Sublattice entanglement in an exactly solvable anyonlike spin ladder

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We introduce an integrable spin ladder model and study its exact solution, correlation functions, and entanglement properties. The model supports two particle types (corresponding to the even and odd sublattices), such that the scattering phases are constants: Particles of the same type scatter as free fermions, whereas the interparticle phase shift is a constant tuned by an interaction parameter. Therefore, the spin ladder bears similarities with anyonic models. We present exact results for the spectrum and correlation functions, and we study the sublattice entanglement by numerical means.

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I. INTRODUCTION

Classical simulations of quantum many-body systems are limited by the growth of entanglement, both in equilibrium and out-of-equilibrium situations. This motivates the study of models where exact solutions can be found, at least for certain physical quantities. Important classes of solvable many-body systems are the free theories, the one-dimensional integrable models [1,2], and also the recently discovered dual unitary quantum circuits [3].

Integrable models have been studied extensively over many decades, and in the last ten years their nonequilibrium dynamics also received considerable attention. Now it is understood that isolated integrable models equilibrate to the generalized Gibbs ensemble [4,5], and their transport properties are described by generalized hydrodynamics [6]. However, these results describe only the large time limit for large system sizes, and generally they lack a completely rigorous proof. Furthermore, they do not provide access to certain exotic features of the dynamics, such as anomalous current fluctuations [7–11].

This motivates the study of selected integrable models with even simpler dynamics, where there is some interaction in the system, nevertheless closed-form results can be derived for the real time evolution of certain physical quantities. Such models include the Rule 54 cellular automaton [12–17], the box-ball systems [10,18,19], classical cellular automata of the XXC type [9,20–24], nontrivial strong-coupling limits of known models [25–27] including the folded XXZ model [28–31], or quantum circuits that are both integrable and dual unitary [32]. A common property of these models is that the scattering of the particles (either classical or quantum) is rather simple compared to a generic integrable model.

What are the simplest possible interacting scattering matrices for integrable (Hermitian) quantum spin chains? An especially simple case is when the scattering matrix is diagonal and the scattering phases are constant (momentum independent). For a single particle species the only possibilities are phase factors ± 1 corresponding to free bosons/fermions. However, for multiple particle species we can explore a wider range of options.

In this paper we introduce a spin ladder model, which supports two particle types which propagate on the two legs of the spin ladder. In this model the scattering phases are constant statistical factors. This bears a strong similarity with anyonic models [33–37], or parafermionic chains [38–40]. A crucial difference is that in our model the anyonlike phases arise from a local interaction defined in the standard spin basis. This provides a unique opportunity to study the entangling effects of constant scattering phases.

II. THE MODEL AND ITS INTEGRABILITY

We consider a spin chain made of qubits, using the notation X_j, Y_j, and Z_j for operators given by the Pauli matrices, acting on site $j = 1 \dots L$, where L (even) is the length of the chain.

Our model can be seen as a spin chain or as a spin ladder in a zigzag geometry (see Fig. 1). We consider a hopping model on the ladder, such that the two legs have a minimal coupling between them. Particles can propagate on the two legs separately, but the local hopping phases on one leg also depend on the local occupation numbers on the other leg, leading to a model Hamiltonian with a three-site interaction:

$$H(\gamma) = \sum_{j=1}^{L/2} h_{2j,2j+1,2j+2}(\gamma) + h_{2j+1,2j+2,2j+3}(-\gamma). \quad (1)$$

Here, $h_{a,b,c}(\gamma)$ is the Hamiltonian density with a real coupling constant $\gamma \in \mathbb{R}$, given by

$$h_{1,2,3}(\gamma) = -[\sigma_1^- e^{i\gamma Z_2}\sigma_3^+ + \sigma_1^+ e^{-i\gamma Z_2}\sigma_3^-], \quad (2)$$

where $\sigma_j^{\pm} = (X_j \pm iY_j)/2$ are the standard raising/lowering operators. The Hamiltonian is invariant with respect to a global spin flip in the Z basis, and it is also space reflection invariant for reflections that also exchange the two sublattices.

In the following we interpret the up spins as a vacuum and the down spins as particles. The Hamiltonian generates hopping on the two sublattices, so that the sublattice magnetizations

$$S^A = \sum_{j=1}^{L/2} Z_{2j}, \quad S^B = \sum_{j=1}^{L/2} Z_{2j+1} \tag{3}$$

are separately conserved. Here and in the following, A and B stand for the even and odd sublattices, respectively. Furthermore, the hopping phase is $e^{\pm i\gamma}$ depending on the position and the occupation of the sites involved.

Similar models include the so-called extended XX model [41,42], a supersymmetric hopping model treated in Ref. [35], or the Bariev model [43]. However, our model is different, as the exact solution below shows.

The model has two free-fermion points, with a different physical interpretation. For $\gamma = 0$ the interaction between the two legs disappears, leading to two uncoupled XX spin chains. For $\gamma = \pi/2$ the model becomes a special case of the extended XX models [42], which can be solved by a single Jordan-Wigner transformation (see below).

The model is integrable for any coupling γ. It has an infinite set of conserved charges, which are given by a diagonal dressing of the known conserved charges of the XX spin chains [44]. The charges are organized into four infinite families, corresponding to the two sublattices and two "chiralities." The charges can be expressed in terms of densities as

$$Q_\alpha^{A+} = \sum_{j=1}^{L/2} q_\alpha^{A+}(2j), \quad Q_\alpha^{B+} = \sum_{j=1}^{L/2} q_\alpha^{B+}(2j+1), \tag{4}$$

together with $Q_\alpha^{A-} = (Q_\alpha^{A+})^\dagger$ and $Q_\alpha^{B-} = (Q_\alpha^{B+})^\dagger$. The index α denotes the range of the given operator density.

The shortest charge densities appear for $\alpha = 3$ and they are simply just terms from the Hamiltonian,

$$\begin{aligned} q_3^{A+}(2k) &= \sigma_{2k}^+ D_{2k+1}^\dagger \sigma_{2k+2}^-, \\ q_3^{B+}(2k+1) &= \sigma_{2k+1}^+ D_{2k+2} \sigma_{2k+3}^-, \end{aligned} \tag{5}$$

where we defined

$$D_j = e^{i\gamma Z_j} = \cos(\gamma) + i \sin(\gamma) Z_j. \tag{6}$$

Higher charges are constructed by a mixed diagonal dressing of the hopping terms of the type $\sigma_k^+ \sigma_{k+\alpha-1}^-$. The dressing is such that for each site between k and $k+\alpha-1$ we include a Z operator if the site is from the same sublattice, and a D operator otherwise. For example, for range $\alpha = 5$ we get

$$\begin{aligned} q_5^{A+}(2k) &= \sigma_{2k}^+ D_{2k+1}^\dagger Z_{2k+2} D_{2k+3}^\dagger \sigma_{2k+4}^-, \\ q_5^{B+}(2k+1) &= \sigma_{2k+1}^+ D_{2k+2} Z_{2k+3} D_{2k+4} \sigma_{2k+5}^-, \end{aligned} \tag{7}$$

together with $q_5^{A-}(2k) = [q_5^{A+}(2k)]^\dagger$ and $q_5^{B-}(2k+1) = [q_5^{B+}(2k+1)]^\dagger$. Higher charges can be constructed in an analogous way. The commutativity of all of these charges is proven as follows. First of all, all charges belonging to either sublattice A or B necessarily commute with each other, because they are just diagonal dressings of the charges of an XX model localized on one of the sublattices [44]. The commutativity of the charges corresponding to different sublattices is less obvious. However, in this case the charge densities actually commute, for example, direct computation gives

$$\left[q_3^{A+}(2k), q_3^{B+}(2l+1)\right] = 0 \tag{8}$$

for all k, l. A more complete proof can be given using a similarity transformation discussed below.

The existence of an infinite family of commuting charges implies that the model is integrable and it has a completely elastic and factorized S matrix [45,46]. We derive this S matrix below. The charges above can be embedded into a transfer matrix constructed from local Lax operators, using the framework of Ref. [47], but we do not treat this approach here.

III. THE SOLUTION OF THE MODEL

For $\gamma = 0$ the model can be solved by two independent Jordan-Wigner (JW) transformations performed on the two sublattices. It is then a natural idea to construct a generalized JW transformation also for finite γ. To this order let us consider the model with free-boundary conditions (or alternatively, a half-infinite chain). We introduce creation and annihilation operators for the two sublattices as

$$\begin{aligned} c^A(2j) &= D_1 Z_2 D_3 Z_4 \cdots Z_{2j-2} D_{2j-1} \sigma_{2j}^+, \\ c^B(2j+1) &= Z_1 D_2^\dagger Z_3 D_4^\dagger \cdots Z_{2j-1} D_{2j}^\dagger \sigma_{2j+1}^+, \end{aligned} \tag{9}$$

together with their adjoints.

Direct computation shows that

$$\{c^A(2j), c^A(2k)\} = 0, \quad \{c^A(2j), c^{A\dagger}(2k)\} = \delta_{jk}, \tag{10}$$

and similarly for the B sublattice. However, for the cross-commutation terms we get for example

$$c^A(2j) c^B(2k+1) = e^{2i\gamma} c^B(2k+1) c^A(2j), \quad j < k. \tag{11}$$

This means that the model is partially anyonic. We remind that even though anyonic statistics is defined originally in two space dimensions, it can be also be observed in one dimension (1D) (see, for example, Ref. [48]), and it amounts to a commutation relation of the type (11), but with one type of creation/annihilation operators [36].

The Hamiltonian can be rewritten for all γ as

$$H(\gamma) = -\sum_{j=1}^{L/2-1} \left[\left(c_{2j+2}^A\right)^\dagger c_{2j}^A + \left(c_{2j+3}^B\right)^\dagger c_{2j+1}^B\right] + \text{c.c.} \tag{12}$$

This means that all the interaction is included now in the definition of the creation/annihilation operators. This is similar to parafermionic models [38–40], but in our case the nontrivial commutation relations arise from the local interaction in the model and not from predefined operator algebras.

We recover the usual JW transformation in two special cases. If $\gamma = 0$ then we get two independent JW transformations on the two sublattices. In contrast, for $\gamma = \pi/2$ we get a single JW transformation on the whole chain. Thus the model interpolates between two uncoupled XX chains and a

single XX chain, although in the latter case the Hamiltonian is actually a higher charge of the usual XX model [42].

While for the free cases the model can be solved by a Fourier transform, this does not work for generic γ due to the mixed anyonlike interactions. In these cases we can use the fact that the phase factors do not depend on the momenta of the particles, only on the relative position of the particles on the two sublattices. This leads to a simple explicit construction of the wave functions.

Let us consider a state with N_A and N_B particles on the two sublattices. Lattice momenta of the particles will be denoted as p_j^A and p_k^B. For writing down the wave function we will use coordinates $\mathbf{a} = \{a_j\}_{j=1,\dots,N_A}$ and $\mathbf{b} = \{b_k\}_{j=1,\dots,N_B}$ which run over the even and odd numbers, respectively. First, we consider periodic boundary conditions. The wave function is then given by

$$\Psi(\mathbf{a},\mathbf{b}) = \det(\mathcal{A})\det(\mathcal{B})\prod_{a<b} e^{-i\gamma}\prod_{b<a} e^{i\gamma}. \tag{13}$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are matrices of sizes $N_A \times N_A$ and $N_B \times N_B$, respectively. They describe free-fermionic wave functions localized on the two sublattices, with components given by

$$\mathcal{A}_{jk} = e^{i(p_j^A-\gamma)a_k/2}, \quad \mathcal{B}_{jk} = e^{i(p_j^B+\gamma)b_k/2}. \tag{14}$$

The interpretation of this wave function is the following: The model supports two particle species moving on the two sublattices, with momenta p_j^A and p_j^B. The scattering in the model is factorized and diagonal, with momentum-independent phase shifts given by

$$\mathcal{S}_{AA} = \mathcal{S}_{BB} = -1, \quad \mathcal{S}_{AB} = (\mathcal{S}_{BA})^{-1} = e^{2i\gamma}. \tag{15}$$

The phase shifts reflect the commutation relations (10) and (11).

Periodicity implies that the momenta have to satisfy the Bethe equations

$$\begin{aligned} e^{ip_j^A L/2} &= (-1)^{N_A-1} e^{i\gamma(L/2-2N_B)}, \\ e^{ip_j^B L/2} &= (-1)^{N_B-1} e^{-i\gamma(L/2-2N_A)}. \end{aligned} \tag{16}$$

These equations are almost free: The only coupling between the two sets of momenta is simply just a twist, which depends on the overall particle numbers. The energy eigenvalues are then

$$E = \sum_{j=1}^{N_A} e\big(p_j^A\big) + \sum_{j=1}^{N_B} e\big(p_j^B\big), \tag{17}$$

with $e(p) = -2\cos(p)$.

Now we consider the model with free-boundary conditions, and show that in this case the wave functions are found simply using a global similarity transformation. The diagonal operator

$$\mathcal{D} = \prod_{2j<2k+1} e^{i\gamma Z_{2j}Z_{2k+1}/4} \prod_{2j>2k+1} e^{-i\gamma Z_{2k+1}Z_{2j}/4} \tag{18}$$

completely decouples the two legs of the ladder:

$$\mathcal{D}H(\gamma)\mathcal{D}^{-1} = H(0). \tag{19}$$

This implies that the spectrum of the open chain is the same as that of two uncoupled XX chains for all α. In this sense the model is free, and it belongs to the class of models investigated in Refs. [49,50]. Nevertheless, the operator $\mathcal{D}$ is highly nonlocal, and it makes the two legs of the ladder highly entangled, both in equilibrium and out-of-equilibrium situations.

We also study the thermodynamic limit (TDL). We introduce root densities $\rho^A(p)$ and $\rho^B(p)$, and the limiting value of the ground state energy density becomes

$$\lim_{\mathrm{TDL}} \frac{E}{L} = \int_{-\pi}^{\pi} \frac{dp}{4\pi} e(p)[\rho^A(p) + \rho^B(p)]. \tag{20}$$

The ground state is given by the half-filled state

$$\rho^{A,B}(p) = \begin{cases} 1 & \text{for } |p| < \pi/2, \\ 0 & \text{for } |p| > \pi/2. \end{cases} \tag{21}$$

The ground state energy density is $-2/\pi$, which is identical to that of the XX model.

IV. CORRELATION FUNCTIONS

The coupling between the two sublattices makes them strongly entangled. We demonstrate this by computing a selected short-range correlation function. Due to the similarity transformation (19) the correlation functions of Z operators will be the same as in two uncoupled XX chains. Therefore, nontrivial information is seen in correlation functions with hopping terms. We choose the following connected correlation function:

$$C_\Psi \equiv \langle\Psi|\sigma_0^- Z_1 \sigma_2^+|\Psi\rangle - \langle\Psi|\sigma_0^-\sigma_2^+|\Psi\rangle\langle\Psi|Z_1|\Psi\rangle. \tag{22}$$

Here, $|\Psi\rangle$ is an arbitrary eigenstate with N_A and N_B particles on the two sublattices. A nonzero value of C_Ψ demonstrates the entanglement between the two sublattices.

We introduce the magnetization on the odd sublattice:

$$m_B = \langle\Psi|Z_1|\Psi\rangle = \frac{L-4N_B}{L}. \tag{23}$$

A certain combination of the operators above is simply the density of a conserved charge (one term in the Hamiltonian), therefore we get

$$\langle\Psi|\cos(\gamma)\sigma_0^-\sigma_2^+ + i\sin(\gamma)\sigma_0^- Z_1\sigma_2^+|\Psi\rangle = 2W, \tag{24}$$

with

$$W = \frac{1}{L}\sum_{j=1}^{N_A} e^{ip_j^A}. \tag{25}$$

Now we apply the Hellmann-Feynman theorem for the corresponding charge, from which we can obtain the mean values of the γ derivative of the operators on the left-hand side above. Combining this with the mean value above and with the Bethe equations (16) we get the result

$$C_\Psi = 2i\sin(\gamma)W(m_B^2 - 1). \tag{26}$$

In the thermodynamic limit we get

$$W \to \int_{-\pi}^{\pi} \frac{dp}{4\pi}\rho^A(p)e^{ip}. \tag{27}$$

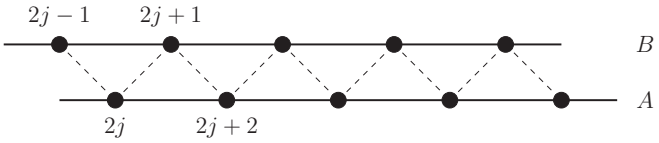

FIG. 1. The geometry of the spin zigzag ladder. A and B denote the two sublattices on which the particles can hop

This retains a finite value unless the root distribution is constant, therefore we obtain a finite correlation between the sublattices for almost all states.

The correlation function vanishes if either lattice is fully polarized, having magnetization equal to ± 1. If the odd sublattice is polarized, then the vanishing is guaranteed by the factor $(m_B^2 - 1)$, whereas if the even sublattice is completely polarized, then $W = 0$ (because for a fully polarized state ρ is constant).

Analogous results can be obtained for combinations similar to (22).

V. SUBLATTICE ENTANGLEMENT

We also study the entanglement properties of the model, in both equilibrium and out-of-equilibrium situations. In the literature the most often studied entanglement is that of connected subsystems [51–56]. However, it is expected that in this model the usual bipartite entanglement behaves very similar to that of the XX model. Therefore we focus on the sublattice entanglement, which is a highly nontrivial quantity that can be tuned by the coupling constant γ. Sublattice entanglement was studied earlier in a number of situations [57–62].

As before, let A and B denote the sites of the even and odd sublattices, and we define the sublattice (von Neumann) entanglement entropy as

$$S = -\mathrm{Tr}(\rho_A \log \rho_A), \tag{28}$$

where $\rho_A = \mathrm{Tr}_B\, \rho$, with ρ being the density matrix of the system, either in equilibrium or in an out-of-equilibrium process. The entanglement entropy is expected to be extensive, and we introduce the entropy density

$$s = \frac{S}{L}. \tag{29}$$

It is expected that s should not depend on L apart from minor finite-size effects.

We numerically study s in the ground state for different values of γ and L, and the results are plotted in Fig. 2. We use exact diagonalization methods implemented using sparse matrices in Wolfram *Mathematica* [63]. We find that finite-size effects are indeed small, and the entropy density obtains its maximum value at the free-fermion point $\gamma = \pi/2$.

We also consider nonequilibrium time evolution started from a selected initial state, namely the ferromagnetic state with polarization in the x direction:

$$|X\rangle = \bigotimes_{j=1}^{L} \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle). \tag{30}$$

We study the real time evolution of s, and the results are plotted in Fig 2. It can be seen that the entanglement is indeed extensive, but now there are bigger finite-size effects. For the largest system size $L = 24$ we see that entanglement reaches a plateau relatively soon for all values of γ, with the height of the plateau behaving in a similar way as before: The maximum entanglement is seen for $\gamma = \pi/2$. Qualitatively similar behavior can also be found for other initial states.

VI. DISCUSSION

We introduced an exactly solvable spin ladder, which is one of the simplest quantum integrable models with a tunable coupling between particles. The model interpolates between two free-fermion points, in which the two sublattices are either uncoupled or maximally coupled. The entanglement between the sublattices was demonstrated analytically by an exact result for a correlation function, and numerically by the sublattice entanglement (28) defined in the real-space basis, which was examined both in and out of equilibrium.

In the special case $\gamma = \pi/2$ the entanglement can also be studied in terms of the fermionic degrees of freedom. It was pointed out in Ref. [64] that for free-fermionic chains these two definitions of entanglement give generally different results. They only agree for connected subsystems, because for disconnected subsystems the Jordan-Wigner transformation causes differences between the two types of entanglement. Our model can be regarded as an extreme example for this phenomenon: For $\gamma = \pi/2$ the two sublattices are completely decoupled if one considers the fermions [see Eq. (12)]. Therefore, the sublattice entanglement in terms of the fermions is

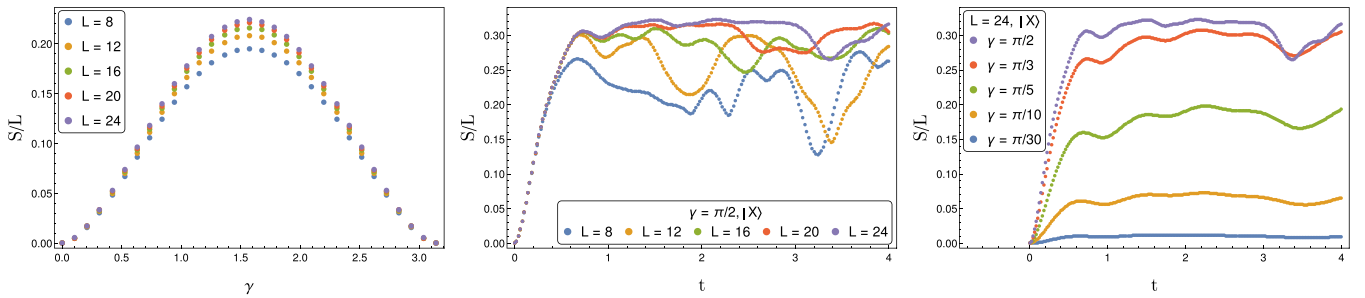

FIG. 2. Numerical results for the sublattice entanglement density $s = S/L$. Left: Ground state values as a function of the coupling γ for different values of L. Middle: Time evolution of the entanglement in a quench problem (see main text), for different volumes and $\gamma = \pi/2$. Right: Time evolution in a quench problem for $L = 24$ and different values of γ.

exactly zero. In contrast, we find that the real-space entanglement is nonzero, and in fact it is maximal for the free-fermion point $\gamma = \pi/2$.

Finally, we note that the model is partially anyonic for a generic γ, and it seems to be one of the simplest nontrivial scattering theories. This could lead to interesting applications, for example, in the realization of interacting Bethe states in quantum computers [65–69].

Note added. Recently, we became aware of Ref. [70] which treats a closely related model. Our results about the integrability and exact solvability of the model (together with the exact result for a correlation function) appear to be different, whereas the results for sublattice entanglement are partly overlapping.

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