Exact relations for energy transfer in simple and active binary fluid turbulence

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Inertial range energy transfer in three-dimensional fully developed binary fluid turbulence is studied under the assumption of statistical homogeneity. Using two-point statistics, exact relations corresponding to the energy cascade are derived in terms of (i) two-point increments and (ii) two-point correlators. Despite having some apparent resemblances, the exact relation in binary fluid turbulence is found to be different from that of the incompressible magnetohydrodynamic turbulence [H. Politano and A. Pouquet, Geophys. Res. Lett. **25**, 273 (1998)]. Besides the usual direct cascade of energy, under certain situations, an inverse cascade of energy is also speculated depending upon the strength of the activity parameter and the interplay between the two-point increments of the fluid velocity and the composition gradient fields. An alternative form of the exact relation is also derived in terms of the "upsilon" variables and a subsequent phenomenology is proposed predicting a $k^{-3/2}$ law for the turbulent energy spectrum.

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I. INTRODUCTION

Turbulence is a ubiquitous phenomenon in classical fluids, typically characterized by its multiscale vortexlike structures and a universal cascade of inviscid invariants (energy, helicity, etc.) within the inertial range. For homogeneous and isotropic turbulence of an incompressible fluid, the energy cascade leads to a $k^{-5/3}$ spectrum [1], whereas for isotropic magnetohydrodynamic (MHD) turbulence, a $k^{-3/2}$ power spectrum is predicted [2,3]. In real space, the universality of turbulence can be understood in terms of exact scaling relations which express the average flux rate (ε) of a cascading invariant in terms of the statistical moments of two-point fluctuations of the relevant field variables. Under the assumption of homogeneity and isotropy, a few algebraic exact relations are obtained for incompressible hydrodynamic (HD) and MHD turbulence [4–6]. However, for homogeneous (and not necessarily isotropic) turbulent flows, differential exact laws have been derived with the generic expression

$$\nabla_r \cdot \boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{S}} = -4\varepsilon, \tag{1}$$

where $\boldsymbol{\mathcal{F}}$ denotes the flux term involving the statistical moments of the two-point increments with $\boldsymbol{r}$ being the two-point separation and $\boldsymbol{\mathcal{S}}$ denotes the source term which cannot be expressed as a divergence of a function of two-point increments in the differential form of the exact relations.

For incompressible HD and MHD turbulence, it has been found [7,8] that $\boldsymbol{\mathcal{S}} = 0$, and

$$\boldsymbol{\mathcal{F}}_{\mathbf{HD}} = \langle (\delta \boldsymbol{u})^2 \delta \boldsymbol{u} \rangle, \tag{2}$$

$$\boldsymbol{\mathcal{F}}_{\mathbf{MHD}} = \langle [(\delta \boldsymbol{u})^2 + (\delta \boldsymbol{b})^2] \delta \boldsymbol{u} - 2(\delta \boldsymbol{u} \cdot \delta \boldsymbol{b}) \delta \boldsymbol{b} \rangle, \tag{3}$$

where $\boldsymbol{u}$ and $\boldsymbol{b}$ denote the fluid velocity and the magnetic field (normalized to a velocity), respectively, and $\delta \boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{a}(\boldsymbol{x})$ represents the two-point increment for the variable $\boldsymbol{a}$. The corresponding algebraic exact relations can finally be obtained just by integrating the divergence term under spherical symmetry [9,10]. In addition to the above simple turbulent flows, Eq. (1) represents the turbulent energy transfer for a broader range of fluids including incompressible Hall-MHD plasma and compressible fluids and plasmas where the source term is nonvanishing and the derivation of isotropic algebraic forms is not straightforward [11–16]. However, algebraic exact relations for a large number of homogeneous turbulent flows can be derived following an alternative formulation proposed recently for incompressible HD and MHD (including Hall MHD) turbulence [17,18] and then generalized to more complicated systems, e.g., rotating fluid, compressible fluids, and plasmas [19,20], and even ferrofluids [21]. Besides the vector fields (velocity and magnetic fields), often it is interesting to study the behavior of a scalar field ϕ in a turbulent flow. In general, the scalar field is advected by the velocity field usually satisfying the advection-diffusion equation. Furthermore, it can be passive (e.g., dust particles in a turbulent flow) or active (e.g., stratified flows) depending on whether it provides feedback to the momentum equation [22]. Incompressible flows with a passive scalar possess two independent quadratic inviscid invariants, i.e., the kinetic energy $\frac{1}{2}\langle \boldsymbol{u}^2 \rangle$ and the scalar energy $\frac{1}{2}\langle \phi^2 \rangle$. While the first conservation leads to an exact relation similar to incompressible HD turbulence, the second one leads to a separate differential exact relation with $\boldsymbol{\mathcal{F}} = \langle (\delta\phi)^2 \delta \boldsymbol{u} \rangle$ and $\boldsymbol{\mathcal{S}} = 0$ [23].

For active scalars, the kinetic energy is no longer conserved. The simplest active scalar flow is represented by a stably stratified fluid where the active scalar ϕ denotes the density perturbation and provides a feedback force density proportional to $\phi \hat{z}$, with $\hat{z}$ being the direction of stratification. For such flows, $\frac{1}{2}\langle \phi^2 \rangle$ remains an inviscid invariant. However, instead of the kinetic energy, the total energy $\frac{1}{2}\langle u^2 + \phi^2/N^2 \rangle$ is found to be an inviscid invariant and the correspond-

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ing sourceless differential exact relation is given by $\mathcal{F} = \langle[(\delta \boldsymbol{u})^2 + (\delta\phi/N)^2]\delta \boldsymbol{u}\rangle$ [24], where N is the Brunt-Väisälä frequency.

A more complex feedback force arises if we consider a system of binary fluid mixtures. Binary fluid is a two-component system ranging from a mixture of two simple fluids, e.g., oil and water, to complex systems such as "active" binary fluids [25–29]. For such systems, the active scalar ϕ represents the local molecular composition of the binary mixture [27]. Simple binary fluids contain microstructures such as droplets or domains of one fluid into the sea of the other, thus producing interfaces or domain walls. Sharp variation in ϕ across the interfaces generates diffusion currents which, in turn, drive the velocity field by exerting feedback stress proportional to $\kappa(\nabla\phi \otimes \nabla\phi)$ [27,30], where κ is a positive constant. However feedback stress does not necessarily take the same form if one of the binary-fluid components is "active," i.e., the constituent particles are equipped with intrinsic mechanisms (e.g., self-propulsion, body deformation [31,32], etc.) of transmuting ambient free energy into directed motion thereby breaking the time reversal symmetry at the scale of each constituent [33]. Nevertheless, if the active particles are either spherically symmetric or possess a weak orientational order (e.g., artificial active colloids, dilute bacterial suspensions, etc.), the feedback stress due to the activity of the particles can be written as $\eta(\nabla\phi \otimes \nabla\phi)$, where η is the activity parameter [34]. The total feedback stress is finally obtained by adding the active stress to the stress due to the interfacial tension, and is given by $(\kappa + \eta)(\nabla\phi \otimes \nabla\phi)$ [27,28].

Although individual components of a binary fluid tend to phase separate (coarse graining) below a critical temperature (T_c) [35], it is often useful to have them in the form of steady emulsions, i.e., a homogeneous phase-mixed state [36]. This can be achieved through the generation of turbulence which, in effect, prevents the phase separation owing to its enhanced mixing property [37–40]. In a physical system, turbulence can be attained in two ways, i.e., either by large nonlinear perturbations (e.g., MHD, stably stratified flows, etc.) or due to the growth of linear instability (e.g., Rayleigh-Benárd convection). Simple binary fluid and active fluid with extensile stress ($\eta > 0$) belong to the former category [26,28], whereas active fluid with contractile stress ($\eta < 0$) having large activity [i.e., $\xi = (\kappa + \eta) < 0$] belongs to the latter one [41,42]. Turbulence in an active binary fluid with $\xi < 0$ finds its application in the study of bioconvective plumes, accumulation of *Dinoflagellates* in turbulent flows, etc. [43]. Note that here we are referring to high Reynolds number (Re) turbulence in active fluids, which is different from the self-sustained pattern formation by dense active fluids at low Re ($\sim 10^{-5}$) [44–51]. The effect of turbulence on domain growth, energy spectra, etc. has been explored both for simple and active binary fluids [26,28,40,52–54]. However, no exact relation has been derived for such systems.

In this paper, using two-point statistics, we derive exact relations for the inertial range energy transfer in binary fluid turbulence (BFT). Under the assumption of statistical homogeneity, first we derive a differential exact relation as in Eq. (1) in terms of the two-point fluctuations of $\boldsymbol{u}$ and $\nabla\phi$. Then the same exact relation is also expressed in terms of (i) two-point correlators and (ii) newly defined upsilon variables. The derivation of the results is complemented by comparative statements on incompressible MHD turbulence which seems to have some interesting resemblances with our system [26,55].

The paper is organized as follows: in Sec. II, we describe the model and the basic equations of dynamics followed by the derivation of total energy conservation in the inviscid limit. Section III contains the detailed derivation of the exact relation both in terms of the two-point increments and two-point correlators. We then introduce a set of variables ($\boldsymbol{\Upsilon}^{\pm}$) and express the inviscid invariant and the corresponding exact relation in terms of those variables in Sec. III C. Finally, in Sec. IV, we discuss our results and conclude.

II. MODEL AND GOVERNING EQUATIONS

A binary fluid, composed of two components A and B, is usually defined in terms of their mean velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = (\rho_A \boldsymbol{u}_A + \rho_B \boldsymbol{u}_B)/(\rho_A + \rho_B)$ and the local molecular composition $\phi(\boldsymbol{x}, t) = (\rho_A - \rho_B)/(\rho_A + \rho_B)$, where (ρ_A, ρ_B) and $(\boldsymbol{u}_A, \boldsymbol{u}_B)$ are the densities and the velocities of the components A and B, respectively [27]. Here we are considering systems where the velocity fluctuations are much less than the sound speed, thus justifying the assumption of incompressibility for the resultant fluid [26]. The corresponding continuity and momentum evolution equations are given by

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4}$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nabla \cdot \boldsymbol{\Sigma} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \tag{5}$$

where p is the fluid pressure, $\boldsymbol{\Sigma}$ the feedback stress tensor due to the active scalar ϕ, ν the kinematic viscosity, and, finally, $\boldsymbol{f}$ a large scale forcing. For a general phase-separating binary fluid mixture, ϕ satisfies the Cahn-Hilliard equation given by

$$\partial_t \phi + (\boldsymbol{u} \cdot \nabla)\phi = \mathcal{M}\nabla^2 \mu, \tag{6}$$

$$\mu = \frac{\delta \mathcal{F}[\phi]}{\delta \phi} = a\phi + b\phi^3 - \kappa \nabla^2 \phi, \tag{7}$$

where $\mathcal{M}$ is the mobility coefficient, μ the chemical potential (or exchange potential), and $\mathcal{F} = \int[\frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \kappa(\nabla\phi)^2]d\tau$ a free-energy functional, where $a < 0$, while b and κ are positive constants and τ represents the volume [25,27,56]. For a phase-mixed binary fluid, $a > 0$ and $\mathcal{F} = \int[\frac{a}{2}\phi^2 + \kappa(\nabla\phi)^2]d\tau$ and Eq. (6) simply becomes

$$\partial_t \phi + (\boldsymbol{u} \cdot \nabla)\phi = \mathcal{M}\nabla^2(a\phi - \kappa\nabla^2\phi). \tag{8}$$

In addition to the force equation (5), one can also force Eq. (6) or Eq. (8) by an appropriate large scale forcing g_ϕ [40]. The corresponding evolution equation of $\phi(\boldsymbol{x}, t)$ is therefore given by

$$\partial_t \phi + (\boldsymbol{u} \cdot \nabla)\phi = \mathcal{M}\nabla^2 \mu + g_\phi. \tag{9}$$

The final step is to express the feedback stress tensor $\boldsymbol{\Sigma}$ in terms of the other field variables. Both for the phase-separating and phase-mixed binary fluids, the thermodynamic stress $\boldsymbol{\Sigma}$ is developed due to the departure of the interfacial profile from the equilibrium. This local inhomogeneity in ϕ leads to a feedback force density $\nabla \cdot \boldsymbol{\Sigma} = -\phi\nabla\mu$ [27,30].

Using the expressions of μ and following a few steps of straightforward algebra, the feedback stress can be put in the form given below,

$$\boldsymbol{\Sigma} = -[\kappa(\nabla\phi \otimes \nabla\phi) + \Lambda\mathbb{I}], \tag{10}$$

where $\sqrt{\kappa}$ is dimensionally homogeneous to the kinematic viscosity [26] and

$$\Lambda = \left(\phi\frac{d\Gamma}{d\phi} - \Gamma - \kappa\phi\nabla^2\phi - \kappa\frac{(\nabla\phi)^2}{2}\right), \tag{11}$$

with Γ being the ϕ dependent part of the free-energy density. Evidently, the term $\Lambda\mathbb{I}$ is proportional to the unit tensor and can therefore be absorbed under a gradient in the force equation (5), which can now be expressed as

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P^* - \kappa\nabla\cdot(\nabla\phi\otimes\nabla\phi) + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \tag{12}$$

where $P^* = p + \Lambda$ can be understood as an effective pressure. The aforesaid equations of dynamics, often called "model H," provides us with a hydrodynamic description of a simple binary fluid mixture.

When one of the binary fluid components is active, the additional stress arises due to the intrinsic swimming mechanism of the active particles [28,57]. Self-motility, i.e., the ability of active particles to self-propel, produces a dipolar force field which yields the additional stress for active fluids. If the particles are either spherically symmetric or having negligible orientational order, the active stress becomes proportional to $\eta(\nabla\phi\otimes\nabla\phi)$, where η can be both positive or negative [40]. Equation (12) now becomes

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P^* - \xi\nabla\cdot(\nabla\phi\otimes\nabla\phi) + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \tag{13}$$

where $\xi = \kappa + \eta$. Note that unlike the feedback stress for simple binary fluids, the additional "active" stress is not derived from any free-energy functional [27]. For the sake of algebraic manipulation, it is convenient to consider $\nabla\phi$ as a field variable rather than ϕ itself. The respective evolution equation is given by

$$\partial_t(\nabla\phi) + \nabla(\boldsymbol{u}\cdot\nabla\phi) = \mathcal{M}\nabla^2(\nabla\mu) + \nabla g_\phi. \tag{14}$$

In the current study, we are interested in the turbulent energy transfer which necessarily involves the fluctuations with respect to the mean fields. By choosing an appropriate Galilean transformation, one can eliminate the mean velocity field. However, the mean composition gradient field cannot be eliminated by such transformations. It is then useful to decompose $\nabla\phi$ as

$$\nabla\phi = S\hat{z} + \nabla\psi = S\hat{z} + \boldsymbol{q}, \tag{15}$$

where $S\hat{z}$ denotes the mean composition gradient field and $\nabla\psi$ or $\boldsymbol{q}$ denotes the corresponding fluctuating field. The corresponding evolution equations of $\boldsymbol{u}$ and $\boldsymbol{q}$ will then be given by

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P - \xi(S\hat{z}+\boldsymbol{q})\nabla\cdot\boldsymbol{q} + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \tag{16}$$

$$\partial_t\boldsymbol{q} + \nabla(Su_z + \boldsymbol{u}\cdot\boldsymbol{q}) = \mathcal{M}\nabla^2(\nabla\mu) + \boldsymbol{g}, \tag{17}$$

where $\nabla\cdot\boldsymbol{u} = 0$, $\nabla\times\boldsymbol{q} = 0$, $\nabla g_\phi = \boldsymbol{g}$, $P = P^* + \xi(q^2/2 + Sq_z)$, and $\nabla\cdot[(S\hat{z}+\boldsymbol{q})\otimes(S\hat{z}+\boldsymbol{q})] = (S\hat{z}+\boldsymbol{q})\nabla\cdot\boldsymbol{q} + \nabla(q^2/2 + Sq_z)$. For such a system, the total turbulent energy is composed of the kinetic and the active energy and can be written as

$$E = \int\frac{1}{2}(u^2 + \xi q^2)d\tau. \tag{18}$$

In the following, we show E is an inviscid invariant of the flow. For that, we simply neglect the large scale forcing and small scale dissipation terms in Eqs. (16) and (17), which gives

$$\partial_t\left(\frac{u^2}{2}\right) = \boldsymbol{u}\cdot\partial_t\boldsymbol{u} = -\nabla\cdot\left(P + \frac{u^2}{2}\right)\boldsymbol{u} - \xi(Su_z + \boldsymbol{u}\cdot\boldsymbol{q})\nabla\cdot\boldsymbol{q}, \tag{19}$$

$$\partial_t\left(\frac{q^2}{2}\right) = \boldsymbol{q}\cdot\partial_t\boldsymbol{q} = -\xi\boldsymbol{q}\cdot\nabla(Su_z + \boldsymbol{u}\cdot\boldsymbol{q}). \tag{20}$$

Now combining Eqs. (18)–(20), we obtain

$$d_tE = \int\partial_t\left(\frac{u^2}{2} + \xi\frac{q^2}{2}\right)d\tau = -\int\nabla\cdot\left[P\boldsymbol{u} + \frac{u^2}{2}\boldsymbol{u} + \xi(Su_z + \boldsymbol{u}\cdot\boldsymbol{q})\boldsymbol{q}\right]d\tau. \tag{21}$$

Finally, using the Gauss-divergence theorem with periodic or vanishing boundary conditions, one can show the total energy to be an inviscid invariant of the flow.

III. DERIVATION OF EXACT RELATION

Here we derive the two-point exact relation corresponding to the inertial range energy transfer in statistically homogeneous BFT. Following [10,17], one can first define the two-point correlator associated with the total energy [Eq. (18)] as

$$\mathcal{R}_E = \mathcal{R}'_E = \left\langle\frac{\boldsymbol{u}\cdot\boldsymbol{u}' + \xi\boldsymbol{q}\cdot\boldsymbol{q}'}{2}\right\rangle, \tag{22}$$

where the unprimed and primed quantities represent the corresponding field properties at point $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$, respectively.

Now, we calculate the time evolution of the energy correlators. Similar to Eq. (16), we can also write the evolution equation for $\boldsymbol{u}'$. Now combining $\boldsymbol{u}\cdot\partial_t\boldsymbol{u}'$ and $\boldsymbol{u}'\cdot\partial_t\boldsymbol{u}$, we obtain

$$\begin{aligned}\partial_t\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle &= \langle\boldsymbol{u}'\cdot[-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla P - \xi(S\hat{z}+\boldsymbol{q})\nabla\cdot\boldsymbol{q}] \\ &\quad + \boldsymbol{u}\cdot[-(\boldsymbol{u}'\cdot\nabla')\boldsymbol{u}' - \nabla'P' - \xi(S\hat{z}+\boldsymbol{q}')\nabla'\cdot\boldsymbol{q}']\rangle \\ &\quad + D_u + F_u \end{aligned} \tag{23}$$

$$\begin{aligned}&= -\langle\nabla\cdot[(\boldsymbol{u}\cdot\boldsymbol{u}')\boldsymbol{u} + \xi(Su'_z\boldsymbol{q} + (\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{q})] \\ &\quad + \nabla'\cdot[(\boldsymbol{u}\cdot\boldsymbol{u}')\boldsymbol{u}' + \xi(Su_z\boldsymbol{q}' + (\boldsymbol{u}\cdot\boldsymbol{q}')\boldsymbol{q}')]\rangle + D_u + F_u\end{aligned} \tag{24}$$

$$= -\nabla_{\boldsymbol{r}} \cdot \langle \boldsymbol{u}'(\boldsymbol{u}\cdot\boldsymbol{u}') - \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{u}') + \xi S u_z \boldsymbol{q}' - \xi S u_z' \boldsymbol{q} + (\boldsymbol{u}\cdot\boldsymbol{q}')\boldsymbol{q}' - (\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{q}\rangle + D_u + F_u, \quad (25)$$

where $D_u = \langle \boldsymbol{u}' \cdot \nu\nabla^2 \boldsymbol{u} + \boldsymbol{u}\cdot\nu\nabla^2\boldsymbol{u}'\rangle$ and $F_u = \langle \boldsymbol{u}'\cdot\boldsymbol{f} + \boldsymbol{u}\cdot\boldsymbol{f}'\rangle$ represent the effective dissipation and forcing contributions in $\partial_t\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$. To obtain Eq. (25), we also use the property of statistical homogeneity,

$$\nabla\cdot\langle(\cdot)\rangle = -\nabla_{\boldsymbol{r}}\cdot\langle(\cdot)\rangle = -\nabla'\cdot\langle(\cdot)\rangle, \quad (26)$$

and the following relations:

$$\text{(i) } \boldsymbol{u}'\cdot\boldsymbol{q}(\nabla\cdot\boldsymbol{q}) = \nabla\cdot[(\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{q}] - \nabla\cdot\left(\boldsymbol{u}'\frac{q^2}{2}\right),$$

$$\text{(ii) } \langle \boldsymbol{u}\cdot\nabla' P'\rangle = -\langle P'(\nabla\cdot\boldsymbol{u})\rangle = 0,$$

$$\text{(iii) } \left\langle \nabla\cdot\left(\boldsymbol{u}'\frac{q^2}{2}\right)\right\rangle = -\left\langle \frac{q^2}{2}(\nabla'\cdot\boldsymbol{u}')\right\rangle = 0.$$

Again, similar to Eq. (17), one can also obtain an evolution equation for $\boldsymbol{q}'$. Combining $\boldsymbol{q}\cdot\partial_t\boldsymbol{q}'$ and $\boldsymbol{q}'\cdot\partial_t\boldsymbol{q}$, we get

$$\partial_t\langle \boldsymbol{q}\cdot\boldsymbol{q}'\rangle = \langle -\boldsymbol{q}'\cdot\nabla(S u_z + \boldsymbol{u}\cdot\boldsymbol{q}) - \boldsymbol{q}\cdot\nabla'(S u_z' + \boldsymbol{u}'\cdot\boldsymbol{q}')\rangle + D_q + F_q \quad (27)$$

$$= -\langle \nabla\cdot[S u_z \boldsymbol{q}' + (\boldsymbol{u}\cdot\boldsymbol{q})\boldsymbol{q}'] + \nabla'\cdot[S u_z'\boldsymbol{q} + (\boldsymbol{u}'\cdot\boldsymbol{q}')\boldsymbol{q}]\rangle + D_q + F_q \quad (28)$$

$$= -\nabla_{\boldsymbol{r}}\cdot\langle S u_z'\boldsymbol{q} - S u_z\boldsymbol{q}' + (\boldsymbol{u}'\cdot\boldsymbol{q}')\boldsymbol{q} - (\boldsymbol{u}\cdot\boldsymbol{q})\boldsymbol{q}'\rangle + D_q + F_q, \quad (29)$$

where $D_q = \langle \boldsymbol{q}'\cdot\mathcal{M}\nabla^2(\nabla\mu) + \boldsymbol{q}\cdot\mathcal{M}\nabla'^2(\nabla'\mu')\rangle$ and $F_q = \langle \boldsymbol{q}'\cdot\boldsymbol{g} + \boldsymbol{q}\cdot\boldsymbol{g}'\rangle$. In the following, we derive the exact relation in two ways [58].

A. In terms of two-point increments

Adding Eqs. (25) and (29), we get

$$\partial_t\mathcal{R} = -\tfrac{1}{2}\nabla_{\boldsymbol{r}}\cdot\langle(\boldsymbol{u}\cdot\boldsymbol{u}')\delta\boldsymbol{u} + \xi[(\boldsymbol{u}\cdot\boldsymbol{q}')\boldsymbol{q}' - (\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{q} + (\boldsymbol{u}'\cdot\boldsymbol{q}')\boldsymbol{q} - (\boldsymbol{u}\cdot\boldsymbol{q})\boldsymbol{q}']\rangle + D + F, \quad (30)$$

where $\mathcal{R} = (\mathcal{R}_E + \mathcal{R}_E')/2$, $D = (D_u + D_q)/2$, and $F = (F_u + F_q)/2$. Furthermore, under statistical homogeneity, we obtain

$$\nabla_{\boldsymbol{r}}\cdot\langle(\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{q}'\rangle = -\langle \boldsymbol{q}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{q}\rangle = \nabla_{\boldsymbol{r}}\cdot\langle \boldsymbol{u}'(\boldsymbol{q}\cdot\boldsymbol{q}')\rangle, \quad (31)$$

where we use $\nabla\times\boldsymbol{q} = \boldsymbol{0}$ along with the identity $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A})$. Now, we consider a statistically stationary state where the left-hand side of Eq. (30) vanishes. In the limit of infinite Reynolds number, within the inertial range, the dissipative effects can also be neglected and the corresponding energy flux rate ε can be associated with the total energy injection rate as $F = \varepsilon$. Finally, using Eq. (31) and following some straightforward algebra, we can express the two-point correlations of Eq. (30) in terms of the two-point increments whence the final exact relation can be obtained as

$$\nabla_{\boldsymbol{r}}\cdot\langle[(\delta\boldsymbol{u})^2 - \xi(\delta\boldsymbol{q})^2]\delta\boldsymbol{u} + 2\xi(\delta\boldsymbol{u}\cdot\delta\boldsymbol{q})\delta\boldsymbol{q}\rangle = -4\varepsilon. \quad (32)$$

Equation (32) is the main result of our paper. As mentioned in Sec. I, here the exact relation is cast in a differential form with

$$\boldsymbol{\mathcal{F}} \equiv \langle[(\delta\boldsymbol{u})^2 - \xi(\delta\boldsymbol{q})^2]\delta\boldsymbol{u} + 2\xi(\delta\boldsymbol{u}\cdot\delta\boldsymbol{q})\delta\boldsymbol{q}\rangle. \quad (33)$$

It expresses the inertial range energy flux rate ε purely in terms of two-point increments of the field variables of BFT. Therefore, ε remains unchanged if $\boldsymbol{q}$ is replaced by $\nabla\phi$ in Eq. (32). For all simple and active binary fluids with extensile stress, ξ is positive and so the form of conserved energy remains the same as Eq. (18),

$$E = \int \frac{1}{2}(u^2 + |\xi| q^2)d\tau, \quad (34)$$

and the corresponding exact law is given by

$$\nabla_{\boldsymbol{r}}\cdot\langle[(\delta\boldsymbol{u})^2 - |\xi|(\delta\boldsymbol{q})^2]\delta\boldsymbol{u} + 2|\xi|(\delta\boldsymbol{u}\cdot\delta\boldsymbol{q})\delta\boldsymbol{q}\rangle = -4\varepsilon. \quad (35)$$

However, active fluids with large contractile stress ($\eta < 0$ and $|\eta| > |\kappa|$) have ξ to be negative. In that case, the total energy becomes

$$E = \int \frac{1}{2}(u^2 - |\xi| q^2)d\tau, \quad (36)$$

and the corresponding exact relation becomes

$$\nabla_{\boldsymbol{r}}\cdot\langle[(\delta\boldsymbol{u})^2 + |\xi|(\delta\boldsymbol{q})^2]\delta\boldsymbol{u} - 2|\xi|(\delta\boldsymbol{u}\cdot\delta\boldsymbol{q})\delta\boldsymbol{q}\rangle = -4\varepsilon. \quad (37)$$

Surprisingly, Eq. (37) looks very similar to the exact relation for energy transfer in incompressible MHD turbulence if one replaces the field $\sqrt{|\xi|}\boldsymbol{q}$ by the local magnetic field in Alfvén units [8]. This can be a bit misleading as, for MHD, the same substitution in Eq. (36) would actually correspond to the residual energy which is not an inviscid invariant of incompressible MHD.

Passive scalar flow. In the limit where the activity coefficient ξ tends to zero, Eqs. (32) and (37) simply reduce to

$$\nabla_{\boldsymbol{r}}\cdot\langle(\delta\boldsymbol{u})^2\delta\boldsymbol{u}\rangle = -4\varepsilon, \quad (38)$$

which represents the inertial range energy transfer in a passive scalar flow and is identical to that in incompressible HD turbulence. In fact, by putting $\xi = 0$, one turns off the feedback force in the momentum evolution equation due to the scalar field ϕ, thus leading to the individual conservation of kinetic energy similar to the incompressible HD case.

Weakly correlated fluctuations. In case $\delta\boldsymbol{u}$ and $\delta\boldsymbol{q}$ are weakly correlated, we have $\langle(\delta\boldsymbol{u}\cdot\delta\boldsymbol{q})\delta\boldsymbol{q}\rangle \simeq 0$ and the exact relation (32) practically becomes

$$\nabla_{\boldsymbol{r}}\cdot\langle[(\delta\boldsymbol{u})^2 - \xi(\delta\boldsymbol{q})^2]\delta\boldsymbol{u}\rangle = -4\varepsilon. \quad (39)$$

Further, if we assume $|\delta\boldsymbol{u}| \sim |\delta\boldsymbol{q}|$, then we can encounter two interesting situations. First, when $\xi > 0$ but very small (e.g., simple binary fluids [26]), then the flux $\boldsymbol{\mathcal{F}} \simeq \langle(\delta\boldsymbol{u})^2\delta\boldsymbol{u}\rangle$ and hence corresponds to the usual Kolmogorov case. However, if ξ becomes sufficiently large, e.g., for active binary fluids with extensile stress, $\boldsymbol{\mathcal{F}} \simeq \langle -\xi(\delta\boldsymbol{q})^2\delta\boldsymbol{u}\rangle$ thereby leading to the possibility of an inverse cascade of energy. In contrast, for active binary fluids with contractile stress where $\xi < 0$, we have $\boldsymbol{\mathcal{F}} \simeq \langle[(\delta\boldsymbol{u})^2 + |\xi|(\delta\boldsymbol{q})^2]\delta\boldsymbol{u}\rangle$, and a direct cascade of energy is always expected. Note that unlike two-dimensional hydrodynamics, here we are not talking about the simultaneous forward and inverse cascades of two invariants. In the present case, the energy cascade is forward or inverse depending upon the activity parameter ξ. For a system with given ξ, the cascade direction is therefore automatically determined.

However, such type of speculations can be nontrivial if the correlation between $\delta\boldsymbol{u}$ and $\delta\boldsymbol{q}$ is not negligible. The direction of the cascade then depends on the mutual competition of the various terms in the flux and can be explored numerically. Such investigation certainly demands a separate study and is beyond the scope of the present paper, which presents a systematic analytical approach for an exact calculation of the inertial range energy transfer rate in homogeneous BFT.

B. In terms of two-point correlators

In order to obtain the exact relations in terms of two-point correlators, we start from Eqs. (23) and (27). Splitting the two-point energy correlator $\mathcal{R}$ in the kinetic and active energy correlators $\mathcal{R}_u$ and $\mathcal{R}_q$, respectively, we can write the evolution equations as follows:

$$\partial_t \mathcal{R}(\boldsymbol{r},t) = \mathcal{T}_u + \mathcal{T}_q + D + F, \tag{40}$$

$$\partial_t \mathcal{R}_u(\boldsymbol{r},t) = \mathcal{T}_u + \chi_{qu} + D_u + F_u, \tag{41}$$

$$\partial_t \mathcal{R}_q(\boldsymbol{r},t) = \mathcal{T}_q - \chi_{qu} + D_q + F_q, \tag{42}$$

where

$$\mathcal{T}_u(\boldsymbol{r},t) = \tfrac{1}{2}\langle -\boldsymbol{u}'\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{u}\cdot(\boldsymbol{u}'\cdot\nabla')\boldsymbol{u}' - \xi(\boldsymbol{u}'\cdot\boldsymbol{q})(\nabla\cdot\boldsymbol{q}) - \xi(\boldsymbol{u}\cdot\boldsymbol{q}')(\nabla'\cdot\boldsymbol{q}')\rangle, \tag{43}$$

$$\chi_{qu}(\boldsymbol{r},t) = \tfrac{1}{2}\langle -\xi S u_z'(\nabla\cdot\boldsymbol{q}) - \xi S u_z(\nabla'\cdot\boldsymbol{q}')\rangle, \tag{44}$$

$$\mathcal{T}_q(\boldsymbol{r},t) = \tfrac{1}{2}\langle -\boldsymbol{q}'\cdot\nabla(\boldsymbol{u}\cdot\boldsymbol{q}) - \boldsymbol{q}\cdot\nabla'(\boldsymbol{u}'\cdot\boldsymbol{q}')\rangle, \tag{45}$$

with $\mathcal{T}_u(\boldsymbol{r},t)$ and $\mathcal{T}_q(\boldsymbol{r},t)$ being the scale-to-scale kinetic and active energy transfer terms and $\chi_{qu}(\boldsymbol{r},t)$ being the active to kinetic energy conversion term. Similar to Eqs. (32) and (37), it is evident to see that the mean gradient field $(S\hat{z})$, which appears only in the conversion terms, cannot affect the scale-to-scale energy transfer rate ε. This is similar to incompressible MHD turbulence, where the mean magnetic field cannot alter the turbulent energy transfer. Reasoning as in the Sec. IIIA, here the final expression of the inertial range exact relation can be written as

$$\mathcal{T}_u + \mathcal{T}_q = -\varepsilon. \tag{46}$$

Equation (46) is another important result of this paper. This form is particularly useful for calculating ε using the spectral method [58].

C. In terms of new variables $\boldsymbol{\Upsilon}^{\pm}$

Instead of $\boldsymbol{u}$ and $\boldsymbol{b}$, the equations for incompressible MHD can also be written in terms of Elsässer variables, $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$ [59]. In a similar way, the basic equations for BFT can also be written in terms of "upsilon" variables $\boldsymbol{\Upsilon}^{\pm} = \boldsymbol{u} \pm i\boldsymbol{Q}$, where $\boldsymbol{Q} = \sqrt{\xi}\nabla\phi$. This can be obtained by writing Eqs. (13) and (14) in the following form:

$$\partial_t \boldsymbol{u} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{Q}\times\nabla)\times\boldsymbol{Q} - \nabla P_T + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \tag{47}$$

$$\partial_t \boldsymbol{Q} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{Q} - (\boldsymbol{Q}\times\nabla)\times\boldsymbol{u} + D\nabla^2\boldsymbol{Q} + \boldsymbol{G}, \tag{48}$$

where $P_T = P^* + Q^2$, $\boldsymbol{G} = \sqrt{\xi}\boldsymbol{g}$, and we use the following vector-calculus identity: $(\boldsymbol{A}\times\nabla)\times\boldsymbol{B} = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} - \boldsymbol{A}(\nabla\cdot\boldsymbol{B})$. Note that here we are considering the chemical potential $\sqrt{\xi}\nabla\mu \sim \boldsymbol{Q}$, which is practically true for phase-mixed or nearly phase-separating binary fluid. Finally, combining Eqs. (47) and (48), one can write the basic equations of BFT in terms of the $\boldsymbol{\Upsilon}^{\pm}$ variables as

$$\partial_t \boldsymbol{\Upsilon}^+ = \tfrac{1}{2}[-\{(\boldsymbol{\Upsilon}^+ + \boldsymbol{\Upsilon}^-)\cdot\nabla\}\Upsilon^+ - \{(\boldsymbol{\Upsilon}^+ - \boldsymbol{\Upsilon}^-)\times\nabla\}\boldsymbol{\Upsilon}^+ + \nabla^2(\nu_+\boldsymbol{\Upsilon}^+ + \nu_-\boldsymbol{\Upsilon}^-)] - \nabla P_T + \boldsymbol{f}_+, \tag{49}$$

$$\partial_t \boldsymbol{\Upsilon}^- = \tfrac{1}{2}[-\{(\boldsymbol{\Upsilon}^+ + \boldsymbol{\Upsilon}^-)\cdot\nabla\}\Upsilon^- + \{(\boldsymbol{\Upsilon}^+ - \boldsymbol{\Upsilon}^-)\times\nabla\}\boldsymbol{\Upsilon}^- + \nabla^2(\nu_-\boldsymbol{\Upsilon}^+ + \nu_+\boldsymbol{\Upsilon}^-)] - \nabla P_T + \boldsymbol{f}_-, \tag{50}$$

where $\nu_\pm = (\nu \pm D)/2$ and $\boldsymbol{f}_\pm = \boldsymbol{f} \pm i\boldsymbol{G}$. The above equations show that unlike the Elsässer variables in incompressible MHD, $\boldsymbol{\Upsilon}^{\pm}$ undergo both cross and self-deformation through two types of nonlinear interactions. In addition to the usual advective nonlinear interactions $(\boldsymbol{u}\cdot\nabla)\boldsymbol{\Upsilon}^{\pm}$, here we also have other nonlinear terms proportional to $(\boldsymbol{Q}\times\nabla)\times\boldsymbol{\Upsilon}^{\pm}$. For a given direction of $\boldsymbol{Q}$ (say $\hat{z}$), $(\boldsymbol{Q}\times\nabla)\times\boldsymbol{\Upsilon}^{\pm}$ calculate the variation of $\boldsymbol{\Upsilon}^{\pm}$ in a plane perpendicular to that direction. In contrast to the Iroshnikov-Kraichnan (IK) phenomenology [2,3] for incompressible MHD turbulence, the coexistence of the two aforesaid nonlinear interactions and their possible entanglement does not provide a simple phenomenological image for BFT. However, in Sec. IV, we predict a power law dependence of the conserved energy analogous to IK phenomenology based on the relative importance of various timescales corresponding to different interactions. From the definition, one can immediately show

$$\nabla\cdot\boldsymbol{\Upsilon}^{\pm} = \pm i\nabla\cdot\boldsymbol{Q}, \quad \nabla\times\boldsymbol{\Upsilon}^{\pm} = \nabla\times\boldsymbol{u}. \tag{51}$$

This is also in contrast to the Elsässer variables which are divergence free, similar to the $\boldsymbol{u}$ and $\boldsymbol{b}$ fields of incompressible MHD.

Previously, we showed that the turbulent energy is an inviscid invariant of a binary fluid flow. By the same method, one can also show that the total energy,

$$E = \int \tfrac{1}{2}(u^2 + Q^2)d\tau, \tag{52}$$

is (as expected) an inviscid invariant. In terms of the upsilon variables, the total energy becomes $E = \int \frac{1}{2}(\boldsymbol{\Upsilon}^+\cdot\boldsymbol{\Upsilon}^-)d\tau$. Furthermore, the exact relation derived in Eq. (32) can be written as

$$-4\varepsilon = \nabla_{\boldsymbol{r}}\cdot\langle[(\delta\boldsymbol{u})^2 - (\delta\boldsymbol{Q})^2]\delta\boldsymbol{u} + 2(\delta\boldsymbol{u}\cdot\delta\boldsymbol{Q})\delta\boldsymbol{Q}\rangle$$
$$= \tfrac{1}{8}\nabla_{\boldsymbol{r}}\cdot\langle 2[(\delta\boldsymbol{\Upsilon}^+)^2 + (\delta\boldsymbol{\Upsilon}^-)^2](\delta\boldsymbol{\Upsilon}^+ + \delta\boldsymbol{\Upsilon}^-)$$

$$- 2[(\delta\boldsymbol{\Upsilon}^+)^2 - (\delta\boldsymbol{\Upsilon}^-)^2](\delta\boldsymbol{\Upsilon}^+ - \delta\boldsymbol{\Upsilon}^-)\rangle$$
$$= \tfrac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \langle(\delta\boldsymbol{\Upsilon}^-)^2\delta\boldsymbol{\Upsilon}^+ + (\delta\boldsymbol{\Upsilon}^+)^2\delta\boldsymbol{\Upsilon}^-\rangle. \quad (53)$$

Note that the same exact relation can also be derived directly from Eqs. (49) and (50). Interestingly, Eq. (53) looks very similar to the exact relation of energy transfer in incompressible MHD turbulence when expressed in terms of the Elsässer variables [10,12]. Here one has to remember that the Elsässer fields $z^{\pm}$ are always real. This is not true for the upsilon variables. For $\xi > 0$, one can write $\boldsymbol{\Upsilon}^{\pm} = \boldsymbol{u} \pm i\sqrt{|\xi|}\nabla\phi$, thus yielding complex upsilon fields. However, for $\xi < 0$, we have $\boldsymbol{\Upsilon}^{\pm} = \boldsymbol{u} \mp \sqrt{|\xi|}\nabla\phi$ and hence the upsilon variables become real in that case. Irrespective of whether the upsilon fields are real or complex, it is straightforward to verify that the flux term in the right-hand side of Eq. (53) is, as expected, always real.

IV. DISCUSSION

In this paper, we derived several exact relations for fully developed, three-dimensional, homogeneous binary fluid turbulence. Using this relation, we can calculate the scale-to-scale transfer rate of total energy (kinetic plus active energy) within the so-called inertial range. Previously, it has been argued [26] that simple binary fluids and incompressible MHD are structurally similar in terms of the equations of dynamics and linear wave modes. Drawing analogy with IK phenomenology, they also predicted a $-3/2$ power law for turbulent energy spectra. However, for fully developed turbulence, our paper shows that the generic form of the exact relation derived in Eq. (32) differs from that of incompressible MHD turbulence [8]. In particular, the flux $\boldsymbol{\mathcal{F}}$ in Eq. (32) has two sign reversals in comparison with that of Eq. (3). Interestingly, active binary fluids with contractile stress ($\xi < 0$) are found to be algebraically identical to the exact relation of incompressible MHD turbulence if we replace the magnetic fields in Eq. (3) with $\sqrt{|\xi|}\boldsymbol{q}$. Nevertheless, these two systems are categorically different with respect to the linear stability analysis. Under weak perturbations, an active binary fluid with contractile stress leads to linear instability, whereas an incompressible MHD fluid responds to weak perturbations in terms of Alfvén waves. From Eq. (32), we retrieved the exact relation for passive scalar turbulence in the limit of vanishing activity parameter. In addition, we have predicted a possible inverse cascade of energy in three-dimensional (3D) active binary fluid turbulence with extensile stress when the correlation between the $\boldsymbol{u}$ and the $\boldsymbol{q}$ fields is sufficiently weak. Inspired by the Elsässer variables, here we introduced the "upsilon" variables and wrote the dynamical equations for binary fluids in a more symmetric form by the introduction of upsilon variables. Finally, from Eq. (32), we also wrote the exact relation in terms of the upsilon variables. Interestingly, this exact relation looks exactly similar to that of incompressible MHD turbulence when expressed in terms of the Elsässer variables. However, unlike incompressible MHD, here the cross helicity $\int(\boldsymbol{u}\cdot\boldsymbol{q})d\tau$ is not an inviscid invariant. A helicity cascade is therefore not guaranteed [60] and hence the derivation of the corresponding exact relation is not useful in BFT.

Based on our previous analysis, here we propose a plausible phenomenology and predict a power law for the turbulent energy spectrum in simple and active binary fluids with extensile stress. Writing Eqs. (49) and (50) in a compact form, we obtain

$$\partial_t\boldsymbol{\Upsilon}^{\pm} = -(\boldsymbol{u}\cdot\nabla)\Upsilon^{\pm} \mp i(\boldsymbol{Q}\times\nabla)\times\boldsymbol{\Upsilon}^{\pm} - \nabla P_T + \nabla^2(\nu_{\pm}\boldsymbol{\Upsilon}^+ + \nu_{\mp}\boldsymbol{\Upsilon}^-) + \boldsymbol{f}_{\pm}. \quad (54)$$

In the above equations, possible nonlinear interactions can be obtained from the terms $(\boldsymbol{u}\cdot\nabla)\Upsilon^{\pm}$ and $(\boldsymbol{Q}\times\nabla)\times\boldsymbol{\Upsilon}^{\pm}$. Whereas $(\boldsymbol{u}\cdot\nabla)\Upsilon^{\pm}$ represent the advection of $\Upsilon^{\pm}$ by the velocity field, the terms $(\boldsymbol{Q}\times\nabla)\times\boldsymbol{\Upsilon}^{\pm}$ represent the variation of $\Upsilon^{\pm}$ in a plane perpendicular to $\boldsymbol{Q}$. Expressing $\boldsymbol{Q}$ as a sum of the mean field $\boldsymbol{Q}_0$ and the fluctuation $\widetilde{\boldsymbol{Q}}$ in Eq. (54), we get three types of interactions. While we associate two kinds of nonlinear timescales τ_ℓ^u and τ_ℓ^Q corresponding to the terms $(\boldsymbol{u}\cdot\nabla)\widetilde{\boldsymbol{\Upsilon}}^{\pm}$ and $(\widetilde{\boldsymbol{Q}}\times\nabla)\times\widetilde{\boldsymbol{\Upsilon}}^{\pm}$, respectively, one linear timescale $\tau_{0\ell}$ corresponds to the term $(\boldsymbol{Q}_0\times\nabla)\times\widetilde{\boldsymbol{\Upsilon}}^{\pm}$, with $\widetilde{\boldsymbol{a}}$ representing the fluctuating part of the vector $\boldsymbol{a}$. This linear time can be associated with the concentration waves as defined in [26] and the corresponding dispersion relation is given by $\omega(k) = \pm|\boldsymbol{k}\times\boldsymbol{Q}_0|$. If the two nonlinear interactions are assumed to be independent, the effective distortion timescale τ_ℓ becomes

$$1/\tau_\ell = 1/\tau_\ell^u + 1/\tau_\ell^Q, \quad (55)$$

where $\ell(\equiv|\boldsymbol{r}|)$ is the characteristic size of the eddies, $u_\ell \sim |\delta\boldsymbol{u}|$, $Q_\ell \sim |\delta\boldsymbol{Q}|$, $\tau_\ell^u \sim \ell/u_\ell$, and $\tau_\ell^Q \sim \ell/\widetilde{Q}_\ell$. Analogous to the Alfvén timescale in incompressible MHD, here we can also define the linear timescale $\tau_{0\ell} \sim \ell/Q_0$. In the presence of strong $\boldsymbol{Q}_0$, we have $\tau_{0\ell} << \tau_\ell$. In that case, the energy transfer timescale $\tau_\ell^{tr} \sim \tau_\ell^2/\tau_{0\ell}$ and the energy flux rate ε would scale as [2,3,61]

$$\varepsilon \sim (\widetilde{\Upsilon}_\ell^+\widetilde{\Upsilon}_\ell^-)/\tau_\ell^{tr}. \quad (56)$$

Under the assumption of weak correlations between u_ℓ and Q_ℓ, we have $\widetilde{\Upsilon}_\ell^+ \sim \widetilde{\Upsilon}_\ell^- \sim \widetilde{\Upsilon}_\ell \sim u_\ell$ and hence $\tau_\ell \sim \tau_\ell^u \sim \ell/\widetilde{\Upsilon}_\ell$. Combining all, finally, we can write

$$\varepsilon \sim \widetilde{\Upsilon}_\ell^4/\ell Q_0 \Rightarrow \widetilde{\Upsilon}_\ell \sim (\varepsilon Q_0)^{1/4}\ell^{1/4}. \quad (57)$$

Again, by definition of energy spectrum $E(k)$,

$$\widetilde{\Upsilon}_\ell^2 \sim E(k)k \Rightarrow E(k) \sim (\varepsilon Q_0)^{1/2}k^{-3/2}, \quad (58)$$

where k is the wave number corresponding to ℓ. This is in agreement with the predictions of [26], which was done for simple binary fluids only.

Similar types of studies can be generalized to the compressible binary fluids as well as to the mixtures of more than two fluids.

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