First-passage process in degree space for the time-dependent Erdős-Rényi and Watts-Strogatz models

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(Received 20 April 2022; accepted 7 September 2022; published 26 September 2022)

In this work, we investigate the temporal evolution of the degree of a given vertex in a network by mapping the dynamics into a random walk problem in degree space. We analyze when the degree approximates a preestablished value through a parallel with the first-passage problem of random walks. The method is illustrated on the time-dependent versions of the Erdős-Rényi and Watts-Strogatz models, which were originally formulated as static networks. We have succeeded in obtaining an analytic form for the first and the second moments of the first-passage time and showing how they depend on the size of the network. The dominant contribution for large networks with N vertices indicates that these quantities scale on the ratio N/p, where p is the linking probability.

DOI: 10.1103/PhysRevE.106.034320

I. INTRODUCTION

The study of the properties of networks led to the development of many mathematical, statistical, and computational tools that can be used to analyze, model, and understand how systems behave in many areas of knowledge such as physics, biology, ecology, and social sciences, to name some of them. Modeling complex systems by networks [1–3] is a natural strategy to investigate a system from a very basic structure composed of agents and interactions among them, represented, respectively, by vertices (or nodes) and links.

In this work, we are mainly interested in the time evolution of the degree of a given node. Concretely, a vertex can gain and/or lose connections during its dynamics, and we investigate when it achieves a preestablished degree for the first time. This is a particularly relevant issue when agents can not afford an indefinite number of connections and some indication of approaching the maximum capacity of the node [4,5] is desired. As an instance, it is known that airports (where the links can be assigned to the routes) have constraints that prevent growth without careful planning [6].

We map the dynamics of increasing or decreasing degrees into a random walk process, as was introduced in Ref. [7], and see if and how long it takes for a vertex with degree k_0 to reach degree k for the first time. This is a one-dimensional random walk in degree space, where the rules of gaining or losing degrees are governed by the dynamics of the network. The random walk is a classical problem [8,9] where a particle moves in random directions and one typically inquires about its statistical properties after a long time. Starting from an origin, one possible question that can be formulated concerns the probability of returning to the starting point, and the first-passage process refers to its return for the first time [10–12]. First-passage processes are seen in many applications, and examples are present in random searches [13–16], cyclization of a polymer [17], integrate-and-fire neurons [18], chemical reactions [19,20], narrow escape problem [21–23], and diffusion on cell membranes [24] to cite some of them. When the random walk is defined on a (hyper)cubic lattice [25], and the particle can move in any direction with the same probability, it is known that the mean first-passage time scales as L^d [26], where L is the linear size of the d-dimensional lattice with periodic boundary conditions; furthermore, in the limit of infinite lattice, this random walk is known to be recurrent (i.e., it returns to the origin with probability 1) for the one-dimensional chains and two-dimensional square lattices, while the process is transient (i.e., there is a positive probability of not returning to the origin) for hypercubic lattices with larger dimensions [12,26].

We investigate how the mean first-passage time of a vertex reaching a preestablished degree scales with the size of the network and other relevant parameters. Our study is based on an extended version of the Watts-Strogatz model [27], which was also introduced in Ref. [7], but we consider first a dynamical version of the Erdős-Rényi model [28] to illustrate and outline the main steps of analysis. An important difference between these two models stems from the locality in the dynamics in the following sense. While the degree dynamics in Erdős-Rényi model depends on two fixed parameters (the total number of vertices and the linking probability, as seen in the next section), the Watts-Strogatz model behaves differently. In the latter, the probability of a vertex gain or losing connections also depends on the degree it has. This means that the number of links of a given vertex is influenced by the actual state of the node, and the dynamics is not trivial as in the Erdős-Rényi case anymore. We analyze the consequence of this property in the present work. On the other hand, both models allow a simplification that arises from a property shared between them, which is the time-translational invariance. Systems that do not have this property, such as the random recursive tree [29,30] or Barabási-Albert network [31], indicate the need for a different approach, and will be examined elsewhere.

This paper is organized as follows. We define the dynamical version of the Erdős-Rényi and Watts-Strogatz models in

Sec. II and the general formalism to investigate the moments of the first-passage time is presented in Sec. III. The results for both models are shown in Sec. IV and some final comments are given in the last section.

II. MODELS

Two models are introduced to test our ideas in this work. Both of them are already well known in the literature [27,28], but were initially defined as static networks.

The first one, which is a minimal model, is the dynamical Erdős-Rényi network: the dynamics is just a simple addition of edges per time unit, and we monitor the increase of degrees only. The second one, the dynamical Watts-Strogatz model, is the simplest network that contains the process where a vertex can gain and/or lose connections randomly. The usual first-passage process (which is concerned with the return to the starting point) in the latter model corresponds to the so-called Motzkin paths [32]. Evaluating the number of such paths is a combinatorial problem and is given by the Motzkin numbers [33,34], which are related to many other problems [35]. There is an interesting example in biology, where the connections in the endoplasmatic reticulum network can be regulated by lysosomes [36].

A. Time-dependent Erdős-Rényi model

In the dynamical version of the Erdős-Rényi model, consider a network with N vertices. At each unitary time step, two vertices are randomly chosen and connected with probability p; this includes the possibility of (i) having a loop (i.e., an edge that connects a vertex to itself) and (ii) having more than one connection between the same pair of vertices. Since there is no preferential attachment, the probability of any vertex being chosen is $1/N$.

Defining $p_s(k,t)$ as the probability that a vertex s has degree k at time t, the dynamics can be represented by the recurrence relation

$$\begin{aligned} p_s(k,t+1) &= \omega_{\mathrm{ER}}(k|k-2)p_s(k-2,t) \\ &\quad + \omega_{\mathrm{ER}}(k|k-1)p_s(k-1,t) \\ &\quad + \omega_{\mathrm{ER}}(k|k)p_s(k,t). \end{aligned} \tag{1}$$

The term $\omega_{\mathrm{ER}}(k|m)$ is the time-independent transition rate of changing the degree of a vertex from m to k; in this time-discrete case with unitary time step, the transition rate coincides numerically to the conditional probability. The right-hand side of the dynamics (1) contemplates three cases:

(i) The degree of vertex s changes from $k-2$ (at time t) to k (at time $t+1$). An edge is introduced, with probability p (there should be no confusion with p_s), and the vertex s is connected to itself (by being chosen twice); this leads to

$$\omega_{\mathrm{ER}}(k|k-2) = \frac{p}{N^2}; \tag{2}$$

(ii) The degree of vertex s changes from $k-1$ (at time t) to k (at time $t+1$). An edge is introduced, with probability p, and links to two different vertices, but the vertex s has to be one of them. This situation is described by

$$\omega_{\mathrm{ER}}(k|k-1) = \frac{2p}{N}\left(1-\frac{1}{N}\right); \tag{3}$$

(iii) The vertex s already has degree k, and one should consider the probability of not changing its degree, i.e., the link is not introduced (with probability $1-p$) or, when the edge joins the network (with probability p), it connects two vertices other than s with probability $(1-1/N)^2$. In this case, one has

$$\omega_{\mathrm{ER}}(k|k) = (1-p) + p\left(1-\frac{1}{N}\right)^2 = 1 - \frac{2p}{N} + \frac{p}{N^2}. \tag{4}$$

B. Time-dependent Watts-Strogatz model

In this version of the Watts-Strogatz model, the network has a fixed number N of vertices and

$$M := cN \tag{5}$$

degrees, where c is the mean degree of the network (therefore, the entire graph has $cN/2$ edges). At each time step, an edge end is chosen at random with uniform probability $1/M$ and reconnected with probability p (and no action takes place with probability $1-p$). This scheme does not forbid loops or multiple connections between the same pair of vertices.

Defining $p_s(k,t)$ as the probability that a vertex s has degree k at time t as before, the dynamics can be represented by

$$\begin{aligned} p_s(k,t+1) &= \omega_{\mathrm{WS}}(k|k-1)p_s(k-1,t) \\ &\quad + \omega_{\mathrm{WS}}(k|k+1)p_s(k+1,t) \\ &\quad + \omega_{\mathrm{WS}}(k|k)p_s(k,t), \end{aligned} \tag{6}$$

where $\omega_{\mathrm{WS}}(k|m)$ represents the time-independent transition rate of a vertex changing its degree from m to k.

There are some different possible scenarios for a given vertex to change its degree from m to k in a single time step:

(i) The degree of vertex s changes from $k-1$ (at time t) to k (at time $t+1$). An edge end not connected to s is chosen with probability $1-\frac{k-1}{M}$, rewired with probability p and connects to s with probability $\frac{1}{N}$. In this situation, one has

$$\omega_{\mathrm{WS}}(k|k-1) = \frac{p}{N}\left(1-\frac{k-1}{M}\right); \tag{7}$$

(ii) The degree of vertex s changes from $k+1$ (at time t) to k (at time $t+1$). An edge end connected to s is chosen with probability $\frac{k+1}{M}$, rewired with probability p and connects to a vertex other than s with probability $1-\frac{1}{N}$, resulting in

$$\omega_{\mathrm{WS}}(k|k+1) = \frac{k+1}{M}p\left(1-\frac{1}{N}\right); \tag{8}$$

(iii) The vertex s has degree k at time t and neither gains nor loses connections. This is represented by the sum of some disjoint cases: (a) there is no rewiring at all in the process with probability $1-p$, or (b) an edge end connected to s is chosen with probability $\frac{k}{M}$, rewired with probability p and connected again to s with probability $\frac{1}{N}$; (c) an edge end not connected to s is chosen with probability $1-\frac{k}{M}$ and rewired (with probability p) to connect to a vertex other than s with probability $1 - 1/N$. The sum of these probabilities results in

$$\omega_{\mathrm{WS}}(k|k) = \left(1 - p\right) + p\frac{k}{M}\frac{1}{N} + p\left(1 - \frac{k}{M}\right)\left(1 - \frac{1}{N}\right) = 1 - \frac{p}{N}\left(1 + \frac{kN}{M} - \frac{2k}{M}\right). \tag{9}$$

III. RANDOM WALK IN DEGREE SPACE

Considering that vertices, in general, gain or lose connections, one can look at these changes in degree (of a specified vertex) as a one-dimensional random walk in degree space [7]. Furthermore, the mean time required by a vertex to reach a certain degree for the first time can be evaluated through a parallel with the first-passage problem of random walks [11,12].

In both models presented in the previous section, there are two important symmetries. First, the particular choice of a vertex s is irrelevant, and this parameter has no role in our work, except for remembering that we are dealing with the time evolution of the degree of a given vertex.

Let us consider a vertex s from the network, and assume that it has degree k_0 at time $t_0 = 0$. The choice of initial time is irrelevant, as we will see later, since the system displays time-translational symmetry. The mean time $\langle t\rangle$ to reach a certain degree k for the first time is given by

$$\langle t\rangle = \sum_{t=0}^{\infty} t f_s(k, t|k_0, 0), \tag{10}$$

where $f_s(k, t|k_0, 0)$ is the probability of vertex s having degree k for the first time at t, given that it had degree k_0 at time $t_0 = 0$. This probability can be obtained from the discrete-time version of the first-passage process equation [11,12], and it obeys the equation

$$p_s(k, t|k_0, 0) = \sum_{t'=0}^{t} f_s(k, t'|k_0, 0)p_s(k, t|k, t'), \tag{11}$$

which describes the probability $p_s(k, t|k_0, 0)$ of the vertex s having degree k at time t (not necessarily for the first time), given that it had degree k_0 at time $t_0 = 0$. This is a sum of all disjoint probabilities where the degree of the vertex reaches k at time t' ($\leqslant t$) for the first time, and then reaches degree k again at instant t. The initial condition $p_s(k, 0|k_0, 0) = \delta_{k,k_0}$ is satisfied by assuming $f_s(k, 0|k_0, 0) = \delta_{k,k_0}$ (an extra term in (11) associated to the initial condition is not required here as it is in the continuous-time version [11,12] of the equation).

The second important symmetry of our models can be seen from the transition rates ω_{ER} and ω_{WS}: they are invariant under time translation. As a consequence, $p_s(k, t|k', t') = p_s(k|k'; t - t')$ and $f_s(k, t|k', t') = f_s(k|k'; t - t')$ depend on the difference $t - t'$ only. Therefore, Eq. (11) can be cast as

$$p_s(k|k_0; t) = \sum_{t'=0}^{t} f_s(k|k_0; t')p_s(k|k; t - t'). \tag{12}$$

As usual, the convolution product in (12) suggests the introduction of the characteristic function

$$p_s^z(k|k_0; z) = \sum_{t=0}^{\infty} z^t p_s(k|k_0; t) \tag{13}$$

and a similar definition for the characteristic function f_s^z of the function f_s. Then, it is immediate that

$$f_s^z(k|k_0; z) = \frac{p_s^z(k|k_0; z)}{p_s^z(k|k; z)}, \tag{14}$$

and we can access the first-time probability f_s (or f_s^z) from the probability p_s (or p_s^z). As stated before, this is a consequence of the time-translation invariance; models that do not have this symmetry (such as the random recursive tree [29,30] or Barabási-Albert network [31]) do not display the form (12).

We are mainly interested in (14) because it provides some quantities of interest. The first one is

$$\mathcal{A} := \lim_{z\to 1} f_s^z(k|k_0; z) = \sum_{t=0}^{\infty} f_s(k|k_0; t), \tag{15}$$

which stands for the arriving probability of a vertex reaching degree k, starting from degree k_0, at some time, while

$$\langle t^n\rangle = \lim_{z\to 1} (z\partial_z)^n f_s^z(k|k_0; z) = \sum_{t=0}^{\infty} t^n f_s(k|k_0; t), \tag{16}$$

where ∂_z stands for the partial derivation in z variable, shows that the quantity f_s^z is also useful to evaluate any moment of the first-passage time. In this work, we are particularly interested in the first and second moments, $\langle t\rangle$ and $\langle t^2\rangle$, respectively; the latter is directly associated with the variance $\sigma^2 = \langle t^2\rangle - \langle t\rangle^2$.

Hence, one can also expand (14) as

$$f_s^z(k|k_0; z) = \mathcal{A} + \langle t\rangle(z - 1) + \left[\frac{\langle t^2\rangle - \langle t\rangle}{2}\right](z - 1)^2 + \cdots \tag{17}$$

and obtain the desired quantities ($\mathcal{A}$, $\langle t\rangle$, and $\langle t^2\rangle$) through this series representation.

IV. RESULTS

In this section, we present the results for the first and second moments of the first-passage time for both models.

A. Time-dependent Erdős-Rényi model

The discrete-time evolution for the dynamical version of Erdős-Rényi model, introduced in Sec. II, is given by (1). Introducing the characteristic function

$$p_s^K(K; t) = \sum_{k=0}^{\infty} K^k p_s(k|k_0; t) \tag{18}$$

into (1) leads to

$$p_s^K(K, t) = \left[1 - p + p\left(\frac{K}{N} + 1 - \frac{1}{N}\right)^2\right]^t K^{k_0}, \tag{19}$$

where the initial condition $p_s(k|k_0; 0) = \delta_{k,k_0}$ or, equivalently, $p_s^K(K; 0) = K^{k_0}$ was adopted. From (18), the probability

$p_s(k|k_0;t)$ is the coefficient of the term K^k in the series; therefore, expanding (19) and organizing the terms implies

$$p_s(k|k_0;t) = \sum_{m=\lceil \frac{\Delta}{2} \rceil}^{t} \binom{t}{m}\binom{2m}{\Delta}(1-p)^{t-m}p^m \times \left(1-\frac{1}{N}\right)^{2m-\Delta}\frac{1}{N^{\Delta}}. \quad (20)$$

From (20), the function p_s depends on the difference $\Delta := k - k_0$ only, and not on the initial and final degrees independently. This property is propagated to the quantities of interest in this work.

Using (20), the characteristic function (in time variable) of p_s is

$$p_s^z(k|k_0;z) = \sum_{t=0}^{\infty} z^t p_s(k|k_0;t) = \sum_{m=\lceil \frac{\Delta}{2} \rceil}^{\infty} \binom{2m}{\Delta}\left(1-\frac{1}{N}\right)^{2m-\Delta} \times \frac{1}{N^{\Delta}}\frac{(zp)^m}{[1-z(1-p)]^{m+1}}, \quad (21)$$

from which one can also evaluate $p_s^z(k|k;z)$ by taking $k_0 = k$ (or $\Delta = 0$). Then, using the relation

$$\sum_{m=\lceil \frac{\Delta}{2} \rceil}^{\infty} \binom{2m}{\Delta} x^{2m} = \frac{x^{\Delta}}{2}[(1-x)^{-\Delta-1} + (-1)^{\Delta}(1+x)^{-\Delta-1}] \quad (\Delta \in \mathbb{N}, x \in (-1,1) \subset \mathbb{R}), \quad (22)$$

which can be deduced by combining the expansion of $(1 \pm x)^{-\Delta-1}$ for $|x| < 1$, it is now possible to obtain

$$f_s^z(k|k_0;z) = \frac{1-\zeta^2}{2(N-1)^{\Delta}}\left[\frac{\zeta^{\Delta}}{(1-\zeta)^{\Delta+1}} + \frac{(-1)^{\Delta}\zeta^{\Delta}}{(1+\zeta)^{\Delta+1}}\right], \quad (23)$$

where

$$\zeta := \zeta(z) = \left(1-\frac{1}{N}\right)\sqrt{\frac{zp}{1-z(1-p)}}. \quad (24)$$

Expanding (23) as in (17) is a tedious but direct procedure. From this operation, the arrival probability can be obtained as being

$$\mathcal{A}_{\text{ER}} = 1 - \frac{1}{2N}\left[1 - \frac{(-1)^{\Delta}}{(2N-1)^{\Delta}}\right]. \quad (25)$$

Although the dynamics suggests that the vertex s can achieve any larger degree if one waits a sufficiently long time, the probability (25) is less than one. However, this odd result is a consequence of the network growing rule, which allows a vertex to increase its degree by two units by forming a loop. In this case, the targeted degree, k, may be surpassed from $k-1$ to $k+1$ without being accessed. For this reason, the arrival probability is not 1. Nonetheless, if one evaluates

$$\sum_{t=0}^{\infty} f_s(\text{degree} \geqslant k|k_0;t) = \sum_{t=0}^{\infty}[f_s(k|k_0;t) + \omega_{\text{ER}}(k+1|k-1)p_s(k-1,t)], \quad (26)$$

which is a correction to (25), the arrival probability is 1, as expected. Note that the arrival probability (25) tends to 1 with the size of the network, which is expected since the loop becomes rare with the number of vertices. One should also note that this result is valid for any positive linking probability p (the case $p = 0$ is trivial), but does not depend explicitly on this parameter. As shown below, this parameter scales the time elapsed until a vertex reaches some degree for the first time, but it does not have any impact on the probability of reaching the preestablished degree (except the trivial case $p = 0$, when $\mathcal{A} = 0$ for $\Delta > 0$).

The first and second first-passage time moments can also be derived from (23). The leading term of the mean first-passage time is

$$\langle t \rangle_{\text{ER}} \simeq \frac{N}{2p}\Delta \quad (27)$$

for $N \gg 1$, while the second moment is

$$\langle t^2 \rangle_{\text{ER}} \simeq \left(\frac{N}{2p}\right)^2 \Delta(\Delta+1). \quad (28)$$

The variance can also be determined from (27) and (28), and depends quadratically on the ratio N/p, but linearly on the difference $\Delta := k - k_0$ as $\sigma_{\text{ER}}^2 := \langle t^2 \rangle_{\text{ER}} - \langle t \rangle_{\text{ER}}^2 \simeq (\frac{N}{2p})^2\Delta$. The results (27) and (28) are supported by numerical simulations, as one can see in Fig. 1.

B. Time-dependent Watts-Strogatz model

The analysis of the dynamical version of the Watts-Strogatz model is much more intricate than the previous model. To convey better the ideas, all the technical details are presented in the Supplemental Material [37], and we will restrict ourselves to highlighting only the important points in this section. It is important to stress that our mains results are all established analytically, and despite the fact that the techniques involved are not new, a careful choice in the management of the expressions involved is required to obtain a compact form of the final result.

The dynamics of this model was already presented in (6), where the transition rates are given in (7), (8), and (9). Introducing a characteristic function that transforms both the degree and time variables [see (18) and (13)] into new ones, the recurrence relation (6) can be converted into the differential equation

$$\frac{\partial}{\partial K}p_s^{Kz}(K,z) = -\frac{M}{p}\left(\frac{1-z^{-1}}{1-K} + \frac{1-p-z^{-1}}{N+K-1}\right)p_s^{Kz}(K,z) - \frac{Mz^{-1}}{p}\left(\frac{1}{1-K} + \frac{1}{N+K-1}\right) \times p_s^K(K,t=0). \quad (29)$$

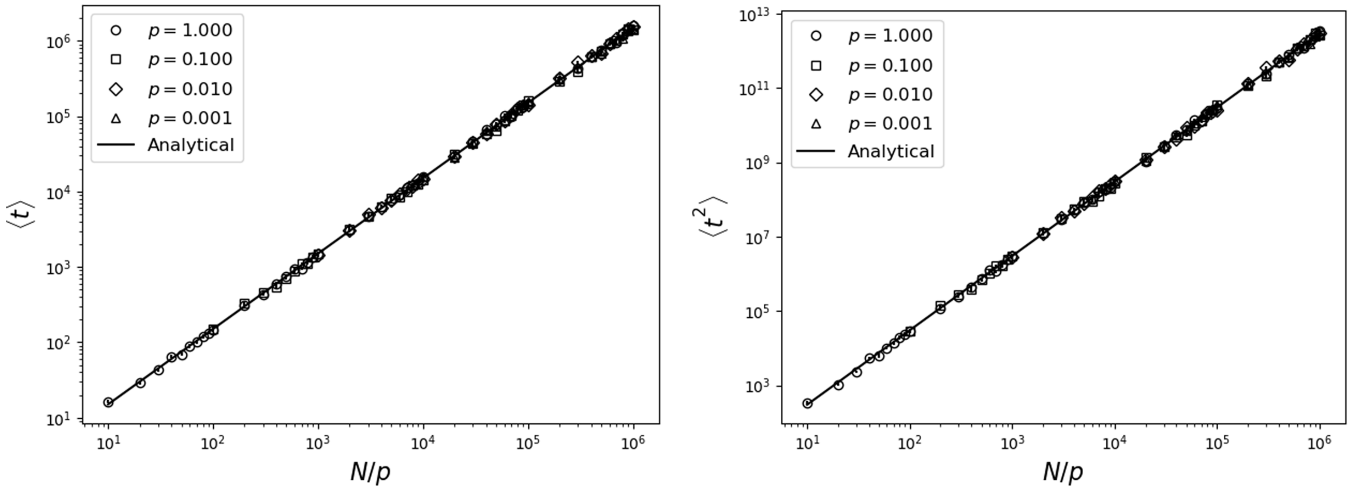

FIG. 1. The mean first (left) and second (right) moments of the first-passage time as a function of the ratio N/p for the dynamical version of the Erdős-Rényi model with $k_0 = 2$ and $k = 5$. The simulations used 100 samples and compared with the asymptotic results (27) and (28); the error bars are smaller than the size of the points.

Using the normalization condition $p_s^{Kz}(K = 1, z) = \frac{1}{1-z}$ and assuming $N \gg 1$, the solution of (29) can be cast as

$$p_s^{Kz}(K, z) = \frac{Mz^{-1}}{p}(1-K)^{-M\alpha}e^{-c(1+\alpha)(1-K)} \times \int_K^1 d\xi\, e^{c(1+\alpha)(1-\xi)}(1-\xi)^{M\alpha-1}p_s^z(\xi, t=0), \tag{30}$$

where

$$\alpha := \frac{1}{p}(z^{-1} - 1). \tag{31}$$

Then, returning back to the degree variable by inverting the transform (18) leads to

$$p_s^z(k|m; z) = \frac{1}{k!}\frac{\partial^k}{\partial K^k}\left[\frac{Mz^{-1}}{p}A_\alpha^{-1}(K) \times \int_K^1 d\xi(1-\xi)^{-1}A_\alpha(\xi)\xi^m\right]_{K\to 0}, \tag{32}$$

for $m \in \{k, k_0\}$ and

$$A_\alpha(K) := e^{c(1+\alpha)(1-K)}(1-K)^{M\alpha}. \tag{33}$$

The probability (32) is the key function to compute (14), which can be used to evaluate some quantities of interest through (17). This procedure is not smooth as it was in the case of the dynamical Erdős-Rényi model and the technicali-

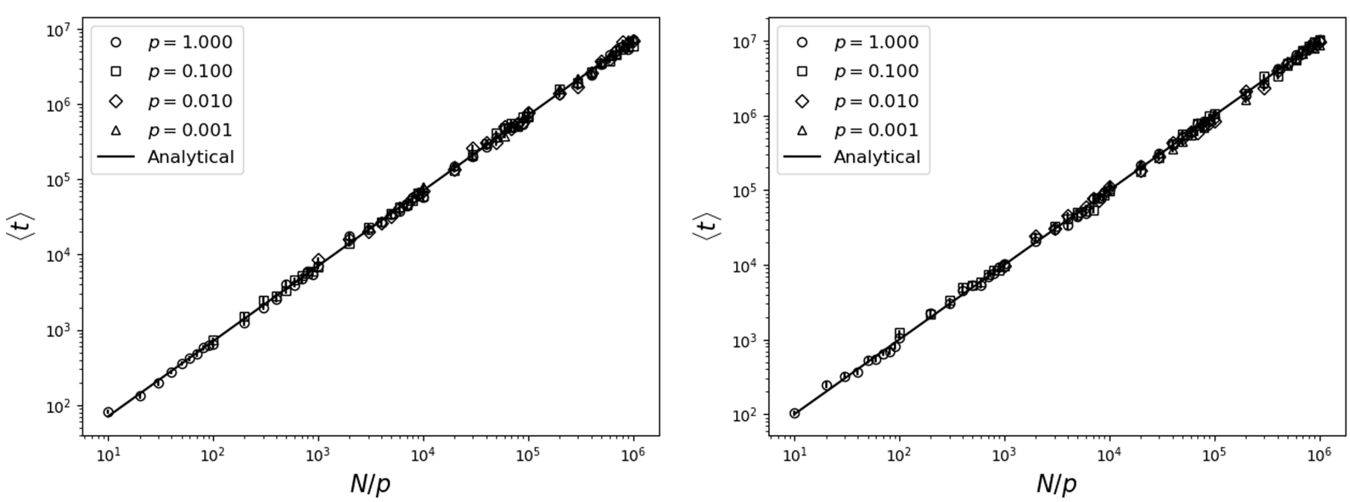

FIG. 2. The mean first-passage time as a function of the ratio N/p. Left: $k_0 = 2$ and $k = 5$; right: $k_0 = 5$ and $k = 2$. In both graphs, the mean degree of the network is $c = 4$ and the results were obtained from 100 samples; the error bar is smaller than the size of the points. These simulations were compared with the analytical result (35).

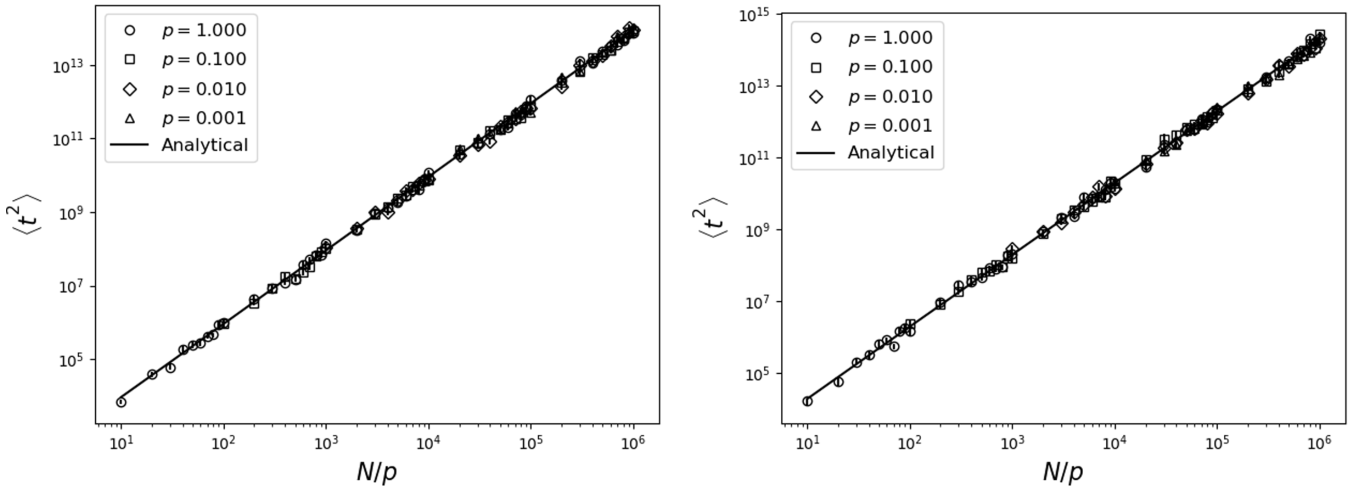

FIG. 3. The (mean) second moment of the first-passage time as a function of the ratio N/p. Left: $k_0 = 2$ and $k = 5$; right: $k_0 = 5$ and $k = 2$. In both graphs, the mean degree of the network is $c = 4$ and the results were obtained from 100 samples; the error bar is smaller than the size of the points. These simulations were compared with the analytical result (36).

ties are exposed in the Supplemental Material [37]. Here, we will show the results only.

The arrival probability in this model is

$$\mathcal{A}_{\mathrm{WS}} = 1, \tag{34}$$

as expected. No anomalous behavior as seen in the previous model is present here, where the degree changes by a single unit only at each time step.

The leading term of the first-passage time is

$$\langle t \rangle_{\mathrm{WS}} \sim \begin{cases} \frac{N}{p} e^c \sum_{n=k_0}^{k-1} \frac{\Gamma(n+1,c)}{c^n}, & k > k_0 \\ \frac{N}{p} e^c \sum_{n=k}^{k_0-1} \frac{\gamma(n+1,c)}{c^n}, & k < k_0 \end{cases}, \tag{35}$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are, respectively, the upper and lower incomplete Γ functions. It is worth mentioning that this time is also proportional to N/p, as in the dynamical Erdős-Rényi model. The simulation of this model supports the analytical expression (35), which is valid for $N \gg 1$, as shown in Fig. 2. One can also see that (35) depends on the initial (k_0) and final (k) degrees independently.

On the other hand, the leading contribution to the second moment is given by

$$\langle t^2 \rangle_{\mathrm{WS}} \sim \begin{cases} 2\left(\frac{N}{p}\right)^2 e^c \sum_{n=k_0}^{k-1} \frac{n!}{c^n} \sum_{\ell=0}^{n} \frac{c^\ell}{\ell!} \sum_{m=\ell}^{k-1} \frac{\Gamma(m+1,c)}{c^m}, & k > k_0 \\ 2\left(\frac{N}{p}\right)^2 e^c \sum_{n=k}^{k_0-1} \frac{n!}{c^n} \sum_{\ell=n+1}^{\infty} \frac{c^\ell}{\ell!} \sum_{m=k}^{\ell-1} \frac{\gamma(m+1,c)}{c^m}, & k < k_0 \end{cases}, \tag{36}$$

and is proportional to $(N/p)^2$. Like the asymptotic expression for the first moment (35), (36) depends also on k_0 and k independently. The validity of (36), which also assumes $N \gg 1$, was tested by comparing to simulation in Fig. 3. There is an alternative representation of (36) in the Supplemental Material [37], but the form given here seems to be the most compact one. Naturally, (36) and (35) can be used to compute the variance, which is also proportional to $(N/p)^2$. Since this expression shows no special aesthetic appeal, it will not be presented here.

V. CONCLUSION

In this work, we investigated the time needed for a vertex to achieve a preestablished degree for the first time. The main strategy was mapping the problem into a first-passage problem in degree space. The gain (loss) of degrees was illustrated by the time-dependent version of the Erdős-Rényi and Watts-Strogatz models, which display time-translational symmetry. This property was explored and analytical results concerning the first and second moments of the first-passage time were obtained. In both cases, the arrival probability ensured that the preestablished degree is achieved with probability 1 (with a careful interpretation in the case of the Erdős-Rényi dynamics). Furthermore, the mean first-passage time is scaled linearly with the ratio N/p for both models in the asymptotic regime of large networks, while this scale is quadratic for the second moment also in both models. Since the probability p controls the change in degrees, the factor $1/p$ is related to the time scale of the dynamics, and is present in our results.

As expected, the dynamics of the Erdős-Rényi model, which depends solely on two fixed parameters (total vertices N and linking probability p), is a trivial one in the sense that the first-passage moments is a function of the difference Δ between the final and initial degrees only. Watts-Strogatz network, on the other hand, does not display such simplicity: the transition rate associated with the degree dynamics of a given vertex depends on its state (degree). We have shown that in this case, the mean first-passage time and the second moment are not a function of the difference Δ anymore, and depend independently on the initial and final degrees. Furthermore, this characterization can be accomplished through a compact formula for the mean first-passage time, which then can be represented as a sum of terms involving upper (lower) incomplete Γ functions when the final degree is larger (smaller) than the initial one.

ACKNOWLEDGMENTS

F.A. was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES).

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