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Comment on "Deformations, relaxation, and broken symmetries in liquids, solids, and glasses: A unified topological field theory"

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We discuss a field-theoretical approach to liquids, solids, and glasses, published recently [Phys. Rev. E **105**, 034108 (2022)], which aims to describe these materials in a common formalism. We argue that such a formalism contradicts the known hydrodynamic theory of classical liquids. In particular, the authors miss the important particle-number conservation law and the density fluctuations as a hydrodynamic slow variable. This results in an exotic mechanism of hydrodynamic sound instead of the standard hydrodynamic one due to combined particle-number and momentum conservation, a fact well documented in fluid-mechanics textbooks.

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In a recent paper [1], Baggioli, Landry, and Zaccone (BLZ) present a formalism aimed to describe crystalline solids, glassy solids, and liquids in a unified way. BLZ obtained a hydrodynamic matrix which in a particular case of simple one-component liquids resulted in three collective modes, one of which showed an obvious nonhydrodynamic behavior in the long-wavelength limit.

Any proposed hydrodynamic approach of a classical liquid must start with naming the conservation laws for a set of relevant hydrodynamic variables [2–5]. The conserved quantities are the particle number, total momentum, and energy. The corresponding hydrodynamic variables, which describe the collective modes, are the number density $n(\mathbf{r},t)$, the mass-current density $\mathbf{j}(\mathbf{r},t)$, and the energy density $e(\mathbf{r},t)$. It is important to notice that the damping of each collective hydrodynamic mode must be proportional to k^2 [2–5], with k being the wave number, i.e., their lifetime tends to infinity in the long-wavelength (continuum) limit as a consequence of local conservation laws.

However, in the treatment of BLZ [1] the density does not appear as a relevant field, nor does the corresponding continuity equation. Instead of the hydrodynamic longitudinal sound mode, which arises from particle number and momentum conservation, BLZ invoke a sophisticated scheme with Goldstone modes responsible for hydrodynamic excitations. This results in a hydrodynamic matrix for longitudinal dynamics [Eq. (38) of BLZ], the eigenmodes of which contradict the known hydrodynamic modes in fluids [2–5].

Here, we review the correct form of the hydrodynamic matrix for longitudinal dynamics, whose eigenmodes are the hydrodynamic modes in the longitudinal channel. For describing the longitudinal excitations of classical liquids one may use the three orthogonal dynamic variables $n(k,t)$, $j^L(k,t)$, $h(k,t)$, where the first two are the Fourier components of the number density and longitudinal mass-current density fluctuations. $h(k,t)$ denotes fluctuations of the heat density, which is the energy density, orthogonalized to the number density [6]

$$h(k,t) = e(k,t) - \frac{\langle e_{-k} n_k \rangle}{\langle n_{-k} n_k \rangle} n(k,t), \tag{1}$$

where the angle brackets denote a statistical average. Using this set of orthogonal hydrodynamic variables, one obtains the following hydrodynamic 3×3 matrix,

$$\mathbf{T}^{(\mathrm{hyd})}(k) = \begin{pmatrix} 0 & -ikc_T & 0 \\ -ikc_T & D_L k^2 & -ikc_T\sqrt{\gamma - 1} \\ 0 & -ikc_T\sqrt{\gamma - 1} & \gamma D_T k^2 \end{pmatrix}, \tag{2}$$

where c_T is the isothermal speed of sound, and $\gamma = C_P/C_V$ is the ratio of the specific heat. D_L is the longitudinal kinematic viscosity, and D_T the thermal diffusivity (diffusivity of the local temperature). One can see that when $\gamma = 1$ (no coupling between the thermal and viscous processes) the eigenmodes of $\mathbf{T}^{(\mathrm{hyd})}(k)$ can be estimated immediately: In this case the hydrodynamic matrix has one purely real eigenvalue

$$z_{th}(k) = D_T k^2, \tag{3}$$

which corresponds to a thermal relaxation mode, and a pair of complex-conjugated eigenvalues

$$z_{\pm}(k) = \frac{D_L}{2}k^2 \pm ic_T k \quad (\text{for } \gamma = 1),$$

which are the sound modes, which are decoupled from the thermal fluctuations with linear dispersion and isothermal speed. In the general case of $\gamma \geqslant 1$ the standard sound modes with an adiabatic speed of sound $c_s = c_T\sqrt{\gamma}$ and correct hydrodynamic damping,

$$z_{\pm}(k) = \frac{D_L + (\gamma - 1)D_T}{2}k^2 \pm ic_s k, \tag{4}$$

are obtained from the hydrodynamic matrix (2). The origin of the long-wavelength sound modes is now clearly seen even in the particular case of $\gamma = 1$: They come from the coupling of density and mass-current fluctuations. At variance, in Ref. [1], BLZ do not account for the density fluctuations as the hydrodynamic variable [the set of dynamic variables in their Eq. (14)], i.e., they ignore the continuity equation

$$\frac{dn(k,t)}{dt} + i\frac{k}{m}j^L(k,t) = 0, \tag{5}$$

where m is molar mass, which is fundamental for all liquids [2–5]. Recently, some of us [7] have demonstrated that one obtains results, which contradict the known dynamic properties of liquids, if the continuity equation is not taken into account.

As a consequence of the absence of the continuity equation (5) in their treatment, BLZ claim that their longitudinal sound would come "from the mixing of energy fluctuations and longitudinal momentum fluctuations" (quoting BLZ [1]), at variance with textbook knowledge [2–5].

Further, BLZ do not obtain the thermal relaxation mode (3) among the eigenmodes of their 3 × 3 "hydrodynamic matrix" [their Eq. (38) and Fig. 9]), which means that they do not recover the standard Rayleigh-Brillouin three-peak shape for the dynamic structure factors $S(k, \omega)$ (see Fig. 4.2 in Forster's book [3]) with the famous Landau-Placzek ratio [2–5]. Their single relaxation mode behaves (in their Fig. 9) as typical nonhydrodynamic relaxation, which has a finite lifetime at large length scales, and cannot contribute to $S(k \to 0, \omega)$ in the hydrodynamic $k \to 0$ limit.

We further mention that the viscoelastic equation [8,9] [similarly as Eq. (96) in Ref. [1]] explicitly contains the Maxwell relaxation time $\tau = \eta/G$,

$$\frac{d}{dt}\gamma(t) = \frac{1}{G}\frac{d}{dt}\sigma(t) + \frac{1}{\eta}\sigma(t) = \frac{1}{G}\left(\frac{d}{dt} + \frac{1}{\tau}\right)\sigma(t), \tag{6}$$

where $\gamma(t)$ is the strain rate, $\sigma(t)$ the stress, G the shear modulus, and η the shear viscosity. BLZ incorrectly call τ the "single-particle relaxation time" (in the caption to their Fig. 5), whereas it describes the relaxation of the *collective macroscopic* shear stress $\sigma(t)$ [10] and by the Kubo-Green relation [4] and definition of correlation times is exactly equal to the correlation time of the shear stress autocorrelation

$$\tau = \frac{\eta}{G} \equiv \frac{1}{\langle\sigma(0)\sigma(0)\rangle}\int_0^{\infty}\langle\sigma(t)\sigma(0)\rangle dt = \tau_{\sigma}^{\text{corr}}. \tag{7}$$

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