with the tightness of the entropic bound when the relative phase is 0 or π . Therefore, we will further study the relationship between the stored energy and the tightness of the entropic bound from the transfer of energy between charger and battery. Here, we study the evolution of the entanglement and tightness of the entropic bound for any case of energy transfer. The energy variation of the charger and battery is associated with the

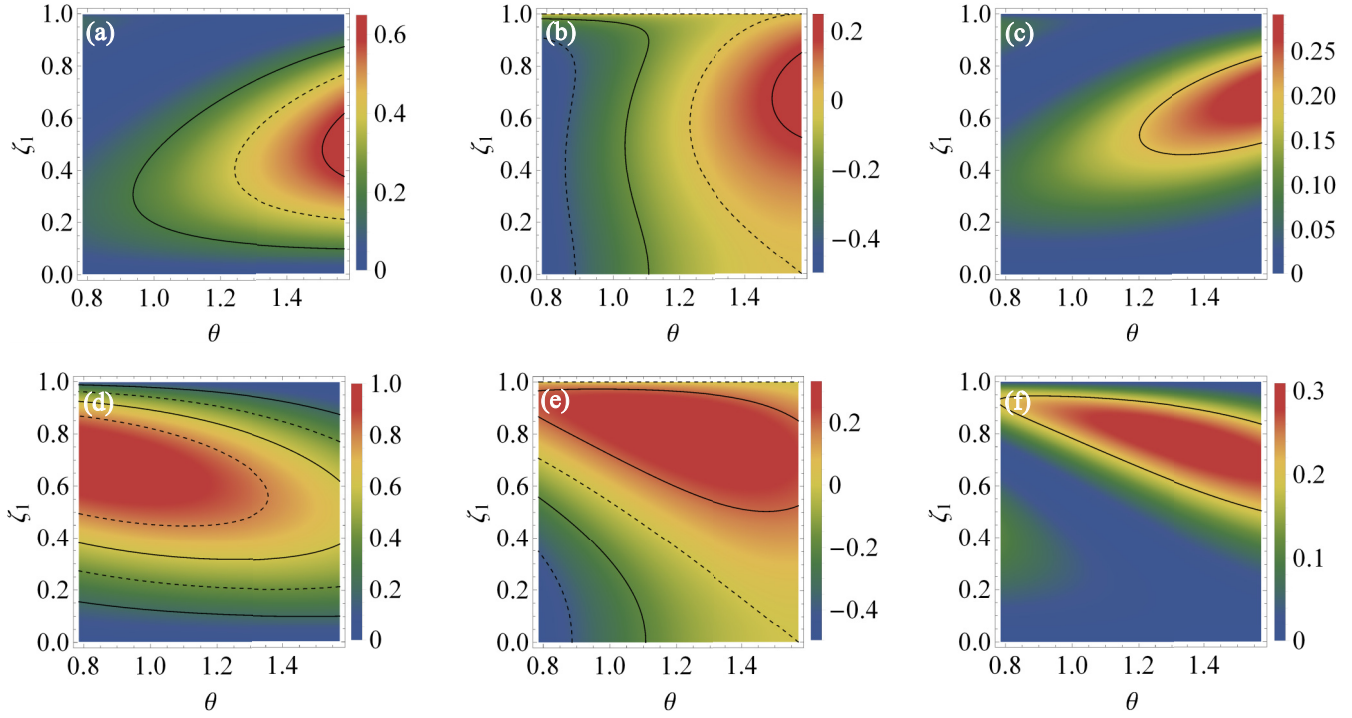


FIG. 4. When the battery is in the steady state, the concurrence $C_{\rho_{AB}(\infty)}$, stored energy, $\Delta E_B(\infty)/\omega_0$ and the tightness of the lower bound $\Delta U^{xz}(\infty)$ are affected by the relative interaction strength ζ_1 and the probability amplitude of the initial state. $C_{\rho_{AB}(\infty)}$ is plotted in (a) and (d); $\Delta E_B(\infty)/\omega_0$ is plotted in (b) and (e); $\Delta U^{xz}(\infty)$ is plotted in (c) and (f); $\theta \in [0.25\pi, 0.5\pi]$ and $\zeta_1 \in [0, 1]$ are given, respectively. In addition, the relative phase is also associated with the concurrence $C_{\rho_{AB}(\infty)}$, stored energy, $\Delta E_B(\infty)/\omega_0$ and the tightness of the lower bound $\Delta U^{xz}(\infty)$, here, we set $\phi = 0$ in Graphs (a), (b), and (c); and $\phi = \pi$ in Graphs (d), (e), and (f).

tightness of the lower bound and the entanglement [according to Eqs. (20)–(23)]. From the relation $|\eta_1(t)|^2 = \Delta E_A(t)/\omega_0 + \sin^2 \theta$ and $|\eta_2(t)|^2 = \Delta E_B(t)/\omega_0 + \cos^2 \theta$, where $|\Delta E_A(t)|$ is the energy lost by the charger, we can see that the energy partly determines the evolutionary behavior of entanglement or tightness. The constraint can then be expressed as $C_{\rho_{AB}(t)} = 2[\Delta E_A(t)/\omega_0 + \sin^2 \theta]^{1/2}[\Delta E_B(t)/\omega_0 + \cos^2 \theta]^{1/2}$. Similarly, the tightness of the entropic bound can be represented by $\Delta U^{xz}(t)$, which is obtained by substituting energy and parameters θ .

Next, we visually depict the relationship between entanglement or tightness and energy storage in Fig. 5. The energy stored by the battery during the charging process has a boundary value, which is the maximum value of the stored energy regardless of the energy lost by the charger, as shown in Fig. 5 as the hypotenuse of a triangle region. The boundary value or constraint can be expressed as $\Delta E_B(t)/|\Delta E_A(t)|=1$. This implies that the energy lost by the charger is completely transferred to the battery. However, we find that at $\Delta E_B(t) \neq |\Delta E_A(t)|$, that is, $\Delta E_B(t)/|\Delta E_A(t)| < 1$ the energy is not completely transferred to the battery, which means that some of the transferred energy flows into the environment. Different energy transfers actually correspond to arbitrary charging processes caused by different parameter settings in the charging model. Therefore, it is meaningful to discuss the effect of energy transfer rate $\Delta E_B(t)/|\Delta E_A(t)|$ on the tightness of entropic bound.

Meanwhile, Fig. 5(a) shows that the relationship between entanglement and energy variation during the arbitrary charging process. It can be seen that an increase in the energy

transfer rate (i.e., the charger loses less energy or the battery stores more energy) contributes to an increase in entanglement. Entanglement disappears when the battery or charger is empty. Interestingly, the energy can be completely transferred for different entanglements, corresponding to the states of the hypotenuse in Fig. 5(a). Therefore, we can conclude that entanglement is not a clear indicator of the maximal energy stored. Besides, we obtain that the maximum energy transfer rate corresponds to the tightest entropic bound except when the charger or battery is completely empty as shown in Fig. 5(b). This indicates that the tightness of the entropic bound can also play a role in indicating the degree of energy transfer in the charging processes, implying that the energy is completely transferred for the tightest entropic bound $\Delta U^{xz}(t) = 0$, which virtually agrees with the previous conclusion.

To explore the role of tightness in improving charging in a more general context. We consider three measurements with $(\sigma_x, \sigma_y, \sigma_z)$. As a result, the entropic uncertainty is

$$\begin{aligned}
 U_l^{xyz}(t) = & -|\eta_1(t)|^2 \log_2 |\eta_1(t)|^2 + 2|\eta_2(t)|^2 \log_2 |\eta_2(t)|^2 \\
 & - 3[|\eta_2(t)|^2 - 1] \log_2 [1 - |\eta_2(t)|^2] - m \log_2 m \\
 & + (D-1) \log_2 [(1-D)/4] \\
 & - (D+1) \log_2 [(1+D)/4], \quad (24)
 \end{aligned}$$

where $m = 1 - |\eta_1(t)|^2 - |\eta_2(t)|^2$ and the corresponding lower bound is

$$U_r^3(t) = 2U_r^2(t) - 1, \quad (25)$$

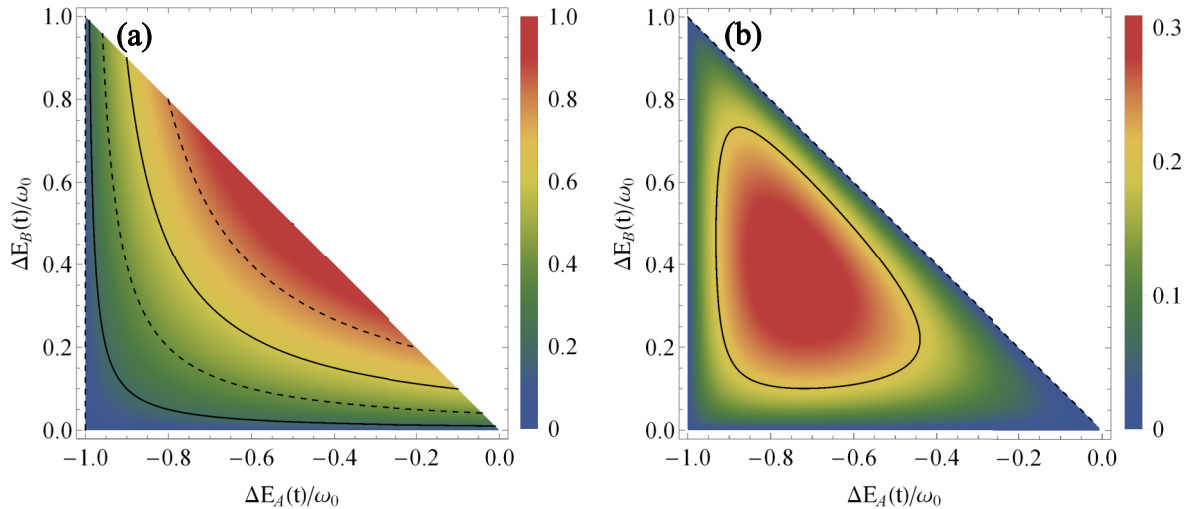


FIG. 5. The concurrence $C_{\rho_{AB}(t)}$ and the tightness $\Delta U^{xyz}(t)$ as functions of energy are plotted in (a) and (b), respectively. The x axis is the energy change in the charger $\Delta E_A(t)/\omega_0$, and the y axis is the stored energy of the battery $\Delta E_B(t)/\omega_0$. In addition, a separable initial state $\theta = 0.5\pi$ is set.

the tightness of the three measurements was $\Delta U^{xyz}(t) = U_l^{xyz}(t) - U_r^3(t)$.

Figure 6 shows the connection between the exchanged energy and the tightness of the entropic uncertainty. Obviously, the complete energy transfer takes place when the tightness is fixed to $\Delta U^{xyz}(t) = 1$ corresponding to the hypotenuse, which also supports tightness can be considered an effective indicator of the maximum energy transfer during the charging of QBs.

V. DISCUSSIONS AND CONCLUSIONS

To summarize, we have investigated the dynamics of the entropic uncertainty of charging quantum batteries in common

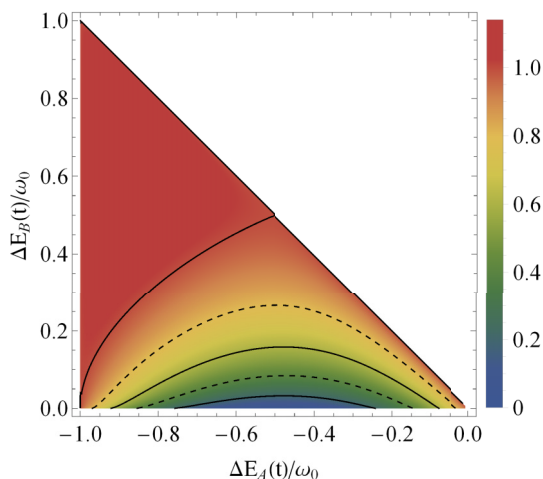


FIG. 6. Tightness $\Delta U^{xyz}(t)$ as functions of energy are plotted. The other settings are the same as those in Fig. 5.

dissipative bosonic environments. We studied two cases of the charging process, and the results show that the coupling enhancement during the Markovian charging process cannot only improve the charging power, but also increase the stored energy of the battery. Stronger coupling in non-Markovian systems can also significantly improve the charging power; in contrast, the battery can be efficiently full during the non-Markovian charging process. When the battery is in a steady state, maximum energy storage is achieved when the charger is more coupled to the reservoir than the battery in the case of a certain initial charger-battery entanglement. It is worth mentioning that a tighter entropic bound has a negative effect on energy storage in the presence of entanglement. We then studied the dynamic behavior of the entanglement and tightness in an arbitrary charging process, and the results revealed that an increase in entanglement can increase the energy transfer rate and degrade the energy storage. Remarkably, it is argued that the tightness of the entropic bound can be regarded as an important indicator of the optimal energy transfer during charging, and the tightest entropic bound corresponds to the complete energy transfer. We believe that our findings will be helpful for the pursuit of high-performance energy transfer in N -cell QB charging in the future.

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