# Ensemble model of turbulence based on states of constant flux in wavenumber space

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An ensemble model of turbulence is proposed. The ensemble consists of flow fields in which the flux of an inviscid conserved quantity, such as energy (or enstrophy in two-dimensional flow fields), across the wave number $k$ is a constant independent of $k$ in an appropriate range. Two-dimensional flow fields of constant enstrophy flux are sampled randomly by a Monte Carlo method. The energy spectra $E_k$ of the flow fields are consistent with the scaling $E_k \propto k^{-3}[\ln(k/k_\mathrm{b})]^{-1/3}$ where $k_\mathrm{b}$ is the bottom wave number of the constant-flux range.

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## I. INTRODUCTION

The motions of viscous fluids can be modeled by the Navier-Stokes (NS) equation. When the Reynolds number $\mathrm{Re} := UL/\nu$, where $U$ and $L$ are typical velocity and length, respectively, of the flow and $\nu$ is the kinematic viscosity, is very large, the individual solutions are sensitive to small disturbances and appear to be irregular in space and time. Motions of fluid in such a situation are called turbulence. It seems natural to employ the concept of statistics or probability in considering turbulence. However, the statistical theory of turbulence is far from being established as we discuss below.

Let us first recall the equilibrium statistical mechanics for comparison. The establishment of the statistical mechanics owes essentially to the ensemble picture which was first introduced in the consistent formalism by Gibbs [1]. See, e.g., Ref. [2] for a historical review. In the ensemble picture, a macroscopic state is modeled by an ensemble of microscopic states. Macroscopic quantities can be derived from the averages of corresponding microscopic quantities over the ensemble. For thermal equilibrium states, the ensemble models can be microcanonical, canonical, or grand canonical, and they can be defined by the Hamiltonian of the system and corresponding thermodynamic variables. Recently, the ensemble picture has been reviewed based on typicality and in the context of the thermalization of isolated quantum systems. See, e.g., Ref. [3] and the references therein. Although an alternative formalism based on so-called individualist views has been discussed especially for taking nonequilibrium processes into consideration [4,5], the ensemble picture provides the complete, concise, and most feasible computational tool of the statistical mechanics at least as far as thermal equilibrium states are concerned.

Since turbulence is a nonequilibrium state in the sense that there is a macroscopic flow of energy coming into the system by external forces and going out by viscosity, the ensemble model for thermal equilibrium states such as the microcanonical or canonical ensemble model cannot be applied. A mathematically rigorous choice of the ensemble is a stationary probability measure on the state space. The analysis related to the stationary probability measure for the NS turbulence is quite difficult, but see Ref. [6] for a recent related analysis on the passive scalar turbulence. There is an idea of representing the stationary probability measure by periodic orbits. See, e.g., Ref. [7]. The periodic orbits were searched numerically; however, the search becomes hard with the increase of the Reynolds number. When we consider the external forcing as a random field in space and time, the probability measure is attributed to trajectories in the state space. The Martin–Siggia–Rose–Janssen–de Dominicis procedure [8–10] may be used to treat the problem in a field-theoretic formalism. Recently, a nonperturbative renormalization group analysis has been attempted within the formalism [11]. The entropy method (EM) is one of the methods that treat the probability measure on the state space in an explicit manner [12]. The relation between EM and the model in this paper will be discussed in Sec. V. In statistical closure approaches, one abandons the idea of specifying the ensemble of states or trajectories and resorts to derive closed relations between low-order moments upon some assumptions for the approximation. Especially, Lagrangian spectral (two-point) closures such as the abridged Lagrangian history direct interaction approximation [13] and the Lagrangian renormalized approximation (LRA) [14] are capable of deriving the Kolmogorov spectrum up to the estimate of the universal constant. See Ref. [15] for a comprehensive review of the statistical closure approaches. Although turbulence has been studied from various aspects, it may be said that there is no established statistical theory of turbulence that can compare with the ensemble models in the equilibrium statistical mechanics as of now.

In this paper, we propose an ensemble model of turbulence expecting its potential to be one of the effective tools for the statistical theory of turbulence. The model incorporates the concept of cascade at the level of its construction. Here, the cascade means successive local transfers of an inviscid conserved quantity from large scales to small scales or vice versa. In the case of three-dimensional turbulence, the energy cascades from large scales to small scales as described by Richardson [16] as early as 1922. When the turbulence is at a stationary state in a statistical sense, the mean energy injection and dissipation rate $\epsilon_{\rm in}$ and $\epsilon_{\rm d}$, respectively, equilibrate and the energy flows with a constant flux independent of the scale $\ell$, i.e., $\Phi_\ell = \epsilon_{\rm in} = \epsilon_{\rm d}$, in the intermediate scale range called the inertial range, where $\Phi_\ell$ is the energy flux from the scales larger than $\ell$ to those smaller than $\ell$. The notion of universality in the turbulence statistics is that statistical quantities in the inertial range are irrelevant to the details of forcing and dissipation outside the range when the inertial range is sufficiently broad, i.e., the Reynolds number is very large. Kolmogorov's hypotheses of similarity claim that the mean energy dissipation rate $\epsilon_{\rm d}$ is the only relevant parameter [17]. Although the hypotheses have been denied in the context of intermittency (see, e.g., Ref. [18]), the significance of the parameter $\epsilon_{\rm d}$ still remains. Since $\epsilon_{\rm d}$ is a quantity associated with the small scales where the viscosity is dominant, it may be appropriate to put the energy flux $\Phi_\ell$, which is a quantity associated with the scales in the inertial range, at the center of the construction of the model and consider $\epsilon_{\rm d}$ as an external parameter. Note that it was pointed out by Onsager [19] that the energy dissipation could take place in the absence of viscosity and the modern analysis of the issue essentially involves the energy flux $\Phi_\ell$. See, e.g., Ref. [20]. In this paper, we formulate the ensemble model of states the energy flux of which is constant, i.e., $\Phi_\ell = \epsilon_{\rm d}$, for the scales $\ell$ in the inertial range. The formulation is given in the wave-vector space.

## II. SETTING OF THE SYSTEM

We consider an incompressible fluid in a $d$-dimensional domain $[0, L]^d$ with periodic boundary conditions, where $d \geqslant 2$ and usually $d = 3$. A state, symbolically denoted by $\boldsymbol{u}$, of the fluid is specified by an incompressible velocity vector field. Let $\boldsymbol{u}_{\boldsymbol{k}} := (2\pi)^{-d} \int_{[0,L]^d} d\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}) (\boldsymbol{k} \in \mathcal{K})$ denote the Fourier coefficients of the velocity field where $\mathcal{K}$ is a set of wave vectors $\mathcal{K} := \{(k_1, \ldots, k_d) | k_j = m\Delta k, m \in \mathbb{Z}, k < k_{\max}\} - \{\boldsymbol{0}\}$, $k := |\boldsymbol{k}|$, $\Delta k := 2\pi/L$, and the cutoff wave number $k_{\max}$ is introduced. The reality of $\boldsymbol{u}$ in the physical space implies $\boldsymbol{u}_{-\boldsymbol{k}} = \boldsymbol{u}_{\boldsymbol{k}}^*$, and the incompressible condition is given by $\boldsymbol{k} \cdot \boldsymbol{u}_{\boldsymbol{k}} = 0$. In the following, $a_j$ denotes the $j$th component of the vector $\boldsymbol{a}$ and the summation over repeated component indices is assumed.

The NS equation in the wave-vector space is given by

$$\frac{d}{dt}\boldsymbol{u}_{\boldsymbol{k}}(t) = \boldsymbol{M}_{\boldsymbol{k}}(\boldsymbol{u}(t)) - \nu k^2 \boldsymbol{u}_{\boldsymbol{k}}(t) + \boldsymbol{f}_{\boldsymbol{k}}(t), \tag{1}$$

where the mass density of the fluid is unity, $\nu$ is the kinematic viscosity constant, $\boldsymbol{f}(t)$ is the external forcing field, $\boldsymbol{M}$ is a map from a vector field to a vector field the component of which is given by

$$M_{\boldsymbol{k},j}(\boldsymbol{u}) = -\frac{i}{2} \sum_{\boldsymbol{p}}^{\Delta} \sum_{\boldsymbol{q}}^{\Delta} \delta^{\Delta}_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}} \left[ k_m \left( \delta_{jn} - \frac{k_j k_n}{k^2} \right) + k_n \left( \delta_{jm} - \frac{k_j k_m}{k^2} \right) \right] u_{\boldsymbol{p},m} u_{\boldsymbol{q},n}, \tag{2}$$

$\sum_{\boldsymbol{k}}^{\Delta} := \sum_{\boldsymbol{k}\in\mathcal{K}} (\Delta k)^d$, $\delta^{\Delta}_{\boldsymbol{k}} = (\Delta k)^{-d}$ for $\boldsymbol{k} = \boldsymbol{0}$ and $\delta^{\Delta}_{\boldsymbol{k}} = 0$ otherwise, and $\delta_{jm}$ is the Kronecker delta.

The energy density per unit volume, or simply energy hereafter, $E(\boldsymbol{u})$, is given by

$$E(\boldsymbol{u}) = \sum_{\boldsymbol{k}}^{\Delta} E_{\boldsymbol{k}}(\boldsymbol{u}), \quad E_{\boldsymbol{k}}(\boldsymbol{u}) := \tfrac{1}{2}(\Delta k)^d |\boldsymbol{u}_{\boldsymbol{k}}|^2, \tag{3}$$

where $E_{\boldsymbol{k}}(\boldsymbol{u})$ is the energy for the wave-vector mode $\boldsymbol{k}$. Hereafter, let $\boldsymbol{u}(t)$ denote the solution of (1) with $\nu = 0$, $\boldsymbol{f}(t) = \boldsymbol{0}$, and the initial condition $\boldsymbol{u}$ at $t = 0$. The energy flux $\Phi_k(\boldsymbol{u})$ from the small-wave-number region $\{\boldsymbol{p}|p < k\}$ to the large-wave-number region $\{\boldsymbol{p}|p \geqslant k\}$ due to the interaction represented by $\boldsymbol{M}$ is given by

$$\Phi_k(\boldsymbol{u}) := -(\Delta k)^d \sum_{\boldsymbol{p}(p<k)}^{\Delta} \mathrm{Re}[\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{u}) \cdot \boldsymbol{u}_{-\boldsymbol{p}}]. \tag{4}$$

## III. ENSEMBLE MODEL

An ensemble of states is specified by a probability density function $P(\boldsymbol{u})$ which satisfies $P(\boldsymbol{u}) \geqslant 0$ and $\int \mathcal{D}\boldsymbol{u} P(\boldsymbol{u}) = 1$, where $\mathcal{D}\boldsymbol{u} := \prod_{\boldsymbol{k}\in\mathcal{K}^+} d\boldsymbol{u}_{\boldsymbol{k}}$, $d\boldsymbol{u}_{\boldsymbol{k}} := \prod_{j=1}^{d-1} d\mathrm{Re}(u^{(j)}_{\boldsymbol{k}}) d\mathrm{Im}(u^{(j)}_{\boldsymbol{k}})$, $\mathcal{K}^+ (\subset \mathcal{K})$ is a set of wave vectors such that either $\boldsymbol{k} \in \mathcal{K}^+$ or $-\boldsymbol{k} \in \mathcal{K}^+$ but not both for all $\boldsymbol{k} \in \mathcal{K}$, and $u^{(j)}_{\boldsymbol{k}} := \boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{e}^{(j)}(\boldsymbol{k})$ with $\boldsymbol{e}^{(j)}(\boldsymbol{k}) (j = 1, \ldots, d-1)$ being an orthonormal basis of the $(d-1)$-dimensional complex vector space perpendicular to $\boldsymbol{k}$. The ensemble average of a function of the state $F(\boldsymbol{u})$ is given by $\langle F(\boldsymbol{u})\rangle := \int \mathcal{D}\boldsymbol{u} P(\boldsymbol{u}) F(\boldsymbol{u})$.

We propose as an ensemble model of turbulence the following probability density function:

$$P_\epsilon(\boldsymbol{u}) := C \prod_{n=0}^{N_t} \prod_{m=0}^{N_k} \delta\big(\Phi_{k_m}(\boldsymbol{u}(t_n)) - \epsilon\big), \tag{5}$$

where $\delta(x)$ is the Dirac delta function, $C$ is the constant for the normalization of probability, $\epsilon$ is a constant corresponding to the energy dissipation rate, $0 < k_0 < \ldots < k_{N_k} < k_{\max}$, and $0 = t_0 < t_1 \ldots < t_{N_t}$. Formally, by taking limits $N_k, N_t \to \infty$ with $k_{\max}, k_{N_k}, t_{N_t} \to \infty$ and $\min_m(k_{m+1} - k_m)$, $\min_n(t_{n+1} - t_n) \to 0$, one obtains a stationary ensemble model of states with the constant energy flux, $\Phi_k = \epsilon$ for $k \geqslant k_0$.

In the ensemble model $P_\epsilon(\boldsymbol{u})$, the states are subject to the conditions $\Phi_{k_m}(\boldsymbol{u}(t_n)) = \epsilon$ and the probability is distributed equally to the possible states in the sense that there is no other constraint. The model is similar to the microcanonical ensemble in which the states are subject to the condition that the energy is equal to a specific value. Behind the construction of the present ensemble model underlies the concept of typicality. The typicality implies that typical states (i.e., almost all states) $\boldsymbol{u}$ in the ensemble already possess some properties of the ensemble average, i.e., $F(\boldsymbol{u}) \approx \langle F(\boldsymbol{u})\rangle$ for the functions $F(\boldsymbol{u})$ of interest. It is supposed in the present ensemble model that each of the states of constant flux such that $\Phi_k(\boldsymbol{u}(t)) = \epsilon$ for $k$ in the inertial range and $t$ in the time interval under consideration typically possesses a considerable part of the characteristics of turbulence. Note that a quasiconstant flux $\Phi_k(\boldsymbol{u}(t)) \approx \epsilon$ is observed in many direct numerical simulations of the NS turbulence in the periodic boundary box, although the inertial range is limited. (See, e.g., Ref. [21].) The fact suggests that the constant flux in the inertial range is one of the essential characteristics of fully developed turbulence. The present ensemble model $P_\epsilon(\boldsymbol{u})$ would be appropriate if a considerable part of the other characteristics of turbulence can be derived from the property of constant flux.

In spite of $P_\epsilon(\boldsymbol{u})$ being a probability density function on the state space, the trajectory $\boldsymbol{u}(t)$ is explicitly involved in the expressions of Eq. (5). For the sake of simplicity, let us replace $\Phi_k(\boldsymbol{u}(t))$ in (5) by its $N_t$th degree Taylor polynomial in $t$. Then, we may rewrite (5) as

$$P_\epsilon^{(N_t)}(\boldsymbol{u}) = C' \prod_{m=0}^{N_k} \prod_{n=0}^{N_t} \delta\big(\Phi_{k_m}^{(n)}(\boldsymbol{u}) - \epsilon\delta_{n0}\big), \tag{6}$$

where

$$\Phi_k^{(n)}(\boldsymbol{u}) := \frac{d^n}{dt^n}\Phi_k(\boldsymbol{u}(t))\bigg|_{t=0}, \tag{7}$$

$C'$ is a normalizing constant, and we now write $N_t$ explicitly in the superscript for this approximation. The expression (6) solely contains the instantaneous $\boldsymbol{u}$. The limit $N_t \to \infty$ should be taken in order that $P_\epsilon^{(N_t)}(\boldsymbol{u})$ is stationary.

Although the model of the ensemble is explicitly given in Eq. (5) or (6), there are some problems regarding the appropriateness of the model. The existence of normalizing constants such that $C, C' > 0$ for fixed $N_k$ and $N_t$ is not clear. The suitable way of taking the limit $N_k, N_t \to \infty$ should be also discussed.

Even if the problems of the appropriateness are solved or avoided in some way, computation of the ensemble average of quantities such as $E_{\boldsymbol{k}}(\boldsymbol{u})$ is difficult for $P_\epsilon(\boldsymbol{u})$ or $P_\epsilon^{(N_t)}(\boldsymbol{u})$ even with $N_t = 0$. This is because $\Phi_k(\boldsymbol{u})$ in Eq. (5) or (6) consists of third-order terms in $\boldsymbol{u}$ such as $\boldsymbol{u_k u_p u_q}\delta^{\Delta}_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}$ and because $\boldsymbol{u_k}$ with different wave vectors $\boldsymbol{k}$ are complexly coupled in $\Phi_k(\boldsymbol{u})$. It is desired to develop some analytical methods for the computation. One candidate may be a method similar to the Martin–Siggia–Rose–Janssen–de Dominicis procedure [8–10]. The model $P_\epsilon(\boldsymbol{u})$ can be expressed in a form that may be more familiar in the field theory by using auxiliary variables $\lambda_{k_m,t_n}$, as

$$P_\epsilon(\boldsymbol{u}) = C''\left(\prod_{n=0}^{N_t}\prod_{m=0}^{N_k}\int_{-\infty}^{\infty} d\lambda_{k_m,t_n}\right) \times \exp\left(\sum_{m=0}^{N_t}\sum_{n=0}^{N_k} i\lambda_{k_m,t_n}\big[\Phi_{k_m}(\boldsymbol{u}(t_n)) - \epsilon\big]\right), \tag{8}$$

where $C''$ is the normalizing constant. One may also consult Ref. [22] for the treatment of probability measures with constraints imposed in the form of the Dirac delta function. However, we will not pursue such analytical methods further in this paper.

## IV. NUMERICAL SAMPLING

If typicality applies to the present ensemble model, some properties of turbulence should be possessed by a single typical state in the ensemble before taking the average. Here, we attempt a random sampling from the ensemble model by a Monte Carlo (MC) method.

For a first trial, we treat the case with $d = 2$ for saving the computational resource. In the case of $d = 2$, the enstrophy

$$\Omega(\boldsymbol{u}) := (\Delta k)^d \frac{1}{2}\sum_{\boldsymbol{k}}^{\Delta} |\omega_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k}}^{\Delta} k^2 E_{\boldsymbol{k}}(\boldsymbol{u}), \tag{9}$$

where $\omega_{\boldsymbol{k}} := i(k_1 u_{\boldsymbol{k},2} - k_2 u_{\boldsymbol{k},1})$ is the vorticity field, is an inviscid conserved quantity as well as the energy. Here, we consider the enstrophy cascade range. The enstrophy flux $\Phi_k^{\Omega}(\boldsymbol{u})$, its time derivatives $\Phi^{\Omega}{}_k^{(n)}(\boldsymbol{u})$, and the probability density function $P_\eta^{(N_t)}(\boldsymbol{u})$ of the constant-enstrophy-flux ensemble model, where $\eta$ is a constant corresponding to the enstrophy dissipation rate, can be defined similarly as in the case of $\Phi_k(\boldsymbol{u})$ in Eq. (4), $\Phi_k^{(n)}$ in Eq. (7), and $P_\epsilon^{(N_t)}(\boldsymbol{u})$ in Eq. (6), respectively.

Let us define the error functions by

$$\Delta^{(n)}(\boldsymbol{u}) := \frac{1}{N_k+1}\sum_{m=0}^{N_k}\big[\Phi^{\Omega}{}_{k_m}^{(n)}(\boldsymbol{u}) - \eta\delta_{n0}\big]^2, \tag{10}$$

for $n = 0, \dots, N_t$. The MC step associated with $\boldsymbol{k}(\in \mathcal{K}^+)$, which updates a given state $\boldsymbol{u}$ to a new one, is given by the following substeps.

(1) Let

$$\boldsymbol{u}'_{\boldsymbol{k}} = \boldsymbol{u_k}\exp(re^{i\theta}), \quad \boldsymbol{u}'_{-\boldsymbol{k}} = (\boldsymbol{u}'_{\boldsymbol{k}})^*, \tag{11}$$

and $\boldsymbol{u}'_{\boldsymbol{p}} = \boldsymbol{u_p}$ for $\boldsymbol{p} \neq \pm\boldsymbol{k}$, where $r$ is a fixed parameter satisfying $0 < r < 1$ and $\theta$ is a uniform random variable on $[0, 2\pi)$.

(2) Accept $\boldsymbol{u}'$ as the new state of $\boldsymbol{u}$ with the probability

$$T(\boldsymbol{u}', \boldsymbol{u}) = \left(\prod_{n=0}^{N_t}\min(e^{-\alpha^{(n)}(\Delta^{(n)}(\boldsymbol{u}')-\Delta^{(n)}(\boldsymbol{u}))}, 1)\right) \times \min\left(\frac{|\boldsymbol{u}'_{\boldsymbol{k}}|^2}{|\boldsymbol{u_k}|^2}, 1\right), \tag{12}$$

and keep $\boldsymbol{u}$ unchanged otherwise, where $\alpha^{(n)} (n = 0, \dots, N_t)$ are parameters satisfying $0 \leqslant \alpha^{(n)} \leqslant \infty$. Since the typical scale of $\boldsymbol{u_k}$ is not known *a priori*, we set a uniform step amplitude $r$ in $(\ln \boldsymbol{u_k})$ space. The transition probability $T(\boldsymbol{u}', \boldsymbol{u})$ is that of the Metropolis algorithm with a modification factor due to the nonuniform step in $\boldsymbol{u_k}$ space. The stationary probability density function concerning the MC steps for all $\boldsymbol{k} \in \mathcal{K}^+$ satisfies

$$P_{\eta,\mathrm{MC}}^{(N_t)}(\boldsymbol{u}) \propto e^{-\sum_{n=0}^{N_t}\alpha^{(n)}\Delta^{(n)}(\boldsymbol{u})}, \tag{13}$$

and $P_{\eta,\mathrm{MC}}^{(N_t)}(\boldsymbol{u})$ tends to $P_\eta^{(N_t)}(\boldsymbol{u})$ in the limit $\alpha^{(n)} \to \infty$.

In this paper, we deal with the ensemble model $P_\eta^{(0)}(\boldsymbol{u})$. The numerical settings are as follows. The number of grid points in the periodic domain $[0, 2\pi]^2$ is $N^2$. A Fourier spectral method with a phase shift is used for the computation of the nonlinear terms and the maximum wave number is $k_{\max} = \sqrt{2}N/3$. The initial state of $\boldsymbol{u}$ is generated randomly with the constraint that the enstrophy is equally distributed to all wave-number modes, i.e., $\boldsymbol{u_k} = (\beta_0)^{-1/2}k^{-1}\exp(i\theta_{\boldsymbol{k}})\boldsymbol{e}^{(1)}(\boldsymbol{k})$ with $0 < \beta_0 < \infty$ and $\theta_{\boldsymbol{k}}$ being uniform random variables on $[0, 2\pi)$. An MC cycle is defined by the performance of the MC steps associated with $\boldsymbol{k}$ for all $\boldsymbol{k} \in \mathcal{K}^+$ in the order of increasing

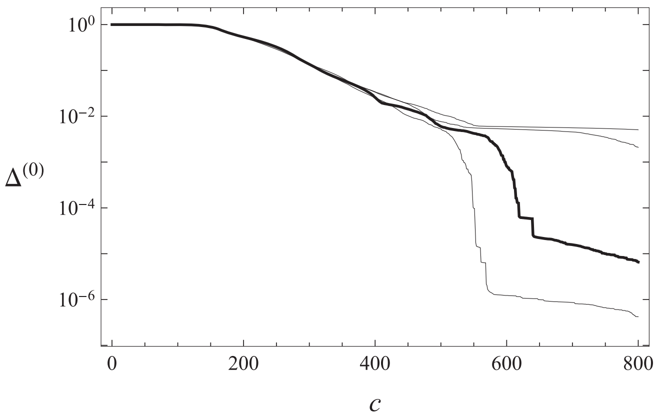


FIG. 1. The error function $\Delta^{(0)}(\boldsymbol{u})$ of the realized states $\boldsymbol{u}$ in the Monte Carlo cycles as a function of the number of cycles $c$. The thick line corresponds to the sequence SEQ0 and the thin lines correspond to the other sequences, SEQ1 to SEQ3.

$k$. The values of the parameters are $N = 512$, $N_k = 239$, $k_m = m + 1.5 (0 \leqslant m \leqslant N_k)$, $\eta = 1$, $r = 0.0625$, $\alpha^{(0)} = \infty$, and $\beta_0 = 10^8$. Four sequences of MC cycles with different random seeds, SEQ0 to SEQ3, are performed up to $c = 800$ where $c$ is the number of MC cycles. We can confirm that the error function $\Delta^{(0)}(\boldsymbol{u})$ decreases with the increase of $c$ in Fig. 1. The enstrophy fluxes $\Phi_k^{\Omega}(\boldsymbol{u})$ of the states $\boldsymbol{u}$ obtained by the MC method are also shown in Fig. 2(a). It is observed that the wave-number region such that $\Phi_k^{\Omega}(\boldsymbol{u}) \approx \eta$ expands from large to small wave numbers with the increase of $c$. The relative discrepancy from the constant flux $|\Phi_k^{\Omega}(\boldsymbol{u}) - \eta|/\eta$ is smaller than 0.025 in the wave-number range $5.5 \leqslant k \leqslant 240.5$ at $c = 800$ for all the sequences.

The energy spectrum of the state $\boldsymbol{u}$ defined by

$$E_k(\boldsymbol{u}) := (\Delta k)^{-1} \sum_{\substack{\boldsymbol{p} \\ (k-\Delta k/2 \leqslant p < k+\Delta k/2)}}^{\Delta} E_{\boldsymbol{p}}(\boldsymbol{u}) \tag{14}$$

is given for the states obtained by the MC method in Fig. 2(b). It is found that $E_k(\boldsymbol{u})$ tends to converge with the increase of $c$ and that $E_k(\boldsymbol{u})$ at $c = 800$ in all the sequences are close to the energy spectrum in the enstrophy cascade range, $E_k = C_K \eta^{2/3} k^{-3} [\ln(k/k_{\mathrm{b}})]^{-1/3}$ with $C_K = 1.81$, estimated in the LRA [23,24], where we put the bottom wave number of the inertial range as $k_{\mathrm{b}} = 1$. It is known that $E_k$ in the LRA is in good agreement with the results from the numerical simulations [25]. See also the Appendix. The ratios of the energy spectra by the MC method $E_k^{(\mathrm{MC})}(\boldsymbol{u})$ at $c = 800$ to that of the LRA $E_k^{(\mathrm{LRA})}$ are confined in the range $0.4 < E_k^{(\mathrm{MC})}(\boldsymbol{u})/E_k^{(\mathrm{LRA})} < 7.5$ for the wave-number range $3 \leqslant k \leqslant 239$. A general tendency is that the ratio increases near the edge $k = k_{\max}$. The vorticity field in the real space of a state obtained at $c = 800$ is given in Fig. 3(a) together with the phase-randomized field with the same energy $E_{\boldsymbol{k}}$ for each wave vector $\boldsymbol{k}$ as the original field in Fig. 3(b). One can observe some organized structures with intense vorticity in the state obtained from the MC method. The maximum absolute value of the vorticity is five times larger than that of the phase-randomized field. Since the structures are absent in the phase-randomized field, it is suggested that the emergence of the structures is due to the constraint of constant enstrophy flux. Apparent anisotropy observed in both fields may be a sign of prominent amplitudes of some specific wave-vector modes.

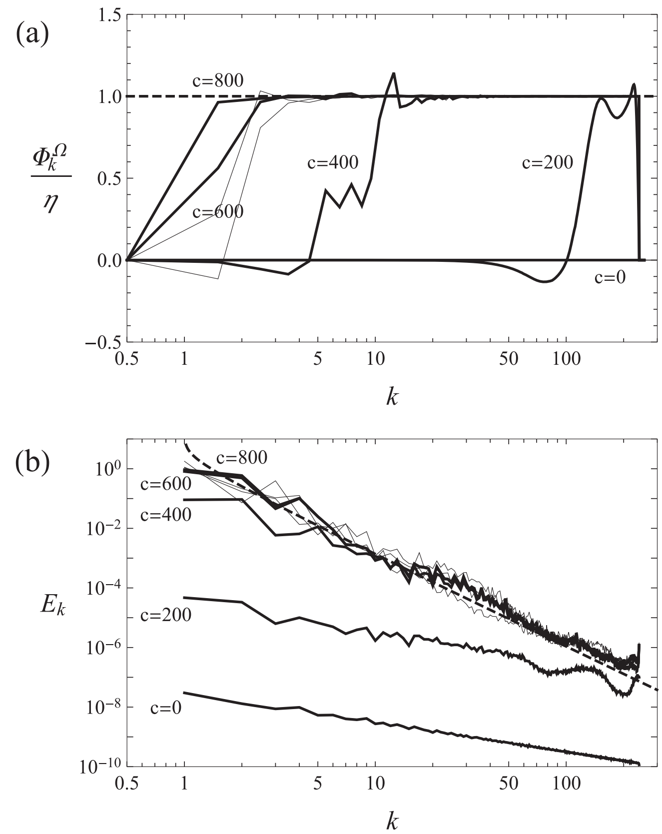


FIG. 2. (a) The enstrophy flux $\Phi_k^{\Omega}(\boldsymbol{u})$ normalized by the parameter $\eta$ corresponding to the enstrophy dissipation rate and (b) the energy spectrum $E_k(\boldsymbol{u})$, of the realized states $\boldsymbol{u}$ in the sequence SEQ0 of the Monte Carlo cycles where $c$ is the number of the cycles (thick lines). The thin lines show results at $c = 800$ from the other sequences, SEQ1 to SEQ3. The dashed line in (b) shows $E_k = C_K \eta^{2/3} k^{-3} [\ln(k/k_{\mathrm{b}})]^{-1/3}$ with $C_K = 1.81$ (LRA) and $k_{\mathrm{b}} = 1$.

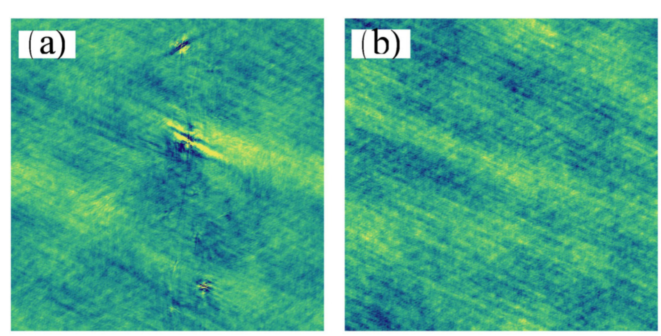


FIG. 3. (a) Vorticity field in the real space of the state $\boldsymbol{u}$ obtained by the Monte Carlo method at $c = 800$ of SEQ0. (b) Phase randomized vorticity field with the same energy $E_{\boldsymbol{k}}$ for each wave vector $\boldsymbol{k}$ as (a). Bright (dark) regions correspond to positive-vorticity (negative-vorticity) regions.

## V. DISCUSSIONS

We proposed an ensemble model of turbulence $P_\epsilon(\boldsymbol{u})$ which can be constructed explicitly by Eq. (5). Since the probability measure is uniformly distributed for the states of constant flux with the value of flux $\epsilon$, the ensemble model $P_\epsilon(\boldsymbol{u})$ maximizes the entropy $S(P) = -\int \mathcal{D}\boldsymbol{u} P(\boldsymbol{u}) \ln P(\boldsymbol{u})$ within the constraint that $P(\boldsymbol{u}) \neq 0$ only for $\boldsymbol{u}$ being one of the states of constant flux. Here we recall the entropy method for turbulence proposed by Edwards and McComb [12]. The model probability density function $P(\boldsymbol{u})$ in the method is parametrized by $\phi_l$ and $\eta_l$, where $l$ runs over all possible components of the velocity field modes $\boldsymbol{u}_{\boldsymbol{k},j}$. (See Ref. [26] for the notations.) The parameters $\phi_l$ and $\eta_l$ are associated with the intensity and damping rate, respectively, of the corresponding mode $\boldsymbol{u}_{\boldsymbol{k},j}$. The model $P(\boldsymbol{u})$ in EM is obtained by a perturbation from a multivariate normal distribution such that $P(\boldsymbol{u}) > 0$ for all $\boldsymbol{u}$ provided that $\phi_l \neq 0$ for all $l$. The model $P_\epsilon(\boldsymbol{u})$ in Eq. (5) is quite different from the multivariate normal distribution since $P(\boldsymbol{u}) = 0$ for all $\boldsymbol{u}$ unless $\boldsymbol{u}$ is one of the states of constant flux. Therefore, it is likely that $P_\epsilon(\boldsymbol{u})$ is not included in the class of probability density function $P(\boldsymbol{u})$ considered in EM. We do not attempt the proof here. It is argued in Ref. [27] that the maximum entropy state under the constraint of constant flux does not exist in the formalism of EM. It is not obvious that the argument can be extended to the present model, $P_\epsilon(\boldsymbol{u})$ in Eq. (5). In the context of the present paper, the existence of $P_\epsilon(\boldsymbol{u})$ that maximizes the entropy under the constraint would be related to the existence of the normalizing constant $C > 0$ that is referred to in Sec. III. The related analysis will be left for a future study.

One method for the validation of the present ensemble model $P_\epsilon(\boldsymbol{u})$ in Eq. (5) is to compute the ensemble averages of some quantities and then compare them with known results in the turbulence statistics. However, the analytical methods for the computation are yet to be developed even for the approximate expression $P_\epsilon^{(0)}(\boldsymbol{u})$ in Eq. (6). The method we adopted in this paper is the numerical sampling from the ensemble, which is rather accessible.

It should be noted that the numerical sampling in this paper is at a beginning stage and that the possibility of some bias in the sampling is not excluded. We cannot conclude whether the anisotropy in the sampled vorticity field is a genuine feature of the constant flux states or an artifact of the sampling at the current stage. In the present analysis, the amplitude of the initial states is chosen to be small so that the energy spectra $E_k(\boldsymbol{u})$ of the constant flux states are approached from below. The MC sequences with an initially large amplitude are not satisfactory so far. Shortage of the analysis aside, it is remarkable that the energy spectra $E_{\boldsymbol{k}}(\boldsymbol{u})$ of the states sampled out from the ensemble $P_\eta^{(0)}(\boldsymbol{u})$ of the constant enstrophy flux states in two-dimensional turbulence are consistent with the form $E_{\boldsymbol{k}} = C_K \eta^{2/3} k^{-3} [\ln(k/k_{\mathrm{b}})]^{-1/3}$ that is obtained in the closure theories and verified in the numerical simulations of the NS equation. A positive prospect is that the ensemble model $P_\eta^{(N_t)}(\boldsymbol{u})$ and the associated random sampling can be useful to analyze some turbulence statistics at relatively low approximation levels such as $N_t = 0$ or 1. Although the constraint of the constant flux $\Phi^{\Omega}{}_k = \eta$ in wave-number space, i.e., the ensemble model $P_\eta^{(0)}(\boldsymbol{u})$, yields some spatial structures of the vorticity field, they do not resemble those in the numerical simulations of the two-dimensional turbulence. See Appendix or, e.g., Ref. [28]. It would be of interest to investigate how the further constraints on the time derivatives of the flux $\Phi^{\Omega}{}_k^{(N_t)} (N_t \geqslant 1)$, i.e., the applications of the ensemble models $P_\eta^{(N_t)} (N_t \geqslant 1)$, affect the structures of the vorticity field as well as the spectrum and higher-order moments.

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## APPENDIX: NUMERICAL SIMULATIONS OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATION

We performed numerical simulations of two-dimensional turbulence with random forcing and hyperviscosity in a periodic boundary box. Basically, we followed the setting of the simulations in Ref. [25]. The governing equation of the simulations is given by

$$\frac{\partial}{\partial t}\omega_{\boldsymbol{k}} = J_{\boldsymbol{k}} + d_{\boldsymbol{k}} + f_{\boldsymbol{k}}, \tag{A1}$$

where $J_{\boldsymbol{k}}$ is the Fourier transform of the Jacobian given by $J(\psi, \omega) = \partial_1\psi\partial_2\omega - \partial_2\psi\partial_1\omega$ in the physical space, $\psi_{\boldsymbol{k}}$ is the stream function related to the velocity field by $\boldsymbol{u}_{\boldsymbol{k}} = (ik_2\psi_{\boldsymbol{k}}, -ik_1\psi_{\boldsymbol{k}})$ and to the vorticity field by $\omega_{\boldsymbol{k}} = k^2\psi_{\boldsymbol{k}}$, $d_{\boldsymbol{k}}$ is the dissipation term, and $f_{\boldsymbol{k}}$ is the forcing term. The nonlinear term $J_{\boldsymbol{k}}$ is computed in the same way as the numerical sampling in Sec. IV. The dissipation term $d_{\boldsymbol{k}}$ is given by

$$d_{\boldsymbol{k}} = -\nu(2\Omega)^{\frac{1}{2}}\left(\frac{k}{k_{\max}}\right)^6 \omega_{\boldsymbol{k}} - \gamma\,\chi_{(0,k_\gamma)}(k)\omega_{\boldsymbol{k}}, \tag{A2}$$

where $\nu$ is the coefficient for the hyperviscosity, $\Omega$ is the enstrophy calculated at every time step, $\gamma$ is the coefficient of the drag applied in the wave-number range $0 < k < k_\gamma$, and $\chi_A(k)$ is a function such that $\chi_A(k) = 1$ for $k \in A$ and $\chi_A(k) = 0$ otherwise. The drag in the small-wave-number range prevents the accumulation of the energy that cascades inversely into the range. The forcing term $f_{\boldsymbol{k}}$ is given by

$$f_{\boldsymbol{k}} = \chi_{[k_{\mathrm{fmin}},k_{\mathrm{fmax}})}(k)(2\eta_{\mathrm{f}})^{\frac{1}{2}} N_{\mathrm{f}}^{-\frac{1}{2}} (\Delta t)^{-\frac{1}{2}} (\Delta k)^{-2} e^{i\varphi_{\boldsymbol{k}}}, \tag{A3}$$

where $\eta_{\mathrm{f}}$ is the average enstrophy injection rate by the forcing, $\Delta t$ is the time increment in the simulations, $N_{\mathrm{f}}$ is the number of wave vectors $\boldsymbol{k}$ satisfying $k_{\mathrm{fmin}} \leqslant k < k_{\mathrm{fmax}}$, $\varphi_{\boldsymbol{k}} (\boldsymbol{k} \in \mathcal{K}')$ is a uniform random variable on $[0, 2\pi)$ generated at every time step, and $f_{-\boldsymbol{k}} = f_{\boldsymbol{k}}^*$.

The values of the parameters in the simulation are as follows. The number of grid points along one coordinate direction is $N = 4096$, the length of the sides of the domain is $L = 2\pi$ which implies $\Delta k = 1$, $k_{\max} = (\sqrt{2}N/3)\Delta k = 1931$, $\Delta t = 0.16 \times 10^{-3}$, $\nu = 1.0$, $\gamma = 1.0$, $k_\gamma = 2.5$, $\eta_{\mathrm{f}} = 1.0$, $k_{\mathrm{fmin}} = 4.5$, and $k_{\mathrm{fmax}} = 7.5$. The initial state $\omega_{\boldsymbol{k}}(t = 0)$ was generated under the conditions $|\omega_{\boldsymbol{k}}(t =$

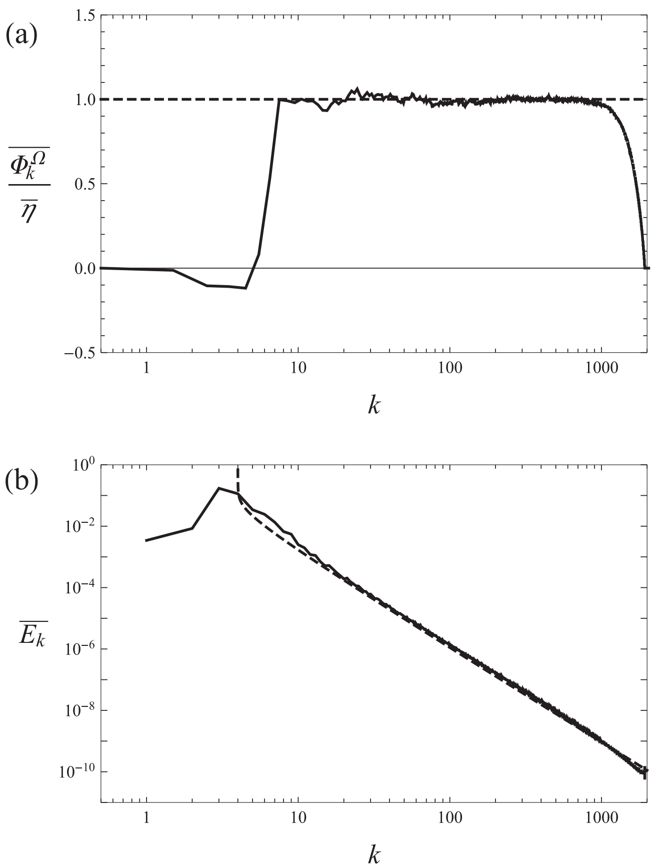


FIG. 4. (a) The time-averaged enstrophy flux $\overline{\Phi_k^\Omega}$ normalized by the time-averaged enstrophy dissipation rate $\overline{\eta}$ and (b) the time-averaged energy spectrum $\overline{E_k}$ in the numerical simulation. The dashed line in (b) shows $E_k = C_K \eta^{2/3} k^{-3}[\ln(k/k_\mathrm{b})]^{-1/3}$ with $C_K = 1.81$ (LRA) and $k_\mathrm{b} = 4.0$.

$0)| \propto k^2 \exp(-3k^2/2k_a^2)$, $k_a = 8$, and the enstrophy $\Omega(t = 0) = 1$. The phase of $\omega_{\boldsymbol{k}}(t = 0)$ was determined randomly.

The enstrophy dissipation rate $\eta(t)$ become quasistationary for $t \gtrsim 64$. Hereafter, $\overline{x}$ denotes the time average of $x$ in the time interval $64 \leqslant t \leqslant 128$. It is observed that $\overline{\eta} = 0.867$ and the normalized standard deviation $\overline{[(\eta - \overline{\eta})^2]}^{1/2}/\overline{\eta}$ is 0.063. The time-averaged enstrophy flux $\overline{\Phi_k^\Omega}$ is given in Fig. 4(a). The normalized deviation of the time-averaged enstrophy flux from the time-averaged enstrophy dissipation rate $|\overline{\Phi_k^\Omega} - \overline{\eta}|/\overline{\eta}$ is smaller than 0.05 in the wave-number range $26 \leqslant k \leqslant 1100$. Here, we consider the wave-number range as the enstrophy cascade range. The normalized standard deviation $[\overline{(\Phi_k^\Omega - \overline{\Phi_k^\Omega})^2}]^{1/2}/\overline{\Phi_k^\Omega}$ is smaller than 0.25 in the enstrophy cascade range.

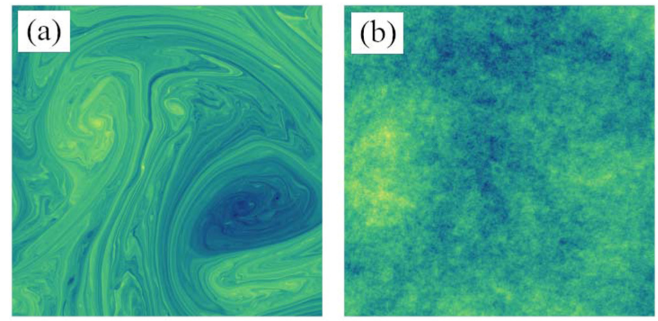


FIG. 5. (a) Vorticity field in a subdomain with sides $\pi/2 \times \pi/2$ of the real space at $t = 128$ of the simulation. (b) Phase randomized vorticity field with the same energy $E_{\boldsymbol{k}}$ for each wave vector $\boldsymbol{k}$ as (a). Bright (dark) regions correspond to positive-vorticity (negative-vorticity) regions.

The time-averaged energy spectrum $\overline{E_k}$ is given in Fig. 4(b). The normalized standard deviation $[\overline{(E_k - \overline{E_k})^2}]^{1/2}/\overline{E_k}$ is smaller than 0.15 in the enstrophy cascade range. The slope of $\overline{E_k}$ is slightly steeper than $k^{-3}$ and the functional form of $E_k(C_K, k_\mathrm{b}) = C_K \overline{\eta}^{2/3} k^{-3}[\ln(k/k_\mathrm{b})]^{-1/3}$ can be fitted to $\overline{E_k}$ with $C_K = 2.20$ and $k_\mathrm{b} = 4.0$ by minimizing the function $\sum_k \{\ln[E_k(C_K, k_\mathrm{b})/\overline{E_k}]\}^2(\Delta k/k)$ where the summation is taken over $k$ in the enstrophy cascade range. The estimate of $C_K^\mathrm{LRA} = 1.81$ in the LRA is in good agreement with that of the simulation in the sense that $|C_K^\mathrm{LRA} - C_K|/C_K < 0.2$. The energy spectrum $E_k^\mathrm{LRA}$ for the enstrophy cascade range estimated in LRA with $k_\mathrm{b} = 4.0$ is also plotted with a dashed line in Fig. 4(b). Note that $\overline{E_k} \propto k^{-3}$ without logarithmic correction may be observed when there is a sufficient amount of energy in the wave-number range $k < k_\mathrm{b}$. See Ref. [29] for details. Since the energy outside the inertial range is not considered in the numerical sampling in Sec. IV, the present setting of the numerical simulation of the NS equation and the spectrum with the logarithmic correction may be appropriate for comparison. The vorticity field in the real space is given in Fig. 5(a) for the simulated field at $t = 128$. A subdomain with sides $\pi/2 \times \pi/2$ is displayed as a representative. One can observe stretched and folded structures of isovorticity regions. The structures disappear in the phase-randomized vorticity fields with the same energy $E_{\boldsymbol{k}}$ for each wave vector $\boldsymbol{k}$ as shown in Fig. 5(b).


[1] J. W. Gibbs, *Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundation of Thermodynamics* (Charles Scribner's Sons, New York, 1902).
[2] H. Inaba, *The Making of Statistical Mechanics* (The University of Nagoya Press, Nagoya, 2021) (in Japanese).
[3] H. Tasaki, J. Stat. Phys. **163**, 937 (2016).
[4] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, in *Statistical Mechanics and Scientific Explanation, Determinism, Indeterminism and Laws of Nature*, edited by V. Allori (World Scientific, Singapore, 2020), pp. 519–581.
[5] K. Yoshida, Phys. Rev. A **101**, 032110 (2020).
[6] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith, Commun. Pure Appl. Math. **75**, 1237 (2022).
[7] G. Kawahara and S. Kida, J. Fluid Mech. **449**, 291 (2001).
[8] P. Martin, E. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).
[9] H.-K. Janssen, Z. Phys. B **23**, 377 (1976).
[10] C. De dominicis, J. Phys. Colloques **37**, C1-247 (1976).

[11] L. Canet, B. Delamotte, and N. Wschebor, Phys. Rev. E **93**, 063101 (2016).
[12] S. Edwards and W. McComb, J. Phys. A: Gen. Phys. **2**, 157 (1969).
[13] R. H. Kraichnan, Phys. Fluids **8**, 575 (1965).
[14] Y. Kaneda, J. Fluid Mech. **107**, 131 (1981).
[15] Y. Zhou, Phys. Rep. **935**, 1 (2021).
[16] L. F. Richardson, *Weather Prediction by Numerical Processes* (Cambridge University, Cambridge, England, 1922), p. 66.
[17] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941) [Proc. R. Soc. A **434**, 9 (1991)].
[18] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University, Cambridge, England, 1995).
[19] L. Onsager, Nuovo Cim **6**, 279 (1949).
[20] G. L. Eyink, arXiv:1803.02223 (2018).
[21] T. Ishihara, K. Morishita, M. Yokokawa, A. Uno, and Y. Kaneda, Phys. Rev. Fluids **1**, 082403(R) (2016).
[22] V. Šverák, in *Vector-Valued Partial Differential Equations and Applications*, edited by J. Ball and P. Marcellini (Springer, New York, 2017), pp. 195–248.
[23] Y. Kaneda, Phys. Fluids **30**, 2672 (1987).
[24] Y. Kaneda, Fluid Dyn. Res. **39**, 526 (2007).
[25] T. Ishihara and Y. Kaneda, Phys. Fluids **13**, 544 (2001).
[26] D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon, Oxford, 1973), Chap. 7.
[27] J. Qian, J. Phys. A: Math. Gen. **29**, 1305 (1996).
[28] G. Boffetta and R. E. Ecke, Annu. Rev. Fluid Mech. **44**, 427 (2012).
[29] Y. Kaneda and T. Ishihara, Phys. Fluids **13**, 1431 (2001).