Microscopic kinetic theory of the mean collision force of a particle moving in rarefied gases

Tongli Wei ,[1,2,*] Xiansheng Cao,[3] Decheng Ma,[4] Yaojin Li ,[5] and Chenglong Jia [2]

[1] *Key Laboratory of Physics and Photoelectric Information Functional Materials Sciences and Technology, North Minzu University, Yinchuan 750021, China*

[2] *Key Laboratory for Magnetism and Magnetic Materials of Ministry of Education, Lanzhou University, Lanzhou 730000, China*

[3] *School of Microelectronics and Control Engineering, Changzhou University, Changzhou 213164, China*

[4] *Department of Physics, Shanghai University, Shanghai 200444, China*

[5] *Electronic Materials Research Laboratory International Center for Dielectric Research, Xi'an Jiaotong University, Xi'an 710049, China*

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The friction of an incident particle interacting with the background molecules is a cornerstone in the nonequilibrium dynamics and statistics. It is reported that the Stokes force may fail while the Brown particle's size is small enough. In this work, the mean collision force of a small classical particle moving through the rarefied gases is analyzed by the direct calculation of the mean decrease of particle's velocity by elastic collisions. As an example, a whole velocity space applicable mean collision force in Maxwell gas is obtained. A self-consistent solution is further provided based on the numerical simulations. Within the low speed limit, comparison of the friction and the Stokes force has been demonstrated. Although the two forces are both proportional to the speed of the particle, their coefficients are different. Unlike the linear speed dependence of Stokes force, the linear behavior in rarefied gases is broken with increasing the speed of incident particle, and a quadratic speed dependence is resulted in high speed. This work clarifies the nonequilibrium dynamics of microscopic particles moving in rarefied gases, and can improve our microscopic understanding of the collision force.

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I. INTRODUCTION

The force of a particle moving in the background fluid or gas is a cornerstone of the relevant non-equilibrium diffusion processes [1–22]. The dynamical Langevin equation suggested the force can be split into two parts: a mean term and a random term [1,2]. Given that a mean value of the Stokes force, $f_s = 6\pi\eta_s\mathcal{R}v_1$ with η_s, $\mathcal{R}$ and v_1 being the viscosity, the radius of the Brownian particle and the speed of the particle, the Stokes-Einstein (SE) relation $\mathcal{DR} = k_BT/6\pi\eta_s$ is deduced [3–5]. Here, $\mathcal{D}$ is the diffusion coefficient, k_B is the Boltzmann constant, T is the fluid/gas temperature. It is believed that for a simple fluid the SE relation holds well even when the size of the Brownian particle decreases to the molecular level [6–17]. However, a detailed investigation of $\mathcal{DR}$ in Ref. [18] suggested different size/mass dependencies of the kinetic friction force $\overline{f}$ and the hydrodynamic Stokes force f_s.

The Stokes hydrodynamics model is based on a classic interacting system: the fluid near the sphere is distinguished as layers with different speeds. A corresponding kinetic interpretation of the viscosity was discovered and introduced by Maxwell [19], in which $\eta_s = n_Bm_2\overline{l}\,\overline{v_2}/3$. n_B, m_2, $\overline{l}$, and $\overline{v_2}$ are the number density, the molecule mass, the mean free path, and the mean speed of molecules of the fluid/gas respectively. For small enough particles, however, the Stokes model may fail because the interaction is too weak to form laminar flow [20]. As the particle's size is much smaller than $\overline{l}$ (the Knudsen number is larger than 10) [21], statistics of the fluid can be assumed unchanged, and a force theory that the particle collides with equilibrated molecules is needed.

In the present study, the mean collision force of a small classical particle moving through the rarefied gas at different temperatures is solved under a kinetic consideration. The classical elastic collision is used, particles are taken as hard spheres [1,2,4,18–22], and the background molecules are assumed to be in equilibrium. By the direct calculation of the mean collision rate and mean decrease of particle's velocity, the friction is obtained. As an important example, a particle moving in Maxwell gas is provided. Further numerical simulations demonstrate a self-consistent conclusion. In the low speed ($v_1 \ll \overline{v_2}$) cases, we show $\overline{f} = -\eta v_1$, where the friction coefficient $\eta \propto n_BD^2\sqrt{T}$ and D is the radius of the scattering cross-section. The speed/temperature dependencies of $\overline{f}$ and f_s are found to be same [19]; however, their size/density dependencies are different. $\overline{f} \propto D^2$ suggested in Ref. [18] is affirmed by our analytic study. Clearly, it is different with the linear dependence of $f_s \propto \mathcal{R}$. Beyond the low speed cases, our theory can be applied to the whole space of velocity. In particular, we found that the kinetic friction force in the high ($v_1 \gg \overline{v_2}$) speed cases changes from the linear function to a quadratic function of the speed of incident particles, $\overline{f} = -\eta' v_1^2$, where η' is the speed-independent coefficient.

*weitl@nun.edu.cn

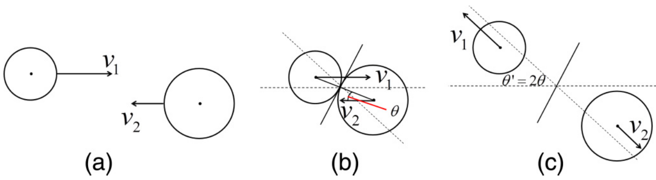

FIG. 1. Collision sketches. The sketches are pictured in center-of-mass frame, v_1 and v_2 are the speeds of the two particles, θ is the angle of the center line and the moving direction, $\theta' = 2\theta$ is the angle of the incident line and exit line.

II. COLLISION MODEL AND STATISTICAL HYPOTHESIS

The classical two-particle elastic collision model has been employed in this work, and the collision process (the change of the particle's velocity) is done instantaneously. A picture to depict the model in the center-of-mass frame is shown in Fig. 1. Details of the present model are set as follows: (i) Energy and momentum are conserved; (ii) Particles are assumed as hard spheres, mass center is at the geometric center of each particle; (iii) The velocities of the particles remain unchanged except for the moments of collisions. It is easy to prove that the exiting speeds [as shown in Fig. 1(c)] in the center-of-mass frame remain unchanged, and the following property is true: In the center-of-mass frame, the magnitudes of the velocities (speed) of the two particles remain unchanged in the collision process. [4,19,23]. Two statistical hypothesizes are proposed. Hypothesis *(i) In the laboratory reference frame, the probability density function of moving directions of the background molecules is a constant, with*

$$\varrho_1(\alpha, \phi) = \frac{1}{4\pi}, \tag{1}$$

where $\varrho_1(\alpha, \phi)$ is the probability density of the moving direction of the background molecules, $\alpha \in [0, \pi]$, and $\phi \in [0, 2\pi]$ are the angles of the moving direction in spherical coordinate, which is shown in Fig. 2. As the solid angle $d\Omega = 2\pi \sin\alpha d\alpha$ in the range of $\alpha \to \alpha + d\alpha$ is dependent on α, the probability

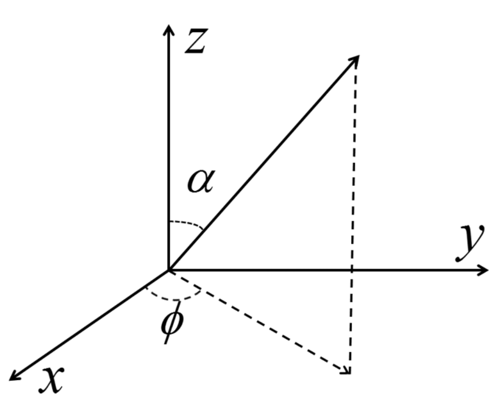

FIG. 2. Spherical coordinate frame. Where α is the angle of the z-axis and the moving direction, ϕ is the angle of the x-axis and the projection of the moving direction in xy plane.

density function of α can be derived as follows

$$\begin{aligned}\varrho_2(\alpha) &= \frac{dP}{d\alpha} = \frac{\varrho_1(\alpha,\phi)d\Omega}{d\alpha} = \frac{1}{4\pi}\frac{2\pi r\sin\alpha r d\alpha}{r^2 d\alpha}\\ &= \frac{2\pi r\sin\alpha r d\alpha}{4\pi r^2 d\alpha} = \frac{1}{2}\sin\alpha,\end{aligned} \tag{2}$$

where r is an arbitrary radius to get the differential solid angle, dP is the differential probability. Reversely, if $\varrho_1(\alpha, \phi)$ is independent of ϕ, and $\varrho_2(\alpha) = \sin\alpha/2$, statistical hypothesis i is true. Hypothesis *ii. The occupied probabilities of every spatial position by gas molecules are equal.* By this hypothesis, for a random collision, the incident particle (center) distributes on the area of the scattering cross-section evenly. If r_1/r_2 is the radius of the incident/target particle, we have $D = r_1 + r_2$, and the area of scattering cross section is πD^2, then the probability density function of the areas is

$$\varrho_3 = 1/\pi D^2.$$

As shown in Fig. 1(b), $\theta \in [0, \pi/2]$ is the angle between the incident line and the line through the center of masses at the colliding moment which is perpendicular to the contact surface. After collision [Fig. 1(c)], the angle of the exiting line and the incident line, θ' ($\theta' \in [0, \pi]$), will be twice θ, with $\theta' = 2\theta$. According to statistical hypothesis ii, the probability density of θ' is

$$\varrho_4(\theta') = \frac{dP}{d\theta'} = \frac{1}{\pi D^2}\frac{2\pi D\sin\theta D\cos\theta d\theta}{d\theta'} = \frac{1}{2}\sin\theta'. \tag{3}$$

By using the converse of Hypothesis i as we have mentioned, another property about the exiting directions in the center-of-mass frame can be given as: *In the center-of-mass frame, exiting direction is likely equal to all spatial directions* [4,19,21,23]. Note that the background molecules are assumed in equilibrium, and the molecular chaos hypothesis is used [21,22].

III. GENERAL FORM OF THE MEAN COLLISION FORCE

The mean collision force of the incident particle (set as particle A in the next) $\bar{\mathbf{f}}$, according to Newton's law, can be obtained by multiplying the mean collision rate $\overline{Z_{AB}}$ and the mean change of the momentums $\overline{\Delta\mathbf{p}_1}$ in one collision, with [19,24]

$$\begin{aligned}\bar{\mathbf{f}} &= \overline{\frac{d\mathbf{p}_1}{dt}} = \overline{Z_{AB}\,\Delta\mathbf{p}_1} = m_1\int d\overline{Z_{AB}\Delta\mathbf{v}_1}\\ &= m_1\overline{Z_{AB}}\int d\overline{Z_{AB}\Delta\mathbf{v}_1}/\overline{Z_{AB}} = m_1\overline{Z_{AB}}\;\overline{\Delta\mathbf{v}_1},\end{aligned} \tag{4}$$

where $\mathbf{p}_1/m_1$ is the momentum/mass of particle A, overlines of the parameters denote the ensemble averages, $\overline{\Delta\mathbf{v}_1}$ is the mean velocity change of the particle in one collision, defined as

$$\overline{\Delta\mathbf{v}_1} = \int d\overline{Z_{AB}\Delta\mathbf{v}_1}/\overline{Z_{AB}},$$

and $d\overline{Z_{AB}\Delta\mathbf{v}_1}/\overline{Z_{AB}}$ is the differential contribution of collisions of specific parameters (speeds/angles of the gas molecules). $\overline{Z_{AB}}$ and $\overline{\Delta\mathbf{v}_1}$ are the keys to be studied in this article.

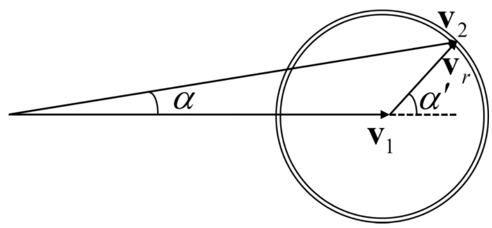

FIG. 3. The relations between $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_r$. $\mathbf{v}_1$ is the velocity of particle A, $\mathbf{v}_2$ is the velocity of particle B (the gas molecule), and $\mathbf{v}_r$ is the relative velocity of the two particles.

A. Mean collision rate and mean relative speed

The familiar mean collision rate is [4,19,25]

$$\overline{Z_{AB}} = n_B \pi D^2 \overline{v_r}, \tag{5}$$

$\overline{v_r}$ is the mean relative speed of particle A to background molecules (set as particle B), defined as

$$\overline{v_r} = \frac{1}{N}\int_0^\infty v_r dn_B(v_r) = \int_0^\infty v_r \rho_{v_1}(v_r) dv_r,$$

where N is the total number of the background molecules, $dn_B(v_r)$ is the number of the molecules in the relative speed range $v_r \to v_r + dv_r$, and $\rho_{v_1}(v_r) = dn_B(v_r)/N$ is the density function of the relative speed while particle A's speed is v_1. As shown in Fig. 3, $\mathbf{v}_1/\mathbf{v}_2$ is the velocity of particle A/B in laboratory frame, $\mathbf{v}_r = \mathbf{v}_2 - \mathbf{v}_1$ is the relative velocity, v_r is the magnitude of $\mathbf{v}_r$, and α/α' is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2/\mathbf{v}_r$. It is easy to see in Fig. 3 that $v_2 = \sqrt{v_1^2 + v_r^2 + 2v_1 v_r \cos\alpha'}$ is changing with α', and $\rho_{v_1}(v_r)$ should be obtained by integrating $\varrho_{v_1}(v_r, \alpha')$ over α', providing

$$\rho_{v_1}(v_r) = \int_0^\pi \varrho_{v_1}(v_r, \alpha') d\alpha',$$

where $\varrho_{v_1}(v_r, \alpha')$ is the density function of v_r and α'. The volume of $\alpha' \to \alpha' + d\alpha'$, $v_r \to v_r + dv_r$ in velocity space is $2\pi v_r^2 \sin\alpha' dv_r d\alpha'$. Let $\rho_{\mathrm{sd}}(v_2)$ be the speed distribution function, the probability of particle B in $v_r \to v_r + dv_r$, $\alpha' \to \alpha' + d\alpha'$ is $2\pi v_r^2 \rho_{\mathrm{sd}}(v_2) \sin\alpha' dv_r d\alpha'$, and $\varrho_{v_1}(v_r, \alpha')$ can be obtained,

$$\varrho_{v_1}(v_r, \alpha') = 2\pi v_r^2 \rho_{\mathrm{sd}}\left(\sqrt{v_1^2 + v_r^2 + 2v_1 v_r \cos\alpha'}\right) \sin\alpha'. \tag{6}$$

Integrating the formula over α', the relative speed distribution function in speed space is given as

$$\begin{aligned}\rho_{v_1}(v_r) &= \int_0^\pi \varrho_{v_1}(v_r, \alpha') d\alpha' \\ &= 2\pi v_r^2 \int_0^\pi \rho_{\mathrm{sd}}\left(\sqrt{v_1^2 + v_r^2 + 2v_1 v_r \cos\alpha'}\right) \sin\alpha' d\alpha',\end{aligned} \tag{7}$$

with the dimension of $[v^{-1}]$. The mean relative speed can be written as

$$\overline{v_r} = \int_0^\infty v_r dv_r \int_0^\pi \varrho_{v_1}(v_r, \alpha') d\alpha'. \tag{8}$$

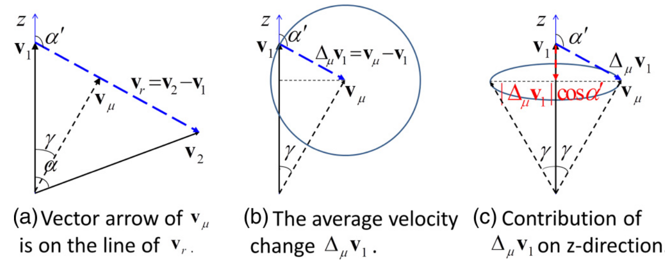

FIG. 4. **z-component of the velocity change.** Where γ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_\mu$, $\Delta_\mu \mathbf{v}_1 = \mathbf{v}_\mu - \mathbf{v}_1$ is the difference between $\mathbf{v}_\mu$ and $\mathbf{v}_1$ and is the average velocity change of particle A in the center-mass-frame, $\mathbf{v}_\mu$ is the velocity of the two particle's mass center.

Combining Eq. (5) and Eq. (8), the general mean collision rate can be obtained.

B. Mean velocity change

The mean velocity change can be deduced by the sketches in Fig. 4. Where $\mathbf{v}_1$ is along the z axis, $\mathbf{v}_\mu = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/(m_1 + m_2)$ is the velocity of the two particle's mass center, γ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_\mu$. It should be noted that $\mathbf{v}_\mu$'s vector arrow is on the line of $\mathbf{v}_r$ [see Fig. 4(a)], cause $\mathbf{v}_\mu - \mathbf{v}_1 = m_2\mathbf{v}_r/(m_1 + m_2)$. According to Eq. (3) and the properties mentioned in Sec. II, after collision, moving direction of particle A is likely equal for all spatial directions, and the speed of particle A remains unchanged in the center-mass-frame. Thus we get the conclusion that, after the collision, velocity of particle A distributes on the surface of the sphere [which has been shown in Fig. 4(b)] evenly. The average velocity change of particle A in Fig. 4(b) is

$$\Delta_\mu \mathbf{v}_1 = \mathbf{v}_\mu - \mathbf{v}_1. \tag{9}$$

The distribution of $\mathbf{v}_2$ is axial symmetry along the z-axis. If all the possibilities of particle B with parameters (v_2, α, ϕ) are considered [see Fig. 4(c), where $\phi \in [0, 2\pi]$], the average velocity change of particle A must be directed along the z axis. Thus only the z component of $\Delta_\mu \mathbf{v}_1$ is necessary to be considered:

$$\begin{aligned}\Delta v_{v_1, v_r, \alpha'} &= \Delta_\mu \mathbf{v}_1 \cdot \mathbf{e}_z = (\mathbf{v}_\mu - \mathbf{v}_1) \cdot \frac{\mathbf{v}_1}{v_1} \\ &= \frac{m_2\left(\mathbf{v}_1 \cdot \mathbf{v}_2 - v_1^2\right)}{(m_1 + m_2)v_1} = \frac{m_2 v_r \cos\alpha'}{m_1 + m_2}\end{aligned} \tag{10}$$

is the average contribution of a collision with parameters v_1, v_r and α' to the mean speed change, $\mathbf{e}_z$ is the unit vector of z direction (along $\mathbf{v}_1$).

The possibility of particle A colliding with molecules of definite parameters (v_r and α') can be deduced by the time averaging. The mean collision rate impacted by the particles of $v_r \to v_r + dv_r$, $\alpha' \to \alpha' + d\alpha'$ can be given as follows,

$$d\overline{Z}(v_1, v_r, \alpha') = n_B \pi D^2 v_r \varrho_{v_1}(v_r, \alpha') dv_r d\alpha'.$$

Meanwhile, the total collision number is $\overline{Z_{AB}}$. Thus the differential probability of collisions is

$$dP_\mu = \frac{d\overline{Z}(v_1, v_r, \alpha')}{\overline{Z_{AB}}} = \frac{n_B\pi D^2\varrho_{v_1}(v_r, \alpha')v_r dv_r d\alpha'}{n_B\pi D^2\overline{v}_r} = \frac{v_r}{\overline{v}_r}\varrho_{v_1}(v_r, \alpha')dv_r d\alpha'.$$

The density function can be defined as

$$\varrho_{\rm coll}(v_r, \alpha') = \frac{dP_\mu}{dv_r d\alpha'} = \frac{v_r}{\overline{v}_r}\varrho_v(v_r, \alpha'). \tag{11}$$

Integrating the formula over α' gives

$$\rho_{\rm coll}(v_r) = \int_0^\pi \frac{v_r}{\overline{v}_r}\varrho_{v_1}(v_r, \alpha')d\alpha' = \frac{v_r}{\overline{v}_r}\rho_{v_1}(v_r), \tag{12}$$

is the collision's possible distribution function of relative speed. By the definition of $\overline{\Delta \mathbf{v}_1}$, here $\overline{\Delta \mathbf{v}_1} = \int dP_\mu \overline{\Delta_\mu \mathbf{v}_1} = \int \overline{\Delta_\mu \mathbf{v}_1} dP_\mu$. The mean velocity change can be obtained by integrating the product of Eq. (10) and Eq. (11) over α' and v_r,

$$\overline{\Delta \mathbf{v}_1} = \overline{\Delta v_1}\mathbf{e}_{v_1} = \mathbf{e}_{v_1}\int_0^\infty dv_r \int_0^\pi \Delta v_{v_1,v_2,\alpha'}\varrho_{\rm coll}(v_r, \alpha')d\alpha' = \mathbf{e}_{v_1}\int_0^\infty dv_r \int_0^\pi \frac{m_2 v_r^2\cos\alpha'}{(m_1+m_2)\overline{v_r}}\varrho_{v_1}(v_r, \alpha')d\alpha', \tag{13}$$

where $\mathbf{e}_{v_1}$ is the unit vector along $\mathbf{v}_1$, and $\overline{\Delta v_1}$ is the scalar of $\overline{\Delta \mathbf{v}_1}$. In the next section, we will focus our sight on $\overline{\Delta v_1}$. With the general mean velocity change given above and the mean collision rate depicted by Eq. (5), if $\rho_{sd}(v_2)$ is known, the mean collision force can be obtained by Eq. (4).

IV. MEAN COLLISION FORCE OF SPHERE PARTICLES MOVING IN MAXWELL INERT-GAS

In this section, the mean collision force of a particle moving in Maxwell gas is derived by the general Eqs. (4)-(13) and the Maxwell distribution $\rho_{sd}(v_2) = (m_2/2\pi k_BT)^{3/2}\exp(-m_2v_2^2/2k_BT)$.

A. Mean collision rate and mean relative speed of the particle to Maxwell molecules

Substituting the Maxwell distribution into Eq. (6) gives

$$\varrho_{v_1}(v_r, \alpha') = 2\pi v_r^2\left(\frac{m_2}{2\pi kT}\right)^{3/2} e^{-\frac{m_2(v_1^2+v_r^2+2v_1v_r\cos\alpha')}{2k_BT}}\sin\alpha'. \tag{14}$$

Integrating the formula over α' gives

$$\rho_{v_1}(v_r) = \frac{v_r}{v_1}\left(\frac{m_2}{2\pi k_BT}\right)^{\frac{1}{2}}\left[e^{-\frac{m_2(v_1-v_r)^2}{2k_BT}} - e^{-\frac{m_2(v_1+v_r)^2}{2k_BT}}\right], \tag{15}$$

is the distribution function of v_r in Maxwell gas. Substituting Eq. (14) into Eq. (8), integrating over v_r, the mean relative speed is derived as

$$\overline{v_r} = \sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2v_1^2}{2k_BT}} + \left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right). \tag{16}$$

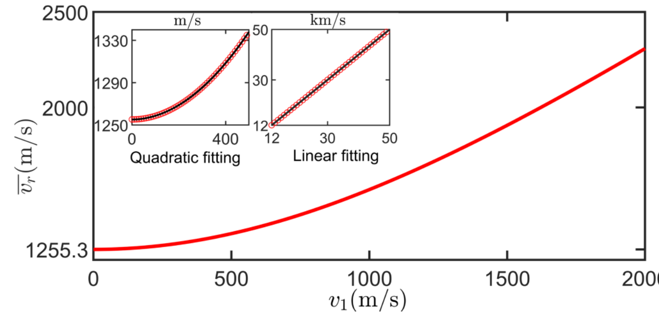

FIG. 5. **Mean relative speed of particle A to the Maxwell gas molecules.** Where $T = 300$ K, $m_1 = 1.672623 \times 10^{-27}$ kg, is approximately the mass of the proton and $m_2 = 4m_1$ is approximately the mass of the helium atom, $\overline{v_2} = 1255.3$ m/s. The two insets are the quadratic and linear fittings of limiting low and high speed cases respectively.

A similar conclusion has been obtained in Refs. [24,26,27]. The dependence of $\overline{v_r}$ on v_1 has been plotted in Fig. 5, where $T = 300$ K, $m_1 = 1.672623 \times 10^{-27}$ kg is approximatively the mass of the proton, $m_2 = 4m_1$ is approximately the mass of the helium atom. The two insets correspond to the low and high speed cases, respectively. The fittings are pretty well. In the low speed cases (the first inset of Fig. 5), v_r can be approximated as the addition of $\overline{v_2} = \sqrt{8k_BT/\pi m_2}$ and a small quadratic v_1 term. In the high speed cases (the second inset of Fig. 5), $\overline{v_r} \approx v_1$, the contribution of the molecules's velocities vanishes.

The collision possible distribution function of the particle moving in Maxwell gas can be obtained by Eq. (15) and Eq. (16), which gives

$$\rho_{\rm coll}(v_r) = \frac{v_r}{\overline{v_r}}\rho_{v_1}(v_r) = \frac{v_r^2}{v_1}\left(\frac{m_2}{2\pi k_BT}\right)^{\frac{1}{2}} \times \frac{e^{-\frac{m_2(v_1-v_r)^2}{2k_BT}} - e^{-\frac{m_2(v_1+v_r)^2}{2k_BT}}}{\sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2v_1^2}{2k_BT}} + \left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}. \tag{17}$$

Substituting Eq. (16) into Eq. (5), the collision rate can thus be obtained,

$$\overline{Z_{AB}} = n_B\pi D^2\left[\sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2v_1^2}{2k_BT}} + \left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)\right]. \tag{18}$$

B. Mean velocity change

Substituting Eq. (14) into Eq. (13), and integrating the formula, we have

$$\overline{\Delta v_1} = -\frac{m_2}{m_1+m_2}\left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\mathbb{R}\right) \tag{19}$$

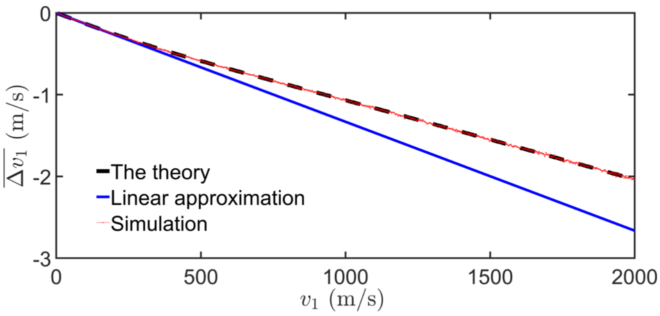

FIG. 6. Comparison of the theory, the simulation, and the linear approximation $\overline{\Delta v_1}$. 10000 runs has been carried in the simulation, with $m_2 = 30 \times 1.6726231^{-27}$ kg, $m_1 = 1000\, m_2$, $T = 300$ K,

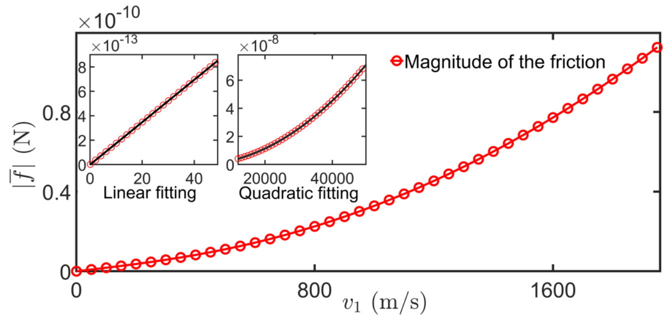

FIG. 7. Dependence of $|\overline{f}| = -\overline{f}$ on the speed of particle A v_1. Here $m_2 = 30 \times 1.6726231^{-27}$ kg, $m_1 = 1000 m_2$, $T = 300$ K, $n_B = 2.68 \times 10^{25}/\mathrm{m}^3$, and $D = 2.59 \times 10^{-9}$ m.

with

$$\mathbb{R} = \frac{\sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2 v_1^2}{2k_BT}} + \left(v_1 - \frac{1}{v_1}\frac{k_BT}{m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}{\sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2 v_1^2}{2k_BT}} + \left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}, \tag{20}$$

where $\mathbb{R}$ is a dimensionless parameter depending on v_1, m_2, and T.

Simulation has been launched to check the calculation. The results have been shown in Fig. 6. Three random numbers of $0 \to 1$ have been used to determine the initial state of particle B: one to determine the relative speed v_r by Eq. (17), one to determine the angle α' by Eq. (14), and the last one to determine the angel ϕ by Eq. (1). Two other random numbers of $0 \to 1$ are used to determine the direction (θ' and ϕ') in the sphere frame of $\mathbf{v}_\mu$ by Eq. (3). The magnitude of the final velocity $\mathbf{v}_{1a}$ is thus determined by $\mathbf{v}_\mu$, θ', ϕ', and $\Delta_\mu \mathbf{v}_1$. The mean velocity change in the simulation is the averaging of the velocity changes of 10000 runs, with $m_2 = 30 \times 1.6726231^{-27}$ kg, $m_1 = 1000\, m_2$, and $T = 300$ K. Some points can be distinguished in Fig. 6: (i) Eq. (19) agrees with the simulation's result perfectly well; (ii) $\overline{\Delta v_1}$ is always negative, implying that $\Delta \mathbf{v}_1$ is always opposite to $\mathbf{v}_1$; (iii) the difference of the theory and the linear approximation of $\Delta \mathbf{v}_1$ increases with the increasing of the speed, and $|\Delta \mathbf{v}_1|$ of the theory is less than the one in linear approximation; (iv) in the limiting cases, while $v_1 \ll \overline{v_2}$ and/or $v_1 \gg \overline{v_2}$, the mean velocity change increases (but opposite the direction) with v_1 linearly. Note that the simulation is only designed to test the self-consistency of the theory, a rigorous check must be done by Monte Carlo collision simulations or direct experiments.

C. Mean collision force of particles moving in Maxwell inert gas

By Eq. (18) and Eq. (19), the friction force (scalar $\overline{\mathbf{f}}$) is thus obtained,

$$\begin{aligned}\overline{f} &= m_1 \overline{Z_{AB}}\, \overline{\Delta v_1} \\ &= -n_B \pi D^2 \frac{m_1 m_2}{m_1 + m_2}\left\{\left(v_1 + \frac{1}{v_1}\frac{k_BT}{m_2}\right)\sqrt{\frac{2k_BT}{\pi m_2}}e^{-\frac{m_2 v_1^2}{2k_BT}}\right. \\ &\quad \left. + \left[v_1^2 + \frac{2k_BT}{m_2} - \frac{1}{v_1^2}\left(\frac{k_BT}{m_2}\right)^2\right]\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)\right\}. \end{aligned} \tag{21}$$

As we know in Fig. 6, $\overline{\mathbf{f}}$ is always opposite the moving direction, with $\overline{f} = -|\overline{f}|$. Dependence of $|\overline{f}|$ on v_1 has been plotted and shown in Fig. 7, and the parameters are the same with Fig. 6. The insets propose linear/quadratic dependence of $\overline{f}$ on v_1 in low/high speed cases.

V. DISCUSSIONS AND APPROXIMATIONS IN THE TWO LIMITING CASES

In this section, Eq. (21) is applied to the limiting low and high speed cases, discussions and approximations are made.

A. Low speed cases

While $v_1 \ll \overline{v_2}$, expand the exponential terms up to third order Taylor's series, we have the approximations

$$e^{-\frac{m_2 v_1^2}{2k_BT}} \approx 1 - \frac{m_2 v_1^2}{2k_BT} + \frac{1}{2}\left(\frac{m_2 v_1^2}{2k_BT}\right)^2 - \frac{1}{6}\left(\frac{m_2 v_1^2}{2k_BT}\right)^6,$$

$$\begin{aligned}\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right) \approx \frac{2}{\sqrt{\pi}}&\left[\sqrt{\frac{m_2 v_1^2}{2k_BT}} - \frac{1}{3}\left(\sqrt{\frac{m_2 v_1^2}{2k_BT}}\right)^3\right. \\ &\left. + \frac{1}{10}\left(\sqrt{\frac{m_2 v_1^2}{2k_BT}}\right)^5 - \frac{1}{42}\left(\sqrt{\frac{m_2 v_1^2}{2k_BT}}\right)^7\right].\end{aligned}$$

The mean relative speed can be approximated as [see Eq. (A1) in Appendix]

$$\overline{v_r} \approx \overline{v_2} + \frac{4v_1^2}{3\pi\overline{v_2}} - \frac{8v_1^4}{15\pi^2\overline{v_2}^3} + \frac{32\pi^2 v_1^6}{105\pi^4\overline{v_2}^5},$$

The quadratic dependence of $\overline{v_r}$ on v_1 in this formula is in agreement with the first inset in Fig. 5. $\overline{\Delta v_1}$ can be approximated as [for details see Eq. (A2) in Appendix]

$$\overline{\Delta v_1} \approx -\frac{m_2}{m_1 + m_2}\left(\frac{4}{3}v_1 - \frac{32v_1^3}{45\pi\overline{v_2}^2} + \frac{256v_1^5}{189\pi^2\overline{v_2}^4}\right).$$

Approximation of $\overline{f}$ in this case is [for details see Eq. (A3)]

$$\overline{f} \approx -n_B \pi D^2 \frac{m_1 m_2}{m_1 + m_2}\left(\frac{4}{3}\overline{v_2}v_1 + \frac{16v_1^3}{15\pi\overline{v_2}} - \frac{32v_1^5}{105\pi^2\overline{v_2}^3}\right). \tag{22}$$

Beyond the linear approximation, odd terms of v_1 should be considered, such as the cubic term $(16v_1^3/15\pi\overline{v_2})$ and the quintic term $(-32v_1^5/105\pi^2\overline{v_2}^3)$ in Eq. (22). In the linear approximation,

$$\overline{f} \approx -\eta v_1, \tag{23}$$

with

$$\eta = \frac{4n_B\pi D^2 m_1 m_2 \overline{v_2}}{3(m_1+m_2)} = \frac{4}{3}\frac{m_1}{m_1+m_2}\rho_B\sigma_{AB}\overline{v_2} \tag{24}$$

is the friction coefficient (here, η is somewhat different with the viscosity coefficient in fluid, as the microscopic mechanism of the force is limited to the present collision model, and the size/boundary dependence is involved) of the collision force at low speed cases, where ρ_B is the mass density of the background gas, and σ_{AB} is the area of the scattering cross section. While $m_1 \gg m_2$, we get

$$\eta = \tfrac{4}{3}\rho_B\sigma_{AB}\overline{v_2}.$$

That is to say, $\overline{f}$ is proportional to v_1 and a linear approximation is proposed.

Since $\overline{f}$ and f_s are all about the force that a particle is moving in fluid/gas, it is necessary to compare the two forces. In some way, the two forces are quite similar. The velocity/temperature dependencies are the same: $\overline{f} = -\eta v_1$ and $f_s = -6\pi\eta_s r_1 v_1$ (here, $\mathcal{R} = r_1$), with $\eta \propto \overline{v_2} \propto \sqrt{T}$ and $\eta_s = n_B m_2 \bar{l}\,\overline{v_2}/3 \propto \overline{v_2} \propto \sqrt{T}$ [19,21,25,28]. Note that a 0.75 power dependence of η on T has been discovered in the air, which was attributed to the increase of the scattering cross section with the temperature increasing for polyatomic molecules[21,23]. The difference of the two forces displays in three aspects as the result of different microscopical mechanisms. The first is about the dependence of n_B, $\eta \propto n_B$ but η_s is independent of n_B (counteracted by n_B and $\bar{l}$ in Maxwell's model); the second is about the dependence of D, $\overline{f} \propto D^2$ but $f_s \propto r_1$; the third is about the mass dependence, $\overline{f} \propto m_1/(m_1+m_2)$ but f_s is independent of m_1. The difference suggests the Stokes force may fail for small particles of a large Knudsen number ($\bar{l}/D \gg 1$), which is supported by the studies in Refs. [18,29,30] where a quadratic size dependence is reported. The D^2 dependence is the essential distinction of the kinetic model compared to the Stokes hydrodynamic model.

B. High speed cases

While $v_1 \gg \overline{v_2}$, the exponent relevant terms can be approximated as

$$e^{-\frac{m_2 v_1^2}{2k_B T}} \sim 0, \qquad \mathrm{erf}\left(\sqrt{\frac{m_2}{2k_B T}}v_1\right) \sim 1,$$

$$\mathbb{R} \approx 1 - \frac{2k_B T}{m_2 v_1^2} + \frac{k_B^2 T^2}{m_2^2 v_1^4}.$$

The mean relative speed can be approximated as

$$\overline{v_r} \approx v_1 + \frac{k_B T}{m_2 v_1} \approx v_1,$$

which is in agreement with the suggestion in the second inset in Fig. 5. Approximation of $\overline{\Delta v_1}$ is

$$\overline{\Delta v_1} \approx -\frac{m_2}{m_1+m_2}\left[v_1 + \frac{1}{v_1}\frac{k_B T}{m_2} - 2\frac{1}{v_1^3}\left(\frac{k_B T}{m_2}\right)^2\right] \approx -\frac{m_2 v_1}{m_1+m_2}.$$

$\overline{f}$ can be approximated as

$$\overline{f} \approx -n_B\pi D^2\frac{m_1 m_2}{m_1+m_2}\left(v_1^2 + \frac{2k_B T}{m_2} - \frac{k_B^2 T^2}{m_2^2 v_1^2}\right) \approx -\eta' v_1^2, \tag{25}$$

which increases with v_1^2 linearly, with

$$\eta' = \rho_B\sigma_{AB}m_1/(m_1+m_2). \tag{26}$$

While $m_1 \gg m_2$, it gives

$$\eta' = \rho_B\sigma_{AB}.$$

That is, the force is proportional to ρ_B, σ_{AB} and v_1^2, but independent of the gas temperature T. Beyond the low order approximation, a small constant resistent force (no speed involved) related to the density and temperature of the background gas can be employed as an addition: $-2n_B\sigma_{AB}k_B T m_1/(m_1+m_2)$.

The SE relation is found to be dependent on the size/mass of the Brownian particle [18,29,30]. A decomposing of the kinetic contribution and the hydrodynamic contribution in the transitional range can be found in Ref. [18] as well. Here, we focus on the friction force of a particle moving through rarefied gases. Under the classical elastic model of hard spheres and the molecular chaos hypothesis, a general formula of the particle's speed, $\overline{f}(v_1)$, is obtained by the direct calculation of $\overline{Z_{AB}}$ and $\overline{\Delta \mathbf{v}_1}$. In the low speed cases, the result is consistent with Ref. [18] and the D^2 dependence is distinguished. Beyond the low speed cases studied in Ref. [18], our results can be applied to all speed ranges. Benefiting from the simplicity of the kinetic model, the analytical Eq. (21) works with a wide range of parameters as long as the sparse requirement is satisfied.

VI. CONCLUSION

Of the explicit force formula Eq. (21), some points are summarized: (i) $\overline{f} = -\eta v_1$ and $\overline{f} = -\eta' v_1^2$ are the approximations for the low and high speed respectively. It shows that the speed dependence is increased with the increasing of v_1. (ii) Throughout the speed ranges, $\overline{f} \propto n_B D^2$ holds well. (iii) The temperature dependence decreases with the increasing of v_1, and in the high speed limiting, the temperature dependence tends to vanish. By the dependencies, using experiments of small particle Brownian motion to verify the theory is becoming possible[18]. Moreover, in the high vacuum environment such as the space tracks, by the direct friction measurement of the macroscopical objects, a verification may also be possible.

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APPENDIX: APPROXIMATIONS AT LOW SPEED LIMITING CASE

While $v_1 \ll \overline{v_2}$, the exponent can be expanded to Taylor's series, here up to the third rank, with

$$e^{-\frac{m_2 v_1^2}{2kT}} \approx 1 - \frac{m_2 v_1^2}{2kT} + \frac{1}{2}\left(\frac{m_2 v_1^2}{2kT}\right)^2 - \frac{1}{6}\left(\frac{m_2 v_1^2}{2kT}\right)^3,$$

$$\begin{aligned}\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_B T}}v_1\right) &= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\frac{m_2 v_1^2}{2kT}}} e^{-\frac{m_2 v_1^2}{2kT}} d\sqrt{\frac{m_2 v_1^2}{2kT}} \\ &\approx \frac{2}{\sqrt{\pi}}\left[\sqrt{\frac{m_2}{2kT}}v_1 - \frac{1}{3}\left(\sqrt{\frac{m_2}{2kT}}\right)^3 v_1^3 + \frac{1}{10}\left(\sqrt{\frac{m_2}{2kT}}\right)^5 v_1^5 - \frac{1}{42}\left(\sqrt{\frac{m_2}{2kT}}\right)^7 v_1^7\right].\end{aligned}$$

The term in the mean relative speed in Eq. (16) at low speed limiting can be approximated as

$$\begin{aligned}\overline{v_r} &= \sqrt{\frac{2kT}{\pi m_2}}e^{-\frac{m_2 v_1^2}{2kT}} + \left(v_1 + \frac{kT}{v_1 m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_B T}}v_1\right) \\
&\approx \sqrt{\frac{2kT}{\pi m_2}}\left[1 - \frac{m_2 v_1^2}{2kT} + \frac{1}{2}\left(\frac{m_2 v_1^2}{2kT}\right)^2 - \frac{1}{6}\left(\frac{m_2 v_1^2}{2kT}\right)^3\right] \\
&\quad + \left(v_1 + \frac{kT}{v_1 m_2}\right)\frac{2}{\sqrt{\pi}}\left[\sqrt{\frac{m_2}{2kT}}v_1 - \frac{1}{3}\left(\sqrt{\frac{m_2}{2kT}}\right)^3 v_1^3 + \frac{1}{10}\left(\sqrt{\frac{m_2}{2kT}}\right)^5 v_1^5 - \frac{1}{42}\left(\sqrt{\frac{m_2}{2kT}}\right)^7 v_1^7\right] \\
&= \sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}} - \sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}\frac{m_2 v_1^2}{2kT} + \frac{1}{2}\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}\left(\frac{m_2 v_1^2}{2kT}\right)^2 - \frac{1}{6}\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}\left(\frac{m_2 v_1^2}{2kT}\right)^3 \\
&\quad + \frac{2v_1}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}v_1 - \frac{1}{3}\frac{2v_1}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3 v_1^3 + \frac{1}{10}\frac{2v_1}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^5 v_1^5 - \frac{1}{42}\frac{2v_1}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^7 v_1^7 \\
&\quad + \frac{kT}{v_1 m_2}\frac{2}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}v_1 - \frac{1}{3}\frac{kT}{v_1 m_2}\frac{2}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3 v_1^3 + \frac{kT}{v_1 m_2}\frac{1}{10}\frac{2}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^5 v_1^5 - \frac{1}{42}\frac{kT}{v_1 m_2}\frac{2}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^7 v_1^7 \\
&= \sqrt{\frac{2kT}{\pi m_2}} - v_1^2\sqrt{\frac{m_2}{\pi 2kT}} + \frac{\pi v_1^4}{2}\left(\sqrt{\frac{m_2}{\pi 2kT}}\right)^3 - \frac{\pi^2 v_1^6}{6}\left(\sqrt{\frac{m_2}{\pi 2kT}}\right)^5 + 2v_1^2\sqrt{\frac{m_2}{2\pi kT}} \\
&\quad - \frac{2\pi v_1^4}{3}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^3 + \frac{\pi^2 v_1^6}{5}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^5 - \frac{\pi^3 v_1^8}{21}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^7 + \sqrt{\frac{2kT}{\pi m_2}} - \frac{1}{3}v_1^2\sqrt{\frac{m_2}{2\pi kT}} \\
&\quad + \frac{\pi v_1^4}{10}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^3 - \frac{\pi^2 v_1^6}{42}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^5 \\
&= 2\sqrt{\frac{2kT}{\pi m_2}} - v_1^2\sqrt{\frac{m_2}{\pi 2kT}} + 2v_1^2\sqrt{\frac{m_2}{2\pi kT}} - \frac{1}{3}v_1^2\sqrt{\frac{m_2}{2kT}} + \frac{\pi v_1^4}{2}\left(\sqrt{\frac{m_2}{\pi 2kT}}\right)^3 - \frac{2\pi v_1^4}{3}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^3 \\
&\quad + \frac{\pi v_1^4}{10}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^3 - \frac{\pi^2 v_1^6}{6}\left(\sqrt{\frac{m_2}{\pi 2kT}}\right)^5 + \frac{\pi^2 v_1^6}{5}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^5 - \frac{\pi^2 v_1^6}{42}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^5 - \frac{\pi^3 v_1^8}{21}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^7 \\
&= 2\sqrt{\frac{2kT}{\pi m_2}} + \frac{2}{3}v_1^2\sqrt{\frac{m_2}{2\pi kT}} - \frac{\pi v_1^4}{15}\left(\sqrt{\frac{m_2}{\pi 2kT}}\right)^3 + \frac{\pi^2 v_1^6}{105}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^5 - \frac{\pi^3 v_1^8}{21}\left(\sqrt{\frac{m_2}{2\pi kT}}\right)^7 \\
&\approx \overline{v_2} + \frac{4v_1^2}{3\pi\overline{v_2}} - \frac{8v_1^4}{15\pi^2\overline{v_2}^3} + \frac{32\pi^2 v_1^6}{105\pi^4\overline{v_2}^5}.\end{aligned} \tag{A1}$$

And mean velocity change in Eq. (19) at low speed limiting can be approximated as

$$
\begin{aligned}
\overline{\Delta v} &= \frac{\left(v_1+\frac{1}{v_1}\frac{kT}{m_2}\right)\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left[v_1^2+2\frac{kT}{m_2}-\frac{1}{v_1^2}\left(\frac{kT}{m_2}\right)^2\right]\frac{2}{\sqrt{\pi}}\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}{\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left(v_1+\frac{kT}{v_1m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}\\
&= v_1+\frac{\frac{1}{v_1}\frac{kT}{m_2}\sqrt{\frac{2kT}{\pi m_2}}e^{-\frac{m_2v_1^2}{2kT}}+\left[\frac{kT}{m_2}-\frac{1}{v_1^2}\left(\frac{kT}{m_2}\right)^2\right]\frac{2}{\sqrt{\pi}}\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}{\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left(v_1+\frac{kT}{v_1m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}\\
&\approx v_1+\frac{1}{\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left(v_1+\frac{kT}{v_1m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}\\
&\quad\times\left\{\frac{v_1}{\sqrt{\pi}}\left[-\frac{1}{2}\left(\sqrt{\frac{2kT}{m_2}}\right)^3\frac{m_2}{2kT}+\frac{2kT}{m_2}\sqrt{\frac{m_2}{2kT}}+\frac{1}{2}\left(\frac{2kT}{m_2}\right)^2\frac{1}{3}\left(\sqrt{\frac{m_2}{2kT}}\right)^3\right]\right.\\
&\quad+\frac{v_1^3}{\sqrt{\pi}}\left[\frac{1}{4}\left(\sqrt{\frac{2kT}{m_2}}\right)^3\left(\frac{m_2}{2kT}\right)^2-\frac{1}{3}\frac{2kT}{m_2}\left(\sqrt{\frac{m_2}{2kT}}\right)^3-\frac{1}{20}\left(\frac{2kT}{m_2}\right)^2\left(\sqrt{\frac{m_2}{2kT}}\right)^5\right]\\
&\quad\left.+\frac{v_1^5}{\sqrt{\pi}}\left[-\frac{1}{2}\left(\sqrt{\frac{2kT}{m_2}}\right)^3\frac{1}{6}\left(\frac{m_2}{2kT}\right)^3+\frac{2kT}{m_2}\frac{1}{10}\left(\sqrt{\frac{m_2}{2kT}}\right)^5+\frac{1}{2}\left(\frac{2kT}{m_2}\right)^2\frac{1}{42}\left(\sqrt{\frac{m_2}{2kT}}\right)^7\right]\right\}\\
&= v_1+\frac{\left[\left(\frac{1}{4}-\frac{1}{3}-\frac{1}{20}\right)\frac{v_1^3}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}\right]+\left(\frac{1}{4}-\frac{1}{3}-\frac{1}{20}\right)\frac{v_1^3}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}}{\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left(v_1+\frac{kT}{v_1m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}+\frac{\left(-\frac{1}{12}+\frac{1}{10}+\frac{1}{84}\right)\frac{v_1^5}{\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3}{\sqrt{\frac{kT}{m_2}}\sqrt{\frac{2}{\pi}}e^{-\frac{m_2v_1^2}{2kT}}+\left(v_1+\frac{kT}{v_1m_2}\right)\mathrm{erf}\left(\sqrt{\frac{m_2}{2k_BT}}v_1\right)}\\
&\approx v_1+\frac{\frac{1}{3}\overline{v_2}v_1-\frac{4}{15}\frac{v_1^3}{\pi\overline{v_2}}+\frac{8}{35}\frac{v_1^5}{\pi^2\overline{v_2}^3}}{\overline{v_2}+\frac{4v_1^2}{3\pi\overline{v_2}}-\frac{8v_1^4}{15\pi^2\overline{v_2}^3}+\frac{32v_1^6}{105\pi^3\overline{v_2}^5}}\\
&\approx v_1+\frac{1}{\overline{v_2}}\left(\frac{1}{3}\overline{v_2}v_1-\frac{4}{15}\frac{v_1^3}{\pi\overline{v_2}}+\frac{8}{35}\frac{v_1^5}{\pi^2\overline{v_2}^3}\right)\\
&\quad\times\left[1-\frac{4v_1^2}{3\pi\overline{v_2}^2}+\frac{8v_1^4}{15\pi^2\overline{v_2}^4}-\frac{32v_1^6}{105\pi^3\overline{v_2}^6}+\left(-\frac{4v_1^2}{3\pi\overline{v_2}^2}+\frac{8v_1^4}{15\pi^2\overline{v_2}^4}-\frac{32v_1^6}{105\pi^3\overline{v_2}^6}\right)^2\right]\\
&\approx \frac{4}{3}v_1-\frac{32}{45}\frac{v_1^3}{\pi\overline{v_2}^2}+\frac{256}{189}\frac{v_1^5}{\pi^2\overline{v_2}^4}.
\end{aligned}\tag{A2}
$$

The brace in Eq. (21) at low speed limiting can be approximated as

$$
\begin{aligned}
&\left(v_1+\frac{1}{v_1}\frac{kT}{m_2}\right)\sqrt{\frac{2kT}{\pi m_2}}\left[1-\frac{m_2v_1^2}{2kT}+\frac{1}{2}\left(\frac{m_2v_1^2}{2kT}\right)^2-\frac{1}{6}\left(\frac{m_2v_1^2}{2kT}\right)^3\right]+\frac{2}{\sqrt{\pi}}\\
&\quad\times\left[v_1^2+2\frac{kT}{m_2}-\frac{1}{v_1^2}\left(\frac{kT}{m_2}\right)^2\right]\left[\sqrt{\frac{m_2}{2kT}}v_1-\frac{1}{3}\left(\sqrt{\frac{m_2}{2kT}}\right)^3v_1^3+\frac{1}{10}\left(\sqrt{\frac{m_2}{2kT}}\right)^5v_1^5-\frac{1}{42}\left(\sqrt{\frac{m_2}{2kT}}\right)^7v_1^7\right]\\
&\quad= v_1\sqrt{\frac{2kT}{\pi m_2}}+\frac{1}{v_1}\left(\sqrt{\frac{kT}{m_2}}\right)^3\sqrt{\frac{2}{\pi}}-v_1\sqrt{\frac{2kT}{\pi m_2}}\frac{m_2v_1^2}{2kT}-\frac{1}{v_1}\left(\sqrt{\frac{kT}{m_2}}\right)^3\sqrt{\frac{2}{\pi}}\frac{m_2v_1^2}{2kT}\\
&\quad+v_1\sqrt{\frac{2kT}{\pi m_2}}\frac{1}{2}\left(\frac{m_2v_1^2}{2kT}\right)^2+\frac{1}{v_1}\left(\sqrt{\frac{kT}{m_2}}\right)^3\sqrt{\frac{2}{\pi}}\frac{1}{2}\left(\frac{m_2v_1^2}{2kT}\right)^2-\frac{v_1}{6}\sqrt{\frac{2kT}{\pi m_2}}\left(\frac{m_2v_1^2}{2kT}\right)^3\\
&\quad-\frac{1}{v_1}\left(\sqrt{\frac{kT}{m_2}}\right)^3\sqrt{\frac{2}{\pi}}\frac{1}{6}\left(\frac{m_2v_1^2}{2kT}\right)^3+\frac{2v_1^3}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}-\frac{2v_1^5}{3\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3+\frac{2v_1^7}{10\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^5
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{v_1^9}{21\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^7+\frac{kT}{m_2}\frac{4v_1}{\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}-\frac{kT}{m_2}\frac{4v_1^3}{3\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3+\frac{kT}{m_2}\frac{4v_1^5}{10\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^5\\
&-\frac{kT}{m_2}\frac{2v_1^7}{21\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^7-\left(\frac{kT}{m_2}\right)^2\frac{2}{v_1\sqrt{\pi}}\sqrt{\frac{m_2}{2kT}}+\left(\frac{kT}{m_2}\right)^2\frac{2v_1}{3\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^3\\
&-\left(\frac{kT}{m_2}\right)^2\frac{v_1^3}{5\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^5+\left(\frac{kT}{m_2}\right)^2\frac{v_1^5}{21\sqrt{\pi}}\left(\sqrt{\frac{m_2}{2kT}}\right)^7\\
=\;& v_1\sqrt{\frac{2kT}{\pi m_2}}+\frac{\pi}{2v_1}\left(\sqrt{\frac{2kT}{\pi m_2}}\right)^3-\frac{v_1^3}{\pi}\sqrt{\frac{\pi m_2}{2kT}}-\frac{v_1}{2}\sqrt{\frac{2kT}{\pi m_2}}+\frac{v_1^5}{2\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3+\frac{v_1^3}{4\pi}\sqrt{\frac{\pi m_2}{2kT}}\\
&-\frac{v_1^7}{6\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5-\frac{v_1^5}{12\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3+\frac{2v_1^3}{\pi}\sqrt{\frac{\pi m_2}{2kT}}-\frac{2v_1^5}{3\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3+\frac{v_1^7}{5\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5\\
&-\frac{v_1^9}{21\pi^4}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^7+2v_1\sqrt{\frac{2kT}{\pi m_2}}-\frac{2v_1^3}{3\pi}\sqrt{\frac{\pi m_2}{2kT}}+\frac{v_1^5}{5\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3-\frac{v_1^7}{21\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5\\
&-\frac{\pi}{2v_1}\left(\sqrt{\frac{2kT}{\pi m_2}}\right)^3+\frac{v_1}{6}\sqrt{\frac{2kT}{\pi m_2}}-\frac{v_1^3}{20\pi}\sqrt{\frac{\pi m_2}{2kT}}+\frac{v_1^5}{84\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3\\
=\;& v_1\sqrt{\frac{2kT}{\pi m_2}}-\frac{v_1}{2}\sqrt{\frac{2kT}{\pi m_2}}+2v_1\sqrt{\frac{2kT}{\pi m_2}}+\frac{v_1}{6}\sqrt{\frac{2kT}{\pi m_2}}\\
&-\frac{v_1^3}{\pi}\sqrt{\frac{\pi m_2}{2kT}}+\frac{v_1^3}{4\pi}\sqrt{\frac{\pi m_2}{2kT}}+\frac{2v_1^3}{\pi}\sqrt{\frac{\pi m_2}{2kT}}-\frac{2v_1^3}{3\pi}\sqrt{\frac{\pi m_2}{2kT}}-\frac{v_1^3}{20\pi}\sqrt{\frac{\pi m_2}{2kT}}\\
&+\frac{v_1^5}{2\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3-\frac{v_1^5}{12\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3-\frac{2v_1^5}{3\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3+\frac{v_1^5}{5\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3+\frac{v_1^5}{84\pi^2}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^3\\
&-\frac{v^7}{6\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5+\frac{v^7}{5\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5-\frac{v^7}{21\pi^3}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^5-\frac{v^9}{21\pi^4}\left(\sqrt{\frac{\pi m_2}{2kT}}\right)^7\\
\approx\;&\left(1-\frac{1}{2}+2+\frac{1}{6}\right)\frac{v_1}{2}\sqrt{\frac{8kT}{\pi m_2}}+\left(-1+\frac{1}{4}+2-\frac{2}{3}\right)\frac{2v_1^3}{\pi}\sqrt{\frac{\pi m_2}{8kT}}\\
&+\left(\frac{1}{2}-\frac{1}{12}-\frac{2}{3}+\frac{1}{5}+\frac{1}{84}\right)\frac{8v_1^5}{\pi^2}\left(\sqrt{\frac{\pi m_2}{8kT}}\right)^3\\
=\;&\frac{4}{3}\overline{v_2}v_1+\frac{16v_1^3}{15\pi\overline{v_2}}-\frac{32v_1^5}{105\pi^2\overline{v_2}^3}.
\end{aligned}
\tag{A3}
$$

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