Fractional centralities on networks: Consolidating the local and the global

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We propose a new centrality incorporating two classical node-level centralities, the degree centrality and the information centrality, which are considered as local and global centralities, respectively. These two centralities have expressions in terms of the graph Laplacian L, which motivates us to exploit its fractional analog L^γ with a fractional parameter γ. As γ varies from 0 to 1, the proposed fractional version of the information centrality makes intriguing changes in the node centrality rankings. These changes could not be generated by the fractional degree centrality since it is mostly influenced by the local aspect. We prove that these two fractional centralities behave similarly when γ is close to 0. This result provides its complete understanding of the boundary of the interval in which γ lies since the fractional information centrality with $\gamma = 1$ is the usual information centrality. Moreover, our computation for the correlation coefficients between the fractional information centrality and the degree centrality reveals that the fractional information centrality is transformed from a local centrality into being a global one as γ changes from 0 to 1.

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I. INTRODUCTION

Centralities of nodes are network measures which quantify the importance of individual nodes [1]. Depending on the meaning of the importance, various centralities have been designed to identify central nodes of a network. The *degree* centrality, defined as the number of the neighbors of a node, is a widely used centrality measure. While the degree centrality only reflects the *local* property, classical centralities (for example, closeness [2], betweenness [3], and information [4] centralities) take into account *global* features. To incorporate network effects, both *direct* and *indirect* interactions should be appropriately taken into consideration. To address the relative importance of local and global influences, centralities with a *free parameter* have been introduced [5–10].

The *global* centrality that we focus on in this paper is the *information centrality* introduced by Stephenson and Zelen [4], which is based on the theory of statistical estimation. Brandes and Fleischer [11] called this centrality the current-flow closeness centrality since its definition is based on the theory of *electrical circuits*. Van Mieghem, Devriendt, and Cetinay used this centrality to identify the best spreader node [12]. The *effective resistance* between two distinct nodes is the reciprocal of the current between the nodes in the electrical flow driven by a unit voltage battery [13]. This quantity obeys the metric axioms and hence the resulting metric is referred to as the *resistance distance* [14]. It can be interpreted as *escape probability* via random walks on networks [15]. Stephenson and Zelen defined the *information* measure to be the reciprocal of the effective resistance between two nodes so that it measures how close each two nodes are, and introduced the *information centrality* of a node defined to be the harmonic sum of the effective resistances between the node and other nodes.

Based on a *fractional* analog L^γ of the graph Laplacian L, we propose a new centrality with a parameter γ, which we call *fractional information centrality*. These L and L^γ are discrete analogs of the Laplace operator [16] and the fractional Laplace operator [17,18], respectively. For each pair of nonadjacent nodes, while the corresponding entry of the graph Laplacian L is zero, that of the fractional Laplacian L^γ is not zero. These nonzero entries generate nonlocal dynamics concerning anomalous diffusion processes and random walks with long-range dynamics like Levy flights [19–21]. Also, the diagonal entry of the fractional Laplacian is called the *fractional degree centrality*, or γ-degree centrality as a fractional analog of the degree centrality. The electrical network associated with the fractional Laplacian gives rise to fractional information diffusion, which induces a fractional effective resistance and henceforth defines our new centrality.

We analyze γ-information centralities varying the parameter γ to elucidate how the degree centrality and the

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information centrality are connected by the new centralities. The γ-information centralities near $\gamma = 1$ behave similarly to the usual information centrality. In Theorem 1, we prove that at the other endpoint $\gamma = 0$, the derivative of the γ-information centrality of a node is equal to that of the fractional degree centrality up to scalar multiplication and addition. We demonstrate that any changes in the rankings with respect to γ-degree centralities are negligible. In Theorem 2, we also show that the ranks according to the fractional degree centrality remain unchanged as γ varies under certain conditions. Hence, we find that the γ-information centralities near $\gamma = 0$ behave similarly to the degree centrality. To show that the local relevance of γ-information centralities decreases as γ increases, we compute the correlations between the degree centrality and γ-information centralities. We provide parameters γ appropriate for a marriage network and a friendship network, where the former is mostly concerned with direct contacts and the latter requires consideration of indirect interactions as in Ref. [7]. We also present a network where the γ-information centrality works best when γ is neither near 0 nor 1.

The rest of the paper is organized as follows. Section II contains background on two classical centralities, the degree and information centralities. In Sec. III, we introduce the fractional versions of these centralities, which lead to Theorem 1 revealing their similar behaviors near $\gamma = 0$. In Sec. IV, we apply our fractional centralities to real-world networks and random networks, linking the two classical centralities. Finally, Sec. V addresses concluding comments and plans for future study. A proof of Theorem 2 and applications to two large real-world graphs are provided in two Appendices.

II. GRAPH LAPLACIANS, EFFECTIVE RESISTANCES, AND INFORMATION CENTRALITIES

Denote the set of $1, \ldots, n$ by $[n] := \{1, 2, \ldots, n\}$, which will be used as the set of nodes, and the diagonal matrix whose diagonal entries are $d_1, \ldots, d_n$ by $\mathrm{diag}(d_1, \ldots, d_n)$.

A. Graphs and Laplacians

Let $G = ([n], E)$ be a simple undirected loopless graph, where $[n]$ is the set of nodes and E is the set of edges. We assume that G is connected. Refer to Ref. [22] for basic notions concerning graphs and matrices.

To each edge $ij \in E$, we assign the weight $w_{ij} = w_{ji}$ which is referred to as the *conductance* between i and j. With this weighting, the graph G will be regarded as a weighted graph. The neighborhood $N(i)$ of a node $i \in [n]$ is $N(i) = \{ j \in [n] \,|\, ij \in E \}$. For each node $i \in [n]$, the degree centrality d_i (or shortly degree) is defined as $d_i = \sum_{j \in N(i)} w_{ij}$. If $w_{ij} = 1$ for all $ij \in E$, then the degree centrality counts the number of the neighbors of a node.

The *Laplacian* matrix L of G is an n-by-n matrix defined by

$$L_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -w_{ij} & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The *degree* matrix D of G is a diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ and the *adjacency* matrix A of G is a matrix whose (i, j) entry is w_{ij} if $ij \in E$, and 0 otherwise. Then the Laplacian matrix L of G is expressed as $L = D - A$. This matrix L is a discrete analog of the Laplace operator $(-\nabla^2)$, where ∇ is the gradient operator [16], Sec. 8.4.

B. Effective resistances and information centralities

Consider the electrical circuit whose underlying graph is G and each edge $ij \in E$ has the resistance given by the reciprocal of the weight w_{ij}. For two nodes $i, j \in [n]$, the *effective resistance* R_{ij} between i and j is defined to be the reciprocal of the current I_{ij} flowing from i to j when a unit potential difference is introduced between i and j [8,13]. This quantity R_{ij} is also referred to as the *resistance distance* between i and j [14]. Its reciprocal is known to be equal to *escape probability*. Refer to Refs. [22–24] for the effective resistance and its combinatorial interpretation, and Ref. [15] for its probabilistic aspects.

Let L^+ be the *pseudoinverse* of the Laplacian matrix L defined so that L^+L is the projector on the subspace spanned by all nonzero eigenvectors of L. Refer to Ref. [25] for the pseudoinverse, which is also called the Moore–Penrose pseudoinverse. The effective resistance R_{ij} is expressed in terms of L^+ as follows:

$$R_{ij} = L^+_{ii} + L^+_{jj} - 2L^+_{ij}.$$

Refer to Ref. [14], Theorem A for this expression.

We review *information centrality* introduced by Stephenson and Zelen [4]. They call the current I_{ij} the *information measure* I_{ij} between i and j for two distinct nodes $i, j \in [n]$ so that $I_{ij} = \frac{1}{R_{ij}}$. Also, note that $I_{ii} = \infty$.

The *information centrality* I_i of node $i \in [n]$ is defined to be the harmonic mean of the information measures between i and the other nodes, i.e.,

$$I_i = n\left(\sum_{j \in [n]} \frac{1}{I_{ij}}\right)^{-1}.$$

Equivalently, the centrality I_i is defined to be the reciprocal of the mean of effective resistances between i and the other nodes, i.e.,

$$I_i = \frac{n}{\sum_{j \in [n]} R_{ij}}.$$

Thus, it is also referred to as the *current-flow closeness centrality* [11] as a current-flow analog of closeness centrality [2].

The centrality I_i can be expressed via the pseudoinverse of L as follows (see Ref. [26], Eq. (15) or Ref. [12], Eq. (27)):

$$I_i = \frac{n}{\sum_{j \in [n]} R_{ij}} = \frac{n}{n L^+_{ii} + \mathrm{tr}(L^+)}. \tag{1}$$

Note that this expression will be employed to prove the theorem (Theorem 1) connecting the degree centrality and the information centrality.

III. FRACTIONAL CENTRALITIES ON NETWORKS

A. Fractional degrees and fractional Laplacians

Let γ be a fractional parameter with $0 \leqslant \gamma \leqslant 1$. A fractional analog L^γ of the Laplacian L is defined as follows [20]. This L^γ is a discrete version of the fractional Laplace operator [17,18].

The Laplacian L has a spectral decomposition since the matrix L is symmetric. Let $\{\Psi_i\}_{i=1}^n$ and $\{0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_n\}$ be orthonormal eigenvectors of L and the corresponding eigenvalues, respectively, i.e.,

$$L\Psi_i = \mu_i \Psi_i, \text{ and } \Psi_i^T \Psi_j = \delta_{ij}, \text{ for } i, j = 1, \dots, n.$$

Since we assume that graphs are connected, the smallest eigenvalue $\mu_1 = 0$ and $\mu_i > 0$ for $i = 2, \dots, n$. Then $\Psi_1 = \mathbf{1}/\sqrt{n}$, where $\mathbf{1}$ is all-ones vector of size n. Let $Q = (q_{ij})_{i,j\in[n]}$ be the matrix whose columns are $\Psi_1, \Psi_2, \dots, \Psi_n$ so that Q is an orthogonal matrix, and let Λ be the diagonal matrix whose diagonal entries are $0, \mu_2, \dots, \mu_n$: $\Lambda = \mathrm{diag}(0, \mu_2, \dots, \mu_n)$. Then $L = Q\Lambda Q^{-1} = Q\Lambda Q^T$ holds. The *fractional* Laplacian L^γ is defined to be the Laplacian L to the power γ, i.e.,

$$L^\gamma = Q\Lambda^\gamma Q^T,$$

where $\Lambda^\gamma = \mathrm{diag}(0, \mu_2^\gamma, \dots, \mu_n^\gamma)$. Since $\mathbf{1}$ is an eigenvector of L^γ with eigenvalue 0, it holds that for each $i \in [n]$,

$$\sum_{j=1}^n (L^\gamma)_{ij} = 0. \tag{2}$$

Another important property [20,27] for L^γ is that its nondiagonal entries are negative, i.e., for distinct nodes $i, j \in [n]$,

$$(L^\gamma)_{ij} < 0.$$

The diagonal entry $(L^\gamma)_{ii}$ of L^γ is called the (fractional) γ-*degree centrality* of node i. This invariant is denoted by $d_i^{(\gamma)}$ and expressed as

$$d_i^{(\gamma)} = \sum_{\alpha=2}^n q_{i\alpha}^2 \mu_\alpha^\gamma. \tag{3}$$

Take distinct integers $i, j \in [n]$. Let $w_{ij}^{(\gamma)}$ be the absolute value of the (i, j) entry of L^γ, i.e.,

$$w_{ij}^{(\gamma)} = |(L^\gamma)_{ij}| = -(L^\gamma)_{ij}.$$

This quantity $w_{ij}^{(\gamma)}$ is a fractional analog of the entry w_{ij} of the adjacency matrix A. Equation (2) can be written as

$$d_i^{(\gamma)} = \sum_{j\neq i} w_{ij}^{(\gamma)}. \tag{4}$$

Let $s = \max_{i\in[n]} d_i$, $\gamma \in [0, 1]$, and $i \in [n]$. To apply a Taylor series approximation, we define

$$l_i^{(\gamma)} = (d_i + 1)^{\gamma-1} d_i \text{ and}$$

$$r_i^{(\gamma)} = s^\gamma \left[1 - \gamma + \frac{\gamma\, d_i}{s} + \binom{\gamma}{2}\left(1 - \frac{2d_i}{s} + \frac{d_i^2 + d_i}{s^2}\right)\right].$$

We show that the γ-degree centrality $d_i^{(\gamma)}$ of node i satisfies the following inequality:

$$l_i^{(\gamma)} \leqslant d_i^{(\gamma)} \leqslant r_i^{(\gamma)}. \tag{5}$$

Its proof using a technique in Refs. [27,28] is given in Appendix A 1. When a graph G is regular (i.e., every node has the same degree), inequality (5) was obtained in Ref. [28].

B. Fractional effective resistances and information centralities

We now introduce fractional analogs of effective resistance and information centrality.

Definition 1 (Fractional effective resistance). For a fractional parameter γ with $0 \leqslant \gamma \leqslant 1$, the γ-*effective resistance* between two nodes $i, j \in [n]$ is defined to be

$$R_{ij}^\gamma = (L^\gamma)_{ii}^+ + (L^\gamma)_{jj}^+ - 2(L^\gamma)_{ij}^+,$$

where $(L^\gamma)^+$ is the pseudoinverse of the fractional Laplacian L^γ.

The *probabilistic interpretation* for R_{ij}^γ as a fractional analog of the escape probability $P^\gamma(i \to j)$ [15] is given as follows. Let x be a particle on nodes. Using Eq. (4), we suppose that at node i, the particle x moves to its neighbor j with probability $w_{ij}^\gamma/(d_i^{(\gamma)})$. Define the *fractional escape probability* $P^\gamma(i \to j)$ to be the probability that x visits j before returning to i. Then the probability $P^\gamma(i \to j)$ is given by

$$P^\gamma(i \to j) = \frac{1}{d_i^{(\gamma)} R_{ij}^\gamma}.$$

We define a new centrality, called *fractional information centrality*, as a fractional analog of information centrality introduced by Stephenson and Zelen [4].

Definition 2 (Fractional information centrality). Let γ be a parameter with $0 \leqslant \gamma \leqslant 1$. For two nodes $i, j \in [n]$, its (fractional) γ-*information measure* $I_{ij}^{(\gamma)}$ between i and j is defined as the reciprocal of the γ-effective resistance R_{ij}^γ between i and j. The (fractional) γ-*information centrality* $I_i^{(\gamma)}$ of node $i \in [n]$ is defined to be the harmonic mean of γ-information measures between i and the other nodes, i.e.,

$$I_i^{(\gamma)} = n\left(\sum_{j\in[n]} \frac{1}{I_{ij}^{(\gamma)}}\right)^{-1} = \frac{n}{\sum_{j\in[n]} R_{ij}^\gamma}.$$

Let us express the fractional information centrality $I^{(\gamma)}$ in terms of the pseudoinverse $(L^\gamma)^+$ of the fractional Laplacian L^γ. Note that $(L^\gamma)^+$ is given by

$$(L^\gamma)^+ = Q\Lambda^{-\gamma} Q^T.$$

For $i, j \in [n]$, its (i, j) component is

$$(L^\gamma)_{ij}^+ = \sum_{\alpha=2}^n q_{i\alpha}\, \mu_\alpha^{-\gamma}\, q_{j\alpha}.$$

It follows from $\sum_{j\in[n]} (L^\gamma)_{ij}^+ = 0$ that

$$\begin{aligned}\sum_{j\in[n]} R_{ij}^\gamma &= \sum_{j\in[n]} [(L^\gamma)_{ii}^+ + (L^\gamma)_{jj}^+ - 2(L^\gamma)_{ij}^+] \\ &= \sum_{j\in[n]} (L^\gamma)_{ii}^+ + \sum_{j\in[n]} (L^\gamma)_{jj}^+ - 2\sum_{j\in[n]} (L^\gamma)_{ij}^+ \\ &= n\,(L^\gamma)_{ii}^+ + \mathrm{tr}[(L^\gamma)^+].\end{aligned}$$

As in Eq. (1), the γ-information centrality $I_i^{(\gamma)}$ of node i is expressed as follows:

$$I_i^{(\gamma)} = \frac{n}{\sum_{j\in[n]} R_{ij}^{\gamma}} = \frac{n}{n\,(L^{\gamma})_{ii}^{+} + \mathrm{tr}[(L^{\gamma})^{+}]} = \frac{n}{n\,\sum_{\alpha=2}^{n} q_{i\alpha}^2\,\mu_\alpha^{-\gamma} + \sum_{\alpha=2}^{n}\mu_\alpha^{-\gamma}}. \tag{6}$$

Note that when $\gamma = 1$, the γ-information centrality $I_i^{(\gamma)}$ of node i is equal to the (usual) information centrality I_i.

We now suppose that $\gamma = 0$. It follows from the fact that $\Psi_1 = \mathbf{1}/\sqrt{n}$ and Q is an orthogonal matrix that

$$L^0 = Q\,\mathrm{diag}(0, 1, \ldots, 1)Q^T = \mathrm{Id} - J/n,$$

where Id denotes an identity matrix and J is the all-ones n-by-n matrix which equals $\mathbf{1}\mathbf{1}^T$. For each $i \in [n]$, the 0-degree $d_i^{(0)}$ of node i equals

$$d_i^{(0)} = 1 - \frac{1}{n}. \tag{7}$$

The pseudoinverse of L^0 is itself

$$L^0 = \mathrm{Id} - J/n.$$

Therefore, when $\gamma = 0$, all pairs of distinct nodes have the *same* effective resistance $R_{ij}^0 = 2$, and all nodes have the *same* information centrality with value

$$I_i^{(0)} = \frac{n}{2n-2}. \tag{8}$$

C. Derivatives of fractional degree centralities and fractional information centralities

We show that the derivative of the γ-information centrality evaluated at $\gamma = 0$ is equal to that of the γ-degree centrality up to scalar multiplication and addition. Our result says that these two centralities, $d_i^{(\gamma)}$ and $I_i^{(\gamma)}$, behave *similarly* near $\gamma = 0$. By Eqs. (7) and (8), all nodes have the same value for both centralities, and hence the derivatives at $\gamma = 0$ can be used as their approximations. Note that the result is motivated by the fact that expressions [Eqs. (3) and (6)] for the fractional degree centralities $d_i^{(\gamma)}$ and information centralities $I_i^{(\gamma)}$ contain terms $\sum_{\alpha=2}^{n} q_{i\alpha}^2\,\mu_\alpha^{\gamma}$ and $\sum_{\alpha=2}^{n} q_{i\alpha}^2\,\mu_\alpha^{-\gamma}$, respectively.

Theorem 1. For each node i, the derivative of the γ-information centrality of node i evaluated at $\gamma = 0$ is equal to

$$\frac{d}{d\gamma}\big(I_i^{(\gamma)}\big)\Big|_{\gamma=0} = \frac{n^2}{(2n-2)^2}\left\{\frac{d}{d\gamma}\big(d_i^{(\gamma)}\big)\Big|_{\gamma=0} + \frac{\log\big(\mathrm{pdet}(L)\big)}{n}\right\}, \tag{9}$$

where pdet(L) is the product of all nonzero eigenvalues of L.

Proof. Let us differentiate $d_i^{(\gamma)}$ [Eq. (3)] and evaluate it at $\gamma = 0$ to obtain

$$\frac{d}{d\gamma}\big(d_i^{(\gamma)}\big)\Big|_{\gamma=0} = \left[\sum_{\alpha=2}^{n} q_{i\alpha}^2 \log(\mu_\alpha)\mu_\alpha^{\gamma}\right]\Bigg|_{\gamma=0} = \sum_{\alpha=2}^{n} q_{i\alpha}^2 \log(\mu_\alpha).$$

Then we differentiate the denominator of Eq. (6) and evaluate it at $\gamma = 0$ to obtain

$$\begin{aligned}\frac{d}{d\gamma}&\left(n\sum_{\alpha=2}^{n} q_{i\alpha}^2\mu_\alpha^{-\gamma} + \sum_{\alpha=2}^{n}\mu_\alpha^{-\gamma}\right)\Bigg|_{\gamma=0}\\ &= \left[-n\sum_{\alpha=2}^{n} q_{i\alpha}^2 \log(\mu_\alpha)\mu_\alpha^{-\gamma} - \sum_{\alpha=2}^{n}\log(\mu_\alpha)\mu_\alpha^{-\gamma}\right]\Bigg|_{\gamma=0}\\ &= -n\sum_{\alpha=2}^{n} q_{i\alpha}^2 \log(\mu_\alpha) - \sum_{\alpha=2}^{n}\log(\mu_\alpha)\\ &= -n\big(d_i^{(0)}\big)' - \log[\mathrm{pdet}(L)].\end{aligned}$$

Since the 0-information centrality I_i^0 of node i equals $I_i^0 = n/(2n-2)$ from Eq. (8), we derive our desired identity (9):

$$\big(I_i^{(0)}\big)' = \frac{n}{(2n-2)^2}\big\{n\cdot\big(d_i^{(0)}\big)' + \log[\mathrm{pdet}(L)]\big\}.$$

■

IV. ANALYZING NETWORKS VIA FRACTIONAL CENTRALITIES

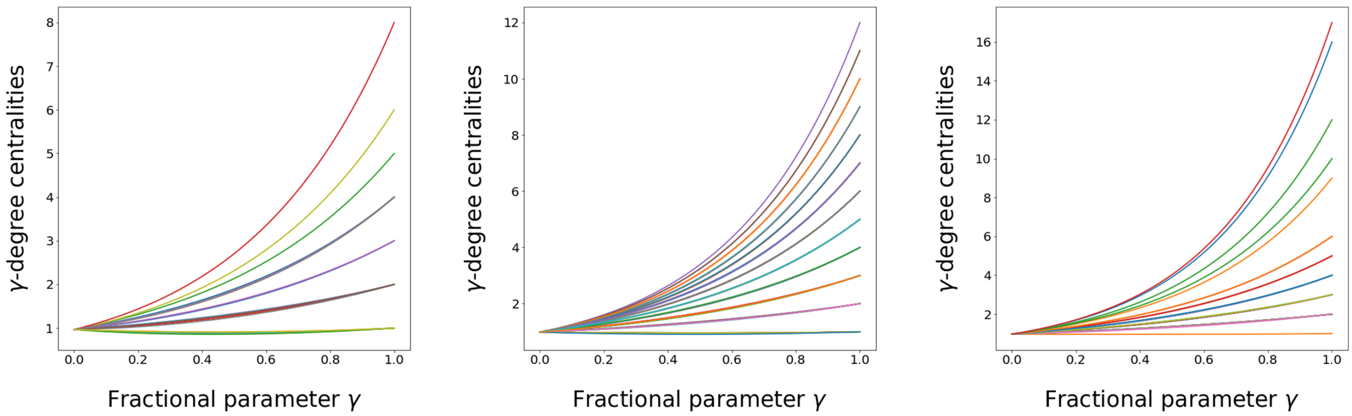
We apply the fractional centralities discussed in the previous section to real-world complex networks and random network models. Note that only connected graphs will be taken from these networks.

The following three networks derived from real-world data will be considered: The friendship network [29] collected by Van de Bunt, the Karate club network [30], and the Dolphin network [31]. For short, these networks will be abbreviated as friendship, karate, and dolphin, respectively.

We *carefully* analyze the Van de Bunt's *friendship network* to elucidate meaning of the fractional centralities. The friendship network depicted in Fig. 2(a) has 32 nodes and 43 edges. The network obtained by ignoring isolated nodes (5, 12, and 18) is a connected network. In what follows, the resulting network will be referred to as the friendship network.

Two more real-world complex networks together with *outcome* variables for nodes of the networks will be also considered. One is the classic *marriage* network among Florentine families with the *attribute wealth* which represents the richness of each family [32]. The other is the *EIES* friendship network with the *citations* of network researchers [33]. Also, we will analyze the Hollywood film music network [34] with 62 producers and 40 composers. In this network, the top 5 composers according to their incomes are identified.

Random network models we will employ are Barabási-Albert graphs [35] and Watts-Strogatz graphs [36]. The notation BA(n, m) stands for a Barabási-Albert graph with



(a) Friendship (b) Karate (c) Dolphin

FIG. 1. Fractional degree centralities of all the nodes of the real-world networks.

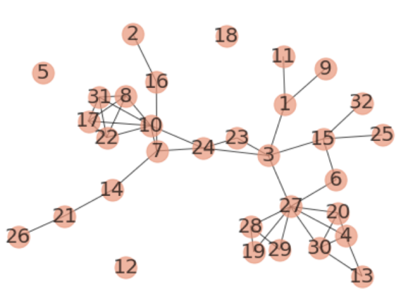
n nodes and m edges attached between a new node and existing nodes, and the notation WS(n, k, p) stands for an Watts-Strogatz graph with n nodes, the number k of nearest neighbors of each node in a ring topology and the probability p of rewiring each edge.

A. Fractional degree centralities

The 1-degree centralities, which are equal to the (usual) degree centralities, for the friendship network are given in Fig. 2. The network have seven different values of degree centralities. Node 27 has the highest degree 8, node 10 has the second highest degree 6, and node 3 has the third highest degree 5. Nodes 4, 7, 8, 15, 17, 22, 24, 30, and 31 have the same degree 4.

Fractional degree centralities for the friendship network, the karate network and the dolphin network are given in Figs. 1(a)–1(c), respectively. Although these figures contain all the curves for the centralities of 29 nodes, 34 nodes, and 62 nodes, they appear to contain just 7 curves, 12 curves, and 11 curves. This is because nodes of the *same* degree have *almost* the same fractional degree centralities. To explain this statement, let us calculate fractional degree centralities of nodes of the friendship network. For each integer d and a parameter $\gamma \in [0, 1]$, denote by $\mathcal{I}(d, \gamma)$ the collection of γ-degree centralities of nodes of degree d, i.e.,

$$\mathcal{I}(d, \gamma) = \{ d_i^{(\gamma)} | d_i = d \}.$$



Degree	Nodes
1	2,9,11,25,26,32
2	6,13,14,16,19,21,23,29
3	1,20,28
4	4,7,8,15,17,22,24,30,31
5	3
6	10
7	
8	27

(a) The friendship network (b) Degree centralities

FIG. 2. The friendship network among 32 Dutch students by Van de Bun.

The *diameter* of a finite subset $\mathcal{I}$ of real numbers is defined to be diam $\mathcal{I} = \max \mathcal{I} - \min \mathcal{I}$. Then we compute

$$\max_{\gamma \in [0,1]} \operatorname{diam} \mathcal{I}(1, \gamma) = 0.053 \cdots,$$
$$\max_{\gamma \in [0,1]} \operatorname{diam} \mathcal{I}(2, \gamma) = 0.080 \cdots,$$
$$\max_{\gamma \in [0,1]} \operatorname{diam} \mathcal{I}(3, \gamma) = 0.023 \cdots,$$
$$\max_{\gamma \in [0,1]} \operatorname{diam} \mathcal{I}(4, \gamma) = 0.042 \cdots,$$

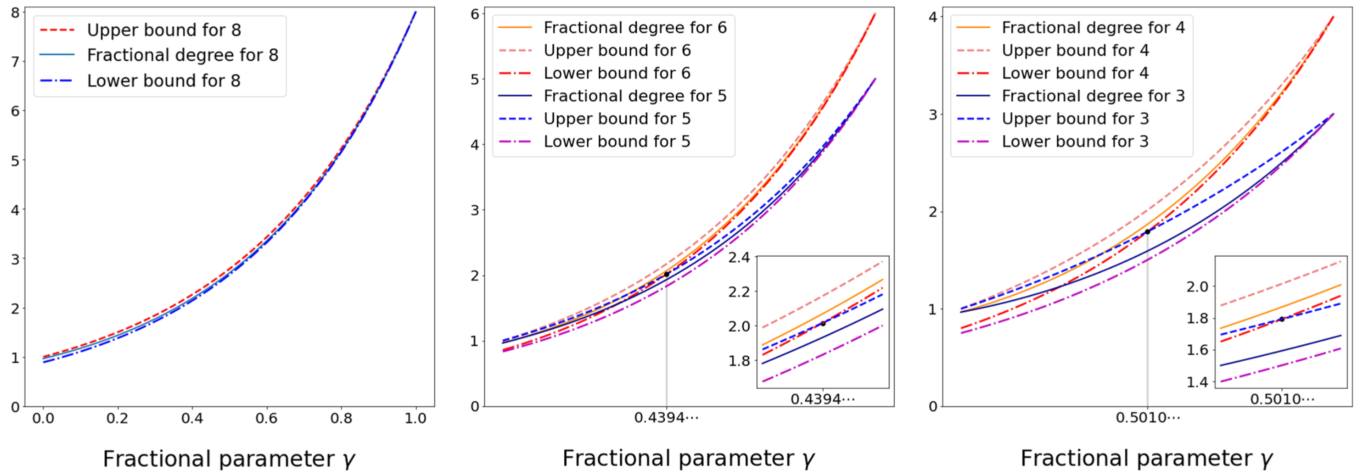
which are significantly smaller than $0.8697 \cdots$, the minimum of fractional degree centralities of all nodes over γ with $0 \leqslant \gamma \leqslant 1$.

The fractional degree centralities have been utilized to distinguish nodes of regular graphs arising in mathematical chemistry as molecules [28]. In Ref. [28], Chapter 2.3, the authors considered a fullerene regarded as a degree-3 graph with 26 nodes, and divided its nodes into four groups according to their fractional degree centralities at $\gamma = 0.9$. However, their fractional degree centralities are approximately 1.625 and their differences are "minuscule," being at most 0.001, as we figured out in Fig. 1. Since the main goal of introducing centralities is to evaluate the importance of nodes, these *slight* differences are not enough to capture the importance of nodes distinct from the degree centrality.

We next explore fractional degree centralities of nodes of *different* degrees. From Figure 1, we observe that a node j having a greater degree centrality than another node i has a greater fractional degree centrality than the node i, i.e.,

$$\text{if } d_j > d_i, \text{ then } d_j^{(\gamma)} > d_i^{(\gamma)},$$

for γ with $0 < \gamma \leqslant 1$. By virtue of inequality (5), the above statement can be shown under some assumptions for degrees d_i and parameter γ (see Theorem 2). Let us take nodes of degrees 8,6,5, and 4 from the Friendship graph. In Figs. 3(a)–3(c), we plot the fractional degree centralities for the node of degree 8, the nodes of degrees 5 and 6 and the nodes of degrees 3 and 4, respectively, together with their lower bounds and upper bounds given in inequality (5). Figure 3(a) shows that the lower bound and upper bound for the highest degree are good estimates. In Fig. 3(b) [respectively, Fig. 3(c)], the



(a) Degree 8 (b) Degrees 5 and 6 (c) Degrees 3 and 4

FIG. 3. Fractional degree centralities, lower bounds, and upper bounds given in inequality (5).

lower bound for the node with a greater degree is greater than the upper bound for the node with a smaller degree, for parameters γ with $0.4395 \leqslant \gamma \leqslant 1$ (respectively, $0.5011 \leqslant \gamma \leqslant 1$).

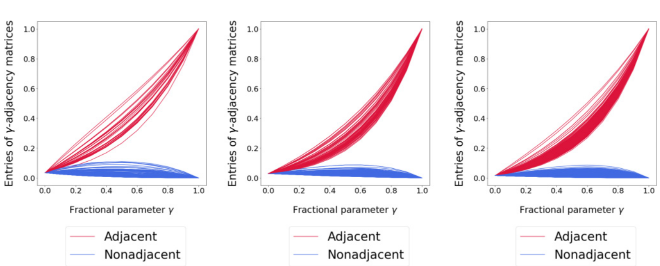
Theorem 2. Let G be an unweighted graph with the maximum degree $s = \max_{i\in[n]} d_i$. Then there is a constant $c(s)$ with $0 < c(s) < 1$ such that for all $1/2 \leqslant \gamma < 1$, if $d_j > d_i \geqslant sc(s)$, then $d_j^{(\gamma)} > d_i^{(\gamma)}$.

Proof. See Appendix A 2. ■

We have seen that although the parameter γ varies, the ranks for fractional degree centralities of nodes of *different* degrees are rarely reversed. Therefore, the fractional degree centrality is similar to the usual degree centrality for the purpose of node centralities.

Finally, we plot entries of fractional adjacency matrices for the real-world complex networks in Figs. 4(a)–4(c). Those entries for adjacent nodes and nonadjacent nodes correspond to red curves and blue curves, respectively. Red curves decrease as γ decreases, which means that *direct* interactions between adjacent nodes are weakened. Blue curves increase as γ decreases to about 0.5, and then decrease as γ decreases to 0. This indicates that *indirect* interactions between nonadjacent nodes are strengthened and then weakened as γ decreases from 1 to 0.

Figure 4 shows that almost all red curves lie above blue curves with a few blue points lying above red points near $\gamma = 0$. This indicates that *nonlocal* dynamics induced by fractional adjacency matrices are not sufficient to alter the importances



(a) Friendship (b) Karate (c) Dolphin

FIG. 4. Entries of the fractional adjacency matrices of the real-world networks.

of nodes govern by the direct interactions. That is, fractional degree centralities of nodes are largely determined by adjacencies of nodes, or their degree centralities.

B. Fractional information centralities

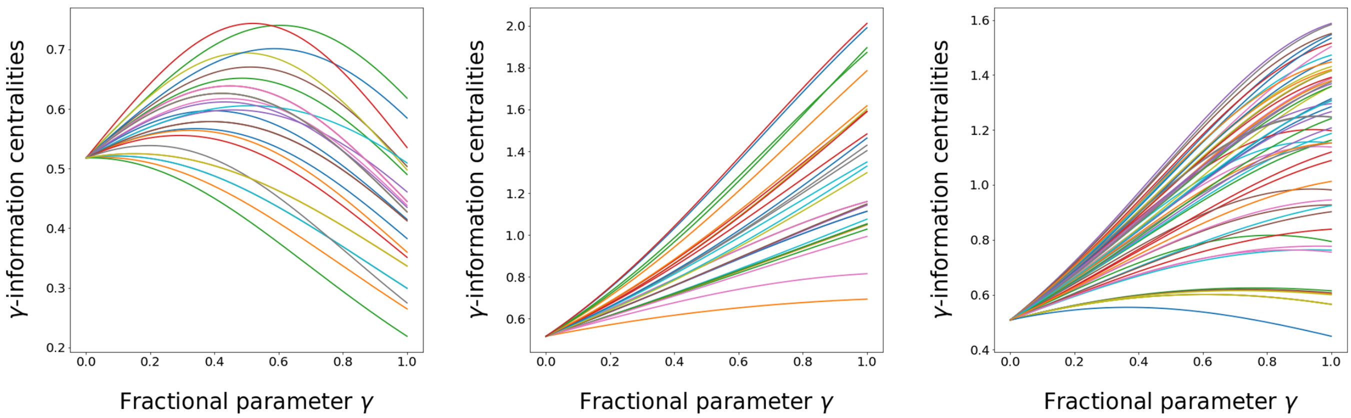
We compute fractional information centralities of nodes of the real-world complex networks and plot these values in each network [Figs. 5(a)–5(c)]. Unlike the case for fractional degree centralities, the ranks of the nodes according to γ-information centralities are *reversed* at myriad values of γ.

To point out the change of ranks according to γ-information centralities, we extract *top* and *bottom* nodes of the friendship network ranked according to information centralities and plot γ-information centralities of those nodes as our parameter γ changes in Figs. 6(a) and 6(b), respectively.

The top 6 nodes ranked according to information centralities are nodes 3, 24, 27, 23, 7, and 10 [Fig. 6(a)]. Node 3 has the largest γ-information centrality for γ with $0.5782 \leqslant \gamma \leqslant 1$, and node 23 has the largest γ-information centrality for γ with $0 < \gamma \leqslant 0.5781$. At $\gamma = 0.7727\cdots$, the ranks of nodes 24 and 27 are reversed. Node 23 ranks fourth for the γ-information centrality with $\gamma = 1$, while the node gets lower ranks as γ decreases, ranking 12th for the γ-information centrality near $\gamma = 0$. Node 10 ranks 6th for the γ-information centrality with $\gamma = 1$, while the node gets a higher rank as γ decreases, ranking second for the γ-information centrality near $\gamma = 0$.

Except for the isolated nodes, nodes 26, 2, 21, 9, 11, 25, and 32 are the bottom 7 nodes ranked according to information centralities [Fig. 6(b)]. The nodes except node 21 are of degree 1, and node 21 is of degree 2. Node 21 ranks third to last for the γ-information centrality with $\gamma = 1$ but gets a higher rank as γ decreases. The rank of node 21 and ranks of node 9 and node 11 are reversed at $\gamma = 0.7583\cdots$, and the rank of node 21 and ranks of node 25 and node 32 are reversed at $\gamma = 0.5141\cdots$. Then at γ less than 0.5141, node 21 of degree 2 precedes all the nodes of degree 1.

From Figs. 6(a) and 6(b) near $\gamma = 0$, we observe that ranks of nodes according to γ-information centrality near $\gamma = 0$ are



(a) Friendship (b) Karate (c) Dolphin

FIG. 5. Fractional information centralities of all the nodes of the real-world networks.

similar to their degree centralities. Three nodes 27, 10, and 3 are ranked at the top three for the γ-information centrality near $\gamma = 0$, and so do they for their degree centralities. Among the nodes ranked at the bottom, the only node of degree 2 is the node 21 which has a higher rank than the others of degree 1. The behavior of the γ-information centralities near $\gamma = 0$ will be further discussed in the next section.

To figure out that *structural* measures of networks are concerned with the changes in the node centrality rankings, we count *intersections* for curves of γ-information centralities with $0 < \gamma < 1$ and compare the numbers of the intersections with those measures. Since it is observed from Figs. 5(a), 5(b) and 5(c) that distinct two curves with $\gamma > 0$ have at most one intersection, we instead consider the cardinality of the following collection as an estimation of the number of intersections:

$$\left\{ \{i, j\} \mid i, j \in [n],\ \left(I_i^{(1)} - I_j^{(1)}\right)\left(I_i^{(0.1)} - I_j^{(0.1)}\right) < 0 \right\},$$

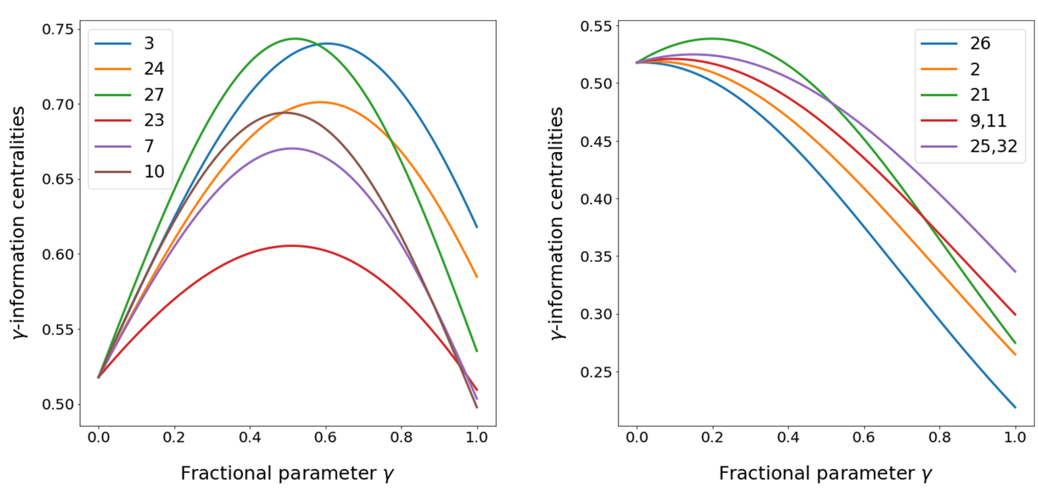
where we restrict the range of γ to $0.1 \leqslant \gamma \leqslant 1$ since γ-information centralities have the same value at $\gamma = 0$. The structural measures we will consider are *network density* and *algebraic connectivity*. The network density of a graph with n nodes and the edge set E is

$$\frac{|E|}{\binom{n}{2}} = \frac{2|E|}{n(n-1)}.$$

The algebraic connectivity of a graph is defined as the *smallest* nonzero eigenvalue of its Laplacian [37].

We observe that the *lower* density a network has, the *more* intersections the network has. In fact, the friendship network and the Karate club network have 32 and 15 intersections, respectively. The network density of the friendship network equaling 0.1059 approximately is less than that of the Karate club network equaling 0.1390 approximately. The numbers of intersections and densities of random network models are documented in Table I. For Barabási-Albert graphs and Watts-Strogatz graphs with the *same probability* p of rewiring each edge, we see that the *lower* density a network has, the *more* intersections the network has. However, since WS(200, 2, 0.001), WS(200, 2, 0.05), and WS(200, 2, 0.8) have the *same* network density, we need to consider *another* measure for explaining the changes in the rankings.

The measure we will employ is *algebraic connectivity*, which serves as *decentralized* estimation [38]. From Table I,



(a) Top rankers (b) Bottom rankers

FIG. 6. Top and bottom nodes ranked according to γ-information centralities on the friendship network.

TABLE I. The number of intersections (IS) for fractional information centralities, edge densities (ED), and algebraic connectivities (AC) with 100 trials for each random network, where these values are represented by (mean ± std) with $n = 200$.

	IS	ED	AC
BA(n,1)	$1.9 \times 10^2 \pm 4.5 \times 10^1$	0.04	$1.1 \times 10^{-2} \pm 3.8 \times 10^{-3}$
BA(n,2)	$1.1 \times 10^1 \pm 5.3 \times 10^0$	0.08	$5.8 \times 10^{-1} \pm 3.1 \times 10^{-2}$
BA(n,3)	$5.7 \times 10^0 \pm 2.7 \times 10^0$	0.12	$1.3 \times 10^{0-} \pm 1.0 \times 10^{-1}$
WS(n,2,0.05)	$4.2 \times 10^3 \pm 1.0 \times 10^3$	0.01	$5.0 \times 10^{-1} \pm 7.8 \times 10^{-2}$
WS(n,12,0.05)	$2.0 \times 10^3 \pm 3.8 \times 10^2$	0.06	$1.7 \times 10^{0-} \pm 1.3 \times 10^{-1}$
WS(n,24,0.05)	$6.5 \times 10^2 \pm 1.8 \times 10^2$	0.12	$3.6 \times 10^{0-} \pm 2.1 \times 10^{-1}$
WS(n,5,0.05)	$3.0 \times 10^3 \pm 3.5 \times 10^2$	0.02	$2.7 \times 10^{-5} \pm 1.3 \times 10^{-5}$
WS(n,5,0.3)	$1.0 \times 10^3 \pm 2.1 \times 10^2$	0.02	$6.3 \times 10^{-5} \pm 2.3 \times 10^{-5}$
WS(n,5,0.8)	$2.1 \times 10^2 \pm 4.2 \times 10^1$	0.02	$2.0 \times 10^{-4} \pm 8.7 \times 10^{-5}$

we see that for Watts-Strogatz graphs WS(n, m, p), as a network has *higher* probability p, the network has *lower* algebraic connectivity and *more* intersections. This indicates that increasing the rewiring probability forces a network to be more centralized and have more changes of ranks of nodes as parameters γ vary.

C. Connecting degree centralities and information centralities

We investigate how the proposed fractional information centralities are related to classical centralities. The γ-information centrality with $\gamma = 1$ is the information centrality, one of well-known global centralities, introduced by Stephenson and Zelen [4]. Thus, γ-information centralities for γ near 1 reflect the global structure of a given network.

To explain the behavior of γ-information centralities $I_i^{(\gamma)}$ of node i for γ near 0, we consider its derivative at $\gamma = 0$. This quantity is denoted by $(I_i^{(0)})'$. Our theorem (Theorem 1) shows that near $\gamma = 0$, γ-information centralities I_i^{γ} behave *similarly* to γ-degrees $d_i^{(\gamma)}$. Since γ-degree centralities $d_i^{(\gamma)}$ are almost the same centralities as degree centralities d_i by the observation in Sec. IV A, we see that γ-information centralities I_i^{γ} near $\gamma = 0$ behave *similarly* to the degree centralities d_i, one of *local* centralities.

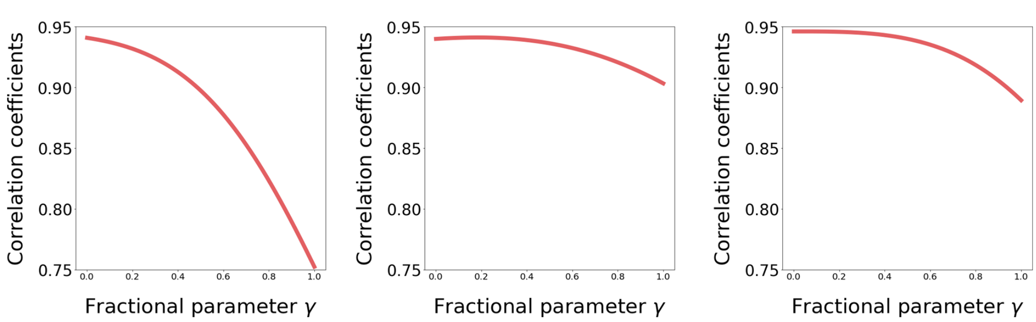
We measure *(Pearson) correlation coefficients* [39] between degree centralities and γ-information centralities on real-world networks (the friendship network, the Karate club network, and the dolphin network [Fig. 7(a)–7(c)]) and random networks (Barabási-Albert graphs and Watts-Strogatz graphs). Graphs plotted in Fig. 8(a) correspond to BA(1000, 1), BA(1000, 2), BA(1000, 3), graphs plotted in Fig. 8(b) correspond to WS(1000, 2, 0.05), WS(1000, 12, 0.05), WS(1000, 24, 0.05) and graphs plotted in Fig. 8(c) correspond to WS(1000, 2, 0.001), WS(1000, 2, 0.05), WS(1000, 2, 0.8).

From these figures, we observe that correlation coefficients between degree centralities and γ-information centralities *decrease* as the parameter γ *increases*. In particular, the γ-information centrality near $\gamma = 0$ is highly correlated with the degree centrality, while the γ-information centrality near $\gamma = 1$ has a lower correlation with the degree centrality. These computations tell us that the γ-information centralities near $\gamma = 0$ and at near $\gamma = 1$ are *local* and *global* centralities, respectively. Therefore, for varying γ, the γ-information centralities have the aspects of the degree centrality and the information centrality.

D. Choosing appropriate parameters γ

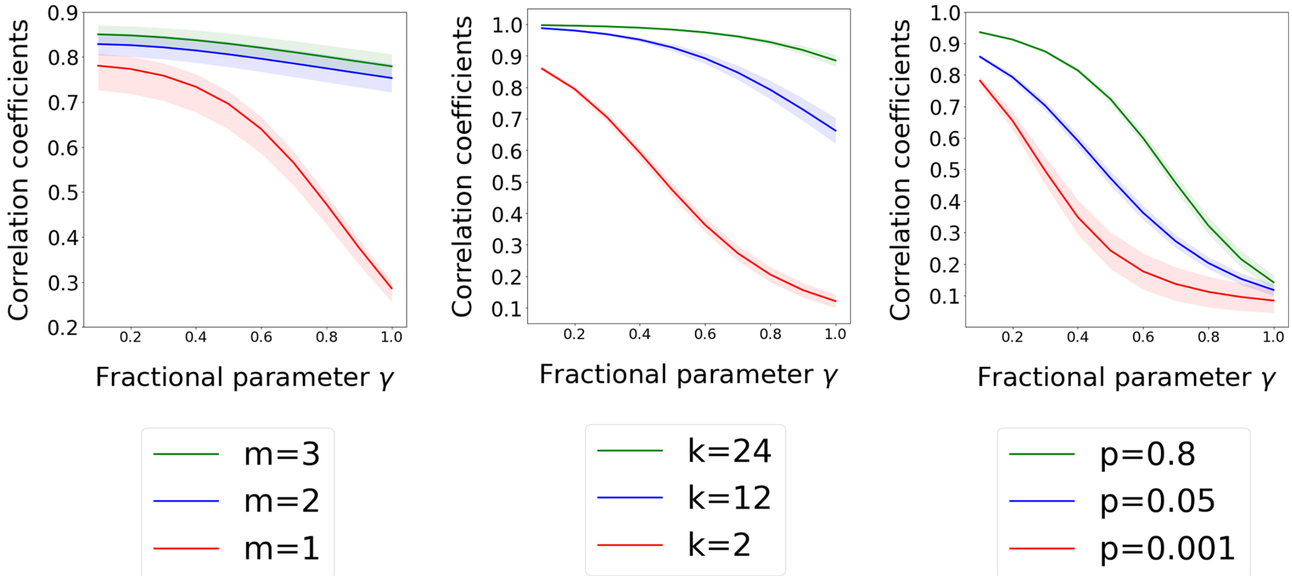
When we apply our γ-information centralities, our task is to choose an appropriate parameter γ according to properties of networks. To provide a guideline for selecting a parameter γ, we first consider two real-world complex networks together with *outcome* variables for nodes of the networks. One is a *marriage* network and the other is a friendship network, which are different from the Van de Bunt's friendship network.

The marriage network is a network among Florentine families with the *attribute wealth* of each family [32]. The friendship network is the *EIES* friendship network whose



(a) Friendship (b) Karate (c) Dolphin

FIG. 7. Correlation coefficients between degree centralities and γ-information centralities of the real-world networks.

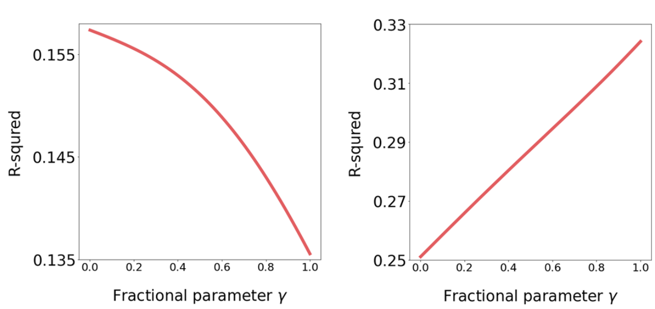


(a) BA_m (b) WS_k (b) WS_p

FIG. 8. Correlation coefficients between degree and γ-information centralities on random networks with each 100 trials. We represent their means with solid lines and their standard deviations with shaded areas.

nodes are early network researchers with the number of the *citations* of the nodes [33]. We focus on the subnetwork consisting of the researchers whose discipline is sociology. As in Ref. [7], Sec. 2.4, the natural log transformation $[\log(x+1)]$ is used for the citations variable. The goal is to find a parameter γ which *maximizes* the explanatory power of the outcome variable based on our γ-information centralities for each network.

We plot the *R-squared* (or coefficient of determination) in Fig. 9 to figure out how γ-information centralities explain the outcome variables for these two networks as γ varies. The *R-squared* for the marriage network attains the maximum value $0.157\cdots$ at γ close to 0 and decreases as γ goes to 1 [Fig. 9(a)]. This tells us that the importance of the family in the marriage network mainly depends on the number of marriages, or the degree centrality meaning the *direct* interaction between families. Unlike the marriage network, the *R-squared* for the EIES friendship network increases as γ increases and attains the maximum value $0.324\cdots$ at $\gamma = 1$



(a) Marriage (b) Friendship

FIG. 9. The R-squared of the outcome variables based on γ-information centralities.

[Fig. 9(b)]. This indicates that *indirect* communication between researchers is a significant factor in predicting citations of their papers. Note that our analysis is consistent with that of Ref. [7], Sec. 2.4 using a generalized measure of centrality based on closeness.

Next, let us consider the Hollywood film music network [34] collected from 1964 to 1976, consisting of sixty-two producers and forty composers. The nodes indexed as $1, 2, \ldots, 62$ are producers and the nodes indexed as $63, \ldots, 102$ are composers. Each edge between a producer A and a composer B is weighted by the number of movies which is produced by A and whose music is created by B. Nodes 63, 79, 80, 81, and 92 are the top 5 composers each of whom gained more than 1.5% of the total income of Hollywood movie score composers in the period.

We investigate changes of ranks of the top 5 composers according to γ-information centralities as γ varies from a positive number close enough to 0 to 1. For any γ, nodes 63, 92 and 81 rank first, fourth, and fifth, respectively. Node 79 ranks second for $\gamma \leqslant 0.89$ and node 80 ranks ninth for $\gamma \geqslant 0.79$ (see Fig. 10). These ranks are the highest among the ranks according to γ-information centralities, respectively. It is worth noting that different from the previous two network, the γ-fractional information centrality in this network is most effective when γ is between $0.79\,(> 0)$ and $0.89\,(< 1)$ in a sense that the sum of ranks of the top 5 nodes is minimum.

V. CONCLUSION

We have studied fractional analogs of two classical centrality measures, degree centrality and information centrality. Since the fractional degree centrality hardly gives rise to any changes in the rankings as a fractional parameter γ varies, the fractional degree centrality does not perform better than the degree centrality in the sense of centrality. However, the frac-

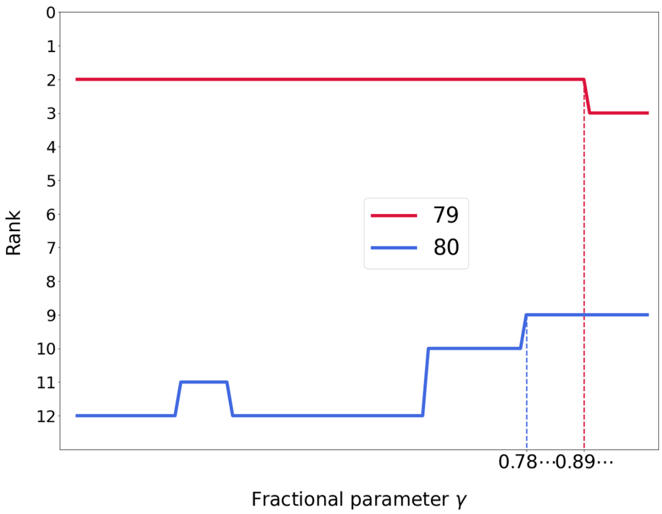


FIG. 10. The changes of ranks of nodes 79 and 80 according to γ-information centralities.

tional information centrality leads to the compelling changes in the rankings as γ goes to 0 from 1. Our theorem (Theorem 1) for the fractional information centrality with γ close to 0 shows that the centrality works like the degree centrality. In addition, correlation coefficients between the fractional information centrality and the degree centrality tell us that the fractional information centrality is getting more global as γ goes to 0 from 1. Therefore, our study has demonstrated that the fractional centralities unify the local centrality (degree centrality) and the global centrality (information centrality).

Our main idea of connecting the degree centrality and the information centrality is the fractional analog of graph Laplacians. In addition to these two centralities, there are centralities which can be expressed in terms of graph Laplacians, including eigenvector centrality [40], pagerank centrality [41], Katz centrality [5], diffusion centrality [42], and Laplacian centrality [43]. It would be interesting to define factional analogs of these centralities and employ these fractional versions to analyze networks. Note that the fractional analog of the *pagerank centrality* is used to improve the performance of graph-based semi-supervised learning [44].

Other than centrality measures, various network measures such as Kirchhoff index [14] are expressed in terms of graph Laplacians. Kirchhoff index has been utilized as an overall structure descriptor and is also interpreted as the average commuting times of random walks. Estrada [21] developed its fractional analog and explored how his measure changes in a fixed network as the fractional parameter varies. It would be interesting to use this fractional analog to compare the robustness of distinct networks and uncover a relation to other robustness measures as we connect the information centrality with the degree centrality via the fractional Laplacian.

Gurfinkel and Rikvold developed a classification system for parametrized centralities [8,10]. One of the parametrized centralities is *reach-parametrized centralities*, whose parameter adjusts the impacts of flows between nodes. As γ goes from 1 to 0, fractional weights for pairs of adjacent vertices decrease while the impacts of those for pairs of nonadjacent vertices increase (see Fig. 5). Hence, we see that the fraction degree centrality is classified as reach-parametrized centralities. The fractional information also seems to be classified as a measure of the same type since the local influence increases as γ goes from 1 to 0. However, to confirm this statement, it still needs a careful analysis for fractional effective resistances, which we leave for future work.

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APPENDIX A: PROOFS OF INEQUALITIES FOR FRACTIONAL DEGREE CENTRALITIES

We provide proofs of inequality (5) and Theorem 2.

1. Inequalities for fractional degree centralities

We first recall inequality (5). Let $s = \max_{i\in[n]} d_i$. For $i \in [n]$ and $\gamma \in [0, 1]$, define

$$l_i^{(\gamma)} = (d_i + 1)^{\gamma-1} d_i, \text{ and}$$

$$r_i^{(\gamma)} = s^\gamma \left[\sum_{k=0}^{2} (-1)^k \binom{\gamma}{k} \left(1 - \frac{d_i}{s}\right)^k + \binom{\gamma}{2} \frac{d_i}{s^2} \right]$$

$$= s^\gamma \left[1 - \gamma + \frac{\gamma\, d_i}{s} + \binom{\gamma}{2} \left(1 - \frac{2d_i}{s} + \frac{d_i^2 + d_i}{s^2} \right) \right].$$

Then the γ-degree centrality $d_i^{(\gamma)}$ of node i satisfies the following inequality:

$$l_i^{(\gamma)} \leqslant d_i^{(\gamma)} \leqslant r_i^{(\gamma)}.$$

Proof. Set $B = s\,\mathrm{Id} - L$. Since $s = \max_{i\in[n]} d_i$, the matrix B is a nonnegative matrix. Indeed, $B_{ii} = s - d_i \geqslant 0$ for $i \in [n]$ and $B_{ij} = A_{ij}$ for $i, j \in [n]$ with $i \neq j$. Let us fix $i \in [n]$. Shortly, denote

$$b_k = (B^k)_{ii},\, d = d_i,\, d^{(\gamma)} = d_i^{(\gamma)},\, l^{(\gamma)} = l_i^{(\gamma)}, \text{ and } r^{(\gamma)} = r_i^{(\gamma)}.$$

First, let us show the right inequality of Eq. (5), i.e.,

$$d^{(\gamma)} \leqslant r^{(\gamma)}.$$

The series expansion of L^γ is given as follows:

$$L^\gamma = s^\gamma \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \frac{B^k}{s^k}$$

$$= s^\gamma \sum_{k=0}^{2} (-1)^k \binom{\gamma}{k} \frac{B^k}{s^k} + s^\gamma \sum_{k=3}^{\infty} (-1)^k \binom{\gamma}{k} \frac{B^k}{s^k}.$$

Since $b_1 = s - d$ and $b_2 = (s-d)^2 + d = s^2 - 2sd + d^2 + d$, we have

$$d^{(\gamma)} = r^{(\gamma)} + s^\gamma \sum_{k=3}^{\infty} (-1)^k \binom{\gamma}{k} \frac{b_k}{s^k}.$$

For a natural number k, from the fact that $(-1)^k\binom{\gamma}{k} < 0$ and $b_k \geqslant 0$, it follows that $(-1)^k\binom{\gamma}{k}(B^k)_{ii} \leqslant 0$. Hence, the desired inequality follows.

Next, we prove the left inequality of Eq. (5):

$$d^{(\gamma)} \geqslant l^{(\gamma)}.$$

Since each row sum of L is 0, i.e., $L\mathbf{1} = 0$, we have $B^k\mathbf{1} = (sI - L)^k\mathbf{1} = s^k\mathbf{1}$. Then

$$\begin{aligned}(B^k)_{ii} &= \sum_{j\in[n]} (B^{k-1})_{ij} b_{ji} \\ &= \sum_{j\in[n],\, j\neq i} (B^{k-1})_{ij} b_{ji} + b_{ii}(B^{k-1})_{ii} \\ &\leqslant \sum_{j\in[n],\, j\neq i} (B^{k-1})_{ij} + b_{ii}(B^{k-1})_{ii} \\ &= \sum_{j\in[n]} (B^{k-1})_{ij} + (b_{ii}-1)(B^{k-1})_{ii} \\ &\leqslant s^{k-1} + (s-d-1)(B^{k-1})_{ii}.\end{aligned}$$

The above inequality can be written as

$$b_k \leqslant s^{k-1} + (s-d-1)b_{k-1},$$

which is equivalent to

$$b_k - \frac{s^k}{d+1} \leqslant (s-d-1)\left(b_{k-1} - \frac{s^{k-1}}{d+1}\right).$$

Solving this recurrence inequality gives

$$b_k \leqslant \frac{s^k}{d+1}\left[1 + d\left(1 - \frac{d+1}{s}\right)^k\right].$$

Using this inequality, we obtain

$$\begin{aligned}d^{(\gamma)} &= s^\gamma \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \frac{b_k}{s^k} \\ &\geqslant s^\gamma \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \frac{1}{d+1}\left[1 + d\left(1 - \frac{d+1}{s}\right)^k\right] \\ &= \frac{s^\gamma}{d+1}[(1-1)^\gamma] + \frac{s^\gamma d}{d+1}\left[1 - \left(1 - \frac{d+1}{s}\right)\right]^\gamma \\ &= \frac{s^\gamma d}{d+1}\left(\frac{d+1}{s}\right)^\gamma = (d+1)^{\gamma-1} d = l^{(\gamma)},\end{aligned}$$

as desired. ■

We remark that the first-order approximations yields the simple estimates given by

$$d_i^\gamma - \frac{1-\gamma}{d_i} \leqslant d_i^{(\gamma)} \leqslant (1-\gamma)s^\gamma + \gamma\, d_i s^{\gamma-1}. \tag{A1}$$

2. A comparison of fractional degree centralities

Using inequality (5), we prove Theorem 2.

italic Proof It suffices to consider the case $d_j = d_i + 1$. Note that $1 \leqslant d_i \leqslant s-1$. By inequality (5), it is enough to show that

$$r_i^{(\gamma)} < l_j^{(\gamma)}, \tag{A2}$$

which is equivalent to

$$\begin{aligned}1 - \gamma\left(1 - \frac{d_i}{s}\right) + \binom{\gamma}{2}\left[\left(1 - \frac{d_i}{s}\right)^2 + \frac{d_i}{s^2}\right] \\ < \frac{d_i+1}{d_i+2}\left(\frac{d_i+2}{s}\right)^\gamma.\end{aligned}$$

To prove this inequality, we will show that the quantity

$$\left(\frac{d_i}{s}\right)^\gamma$$

lies between the above two numbers appearing in the inequality.

We first claim that

$$\frac{d_i+1}{d_i+2}\left(\frac{d_i+2}{s}\right)^\gamma > \left(\frac{d_i}{s}\right)^\gamma,$$

which is equivalent to

$$\gamma > \frac{\log(d_i+2) - \log(d_i+1)}{\log(d_i+2) - \log d_i}.$$

Let $f(x)$ be a function defined by

$$f(x) := \frac{\log(x+2) - \log(x+1)}{\log(x+2) - \log x}.$$

It can be shown by a direct computation that $f(x) < 1/2$ for all $x \in \mathbb{R}$. Hence, the claim follows from the assumption $\gamma \geqslant 1/2$.

Next, we will show that

$$I := \left(\frac{d_i}{s}\right)^\gamma - r_i^{(\gamma)} \geqslant 0.$$

We apply the Taylor expansion to $(d_i/s)^\gamma$ to obtain

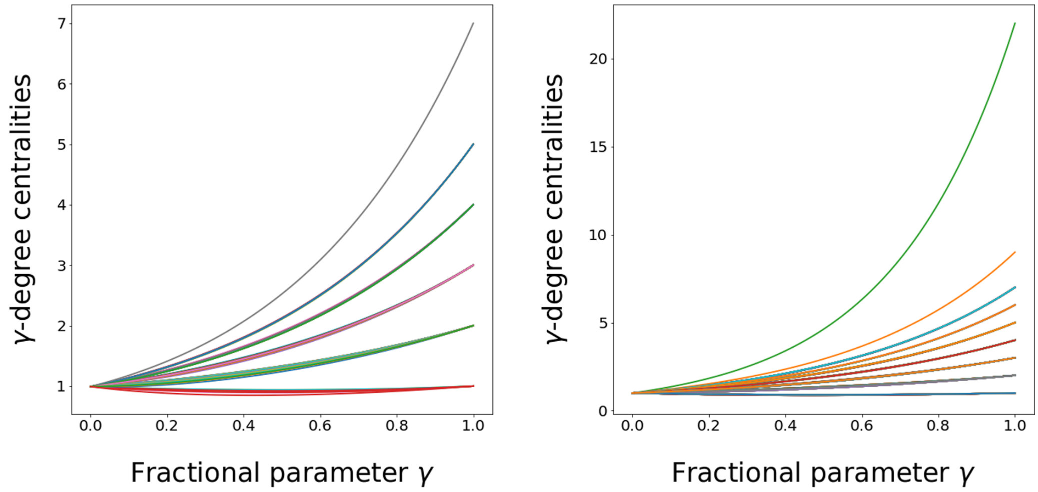
$$\left(1 + \frac{d_i}{s} - 1\right)^\gamma = \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k}\left(1 - \frac{d_i}{s}\right)^k.$$

Then

$$I = \sum_{k=3}^{\infty} (-1)^k \binom{\gamma}{k}\left(1 - \frac{d_i}{s}\right)^k - \binom{\gamma}{2}\frac{d_i}{s^2}.$$

For $k \geqslant 3$, the following inequality holds:

$$\begin{aligned}0 &< (-1)^k \binom{\gamma}{k} \Big/ \binom{\gamma}{2} \\ &= \frac{(2-\gamma)(3-\gamma)\cdots(k-1-\gamma)}{k(k-1)\cdots 3} \leqslant \frac{2}{k}.\end{aligned}$$



(a) The italian power grid (b) The electrical circuit

FIG. 11. Fractional degree centralities of all the nodes.

Using this inequality together with the fact that $(-1)^k\binom{\gamma}{k} < 0$ for $k \geqslant 1$, we get

$$I \geqslant \binom{\gamma}{2}\sum_{k=3}^{\infty}\frac{2}{k}\left(1-\frac{d_i}{s}\right)^k - \binom{\gamma}{2}\frac{d_i}{s^2}$$
$$= -\binom{\gamma}{2}\left[\frac{d_i}{s^2} + 2\log\left(\frac{d_i}{s}\right) + 2\left(1-\frac{d_i}{s}\right) + \left(1-\frac{d_i}{s}\right)^2\right].$$

Let $g(x)$ be a function defined by for $x > 0$,

$$g(x) := \frac{x}{s} + 2\log x + 2(1-x) + (1-x)^2.$$

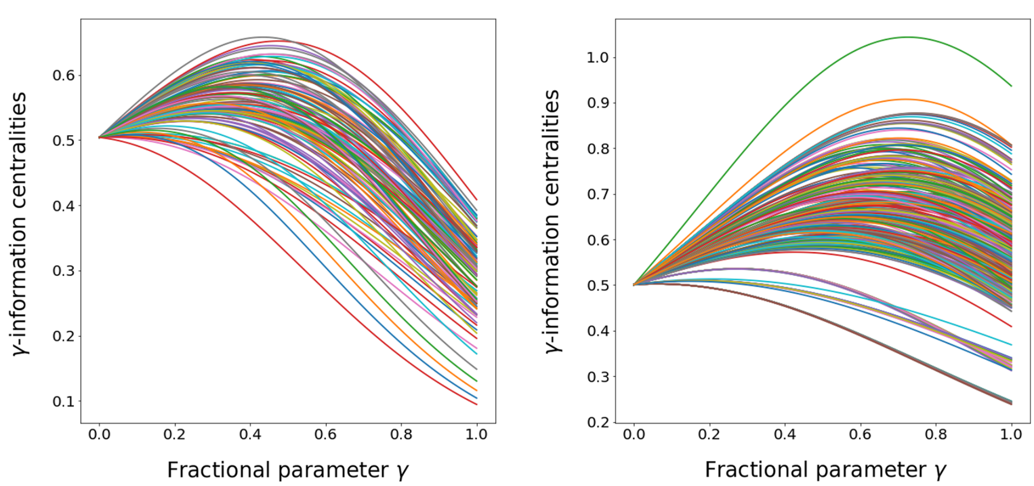
It follows from $g'(x) = 1/s + 2/x - 2 + 2(x-1) \geqslant 1/s > 0$ that the function g is increasing. Since $g(1) > 0$, there exists a unique $c = c(s)$ such that $g(x) > 0$ for $x > c$. From the assumption $d_i \geqslant cs$, we have

$$I = -\binom{\gamma}{2} g\left(\frac{d_i}{s}\right) \geqslant 0,$$

which completes the proof. ■

APPENDIX B: APPLICATIONS TO OTHER REAL-WORLD NETWORKS

We consider the Italian 380 kV power grid [45] (for short, the Italian power grid) and the electrical circuit network, s838, from the ISCAS 89 benchmark set [46] (for short, the electrical circuit) which are larger than those discussed in Sec. IV. The Italian power grid has the largest component which we focus on contains 124 nodes and 169 edges. The electrical circuit is a connected network with 512 nodes and 819 edges.



(a) The Italian power grid (b) The electrical circuit

FIG. 12. Fractional information centralities of all the nodes.

1. Fractional degree centralities

Fractional degree centralities for the Italian power grid and the electrical circuit are given in Figs. 11(a) and 11(b), respectively. We figure out that nodes of the *same* degree have *almost* the same fractional degree centralities as in Figs. 1(a)–1(c).

2. Fractional information centralities

Fractional information centralities for the Italian power grid and the electrical circuit are given in Figs. 12(a) and 12(b), respectively. We see that as parameters γ vary, the ranks of nodes according to γ-fractional information centralities changes.

[1] S. Wasserman, K. Faust *et al.*, (1994).
[2] A. Bavelas, J. Acoust. Soc. Am. **22**, 725 (1950).
[3] L. C. Freeman, Social Netw. **1**, 215 (1978).
[4] K. Stephenson and M. Zelen, Social Netw. **11**, 1 (1989).
[5] L. Katz, Psychometrika **18**, 39 (1953).
[6] E. Estrada, J. Theor. Biol. **263**, 556 (2010).
[7] F. Agneessens, S. P. Borgatti, and M. G. Everett, Social Netw. **49**, 12 (2017).
[8] A. J. Gurfinkel and P. A. Rikvold, Phys. Rev. E **101**, 012302 (2020).
[9] A. Arnaudon, R. L. Peach, and M. Barahona, Phys. Rev. Research **2**, 033104 (2020).
[10] A. J. Gurfinkel and P. A. Rikvold, Phys. Rev. E **103**, 052308 (2021).
[11] U. Brandes and D. Fleischer, in *Proceedings of the Annual Symposium on Theoretical Aspects of Computer Science* (Springer, Berlin, 2005), pp. 533–544.
[12] P. Van Mieghem, K. Devriendt, and H. Cetinay, Phys. Rev. E **96**, 032311 (2017).
[13] G. Kirchhoff, Ann. Phys. Chem. **148**, 497 (1847).
[14] D. J. Klein and M. Randić, J. Math. Chem. **12**, 81 (1993).
[15] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Vol. 42 (Cambridge University Press, Cambridge, UK, 2017).
[16] D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs: Theory and Application*, 3rd ed. (Johann Ambrosius Barth, Heidelberg, 1995).
[17] L. Caffarelli and L. Silvestre, Commun. Partial Dif. Eq. **32**, 1245 (2007).
[18] L. A. Caffarelli, S. Salsa, and L. Silvestre, Invent. Math. **171**, 425 (2008).
[19] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[20] A. P. Riascos and J. L. Mateos, Phys. Rev. E **90**, 032809 (2014).
[21] E. Estrada, New J. Phys. **23**, 073049 (2021).
[22] N. Biggs, Bull. London Math. Soc. **29**, 641 (1997).
[23] W. Kook, Adv. Appl. Math. **46**, 417 (2011).
[24] D. Y. Kho, W. Kook, J. Lee, J. Lee, and K.-J. Lee, Linear Alg. Appl. **524**, 278 (2017).
[25] A. Ben-Israel and T. N. Greville, *Generalized Inverses: Theory and Applications*, Vol. 15 (Springer Science & Business Media, Cham, 2003).
[26] E. Bozzo and M. Franceschet, Social Netw. **35**, 460 (2013).
[27] D. Bianchi, M. Donatelli, F. Durastante, and M. Mazza, J. Math. Anal. Appl. **511**, 126020 (2022).
[28] T. Michelitsch, A. P. Riascos, B. Collet, A. Nowakowski, and F. Nicolleau, *Fractional Dynamics on Networks and Lattices* (John Wiley & Sons, New York, NY, 2019).
[29] G. G. Van de Bunt, M. A. Van Duijn, and T. A. Snijders, Comput. Math. Org. Theory **5**, 167 (1999).
[30] W. W. Zachary, J. Anthropol. Res. **33**, 452 (1977).
[31] D. Lusseau, Evol. Ecol. **21**, 357 (2007).
[32] J. F. Padgett and C. K. Ansell, Amer. J. Sociol. **98**, 1259 (1993).
[33] S. Freeman and L. Freeman, *The Networkers Network: A Study of the Impact of a New Communications Medium on Sociometric Structure*, Social Science Research Reports, No. 46 (School of Social Sciences University of California, Irvine, CA, 1979).
[34] R. R. Faulkner, *Music on Demand: Composers and Careers in the Hollywood Film Industry* (Transaction Books, Piscataway, NJ, 2010).
[35] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[36] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[37] M. Fiedler, Czechoslovak Math. J. **23**, 298 (1973).
[38] M. C. De Gennaro and A. Jadbabaie, in *Proceedings of the 45th IEEE Conference on Decision and Control* (IEEE, Piscataway, NJ, 2006), pp. 3628–3633.
[39] M. F. Triola, *Elementary Statistics*, 13th ed. (Pearson/Addison-Wesley, Boston, MA, 2018).
[40] P. Bonacich, J. Math. Sociol. **2**, 113 (1972).
[41] L. Page, S. Brin, R. Motwani, and T. Winograd, The PageRank citation ranking: Bringing order to the web, Tech. Rep. (Stanford InfoLab, 1999).
[42] A. Banerjee, A. G. Chandrasekhar, E. Duflo, and M. O. Jackson, Science **341**, 1236498 (2013).
[43] X. Qi, E. Fuller, Q. Wu, Y. Wu, and C.-Q. Zhang, Info. Sci. **194**, 240 (2012).
[44] E. Bautista, P. Abry, and P. Gonçalves, Appl. Netw. Sci. **4**, 57 (2019).
[45] P. Crucitti, V. Latora, and M. Marchiori, Fluct. Noise Lett. **05**, L201 (2005).
[46] U. Alon, Uri AlonLab: Collection of complex networks, retrieved from https://www.weizmann.ac.il/mcb/UriAlon/download/collection-complex-networks.