# Method for direct analytic solution of the nonlinear Langevin equation using multiple timescale analysis: Mean-square displacement

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We consider a class of nonlinear Langevin equations with additive, Gaussian white noise. Because of nonlinearity, the calculation of moments poses a serious problem for any direct solution of the Langevin equation. Based on multiple timescale analysis we introduce a scheme for directly solving the equations. We first derive the equations for the fast and slow dynamics, in the spirit of the Blekhman perturbation method in vibrational mechanics, the fast motion being described by the Brownian motion of a harmonic oscillator whose effect is subsumed in the slow motion resulting in a parametrically driven nonlinear oscillator. The multiple timescale perturbation theory is then used to obtain a secular divergence-free analytic solution for the slow nonlinear dynamics for calculation of the moments. Our analytical results for mean-square displacement are corroborated with direct numerical simulation of Langevin equations.

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## I. INTRODUCTION

Nonlinear Langevin equations constitute a class of stochastic differential equations widely used to describe nonequilibrium statistical mechanical systems in the natural sciences [1–4]. The dissipation and thermal noise in the system are connected via a fluctuation-dissipation relation. Their applications cover a wide range, e.g., in the diffusion model of chemical reactions [5], in calculations of transport coefficients like the diffusion coefficient of particles in fluids [2,3,6], and in the calculation of spectra and photon correlation in quantum optics [7], to name a few. A major difficulty that arises in dealing with these problems is the nonlinearity in the dynamics. While the linear stochastic equations are solvable with direct calculation of moments, nonlinearity brings in higher moments, and in general, the equations for the moments are hierarchical and they cannot be closed without approximation [2,4]. In order to circumvent this difficulty one considers linearization of the dynamics around the steady state within the weak noise approximation for which the fluctuations in the system variables in the diffusion coefficients are ignored. Such a procedure is adopted traditionally in dealing with the quantum optical processes described by $c$-number equations using quasiclassical distribution functions [7] and others. Linearization of the potential around the saddle and stable steady states of a double well lies at the heart of the Kramers problem of barrier crossing dynamics [2,5]. The "harmonic linearization" technique for a nonlinear potential for a Langevin equation developed in early 1980s is also worth mentioning [8]. Although the linear Langevin equations are formally exact, they are valid for near-equilibrium processes and for calculations of linear transport laws [2]. For nonlinear transport processes and anomalous behavior of the transport coefficients one must resort to mode-coupling theories [2,9]. In a different context, the stochastic differential equations of nonlinear Langevin type with linear and quadratic noise have been solved for calculation of photoelectron counting probability for amplitude fluctuations in an electromagnetic field [10]. A Langevin equation with multiplicative nonlinear periodic noise has been solved exactly in the form of the periodic solution of the Hill equation [11]. An asymptotic formula for the steady-state distribution for a non-Gaussian nonlinear Langevin equation has been derived [12]. Although a variety of physical systems characterized by multiscale interactions are treated within the framework nonlinear Langevin equations [13–19], we confine ourselves here to a class of Langevin equations for which the potential is nonlinear and the noise is additive, white, and Gaussian in nature.

A common theoretical ground in the traditional treatment of the above-mentioned nonlinear processes is the separation of timescales in which dynamical variables are slow and the reduced distributions of the slow variables are obtained by integrating over the fast variables [2]. A perturbation scheme using this separation to study interacting particles in an open system was developed within a mean-field approximation to derive a nonlinear integro-differential Fokker-Planck equation by Savel'ev *et al.* [20]. In many treatments the separation of timescales is too difficult and often leads to complications due to intermingling timescales. Another difficulty arises in constructing the Fokker-Planck equation from the nonlinear Langevin equation as emphasized by Van Kampen [21]. This is regarding the nonuniqueness of the diffusion term in the associated Fokker-Planck description [21,22]. A direct approach to the solution of a nonlinear Langevin equation without any reference to the equation for the phase space distribution function is therefore worth exploring.

The object of the present work is to address this issue. We explore an approach based on explicit separation of timescales [23–26] in two stages. In the first stage [23,24], we extract

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out the fast motion explicitly after averaging over the noise in the spirit of Blekhman perturbation theory used in vibrational mechanics. The slow part is purely dynamical governed by an ordinary nonlinear differential equation of the system variables, driven by the average dynamics of the fast motion. In addition, the coefficients of the slow equation of motion are renormalized by the characteristics of the fast motion. This equation is amenable to solution order by order using a multiscale secular divergence-free perturbation theory, e.g., the Lindstedt-Poincaré method or dynamical renormalization group technique [27–29]. An interesting offshoot of the separation of timescales is the Brownian dynamics of a harmonic oscillator, a generic feature of the fast motion that appears in all the nonlinear cases treated here and admits an exact solution. We consider three prototypical examples of a nonlinear Langevin equation with Duffing potential, double-well potential, and periodic potential and calculate the first and second moments analytically. Our results are verified with direct numerical simulation of the nonlinear Langevin equations.

## II. THE NONLINEAR LANGEVIN EQUATIONS: THE ANALYTIC SOLUTION

The outline of the present method is described as follows: We consider a wide class of nonlinear Langevin equations with additive, Gaussian, white noise of the following form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \epsilon f(x) = \alpha\xi(t), \tag{2.1}$$

where $f(x)$ is a nonlinear function of the system coordinates $x$ and the overdot represents differentiation with respect to time. $\xi(t)$ is a Gaussian white noise with zero mean,

$$\begin{aligned} &\langle\xi(t)\rangle = 0,\\ \text{and the second moment,}\quad &\langle\xi(t)\xi(0)\rangle = 2D\delta(t), \end{aligned} \tag{2.2}$$

satisfying the fluctuation-dissipation relation with $D = \gamma k_B T$, $\gamma$ being the linear damping coefficient. $k_B$ and $T$ are the Boltzmann constant and temperature, respectively. $\alpha$ denotes the strength of noise. $\epsilon$ is a smallness parameter used to keep track of the order of the perturbation. $\omega_0$ is the linear frequency of the system.

In the first step, we proceed as in vibrational mechanics [23,24] to separate out the fast and slow motion as

$$x = X(t) + \psi(t) \tag{2.3}$$

such that $\langle\psi(t)\rangle = 0$ and $\langle\psi^2(t)\rangle \neq 0$, where $X(t)$ refers to the slow variable with a natural timescale of the system and $\psi(t)$ the fast one with a timescale of the noise. More specifically, one can write the condition as $\tau_s \ll \omega_0^{-1}$, where $\tau_s$ is the correlation time of the noise. For Gaussian white noise $\tau_s$ tends to zero. Substituting Eq. (2.3) in Eq. (2.1) and averaging over the fast part we obtain

$$\ddot{\psi} + \omega_0^2\psi + \gamma\dot{\psi} + \epsilon N(X, \psi, \dot{\psi}, \langle\psi\rangle, \langle\psi^2\rangle) = \alpha\xi(t) \tag{2.4}$$

as a descriptor of the fast motion, and the slow motion is governed by

$$\ddot{X} + \gamma\dot{X} + \omega_{osc}^2 X = \epsilon F(X) \tag{2.5}$$

such that Eqs. (2.4) and (2.5) on addition give back Eq. (2.1). Equation (2.5) contains the frequency $\omega_{osc}^2$ and other coefficients in the nonlinear function $F(X)$ which are renormalized by $\langle\psi^2(t)\rangle$. The terms in $N(X, \psi, \dot{\psi}, \langle\psi\rangle, \langle\psi^2\rangle)$ are nonlinear contributions. We neglect the terms in $N$ from the fast motion and obtain the Langevin dynamics for a harmonic oscillator with frequency $\omega_0$ as an effective description of the fast motion. The solution of this equation when inserted in the nonlinear equation (2.5) for slow motion results in its parametric driving by the average fast motion. This yields

$$\ddot{X} + \gamma\dot{X} + \omega_{osc}^2 X = \epsilon G(X, t), \tag{2.6}$$

where $t$ refers to the explicit time dependence of the driving terms.

Equation (2.6) is a driven nonlinear dynamical system which can be solved analytically order by order to avoid secular divergence due to the nonlinear terms and time-dependent parametric driving using multiple timescales. This constitutes the separation of timescales for the second stage [25,26]. We have employed here Lindstedt-Poincaré perturbation theory to calculate the first and second moments. One can also implement dynamical renormalization group technique [27–29] or other methods for solutions. We now illustrate the scheme with the help of three nonlinear models.

### A. Quartic potential

We begin with the dynamics of a generalized quartic oscillator [25,26] in one dimension in the following form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \epsilon x^3 = \alpha\xi(t), \tag{2.7}$$

where $\gamma$ is the damping constant, $\omega_0$ is the natural frequency, and $\epsilon$ stands for the nonlinearity factor of the oscillator. $\alpha$ represents the strength of the noise and $\xi(t)$ is the Gaussian white noise, with zero mean and the properties defined in Eqs. (2.2). For the sake of generality, we consider $\epsilon > 0$ and $\omega_0^2 > 0$ for monostable oscillation (Duffing oscillator) and $\omega_0^2 < 0$ for bistable (double-well oscillator) oscillation as two distinct cases in the present section.

Let us split the variable $x$ into a slow ($X$) and a fast ($\psi$) part as $x = X + \psi$. Substituting in Eq. (2.7) and averaging over the fast part, we obtain

$$\begin{aligned} &\ddot{X} + \ddot{\psi} + \gamma\dot{X} + \gamma\dot{\psi} + \omega_0^2 X + \omega_0^2\psi + \epsilon X^3\\ &\quad + 3\epsilon X^2\langle\psi\rangle + 3\epsilon X^2(\psi - \langle\psi\rangle) + 3\epsilon X\langle\psi^2\rangle\\ &\quad + 3\epsilon X(\psi^2 - \langle\psi^2\rangle) + \epsilon\psi^3 = \alpha\xi(t). \end{aligned} \tag{2.8}$$

Separating them, we find the slow dynamics

$$\ddot{X} + \gamma\dot{X} + \omega_{osc}^2 X + \epsilon X^3 = 0, \tag{2.9}$$

where $\omega_{osc}^2 = \omega_0^2 + 3\epsilon\langle\psi^2\rangle$; the linear frequency is modified by the noise and $\langle\psi\rangle = 0$. The fast dynamics is given by

$$\begin{aligned} &\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi + 3\epsilon X^2(\psi - \langle\psi\rangle)\\ &\quad + 3\epsilon X(\psi^2 - \langle\psi^2\rangle) + \epsilon\psi^3 = \alpha\xi(t). \end{aligned} \tag{2.10}$$

The above scheme of separation of the fast and slow motion is inspired by Blekhman perturbation theory [23,24] whereby one follows the separation method for the dynamics guided by slow and fast time periodic motion rather than noise. The method has been widely used in the context of vibrational mechanics, particularly in vibrational resonance recently [30–45]. The extension of the Blekhman perturbation theory to the systems governed by stochastic processes is essentially a new element of the present approach.

To proceed further, we first note that Eq. (2.9) possesses one steady state $X_s = 0$ for $\omega_{osc}^2 > 0$ and three steady states $X_s = 0, \pm\sqrt{\frac{|\omega_{osc}^2|}{\epsilon}}$ for $\omega_{osc}^2 < 0$. Furthermore, we introduce the change of variable $Y = X - X_s$ to rewrite Eq. (2.9) as

$$\ddot{Y} + \gamma\dot{Y} + \omega_{osc}^2 Y + \epsilon Y^3 + X_s\left[\omega_{osc}^2 + \epsilon\left(X_s^2 - 3X_sY + 3Y^2\right)\right] = 0 \tag{2.11}$$

and Eq. (2.10) as

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi + 3\epsilon Y^2(\psi - \langle\psi\rangle) + 3\epsilon Y(\psi^2 - \langle\psi^2\rangle) + \epsilon\psi^3 + X_s[3\epsilon(X_s + 2Y)(\psi - \langle\psi\rangle) + 3\epsilon(\psi^2 - \langle\psi^2\rangle)] = \alpha\xi(t). \tag{2.12}$$

The terms in the square brackets in Eqs. (2.11) and (2.12) highlight the role of the steady states $X_s$ in the two distinct cases of oscillations. These are discussed separately in the following.

#### *1. Monostable (Duffing) oscillator*

Since $X_s = 0$ is the only steady state for $\omega_{osc}^2 > 0$, we have $Y = X$, and the slow dynamics (2.11) reduces to the following form:

$$\ddot{Y} + \gamma\dot{Y} + \omega_{osc}^2 Y + \epsilon Y^3 = 0. \tag{2.13}$$

Neglecting the nonlinear contributions, we obtain from Eq. (2.10) the Langevin equation for a harmonic oscillator:

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = \alpha\xi(t). \tag{2.14}$$

This equation plays the key role in the dynamics of fast timescale in our calculation. In what follows we show that such a descriptor for fast motion appears also in other cases of nonlinear potential. The well-known solution for the mean-square displacement $\langle\psi^2\rangle$ [46] (see the Appendix) can be written as follows:

$$\langle\psi^2\rangle = c + \alpha_1 e^{-\gamma t}\cos\Omega' t + \alpha_2 e^{-\gamma t} + \alpha_3 e^{-\gamma t}\sin\Omega' t, \tag{2.15}$$

where

$$c = \frac{\alpha^2 D}{\gamma\omega_0^2}, \quad \alpha_1 = -\frac{\alpha^2\gamma D}{\omega_0^2\left(\gamma^2 - 4\omega_0^2\right)},$$
$$\alpha_2 = \frac{\alpha^2 D}{\gamma\omega_0^2} - \frac{\alpha^2\gamma D}{\omega_0^2\left(\gamma^2 - 4\omega_0^2\right)}, \quad \alpha_3 = \frac{\alpha^2 D}{\omega_0^2\sqrt{\left(4\omega_0^2 - \gamma^2\right)}},$$

where $\Omega' = \sqrt{\left(4\omega_0^2 - \gamma^2\right)}$.

Substituting the expression (2.15) in Eq. (2.13), we obtain the following form of the slow dynamics:

$$\ddot{Y} + \gamma\dot{Y} + \omega_0^2 Y + 3\epsilon(c + \alpha_1 e^{-\gamma t}\cos\Omega' t + \alpha_2 e^{-\gamma t} + \alpha_3 e^{-\gamma t}\sin\Omega' t)Y + \epsilon Y^3 = 0. \tag{2.16}$$

We confine ourselves to the underdamped condition $\gamma \ll \omega_0$ and set $\gamma = \epsilon$ [25]. Thus, Eq. (2.16) becomes

$$\ddot{Y} + \left(\omega_0^2 + 3\epsilon c\right)Y = -\epsilon(\dot{Y} + Y^3) - 3\epsilon(\alpha_1 e^{-\gamma t}\cos\Omega' t + \alpha_2 e^{-\gamma t} + \alpha_3 e^{-\gamma t}\sin\Omega' t)Y. \tag{2.17}$$

It is now apparent that effect of the fast motion on the slow timescale is twofold. First, the linear frequency $\omega_0$ gets modified by a new contribution $3\epsilon c$. Second, the frequency is parametrically modulated by the terms containing $\alpha_1, \alpha_2, \alpha_3$.

Equation (2.17) represents a damped and parametrically driven nonlinear oscillator, an ideal candidate for the solution using a perturbation technique. In what follows, we make use of the Lindstedt-Poincaré perturbation method to avoid secular divergence arising out of the driving and nonlinear terms in Eq. (2.17). To proceed, we introduce a separation of timescales for the second stage which is purely dynamical in origin. To this end, we first define a new timescale $\tau = \omega t$, and write Eq. (2.17) in terms of the new timescale $\tau$ as follows:

$$\omega^2 Y'' + \omega_0^2 Y = -\epsilon\omega Y' - \epsilon Y^3 - 3\epsilon c Y - 3\epsilon[\alpha_1 e^{-\gamma_1\tau}\cos\Omega\tau + \alpha_2 e^{-\gamma_1\tau} + \alpha_3 e^{-\gamma_1\tau}\sin\Omega\tau]Y, \tag{2.18}$$

where $\gamma_1 = \gamma/\omega$, $\Omega = \Omega'/\omega$ and $Y'$ denotes the differentiation with respect to $\tau$, the new timescale, i.e., $\dot{Y} = \omega Y'$ and $\ddot{Y} = \omega^2 Y''$, and so on. Eqation (2.18) is the slow dynamics averaged over the internal noise of the system. Now let us expand $Y$ and $\omega$ in powers of $\epsilon$ as

$$Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$$
$$\text{and} \quad \omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots.$$

Substitution into Eq. (2.18) yields

$$\omega_0^2(Y_0'' + Y_0) + \epsilon\left[\omega_0^2(Y_1'' + Y_1) + 2\omega_0\omega_1 Y_0''\right] + \epsilon^2\left[\omega_0^2(Y_2'' + Y_2) + 2\omega_0\omega_1 Y_1'' + \left(\omega_1^2 + 2\omega_0\omega_2\right)Y_0''\right]$$
$$= -\epsilon(\omega_0 Y_0' + Y_0^3 + 3cY_0) - \epsilon^2\left[(\omega_1 Y_0' + \omega_0 Y_1') + 3Y_1\left(Y_0^2 + c\right)\right]$$
$$- 3\epsilon[\alpha_1 e^{-\gamma_1\tau}\cos\Omega\tau + \alpha_2 e^{-\gamma_1\tau} + \alpha_3 e^{-\gamma_1\tau}\sin\Omega\tau](Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots). \tag{2.19}$$

Comparing the coefficients of various powers of $\epsilon$, on both sides, we write

$$(Y_0'' + Y_0) = 0 \tag{2.20}$$

for the zeroth order of $\epsilon$. Similarly, for the first order of $\epsilon$,

$$\omega_0^2(Y_1'' + Y_1) = -2\omega_0\omega_1 Y_0'' - \left(\omega_0 Y_0' + Y_0^3 + 3cY_0\right) - 3[\alpha_1 e^{-\gamma_1\tau}\cos\Omega\tau + \alpha_2 e^{-\gamma_1\tau} + \alpha_3 e^{-\gamma_1\tau}\sin\Omega\tau]Y_0, \tag{2.21}$$

and for the second order,

$$\omega_0^2(Y_2''+Y_2) = -2\omega_0\omega_1 Y_1'' - (\omega_1^2+2\omega_0\omega_2)Y_0'' - [(\omega_1 Y_0'+\omega_0 Y_1') + 3Y_1(Y_0^2+c)] - 3(\alpha_1 e^{-\gamma_1\tau}\cos\Omega\tau + \alpha_2 e^{-\gamma_1\tau} + \alpha_3 e^{-\gamma_1\tau}\sin\Omega\tau)Y_1. \tag{2.22}$$

From Eq. (2.20), the zeroth-order solution can be written as

$$Y_0 = A\cos\tau + B\sin\tau, \tag{2.23}$$

where $A$ and $B$ are two arbitrary constants. We set the following initial conditions: $x(0) = a$, $\dot{x}(0) = b$ so that at $t = 0$ or $\tau = 0$, we have $Y = a$ and $Y' = b$. It follows that at $\tau = 0$, $Y_0(0) = a$, $Y_0'(0) = b$. Also, $Y_i(0) = 0$ and $Y_i'(0) = 0$, $i \neq 0$. Thus, we obtain

$$Y_0 = a\cos\tau + b\sin\tau.$$

Substituting in Eq. (2.21) leads to

$$\omega_0^2(Y_1''+Y_1) = \left[2\omega_0\omega_1 a - \frac{3a}{4}(a^2+b^2) - \omega_0 b - 3ca\right]\cos\tau + \left[2\omega_0\omega_1 b - \frac{3b}{4}(a^2+b^2) + \omega_0 a - 3cb\right]\sin\tau + \frac{a}{4}(3b^2-a^2)\cos 3\tau + \frac{b}{4}(b^2-3a^2)\sin 3\tau - 3e^{-\gamma_1\tau}\left[\alpha_2(a\cos\tau + b\sin\tau) + \frac{1}{2}(a\alpha_1 - b\alpha_3)\cos\Omega_1\tau + \frac{1}{2}(a\alpha_1+b\alpha_3)\cos\Omega_2\tau + \frac{1}{2}(b\alpha_1+a\alpha_3)\sin\Omega_1\tau + \frac{1}{2}(a\alpha_3 - b\alpha_1)\sin\Omega_2\tau\right], \tag{2.24}$$

where $\Omega_1 = \Omega + 1$ and $\Omega_2 = \Omega - 1$. Being secular terms, the coefficients of $\cos\tau$ and $\sin\tau$ in Eq. (2.24) must vanish for removal of the singularity. Therefore, we must have

$$(2\omega_0\omega_1 - 3c)a - \omega_0 b = \frac{3a}{4}(a^2+b^2), \tag{2.25}$$

$$\text{and}\quad (2\omega_0\omega_1 - 3c)b + \omega_0 a = \frac{3b}{4}(a^2+b^2). \tag{2.26}$$

The solution of Eqs. (2.25) and (2.26) yields the first-order correction to the bare frequency $\omega_0$, so that we have

$$\omega_1 = \frac{1}{2\omega_0}\left(\frac{9}{16}r^4 - \omega_0^2\right)^{1/2} + \frac{3c}{2\omega_0}. \tag{2.27}$$

The frequency corrected up to first order is given by

$$\omega = \omega_0 + \epsilon\omega_1 = \omega_0 + \epsilon\left[\frac{1}{2\omega_0}\left(\frac{9}{16}r^4 - \omega_0^2\right)^{1/2} + \frac{3c}{2\omega_0}\right], \tag{2.28}$$

where $r^2 = a^2 + b^2$. Further, we find the solution $Y_1$ up to the first order:

$$Y_1 = \frac{a}{32}(a^2-3b^2)(\cos\tau - \cos 3\tau) + \frac{b}{8}(3a^2-b^2)(\sin^3\tau) + R_2\cos\tau + R_1\sin\tau + 3\alpha_2 a\beta_1 e^{-\gamma_1\tau}(\gamma_1\cos\tau - 2\sin\tau) + 3\alpha_2 b\beta_1 e^{-\gamma_1\tau}(\gamma_1\sin\tau + 2\cos\tau) + 3T_1\beta_3 e^{-\gamma_1\tau}(\beta_2\cos\Omega_1\tau - 2\gamma_1\Omega_1\sin\Omega_1\tau) + 3T_3\beta_3 e^{-\gamma_1\tau}(\beta_2\sin\Omega_1\tau + 2\gamma_1\Omega_1\cos\Omega_1\tau) + 3T_2\beta_5 e^{-\gamma_1\tau}(\beta_4\cos\Omega_2\tau - 2\gamma_1\Omega_2\sin\Omega_2\tau) + 3T_4\beta_5 e^{-\gamma_1\tau}(\beta_4\sin\Omega_2\tau + 2\gamma_1\Omega_2\cos\Omega_2\tau), \tag{2.29}$$

where

$$T_1 = \frac{1}{2}(\alpha_1 a - \alpha_3 b),\quad T_2 = \frac{1}{2}(\alpha_1 a + \alpha_3 b),\quad T_3 = \frac{1}{2}(\alpha_1 b + \alpha_3 a),\quad T_4 = \frac{1}{2}(\alpha_3 a - \alpha_1 b),$$

$$\text{and}\quad \beta_1 = \frac{1}{\gamma_1(\gamma_1^2+4)},\quad \beta_2 = (1+\gamma_1^2-\Omega_1^2),\quad \beta_3 = \frac{1}{(\beta_2^2+4\gamma_1^2\Omega_1^2)},\quad \beta_4 = (1+\gamma_1^2-\Omega_2^2),\quad \beta_5 = \frac{1}{(\beta_4^2+4\gamma_1^2\Omega_2^2)}.$$

Also,

$$R_1 = 3\alpha_2 a\beta_1(\gamma_1^2+2) + 3\alpha_2 b\beta_1\gamma_1 + 3T_1\beta_3\gamma_1(2\Omega_1^2+\beta_2) + 3T_3\beta_3\Omega_1(2\gamma_1^2-\beta_2) + 3T_2\beta_5\gamma_1(2\Omega_2^2+\beta_4) + 3T_4\beta_5\Omega_2(2\gamma_1^2-\beta_4)$$

$$\text{and}\quad R_2 = -3\alpha_2 a\beta_1\gamma_1 - 6\alpha_2 b\beta_1 - 3T_1\beta_2\beta_3 - 6T_3\beta_3\gamma_1\Omega_1 - 3T_2\beta_4\beta_5 - 6T_4\beta_5\gamma_1\Omega_2.$$

Thus, reverting back to $X$ the solution for the slow dynamics up to first order can be written as

$$X = Y = Y_0 + \epsilon Y_1 = (a\cos\omega t + b\sin\omega t) + \epsilon\left[\frac{a}{32}(a^2-3b^2)(\cos\omega t - \cos 3\omega t) + \frac{b}{8}(3a^2-b^2)(\sin^3\omega t)\right.$$

$$+ R_2 \cos \omega t + R_1 \sin \omega t + 3\alpha_2 a\beta_1 e^{-\gamma_1 \omega t}(\gamma_1 \cos \omega t - 2 \sin \omega t) + 3\alpha_2 b\beta_1 e^{-\gamma_1 \omega t}(\gamma_1 \sin \omega t + 2 \cos \omega t)$$
$$+ 3T_1\beta_3 e^{-\gamma_1 \omega t}(\beta_2 \cos \Omega_1 \omega t - 2\gamma_1 \Omega_1 \sin \Omega_1 \omega t) + 3T_3\beta_3 e^{-\gamma_1 \omega t}(\beta_2 \sin \Omega_1 \omega t + 2\gamma_1 \Omega_1 \cos \Omega_1 \omega t)$$
$$+ 3T_2\beta_5 e^{-\gamma_1 \omega t}(\beta_4 \cos \Omega_2 \omega t - 2\gamma_1 \Omega_2 \sin \Omega_2 \omega t) + 3T_4\beta_5 e^{-\gamma_1 \omega t}(\beta_4 \sin \Omega_2 \omega t + 2\gamma_1 \Omega_2 \cos \Omega_2 \omega t)\Big] \quad (2.30)$$

with the frequency $\omega$ corrected up to first order as given in Eq. (2.28).

A closer look at the expression for the frequency $\omega$ reveals that the correction to frequency (2.27) in the leading order ($\epsilon$) arises from a dynamical contribution $\frac{1}{2\omega_0}(\frac{9}{16}r^4 - \omega_0^2)^{1/2}$ and a stochastic contribution $(3c/2\omega_0)$. It is important to note that both corrections to the frequency are of the order of $\epsilon$. The nonlinearity therefore plays a key role in them. For $\epsilon = 0$, we return to a linear system with no correction to the linear frequency. From Eq. (2.30), we find $X^2$, and hence using Eq. (2.15) and (2.30) we derive the analytical expression for the mean-square displacement as $\langle x^2 \rangle = X^2 + \langle \psi^2 \rangle$ since $x = X + \psi$. Having obtained the moments, we are in a position to calculate the associated distribution function for the stochastic process.

We now digress slightly about the analytical solution [Eq. (2.30)]. The result suggests that in the limit $t$ tending to infinity, after the initial transients die out, $X(t)$ keeps varying in time with frequency $\omega$ as given by the expression (2.27) depending on the initial conditions ($a$, $b$). The origin of this dependence of the frequency $\omega$ on $a$, $b$ lies in the Lindstedt-Poincaré method, which has been used to get rid of the divergence due to resonance (as encountered in the traditional perturbation techniques [25]) arising out of parametric driving and nonlinearity and to obtain a periodic solution with the removal of the secular terms. This forces the correction of the frequency $\omega_1$ to depend on the amplitudes $a$, $b$. In other words, the dependence of $\omega$ on $a$ and $b$ is a reflection of the amplitude dependence of frequency in a nonlinear system.

### 2. Bistable (double-well) oscillator

We now return to Eq. (2.11) and note that $X_s = 0, \pm\sqrt{\frac{|\omega_{osc}^2|}{\epsilon}}$ for $\omega_{osc}^2 < 0$, and with the change of variable $Y = X - X_s$, the slow dynamics [Eq. (2.11)] assumes the form for the nonzero steady state,

$$\ddot{Y} + \gamma \dot{Y} + \left(2\omega_0^2 - 6\epsilon\langle\psi^2\rangle\right)Y + 3\epsilon X_s Y^2 + \epsilon Y^3 = 0, \quad (2.31)$$

and the fast motion [Eq. (2.12)] after neglecting the nonlinear contributions reduces to

$$\ddot{\psi} + \gamma\dot{\psi} + 2\omega_0^2\psi = \alpha\xi(t). \quad (2.32)$$

Now with the solution of Eq. (2.32) as given by Eq. (2.15) where $\omega_0$ must to be replaced by $\sqrt{2}\omega_0$, we rewrite the slow dynamics [Eq. (2.31)] as

$$\ddot{Y} + \gamma\dot{Y} + 2\omega_0^2 Y + 3\epsilon X_s Y^2 + \epsilon Y^3 - 6\epsilon(c + \alpha_1 e^{-\gamma t}\cos\Omega' t + \alpha_2 e^{-\gamma t} + \alpha_3 e^{-\gamma t}\sin\Omega' t)Y = 0. \quad (2.33)$$

Here $\omega_0$ in $c$, $\alpha_1$, $\alpha_2$, $\alpha_3$ of Eq. (2.33) must be replaced by $\sqrt{2}\omega_0$. Proceeding exactly as in the earlier case with the multiple timescale perturbation theory, we obtain the corrected frequency $\omega$ as given by Eq. (2.28) (with $\omega_0$ replaced by $\sqrt{2}\omega_0$).

Finally, the solution for the slow dynamics up to the first-order correction takes the form

$$Y = Y_0 + \epsilon Y_1 = (a\cos\omega t + b\sin\omega t) + \epsilon\bigg[\frac{a}{32}(a^2 - 3b^2)(\cos\omega t - \cos 3\omega t) + \frac{b}{8}(3a^2 - b^2)(\sin^3\omega t) + R_2\cos\omega t + R_1\sin\omega t$$
$$+ \frac{3X_s}{2}(a^2 + b^2)(\cos\omega t - 1) + X_s ab(2\sin\omega t - \sin 2\omega t) + \frac{X_s}{2}(a^2 + b^2)\cos 2\omega t$$
$$- 6\alpha_2 a\beta_1 e^{-\gamma t}(\gamma_1\cos\omega t - 2\sin\omega t) - 6\alpha_2 b\beta_1 e^{-\gamma t}(\gamma_1\sin\omega t + 2\cos\omega t) - 6T_1\beta_3 e^{-\gamma t}(\gamma_1\cos\omega t - 2\sin\omega t)$$
$$- 6T_3\beta_3 e^{-\gamma t}(\beta_2\sin\Omega_1\omega t + 2\gamma_1\Omega_1\cos\Omega_1\omega t) - 6T_2\beta_5 e^{-\gamma t}(\beta_4\cos\Omega_2\omega t$$
$$- 2\gamma_1\Omega_2\sin\Omega_2\omega t) - 6T_4\beta_5 e^{-\gamma t}(\beta_4\sin\Omega_2\omega t + 2\gamma_1\Omega_2\cos\Omega_2\omega t)\bigg]. \quad (2.34)$$

Here $R_1$ and $R_2$ are given by

$$R_1 = -6\alpha_2 a\beta_1\left(\gamma_1^2 + 2\right) - 6\alpha_2 b\beta_1\gamma_1 - 3T_1\beta_3\gamma_1\left(2\Omega_1^2 + \beta_2\right) - 3T_3\beta_3\Omega_1\left(2\gamma_1^2 - \beta_2\right)$$
$$- 3T_2\beta_5\gamma_1\left(2\Omega_2^2 + \beta_4\right) - 3T_4\beta_5\Omega_2\left(2\gamma_1^2 - \beta_4\right)$$

and $R_2 = 6\alpha_2 a\beta_1\gamma_1 + 12\alpha_2 b\beta_1 + 3T_1\beta_2\beta_3 + 6T_3\beta_2\gamma_1\Omega_1 + 3T_2\beta_4\beta_5 + 6T_4\beta_5\gamma_1\Omega_2.$

Two pertinent points are to be noted. First, the nonzero value of $X_s$ makes its appearance explicit in the above solution. Second, $\omega_0$ in all the relevant quantities in Eq. (2.34) must be replaced by $\sqrt{2}\omega_0$. Having determined $Y$, $X$ can be obtained as $X = X_s + Y$. The calculation of the mean-square displacement $\langle x^2 \rangle$ is quite straightforward:

$$\langle x^2 \rangle = X^2 + \langle \psi^2 \rangle,$$

where $\langle \psi^2 \rangle$ is given by Eq. (2.15).

For both the monostable and bistable systems $\langle x^2 \rangle$ approaches an oscillatory solution in the limit $t$ tending to infinity. The analytical expression for $\langle \psi^2 \rangle_{t\to\infty}$ is given by

$$\langle \psi^2 \rangle_{t\to\infty} = \frac{\alpha^2 D}{\gamma \omega_0^2}$$

for the monostable case and

$$\langle \psi^2 \rangle_{t\to\infty} = \frac{\alpha^2 D}{4\gamma \omega_0^2}$$

for the bistable case. For $X(t)^2|_{t\to\infty}$ is given by

$$\begin{aligned} X(t)^2|_{t\to\infty} &= (a\cos\omega t + b\sin\omega t)^2 + 2\epsilon(a\cos\omega t + b\sin\omega t) \\ &\quad\times \left[\frac{a}{32}(a^2 - 3b^2)(\cos\omega t - \cos 3\omega t)\right. \\ &\quad \left. + \frac{b}{8}(3a^2 - b^2)(\sin^3\omega t) + R_2\cos\omega t + R_1 \sin\omega t\right] \end{aligned}$$

for the monostable oscillator, where $R_1$ and $R_2$ are expressed below Eq. (2.29). Similarly for the bistable case, we write

$$\begin{aligned} X(t)^2|_{t\to\infty} &= [X_s + (a\cos\omega t + b\sin\omega t)]^2 \\ &\quad + 2\epsilon[X_s + (a\cos\omega t + b\sin\omega t)] \\ &\quad \times \left[\frac{a}{32}(a^2 - 3b^2)(\cos\omega t - \cos 3\omega t)\right. \\ &\quad + \frac{b}{8}(3a^2 - b^2)(\sin^3\omega t) + R_2\cos\omega t + R_1\sin\omega t \\ &\quad + \frac{3X_s}{2}(a^2 + b^2)(\cos\omega t - 1) \\ &\quad \left. + X_s ab(2\sin\omega t - \sin 2\omega t) + \frac{X_s}{2}(a^2 + b^2)\cos 2\omega t\right] \end{aligned}$$

with $R_1$ and $R_2$ as defined earlier below Eq. (2.34). It is interesting to note that in the long-time limit the system, in general, oscillates in time with frequency $\omega$ and its harmonics generated due to nonlinearity of the potential. When averaged over a time period the resultant $X(t)^2|_{t\to\infty}$ in both cases becomes proportional to $D$.

### 3. Numerical results

The analytically calculated profile for the mean-square displacement $\langle x(t)^2 \rangle$ against time $t$ is plotted in Fig. 1 for the parameter values $\gamma = 0.01, \omega_0 = 1.0, \alpha = 1.0, D = 0.01$, and $\epsilon = 0.01$ (dotted line). This is compared to the corresponding numerically simulated profile (continuous line) after taking the average over $10^5$ number of trajectories obtained by directly solving the Langevin equation (2.7) using the Box-Muller algorithm [47] [unless otherwise stated we use the initial condition $x(0) = 1$, $\dot{x}(0) = 1$]. In order to ensure the thermal equilibrium, we have plotted $\langle v^2 \rangle$ against time ($\dot{x} = v$) in the inset. The agreement between the analytical and numerical profiles for the mean-square displacement is quite satisfactory. The effect of the strength of thermal noise is examined by varying $D$ as shown in Fig. 2 ($D = 0.001$) and in Fig. 3 ($D = 0.005$). We observe better agreement between theory and numerics as the strength of thermal noise is reduced. It can be easily checked that the lines represent oscillations when zoomed appropriately. To examine this aspect in more detail and to determine the contributions of fast and slow motion $X^2(t)|_{t\to\infty}$ and $\langle \psi^2 \rangle_{t\to\infty}$, we show in Fig. 4 the variation of $X^2$ and $\langle \psi^2 \rangle$ with time after the initial transients die down. The inset in the figure clearly indicates complex oscillations. The linear variation of $\langle x(t)^2 \rangle_{t\to\infty}$ with $D$ is displayed in Fig. 5. The agreement between the numerics (cross) and the analytical (line) values is excellent.

In Figs. 6 and 7, we have made a comparison of the analytical and numerical profiles for the mean-square displacement $\langle x^2(t) \rangle$ vs time for two different values of the strength of thermal noise $D = 0.01$ and $D = 0.005$, respectively, in the case of a bistable oscillator ($\omega_0^2 < 0$). Except for the initial

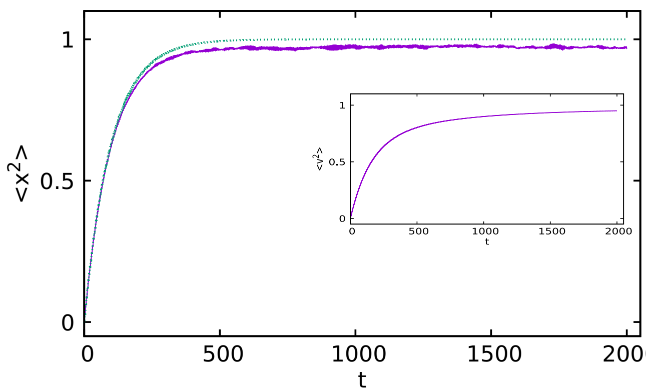


FIG. 1. The mean-square displacement $\langle x^2 \rangle$ is plotted against time $t$ for the Duffing oscillator for the following parameter set: $\omega_0 = 1.0$, $\gamma = 0.01$, $\alpha = 1.0$, $\epsilon = 0.01$, and $D = 0.01$ (analytical, dotted line; numerical, solid line). For numerical simulation averaging is done over $10^5$ trajectories. The inset refers to the variation of $\langle v^2 \rangle$ vs time $t$, calculated numerically to ensure the attainment of thermal equilibrium (units arbitrary).

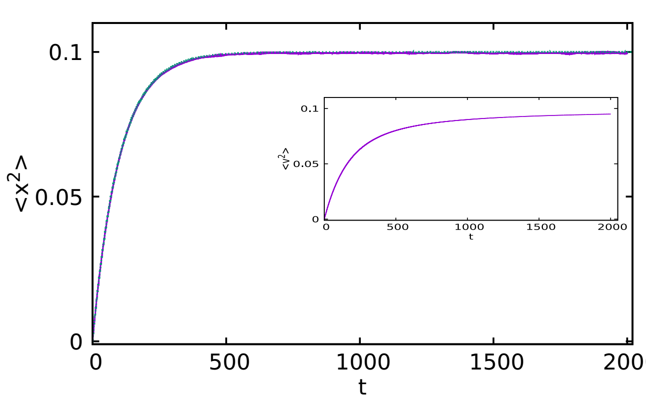


FIG. 2. Same as in Fig. 1 but for $D = 0.001$.

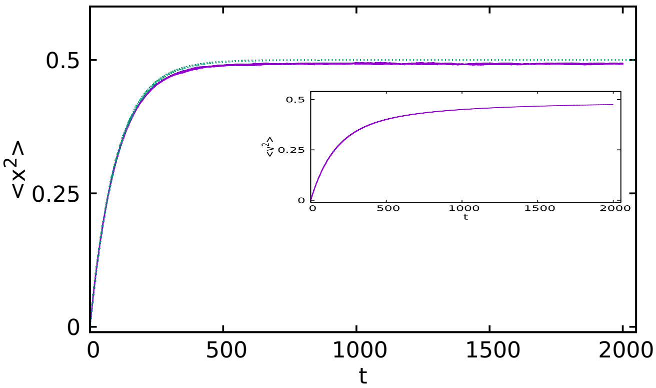


FIG. 3. Same as in Fig. 1 but for $D = 0.005$.

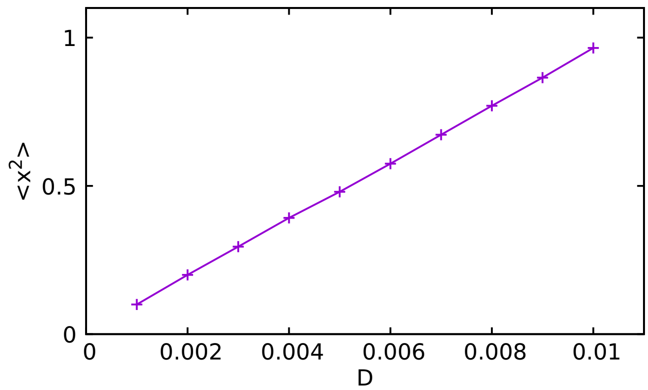


FIG. 5. The variation of $\langle x(t)^2\rangle|_{t\to\infty}$ as a function of $D$ is plotted for $\omega_0 = 1.0$, $\gamma = 0.01, \alpha = 1.0, \epsilon = 0.01$ and for $t = 1800$ corresponding to the Duffing oscillator for numerical (cross) and analytical (line) values.

transients for a brief period the agreement appears to be excellent for the weak noise case. The insets displaying $\langle v^2\rangle$ vs $t$ plots refer to the ensuring thermal equilibration of the system.

### B. Periodic potential

We now consider the case of a periodic potential $V(x)$ and start with the following Langevin equation:

$$\ddot{x} + \gamma\dot{x} + aV'(x) = \alpha\xi(t), \tag{2.35}$$

where $a$ is a constant.

To be specific, we consider the onsite potential as $V(x) = (1 - \cos x)$. Thus, $V'(x) = \sin x$. Substituting in Eq. (2.35) yields

$$\ddot{x} + \gamma\dot{x} + a\sin x = \alpha\xi(t). \tag{2.36}$$

We split the variable $x$ into a slow and a fast part as

$$x = X + \alpha\psi,$$

where the fast part explicitly contains the noise strength parameter $\alpha$. Separating the slow and fast parts, we write the slow dynamics as

$$\ddot{X} + \gamma\dot{X} + a\left[1 - \frac{\alpha^2}{2}\langle\psi^2\rangle\right]\sin X = 0, \tag{2.37}$$

and the fast part takes the form

$$\ddot{\psi} + \gamma\dot{\psi} + (a\alpha\cos X)\psi = \xi(t). \tag{2.38}$$

For the given potential, the steady states are $X = 0, n\pi$. Thus the fast dynamics around the steady state is given by the linear Langevin equation for harmonic oscillator,

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = \xi(t), \tag{2.39}$$

where $\omega_0^2 = a\alpha$. Equation (2.39) describes the fast motion as in the earlier two cases. The solution for the above equation is discussed in the previous section. Having used the standard result for $\langle\psi^2\rangle$, the analytical solution for $X$ in Eq. (2.37) can be worked out following Refs. [48–50]. We make use of the solution given by Salas [48] to calculate $X^2$ analytically and

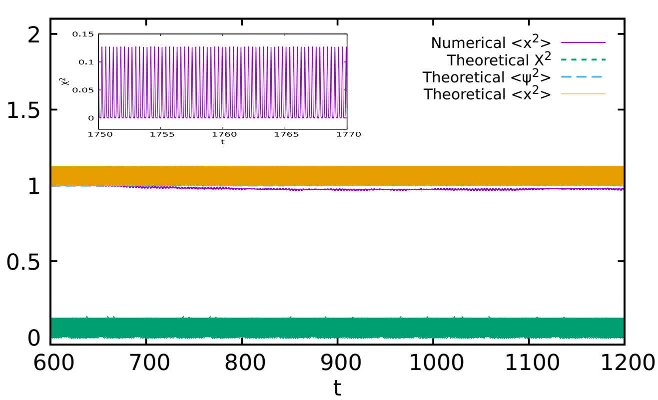


FIG. 4. The variation of $X^2$, $\langle\psi^2\rangle$ along with $\langle x(t)^2\rangle$ (numerical) and $\langle x(t)^2\rangle$ (theoretical) against time is plotted after the initial transients die down for the parameter set $\omega_0 = 1.0$, $\gamma = 0.01, \alpha = 1.0, \epsilon = 0.01$, $a = b = 5.0$, and $D = 0.01$, corresponding to the Duffing oscillator. The inset in the figure refers to the variation of $X^2$ vs $t$ over a short period of time to show the complex oscillations (units arbitrary).

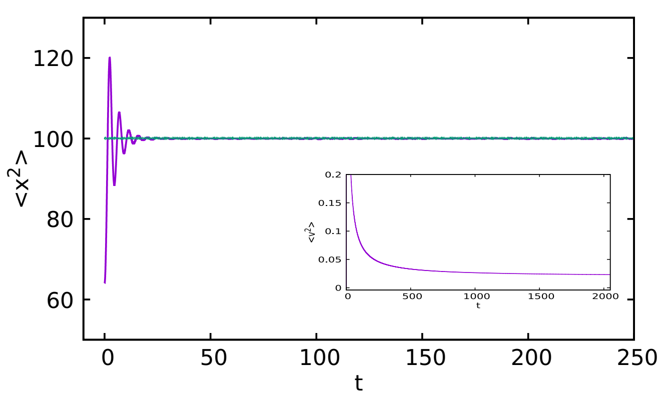


FIG. 6. The mean-square displacement $\langle x^2\rangle$ is plotted against time $t$ for the bistable oscillator for the following parameter set: $\omega_0 = 1.0, \gamma = 0.5, \alpha = 1.0, \epsilon = 0.01$, and $D = 0.01$ (analytical, dotted line; numerical, solid line). For numerical simulation averaging is done over $10^5$ trajectories. The inset in the figure refers to the variation of $\langle v^2\rangle$ vs time $t$, calculated numerically to ensure the attainment of thermal equilibrium (units arbitrary).

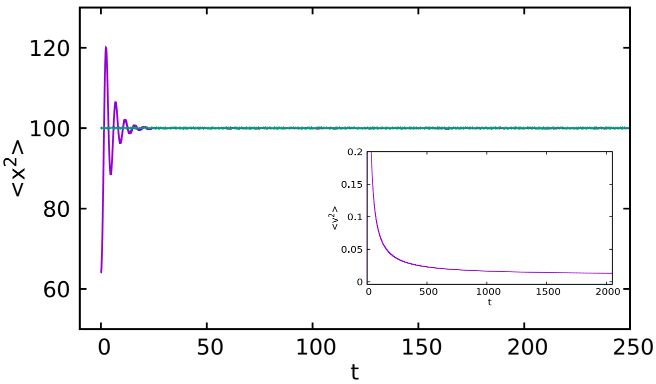


FIG. 7. Same as in Fig. 6 but for $D = 0.005$.

then present it in the inset of Fig. 8. The variation of $\langle x^2(t)\rangle$ vs time $t$ for the numerics (continuous line) and the analytical (dotted line) results for the parameter set $a = 1$, $\gamma = 0.01$, $D = 0.001$ with the initial conditions $x(0) = 0$, $\dot{x}(0) = 0.25$ for the periodic potential are shown in Fig. 8. The agreement is found to be quite satisfactory. It can be noted also that as $t \to \infty$, $X$ (and hence $X^2$) goes to zero [48] while $\langle\psi(t)^2\rangle$ settles down over a constant proportional to $D$. $\langle x(t)^2\rangle_{t\to\infty}$ therefore varies linearly with $D$. Unlike the previous two cases, the approximate solution of Eq. (2.37) used here [48] is nonperturbative. $\alpha$ here is the parameter for strength of noise used to separate out the contributions of fast and slow motion. Since in the long-time limit $X \to 0$, $\langle x(t)^2\rangle_{t\to\infty}$ varies linearly with $D$ as in the earlier cases.

We now demonstrate an immediate application of the treatment for calculation of mobility under a constant external field. In the presence of a constant field $F$, the effective slow dynamics assumes the following form:

$$\ddot{X} + \gamma\dot{X} + d\sin X = F, \tag{2.40}$$

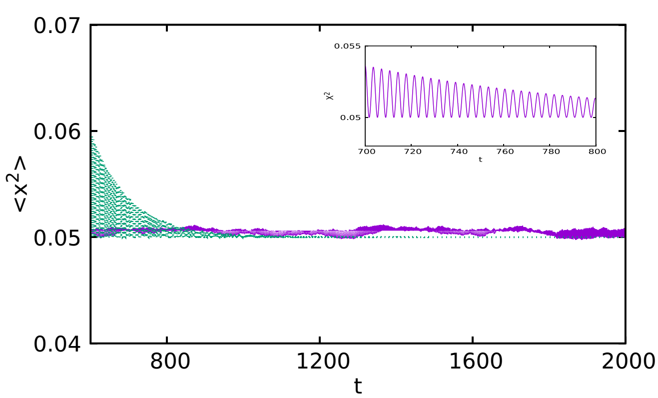


FIG. 8. The mean-square displacement $\langle x^2\rangle$ plotted against time $t$ for the periodic potential for the parameter set $a = 1$, $\gamma = 0.01$, $D = 0.001$ with the initial conditions $x(0) = 0$, $\dot{x}(0) = 0.25$ (analytical, dotted line; numerical, solid line). For the numerical simulation averaging is done over $10^5$ trajectories. The inset displays the variation of $X^2$ vs $t$ calculated analytically using the solution given in [48] (units arbitrary).

where $d = a[1 - \frac{\alpha^2}{2}\langle\psi^2\rangle]$. Following Risken [51], the low-friction limit of the drift velocity and the critical force can be calculated. For this we consider $\langle\psi^2\rangle$ in the limit $t \to \infty$, so that $\langle\psi^2\rangle = \frac{D}{\gamma\omega_0^2}(= \frac{D}{\gamma a\alpha})$ and we have $d = a[1 - \frac{\alpha D}{2\gamma a}]$. The effective barrier height of the periodic potential therefore gets modified by the weak thermal noise. In the very weak friction and noise, we introduce [51] energy $E$ as almost stationary so that $E = P^2/2 - d\cos X$, where $P = \dot{X}$. As a first approximation with respect to friction, we can write $P = P(X, E) = \sqrt{2(\bar{E} + d\cos X)}$, where $E = \bar{E}$ is constant and

$$\frac{dE}{dt} = \gamma(F_0 - P)P, \tag{2.41}$$

where $\dot{X} = P$, $F_0 = F/\gamma$, and Eq. (2.40) is used. The mobility can be calculated as [51,52]

$$\mu = \frac{\langle P\rangle}{F} = \frac{1}{FT}\int_{-\pi}^{\pi} dX = \frac{2\pi}{FT}. \tag{2.42}$$

Here $T$ can be expressed as

$$T = \int_0^T dT = \int_{-\pi}^{\pi}(dX/P)$$
$$= \int_{-\pi}^{\pi}\frac{dX}{\sqrt{2(\bar{E} + d\cos X)}} = 2\pi\frac{d\bar{P}(\bar{E})}{d\bar{E}}, \tag{2.43}$$

where $\bar{P}(\bar{E}) = \int_{-\pi}^{\pi}\frac{dX}{\sqrt{2(\bar{E}+d\cos X)}}$. Making use of the definition for mobility $\mu$, we write

$$\gamma\mu = \left[\frac{\partial\bar{P}(\bar{E})}{\partial\bar{E}}F_0\right]^{-1}. \tag{2.44}$$

The average velocity $\bar{P}(\bar{E})$ in the stationary state $\bar{E}$ is determined by balancing the energy gain due to field and the energy loss due to dissipation over one period. From Eq. (2.41) with $\frac{\partial E}{\partial t} = 0$, we obtain

$$\gamma(F_0 - P)P = 0. \tag{2.45}$$

Integrating over one time period it can be shown from Eq. (2.45) that

$$F_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(PdX) = \bar{P}(\bar{E}). \tag{2.46}$$

For a given $F_0$, we estimate $\bar{E}$, which may be used to calculate $\gamma\mu$ from Eq. (2.44). Equation (2.46) admits solutions only for $F_0 \geqslant \bar{P}(d) = \frac{4\sqrt{d}}{\pi}$. Since $d$ depends on the noise and temperature, the critical force gets lowered for higher temperature. This weak noise assistance of the transport is supported by earlier observations [51].

A closer examination of the present scheme reveals that nonlinearity plays a crucial role in the solution of the nonlinear Langevin equation. First, the interplay of nonlinearity and noise is reflected in the renormalization of linear frequency of the slow dynamics. Second, the linear frequency is also modified due to nonlinear contributions of the potential even in the absence of stochasticity. This is apparent from the expression for frequency correction term in Eq. (2.27). The slow dynamics characteristically evolves in a nonlinear potential field where the coefficients of the linear term takes care of these two modifications. In addition, the system is driven parametrically by the average contribution of the fast motion. This is evident in Eqs. (2.17) and (2.33). We also note in passing that the present scheme makes no explicit reference to the equation for the probability distribution function, e.g., the Liouville or Fokker-Planck equation. The distribution functions can be obtained directly from knowledge of the moments.

## III. CONCLUSION

In this paper we have proposed an analytical approach to the solution of a nonlinear Langevin equation. The scheme is based on a multiple timescale method whereby the separation of timescales is carried out in two stages explicitly. An interesting offshoot of the present treatment is the generic form of the description of fast motion, a Brownian harmonic oscillator. The effect of fast motion is subsumed into the slow motion in two ways. The first one is through the renormalization of the coefficients of the potential of the nonlinear equation for slow variables, and the second one is through parametric driving terms. The scheme relies on Blekhman perturbation theory applied widely in vibrational mechanics to derive the nonlinear equation for slow motion of a system driven by a rapidly varying time-periodic field. We have demonstrated how this theory can be extended to treat stochastic differential equations. Because of parametric driving, the nonlinear equation for slow motion leads to secular divergence, which is avoided by applying the Linstedt-Poincaré perturbation technique. The resulting moments are calculated as a combination of the average of fast and slow variables. Our analytical results for mean-square displacement with time are corroborated by direct numerical simulations of the nonlinear Langevin equations in three distinct cases of nonlinear potential.

Before closing, we would like to make the following pertinent points: First, as already mentioned, the present scheme of multiple timescales is applied in two stages. In the first stage this separation is exact. Rather than integrating over the fast and irrelevant variables, we derive an explicit dynamics for the fast motion which is essentially nonlinear in nature. The linear contribution as a first approximation constitutes the Brownian dynamics of a harmonic oscillator. The dynamics of the slow variables, on the other hand, comes under the purview of nonlinear dynamics. Second, in the traditional treatment of a nonlinear Langevin equation, one must resort to a hybrid approach, i.e., the equation for a phase space probability distribution function, and the Liouville equation is used simultaneously with the the stochastic equation. The present scheme regards the Langevin equation as sufficient and makes no reference to the Liouville equation for calculation of moments and the distribution functions. Third, the present approach, to the best of our knowledge, is the first extension of Blekhman perturbation theory extensively employed to treat the deterministic problems in vibrational mechanics and in allied problems in vibrational resonance involving fast periodic motion to the domain of stochasticity. In this paper we have confined ourselves to additive, Gaussian white thermal noise processes and believe that this method can be applied to multiplicative, non-Gaussian, and nonthermal noise processes as well in the future.

## ACKNOWLEDGMENT

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## APPENDIX: CALCULATION OF $\langle\psi^2(t)\rangle$

To calculate $\langle\psi^2\rangle$, we begin with the Langevin equation, as described in Eq. (2.14), with $\alpha = 1$:

$$\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = \xi(t), \tag{A1}$$

where $\xi(t)$ is the delta-correlated Gaussian white noise, as defined earlier. Taking the ensemble average of Eq. (A1), we obtain

$$\frac{d^2\langle\psi\rangle}{dt^2} + \gamma\frac{d\langle\psi\rangle}{dt} + \omega_0^2\langle\psi\rangle = 0, \tag{A2}$$

where we have used $\langle\xi(t)\psi(t)\rangle = 0$. To derive an equation for $\langle\psi^2\rangle$ we first multiply $\psi$ on both sides of Eq. (A1). After some rearrangement and ensemble averaging, we obtain

$$\frac{d^2\langle\psi^2\rangle}{dt^2} + \gamma\frac{d\langle\psi^2\rangle}{dt} + 2\omega_0^2\langle\psi^2\rangle = 2\langle\dot{\psi}^2\rangle. \tag{A3}$$

From Eq. (A2), we set up the following dynamics after a little algebra:

$$\frac{d^2\langle\psi\rangle^2}{dt^2} + \gamma\frac{d\langle\psi\rangle^2}{dt} + 2\omega_0^2\langle\psi\rangle^2 = 2\langle\dot{\psi}\rangle^2. \tag{A4}$$

Subtraction of Eq. (A4) from Eq. (A3) yields an equation for the variance $V_\psi$, where $V_\psi = \langle\psi^2\rangle - \langle\psi\rangle^2$,

$$\frac{d^2V_\psi}{dt^2} + \gamma\frac{dV_\psi}{dt} + 2\omega_0^2V_\psi = 2k_BT, \tag{A5}$$

where we have used Maxwell's velocity distribution to write $\langle\dot{\psi}\rangle = 0$ and $\langle\dot{\psi}^2\rangle - \langle\dot{\psi}\rangle^2 = k_BT$. Equation (A5) can be solved using a standard technique. For the underdamped case ($\omega_0 \gg \gamma$) as considered here we obtain the solution of Eq. (A2) as follows:

$$\langle\psi(t)\rangle = \psi_0 e^{-\gamma t/2}\left(\cos\frac{\Omega' t}{2} + \frac{\gamma}{\Omega'}\sin\frac{\Omega' t}{2}\right) + \frac{2\dot{\psi}_0}{\Omega'}e^{-\gamma t/2}\sin\frac{\Omega' t}{2}, \tag{A6}$$

where $\Omega' = \sqrt{4\omega_0^2 - \gamma^2}$ and $\psi_0$ and $\dot{\psi}_0$ are the initial values for $\psi$ and $\dot{\psi}$, respectively. We set them equal to zero, so that $\langle\psi(t)\rangle = 0$. The solution for $V_\psi$ can be written as follows:

$$V_\psi = \frac{k_BT}{\omega_0^2}\left\{1 - e^{-\gamma t}\left[\frac{2\gamma^2}{\Omega'^2}\sin^2(\Omega' t/2) + \frac{\gamma}{\Omega'}\sin(\Omega' t) + 1\right]\right\}. \tag{A7}$$

Since $\langle\psi(t)\rangle = 0$, we can write

$$\langle\psi^2\rangle = \frac{k_BT}{\omega_0^2}\left\{1 - e^{-\gamma t}\left[\left(1+\frac{\gamma^2}{\Omega'^2}\right) + \frac{\gamma}{\Omega'}\sin(\Omega' t) - \frac{\gamma^2}{\Omega'^2}\cos(\Omega' t)\right]\right\}. \quad \text{(A8)}$$

Alternatively, Eq. (A8) can be written as

$$\langle\psi^2\rangle = c + \alpha_1 e^{-\gamma t}\cos\Omega' t + \alpha_2 e^{-\gamma t} + \alpha_3 e^{-\gamma t}\sin\Omega' t, \quad \text{(A9)}$$

where we have introduced the strength of noise $\alpha$ as a multiplicative factor to express $c, \alpha_1, \alpha_2, \alpha_3$ as follows:

$$c = \frac{\alpha^2 D}{\gamma\omega_0^2}, \quad \alpha_1 = -\frac{\alpha^2\gamma D}{\omega_0^2(\gamma^2 - 4\omega_0^2)},$$

$$\alpha_2 = \frac{\alpha^2 D}{\gamma\omega_0^2} - \frac{\alpha^2\gamma D}{\omega_0^2(\gamma^2 - 4\omega_0^2)}, \quad \alpha_3 = \frac{\alpha^2 D}{\omega_0^2\sqrt{(4\omega_0^2 - \gamma^2)}},$$

$$\text{where} \quad \Omega' = \sqrt{(4\omega_0^2 - \gamma^2)}.$$

For all our numerical calculations $\alpha$ is set equal to unity. We emphasize that while $\langle\psi(t)\rangle$ depends on the initial conditions on $\psi$ and $\dot\psi$, $\langle\psi(t)^2\rangle$ as given in Eq. (A9) does not depend on any initial value.

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