Positing the problem of stationary distributions of active particles as third-order differential equation

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In this work, we obtain a third-order linear differential equation for stationary distributions of run-and-tumble particles in two dimensions in a harmonic trap. The equation represents the condition $j = 0$, where j is a flux. Since an analogous equation for passive Brownian particles is first-order, a second- and third-order term are features of active motion. In all cases, the solution has a form of a convolution of two distributions: the Gaussian distribution representing the Boltzmann distribution of passive particles, and the beta distribution representing active motion at zero temperature.

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I. INTRODUCTION

Even for the case of an ideal-gas of active particles, exact expressions of stationary distributions remain sparse. One way potentially to advance our understanding of this problem is to try to represent stationary distributions through a differential equation. The structure of such an equation can shed light on the problem and point toward a possible resolution. It could also play a classifying role and be used to relate active particles to other physical systems or, in a more mathematical sense, to other systems of differential equations.

In this work, we obtain a third-order linear differential equation for stationary distributions of run-and-tumble particles in two dimensions in a harmonic trap [1–15]. No exact distributions are known for this system. The terms of an equation represent flux j, and the equation itself represents the condition $j = 0$. In comparison, an equivalent equation for passive Brownian particles is of first order, where the first-order term comes from diffusion. The second- and third-order terms, therefore, are features of active motion.

Third-order differential equations are less commonly encountered in physics than second-order ones [16], and so are the methods to solve them [17,18]. This property can, at least partially, explain some of the difficulties in obtaining exact solutions.

Active particles are formulated and fully described by the Fokker-Planck equation, the solution of which is the distribution ρ in a phase space. For two dimensions and in a steady state, this distribution is a function of the particle position and the orientation of a swimming direction, $\rho \equiv \rho(\mathbf{r}, \theta)$. In this work, however, we are interested in constructing the differential equation for the distribution p as a function of a position only, $p \equiv p(\mathbf{r})$, related to the distribution ρ as $p = \int d\theta\, \rho$. There is no straightforward way of transforming the Fokker-Planck equation into a differential equation for p. Consequently, a considerable part of this work is devoted to the development of a method for constructing such an equation.

Self-propelled particles in a harmonic trap have been the subject of study in numerous papers. To highlight some more recent advancements, in [15] the authors obtained an exact series solution for a stationary distribution of active Brownian particles (ABPs) in a harmonic trap in two dimensions. More recently, an expression for the entropy production rate has been obtained for the run-and-tumble particle model (RTP) in [12–14]. The dynamics of self-propelled particles in harmonic confinement were considered in [4,8].

This paper is organized as follows. In Sec. II we consider the RTP model in one dimension (1D). Given that exact results in 1D are more accessible, the expressions of this section constitute a reference framework for systems in higher dimensions. In Sec. III A, we consider the RTP model in two dimensions in a harmonic potential in the direction along the x-axis, $u = Kx^2/2$. A differential equation for p is obtained together with the solution in the form of a convolution of two distributions. In Sec. III B, we then consider a harmonic trap with isotropic symmetry, for which a differential equation for p is obtained together with a solution. In Sec. IV, we consider a harmonic trap in three dimensions. In this dimension, a simple differential equation for p is no longer possible. Some analysis is carried out to understand why this is so. In Sec. V, the results are consolidated and discussed. The work is concluded with Sec. VI.

II. SETTING UP THE STAGE: THE RTP MODEL IN ONE DIMENSION

In the RTP model, particles, in addition to undergoing diffusion, move with a velocity of constant magnitude v_0 and an orientation that is evolving in time. New orientations are assigned at discrete time intervals drawn from exponential distribution whose average value τ represents the persistence time. Because in one dimension there are two directions, the Fokker-Planck equation (FP) can be represented as a system of two coupled differential equations [1,2,19–24],

$$\dot{p}_+ = -v_0 p_+' + D p_+'' - \frac{1}{2}\frac{1}{\tau}(p_+ - p_-),$$
$$\dot{p}_- = \ \ v_0 p_-' + D p_-'' - \frac{1}{2}\frac{1}{\tau}(p_- - p_+), \quad (1)$$

where $p_+(x,t)$ and $p_-(x,t)$ are the distribution for particles with a forward and backward swimming direction, respectively. The first two terms in each equation are contributions of the flux $j_\pm = -Dp_\pm' \pm v_0$, where D is the diffusion constant. The last term in each equation gives rise to active motion and represents a conversion of particles with one drift direction into particles with another drift direction. The conversion occurs with the rate $1/\tau$.

The two equations in Sec. 1 can be combined into a single differential equation for the total distribution $p = p_+ + p_-$. The procedure outlined in Appendix B leads to the following result:

$$\frac{\partial}{\partial t}[p - 2D\tau p''] + \tau\frac{\partial^2 p}{\partial t^2} = -j', \quad (2)$$

where the flux $j = j_+ + j_-$ on the right-hand side is given by

$$j = -\big(D + \tau v_0^2\big)p' + \tau D^2 p'''. \quad (3)$$

Compared with the equivalent equation for passive Brownian particles governed by $\dot{p} = Dp''$, the first striking difference is a more complex time dependence that involves a second-order time derivative, the signature of ballistic motion. This term is proportional to the persistence time τ. Furthermore, the terms that are first-order in time involve p and p''.

Turning next to the expression of flux in Eq. (3), we find that it is modified in two ways. The first change is the enhanced diffusion constant $D_{\rm eff} = D + \tau v_0^2$. The second and more significant modification is the emergence of a third-order term that has no counterpart in passive Brownian motion and as such can be regarded as a signature of active motion. Note that in the limit $\tau \to 0$, when the drift direction changes extremely fast, active motion becomes negligible and Eq. (2) recovers the standard diffusion equation.

At a steady state, the total flux vanishes, $j = 0$, resulting in the following equation:

$$0 = -\big(D + \tau v_0^2\big)p' + \tau D^2 p'''. \quad (4)$$

A stationary distribution p is then a solution to a third-order differential equation.

For particles confined between two walls at $x = \pm h$, to solve Eq. (4) we need the following boundary conditions:

$$0 = p'(0),$$
$$0 = -v_0^2 p(\pm h) + D^2 p''(\pm h). \quad (5)$$

The first condition ensures that p is symmetric around $x = 0$, and the second one ensures that the fluxes $j_\pm = -Dp' \pm v_0$ vanish separately at the walls. (In contrast to j, the constituent fluxes j_+ and j_- are not required to vanish everywhere in the interval. This gives rise to internal currents responsible for the nonzero entropy production rate [14,21].) The solution to Eq. (4) turns out to have a simple functional form: $p(x) = a + b\cosh(kx)$ [21,23].

A. Harmonic potential

For the case of a harmonic confinement, represented by the potential $u(x) = Kx^2/2$, the two coupled FP equations for the RTP model in one dimension are

$$\dot{p}_+ = [(\mu Kx - v_0)p_+]' + Dp_+'' - \frac{1}{2}\frac{1}{\tau}(p_+ - p_-),$$
$$\dot{p}_- = [(\mu Kx + v_0)p_-]' + Dp_-'' - \frac{1}{2}\frac{1}{\tau}(p_- - p_+). \quad (6)$$

To simplify the expressions, we define the timescales and lengthscales $\tau_k = \frac{1}{\mu K}$, $\lambda_k = v_0\tau_k$, and from now on we work with dimensionless space and time variables $z = \frac{x}{\lambda_k}$, $s = \frac{t}{\tau_k}$. The dimensionless diffusion constant and the rate of orientational change are defined as $B = \frac{D\tau_k}{\lambda_k^2}$ and $\alpha = \frac{\tau_k}{\tau}$. The two equations in (6) become

$$\dot{p}_+ = [(z-1)p_+]' + Bp_+'' - \frac{\alpha}{2}(p_+ - p_-),$$
$$\dot{p}_- = [(z+1)p_-]' + Bp_-'' - \frac{\alpha}{2}(p_- - p_+). \quad (7)$$

Combining the two equations into a single differential equation, following the procedure in Appendix C, leads to

$$\frac{\partial}{\partial s}[(\alpha - 3)p - 2zp' - 2Bp''] + \frac{\partial^2 p}{\partial s^2} = -j', \quad (8)$$

with the flux given by

$$j = -(\alpha - 2)zp - (1 - z^2 - 3B + B\alpha)p' + 2Bzp'' + B^2 p'''. \quad (9)$$

Compared to passive particles, the stiffness parameter is renormalized as $K_{\rm eff} = (\alpha - 2)K$. What is interesting is that for $\alpha < 2$ the stiffness becomes negative, causing particles to be repelled from the trap center rather than being attracted to it. For those values of α, the distribution is bimodal with two symmetric peaks shifted away from the center. If we interpret the coefficient of the first-order term as z-dependent effective diffusion, $B_{\rm eff} = (1 - z^2 - 3B + B\alpha)$, then there is $|z|$ beyond which $B_{\rm eff}$ is negative. Without the second- and third-order terms, the physical distribution does not exist beyond this point. Including these higher-order terms allows the distribution to extend beyond this point. The peak centers, however, remain confined into the region where $B_{\rm eff} > 0$. For $\alpha > 2$, the distribution p becomes unimodal with a single peak at $z = 0$.

At a steady state, $j = 0$ and a stationary distribution is obtained by solving

$$0 = (2 - \alpha)zp - (1 - z^2 - 3B + B\alpha)p' + 2Bzp'' + B^2 p'''. \quad (10)$$

We are next going to consider the solution p in two limiting situations: without thermal fluctuations (with active motion only), and without active motion (with thermal fluctuations only). At zero temperature ($B = 0$) and without thermal fluctuations, Eq. (10) reduces to a first-order differential equation:

$$0 = (2 - \alpha)zp - (1 - z^2)p', \quad (11)$$

for which the solution is a beta distribution on the interval $[-1, 1]$ [1,2,9,11]:

$$p_b \propto (1 - z^2)^{\frac{\alpha}{2} - 1}, \quad (12)$$

where the subscript b indicates that p is represented by a beta distribution. An active motion becomes suppressed in the limit $\alpha \to \infty$ as a result of rapid alternation of a swimming direction. Equation (10) in this situation reduces to

$$0 = -zp - Bp', \tag{13}$$

and the solution is given by the Boltzmann distribution

$$p_g \propto e^{-z^2/2B}, \tag{14}$$

where we use the subscript g to indicate that p is represented by a Gaussian distribution.

There is no closed-form solution to Eq. (10) that we are aware of, but the solution can be expressed as a convolution of two limiting probability distributions discussed above, namely the beta distribution in Eq. (12) and the Boltzmann distribution [25]

$$p(z) \propto \int_{-1}^{1} dz' \, (1 - z'^2)^{\frac{\alpha}{2}-1} e^{-\frac{(z-z')^2}{2B}}. \tag{15}$$

See Appendix A for a derivation.

Before delving into a physical interpretation of the mathematical form of the solution in Eq. (15), we briefly consider some of its mathematical aspects. Because in the limit $B \to 0$ the Gaussian function becomes a δ function, $\lim_{B\to 0} e^{-(z-z')^2/2B} \propto \delta(z - z')$, the solution in Eq. (15) recovers the solution in Eq. (12). And because the beta distribution defined on the interval $[-1, 1]$ in the limit $\alpha \to \infty$ also becomes a δ function, $\lim_{\alpha\to\infty}(1 - z'^2)^{\frac{\alpha}{2}-1} \propto \delta(z')$, the solution in Eq. (15) correctly recovers the Boltzmann distribution.

Next we reflect on physical interpretation of the solution in Eq. (15). The convolution of two distributions arises for the process that involves a sum of two or more independent random variables. For example, if the random variables x [with the associated distribution $p_x(x)$] and y [with the associated distribution $p_y(y)$] are independent, then the distribution for the process $z = x + y$ is $p_z(z) = \int dz' \, p_x(z')p_y(z' - z) = \int dz' \, p_y(z')p_x(z' - z)$. In light of this, the solution in Eq. (15), which can be represented as $p = \int dz' \, p_b(z')p_g(z' - z)$, makes sense, as the two random processes—thermal fluctuations and active motion—are independent, and the distribution p that we are looking for is for the sum of those two processes.

B. Other exact results

In this section, we briefly consider other exact results for the system at hand. These results will be useful when later we consider RTP particles in two dimensions.

One of the quantities that we are going to find useful in analyzing systems in higher dimensions is the even moments $\langle z^{2n}\rangle = \int_{-1}^{1} dz \, z^{2n} p(z)$ for the zero-temperature limit. For the system in 1D we know that the distribution in this limit corresponds to a beta distribution in Eq. (12), and the moments can be calculated as

$$\langle z^{2n}\rangle = \prod_{m=1}^{n} \frac{2m-1}{2m-1+\alpha}, \quad n = 1, 2, \ldots . \tag{16}$$

Alternatively, the moments can be calculated directly from Eq. (11) by operating on it with the integral operator $\int_{-1}^{1} dz \, z^{2n-1}$. This leads to the following iterative expression:

$$\langle z^{2n}\rangle = \frac{2n-1}{2n-1+\alpha}\langle z^{2n-2}\rangle. \tag{17}$$

Another limiting situation of interest is the case $\alpha = 0$, where Eq. (10) reduces to

$$0 = 2zp - (1 - z^2 - 3B)p' + 2Bzp'' + B^2 p'''. \tag{18}$$

The solution for this case can be inferred from physical considerations. Since for $\alpha = 0$ swimming orientations do not evolve in time, each particle with its fixed orientation attains equilibrium. But because there are two swimming orientations, the system is a mixture of particles with different swimming orientations—the case of quenched disorder [14,26]. Accordingly, a distribution is represented as a superposition of Boltzmann distributions of particles with different swimming orientations, in dimensionless units given by $p(z) \propto e^{-\frac{z^2}{2B}} e^{\frac{z}{B}} + e^{-\frac{z^2}{2B}} e^{-\frac{z}{B}}$.

III. RTP MODEL IN 2D

For RTP particles in a harmonic trap in two or higher dimensions, the task of reducing the corresponding stationary FP equation into a single differential equation for the distribution p is more challenging. In 1D, things are simplified on account of there being only two discrete orientations. For higher dimensions, there are infinitely many orientations.

A. Harmonic potential $u = Kx^2/2$

In this section, we consider the RTP particles in 2D trapped in a harmonic potential of the form $u(x) = Kx^2/2$. The system is effectively one-dimensional as it is translationally invariant along the y-axis. The swimming orientation is specified by the unit vector $\mathbf{n} = (\cos\theta, \sin\theta)$, and the relevant distribution $\rho(x, \theta, t)$ is normalized as $\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \, \rho = 1$.

The stationary Fokker-Planck equation that determines the distribution $\rho(x, \theta)$ in dimensionless units is given by

$$0 = [(z - \cos\theta)\rho]' + B\rho'' - \alpha\left(\rho - \int_0^{2\pi} \frac{d\theta}{2\pi}\, \rho\right). \tag{19}$$

See Appendix D for details.

The procedure in Appendix C is not generalizable to continuous orientations, and so another procedure is required. We begin with the zero-temperature limit, or $B = 0$. Equation (19) in this case reduces to

$$0 = z\rho' + \rho - \cos\theta\,\rho' - \alpha\rho + \frac{\alpha}{2\pi}p, \tag{20}$$

where $p = \int d\theta \, \rho$. We next use this equation to generate expressions for even moments $\langle z^{2n}\rangle = \int_{-1}^{1} dz \int_0^{2\pi} d\theta \, z^{2n}\rho$, in analogy to Eq. (17). For example, to obtain an expression for $\langle z^2\rangle$, we operate on Eq. (20) with the integral operator $\int_{-1}^{1} dz \int_0^{2\pi} d\theta \, z^2$. This yields $\langle z^2\rangle = \langle z\cos\theta\rangle$. Then to obtain $\langle z\cos\theta\rangle$, we operate on Eq. (20) with $\int_{-1}^{1} dz \int_0^{2\pi} d\theta \, z\cos\theta$, which yields $\langle z\cos\theta\rangle = 1/2(1+\alpha)$, so that we can write

$$\langle z^2\rangle = \frac{1}{2}\frac{1}{1+\alpha}.$$

A similar procedure can be used to obtain $\langle z^4 \rangle$ and any other moment. From the sequence of such moments, one can then infer a general formula

$$\langle z^{2n} \rangle = \prod_{k=1}^{n} \frac{1}{2} \frac{1}{n+\alpha}, \quad n = 1, 2, \ldots \tag{21}$$

that alternatively can be represented as an iterative relation given by

$$\langle z^{2n} \rangle = \frac{1}{2} \frac{2n-1}{n+\alpha} \langle z^{2n-2} \rangle. \tag{22}$$

We attempt next to infer a differential equation that generates such moments. The similarity between the result in Eq. (22) and that in Eq. (17) suggests that a differential equation ought to have a similar structure to Eq. (11). We determine that the following equation

$$0 = (1 - 2\alpha) zp - (1 - z^2) p' \tag{23}$$

generates the moments in Eq. (22). The solution is a beta distribution on the interval $[-1, 1]$, similar to that in Eq. (12) but that scales differently with α,

$$p(z) \propto (1 - z^2)^{\alpha - \frac{1}{2}}. \tag{24}$$

We next consider the case $\alpha = 0$, where the swimming orientations do not evolve in time, and because particles are confined, the system attains equilibrium. Yet because the system is a mixture of particles with different swimming orientations, it represents quenched disorder where a stationary distribution is a superposition of Boltzmann distributions of particles with different swimming orientations.

For a given orientation θ, a normalized Boltzmann distribution is

$$\rho(x, \theta) = \left[\sqrt{\frac{\beta K}{2\pi}} e^{-\frac{v_0^2 \cos^2 \theta}{2\beta K D^2}} \right] e^{-\frac{\beta K x^2}{2}} e^{\frac{v_0 x \cos \theta}{D}}. \tag{25}$$

Integrating the above expression over all orientations yields the distribution p, which, in dimensionless units, becomes

$$p(z) \propto e^{-\frac{z^2}{2B}} \int_0^{2\pi} d\theta \, e^{-\frac{\cos^2 \theta}{2B}} e^{\frac{z \cos \theta}{B}}. \tag{26}$$

We are now in a situation in which we know the solution but do not have a differential equation that generates it. This constitutes an inverse problem to that for determining a solution from a known equation. We find that the following differential equation generates the solution of interest:

$$0 = zp - (1 - z^2 - 2B) p' + 2Bzp'' + B^2 p'''. \tag{27}$$

The procedure used to infer this equation is described in Appendix E.

At this point we have two equations for two limiting situations, Eq. (23) for $B = 0$ and Eq. (27) for $\alpha = 0$. We combine the two equations in such a way as to avoid repetition of the same terms: zp and $(z^2 - 1)p'$. The resulting equation still fails to recover $0 = -zp - Bp'$ in the limit $\alpha \to \infty$. This can be fixed by including an additional term, $-2\alpha B p'$. Note that this is the only term that couples active and diffusive motion.

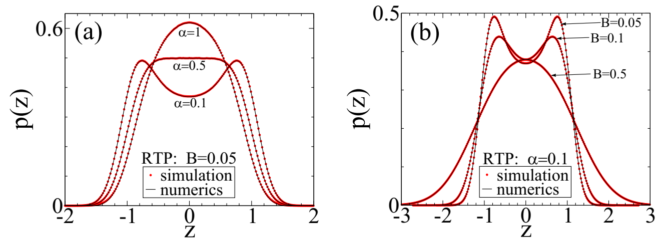

FIG. 1. Stationary distributions for RTP particles in 2D in a harmonic trap $u = Kx^2/2$. Solid lines correspond to the solution in Eq. (30) evaluated numerically. Points represent simulation data. Simulations were carried out using the Euler method for updating particle positions: $z(t + \Delta t) = [\cos\theta(t) - z(t)]\Delta t + \hat{\zeta}(t)\sqrt{2B\Delta t}$, where $\hat{\zeta}$ is a Gaussian noise with zero mean and unit variance.

The complete third-order equation for stationary p is

$$0 = (1 - 2\alpha) zp - (1 - z^2 - 2B + 2\alpha B) p' + 2Bzp'' + B^2 p'''. \tag{28}$$

Note that Eq. (28) has the same structure as Eq. (10) for the true 1D system. In fact, the two equations can be scaled into each other if for every α in Eq. (28) we substitute $(\alpha - 1)/2$. This implies that the solution in Eq. (15) can be scaled to produce a solution to Eq. (28):

$$p(z; \alpha) = p_{1\mathrm{D}}(z; 2\alpha + 1), \tag{29}$$

or more specifically

$$p(z) \propto \int_{-1}^{1} dz' \, (1 - z'^2)^{\alpha - \frac{1}{2}} e^{-\frac{(z - z')^2}{2B}}. \tag{30}$$

The differential equation in Eq. (28) and the solution in Eq. (30) are the main results of this section.

To verify Eq. (28), we compare p obtained from numerical evaluation of Eq. (30) with stationary distributions obtained from simulations. The results are plotted and compared in Fig. 1. The agreement between the two methods provides confirmation for the correctness of Eq. (28).

Note that the differential equation in Eq. (28) and the corresponding solution in Eq. (30) simplify at a specific value of $\alpha_c = 1/2$. At this crossover value of α, the effective stiffness of a trap $K_{\mathrm{eff}} = (2\alpha - 1)K$ changes sign. As a consequence, the zero-order term in Eq. (28) vanishes, and the integral solution in Eq. (30) simplifies to yield

$$p(z) \propto \mathrm{erf}\left[\frac{z + 1}{\sqrt{2B}}\right] - \mathrm{erf}\left[\frac{z - 1}{\sqrt{2B}}\right]. \tag{31}$$

At $B = 0$, the above distribution becomes a rectangular function. Then as B increases, it transforms into a normal distribution. A degree of flatness of this distribution can be quantified from the second derivative at $z = 0$, which is found to be $p''(0) \propto 1/\sqrt{B^3 e^{1/B}}$ and is plotted in Fig. 2. The distribution can be assumed to be flat if $p''(0) \approx 0$. This is the case for $B \lesssim 0.06$; for larger B, $p''(0)$ exhibits a sharp increase.

Figure 1(a) shows distributions before, at, and beyond the crossover α_c. All the plots are for $B = 0.05$, where the distribution can be considered as flat at the crossover. Figure 1(b) shows distributions at a fixed α, below the crossover value, for different values of B. For $B = 0.5$, thermal fluctuations destroy the bimodal structure of a distribution.

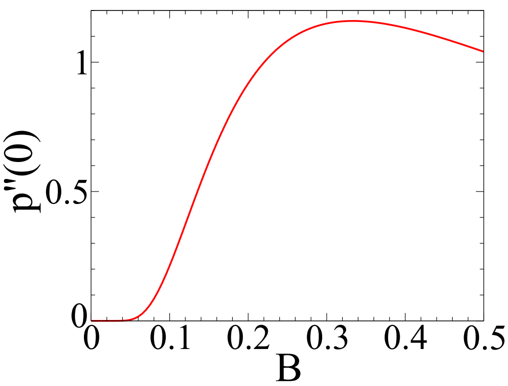

FIG. 2. Second derivative of the distribution p at $z = 0$ and calculated for $\alpha_c = 1/2$, see Eq. (31), as a way to quantify the degree of flatness.

B. Harmonic potential $u = Kr^2/2$

We next consider RTP particles (in 2D) in a harmonic trap with circular symmetry, $u(r) = Kr^2/2$. The stationary Fokker-Planck equation that determines the distribution $\rho(z, \theta)$ in this case is

$$0 = z\rho' + 2\rho - \cos\theta\,\rho' + \frac{\sin\theta}{z}\frac{\partial\rho}{\partial\theta} + B\left[\rho'' + \frac{\rho'}{z} + \frac{1}{z^2}\frac{\partial^2\rho}{\partial\theta^2}\right] - \alpha\left(\rho - \int_0^{2\pi}\frac{d\theta}{2\pi}\,\rho\right), \quad (32)$$

where $z = r/\lambda_k$. In this scenario, θ represents a relative angle between the swimming orientation and the angular position of a particle in a trap. See Appendix D for details.

At zero temperature, or $B = 0$, the stationary distribution ρ is governed by

$$0 = z\rho' + 2\rho - \cos\theta\,\rho' + \frac{\sin\theta}{z}\frac{\partial\rho}{\partial\theta} - \alpha\rho + \frac{\alpha}{2\pi}p, \quad (33)$$

where $p = \int_0^{2\pi} d\theta\,\rho$. By operating on the above equation with integral operators of the form $2\pi\int_0^{2\pi} d\theta\int_0^1 dz\, z^{2n+1}\rho$, we can calculate the moments $\langle z^{2n}\rangle$. For example, to obtain $\langle z^2\rangle = 2\pi\int_0^{2\pi} d\theta\int_0^1 dz\, z^3\rho$, we operate on Eq. (33) with $2\pi\int_0^{2\pi} d\theta\int_0^1 dz\, z^3$. This yields $\langle z^2\rangle = \langle z\cos\theta\rangle$. Then to calculate $\langle z\cos\theta\rangle$, we operate on Eq. (33) with $2\pi\int_0^{2\pi} d\theta\int_0^1 dz\, z^2\cos\theta$. This leads to

$$\langle z^2\rangle = \frac{1}{1+\alpha}.$$

A similar procedure can be used to calculate higher moments, from which it is possible to infer the following general expression:

$$\langle z^{2n}\rangle = \prod_{m=1}^{n}\frac{m}{m+\alpha}, \quad (34)$$

alternatively expressed as an iterative relation

$$\langle z^{2n}\rangle = \frac{n}{n+\alpha}\langle z^{2n-2}\rangle. \quad (35)$$

The first-order differential equation that generates such moments is

$$0 = (2 - 2\alpha)zp + (z^2 - 1)p'. \quad (36)$$

The solution of the above equation is a beta distribution on the interval $[-1, 1]$, similar to that in Eqs. (12) and (24) but that scales differently with α,

$$p(z) \propto (1 - z^2)^{\alpha-1}. \quad (37)$$

We next consider the limiting case $\alpha = 0$. Because swimming orientations in this limit do not change in time, the confined RTP particles attain equilibrium, and because the system is a mixture of particles with different swimming orientations, it represents quenched disorder where a stationary distribution is a superposition of different Boltzmann distributions. For particles with swimming orientation θ, the normalized Boltzmann distribution is the same as that in Eq. (25) with $x \to r$. Integrating this distribution over all orientations in dimensionless units yields

$$p(z) \propto e^{-\frac{z^2}{2B}}\,\mathrm{I}_0\left(\frac{z}{B}\right). \quad (38)$$

The solution is the product of a Gaussian and modified Bessel function of the first kind. The third-order differential equation that generates this solution is determined to be

$$0{=}2zp{+}\left(z^2 - 1 + 4B - \frac{B^2}{z^2}\right)p'{+}\left(\frac{B^2}{z^2} + 2B\right)zp'' + B^2p'''. \quad (39)$$

See Appendix E for details.

To obtain the complete equation, Eqs. (36) and (39) are combined taking care that the similar terms are not repeated. To ensure the correct behavior in the limit $\alpha \to \infty$, we add the term $-2\alpha Bp'$. The complete equation becomes

$$0 = 2(1-\alpha)zp - \left(1 - z^2 - 4B + 2\alpha B + \frac{B^2}{z^2}\right)p' + \left(2B + \frac{B^2}{z^2}\right)zp'' + B^2p'''. \quad (40)$$

Using previously gained insights, it is expected that the solution to the above equation can be represented as a convolution of the beta solution in Eq. (37) and the Gaussian distribution. If we consider that the convolution is done in 2D space, this leads to the following result:

$$p \propto \int_0^1 dz'\,(1 - z'^2)^{\alpha-1}e^{-\frac{(z'-z)^2}{2B}}\left[z'e^{-zz'/B}\mathrm{I}_0\left(\frac{zz'}{B}\right)\right], \quad (41)$$

where $\mathrm{I}_0(x)$ is the modified Bessel function of the first kind. See Appendix A for details.

Equation (40) together with the solution in Eq. (41) are the main results of this section. To verify Eq. (40), in Fig. 3 we compare p obtained from evaluating Eq. (41) with the simulated data points. The agreement is exact apart from statistical noise.

At the crossover value of α, $\alpha_c = 1$, the zero-order term in Eq. (40) vanishes and the solution becomes

$$p(z) \propto e^{-\frac{z^2}{2B}}\int_0^1 dz'\,z'e^{-\frac{z'^2}{2B}}\mathrm{I}_0\left(\frac{zz'}{B}\right). \quad (42)$$

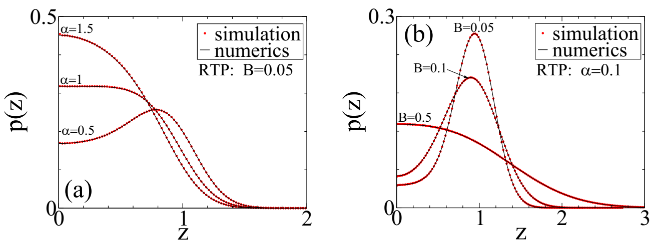

FIG. 3. Stationary distributions of RTP particles in 2D in an isotropic harmonic trap $u = Kr^2/2$. Solid lines correspond to numerical solution of Eq. (41), and the points represent simulation data.

The distribution becomes rectangular for $B = 0$, and with the onset of thermal fluctuations it approaches a normal distribution. Figure 4 shows how the second derivative of this distribution measured at $z = 0$ changes with B.

Figure 3(a) plots the distribution at $\alpha_c = 1$ and two other distributions on both sides of the crossover. All distributions are for $B = 0.05$ when thermal fluctuations are still small. To demonstrate the effect of temperature, Fig. 3(b) shows the distributions for a fixed α, below the crossover value, for three different B.

IV. RTP MODEL IN 3D: CONSIDERATIONS

It turns out that in the case of the RTP model in 3D, it is no longer possible to obtain a simple differential equation for the stationary distribution p, at least within the methodology developed and used in the previous sections. The main obstacle seems to be the fact that it is not obvious how to infer a differential equation for the zero-temperature limit, or $B = 0$, from the moments.

To illustrate those difficulties, we consider a harmonic potential along the x-axis, $u(x) = Kx^2/2$. The stationary FP equation is the same as that in Eq. (19). What is different is how the averaging over a swimming orientation is evaluated:

$$p(z) = \int_0^{\pi} d\theta \, \sin\theta \, \rho(z, \theta). \tag{43}$$

This apparently trivial modification in actuality turns out to complicate things. Recall that for the system in 2D and

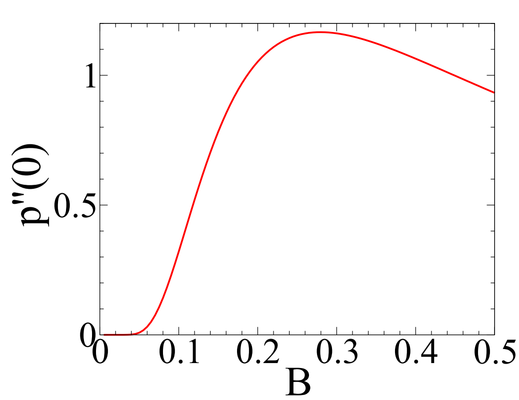

FIG. 4. Second derivative of the distribution p at $z = 0$ and for $\alpha_c = 1$, as a way to quantify the degree of flatness.

for the same potential, the averaging is done as $p(z) = \int_0^{2\pi} d\theta \, \rho(z, \theta)$.

At zero temperature or $B = 0$, the stationary FP equation becomes

$$0 = z\rho' + \rho - \cos\theta \rho' - \alpha\rho + \frac{\alpha}{2}p.$$

The moments generated by this equation are determined to be

$$\langle z^{2n}\rangle = \alpha \sum_{k=0}^{n-1} \frac{(2n-1)!}{2n+1-2k} \frac{\langle z^{2k}\rangle}{(2k)!} \prod_{m=2k}^{2n-1} \frac{1}{m+\alpha}, \tag{44}$$

no longer a simple expression such as that for two dimensions. Compare this expression with that in Eq. (22). To infer from those moments the first-order differential equation that generates them is no longer straightforward, and, at least at this moment, we have no workable methodology to attain this goal.

We may try to get some insights by considering the limit $\alpha = 0$, which is the case of quenched disorder. The distribution for a given drift orientation has a Boltzmann form given in Eq. (25). When integrated over all orientations, $\int_0^{\pi} d\theta \, \sin\theta \, \rho(x, \theta)$, in reduced units it becomes

$$p(z) \propto \text{erf}\left[\frac{z+1}{\sqrt{2B}}\right] - \text{erf}\left[\frac{z-1}{\sqrt{2B}}\right]. \tag{45}$$

The third-order differential equation that generates such a solution is found to be

$$0 = -[1 - z^2 - B]p' + 2Bzp'' + B^2 p'''. \tag{46}$$

Note the absence of the zero-order term.

We return now to the case $B = 0$ for which we propose the following ansatz differential equation:

$$0 = -\alpha z p - (1 - z^2)p' - \left[p \sum_{n=0}^{M} b_n z^{2n}\right]'. \tag{47}$$

The coefficients b_n and their number M still must be determined. Combining the ansatz with Eq. (46) and adding the term $-\alpha B p'$ to ensure correct behavior in the limit $\alpha \to \infty$, the complete equation becomes

$$\begin{aligned} 0 = &-\left(\alpha + 2\sum_{n=0}^{M} b_n n z^{2n-2}\right) zp \\ &-\left(1 - z^2 - B + \alpha B + \sum_{n=0}^{M} b_n z^{2n}\right) p' + 2Bzp'' + B^2 p'''. \end{aligned} \tag{48}$$

What we know about the coefficients b_n is that in the limit $\alpha \to 0$ they vanish if we are to have an agreement with Eq. (46). We also know that in the limit $\alpha \to \infty$ they go to a finite value to ensure convergence to passive Brownian particle behavior.

Finally, since for $B = 0$ the effective diffusion constant, defined as $D_{\text{eff}} = 1 - z^2 + \sum_{n=0}^{M} b_n z^{2n}$, should vanish at $z = 1$ to prevent particles from existing beyond $z > 0$, this suggests

TABLE I. Third-order equations for flux j of RTP particles in a harmonic trap. 2DL denotes a two-dimensional system with harmonic potential in the x-direction, $u = Kx^2/2$, and 2D denotes isometric harmonic potential, $u = Kr^2/2$. For 1D and 2DL, $z = x/\lambda_k$; for 2D, $z = r/\lambda_k$. Apart from different parametrization, the equations 1D and 2DL are the same. The equation 1D is transformed into the equation 2DL using the transformation $\alpha \to 2\alpha + 1$.

RTP	
1D	$j = [2-\alpha]zp - [1 - z^2 - 3B + \alpha B]p' + 2Bzp'' + B^2p'''$
2DL	$j = [1-2\alpha]zp - [1 - z^2 - 2B + 2\alpha B]p' + 2Bzp'' + B^2p'''$
2D	$j = [2-2\alpha]zp - [1 - z^2 - 4B + 2\alpha B + \frac{B^2}{z^2}]p' + [2B + \frac{B^2}{z^2}]zp'' + B^2p'''$

the constraint

$$\sum_{n=0}^{M} b_n = 0.$$

From Eq. (48) in the limit $B = 0$ we obtain the following relation for the moments:

$$0 = \alpha\langle z^{2m}\rangle - (2m-1)\sum_{n=0}^{M} b_n\langle z^{2n+2m-2}\rangle. \tag{49}$$

In combination with Eq. (44), in principle, we should be able to calculate the coefficients b_n.

The hope is that M is finite and the expressions for b_n are simple. This turns out not to be the case. An infinite number b_n is needed if Eq. (49) is to be satisfied for all n. The expressions for those coefficients are not simple and grow in complexity with increasing M.

V. SUMMARY AND DISCUSSION

Table I summarizes the main results of this work: third-order equations for the flux j for RTP particles in a harmonic trap. The stationary distributions are obtained from the condition $j = 0$ together with the boundary conditions

$$p'(0) = 0,$$
$$p(\pm\infty) = 0. \tag{50}$$

The first condition ensures the symmetry of p around $z = 0$, and the second condition ensures that p vanishes far from the trap center. The normalization constraint of p fully defines the solution. The normalization for 1D and 2DL is $\int_{-\infty}^{\infty} dz\, p(z) = 1$ and for 2D it is $2\pi\int_0^{\infty} dz\, zp(z) = 1$.

In the limit $\alpha \to \infty$, where the swimming orientation changes rapidly, all equations become dominated by the terms that are linear in α and reduce to $j = -zp - Bp'$, a system of passive Brownian particles in a harmonic trap.

Third-order equations in Table I do not lend themselves to a closed-form solution, but the solutions can be represented as a convolution of the beta and Gaussian distributions as listed in Table II. Such a form of the solution is intuitively accurate and represents a sum of two independent random processes: thermal fluctuations and active motion. This separation, however, should not be regarded as universal. Most likely, it is a feature of a harmonic trap.

TABLE II. Solutions to the third-order equations in Table I for $j = 0$.

RTP	
1D	$p \propto \int_{-1}^{1} dz'\,(1 - z'^2)^{\frac{\alpha}{2}-1} e^{-\frac{(z-z')^2}{2B}}.$
2DL	$p \propto \int_{-1}^{1} dz'\,(1 - z'^2)^{\alpha-\frac{1}{2}} e^{-\frac{(z-z')^2}{2B}}.$
2D	$p \propto \int_{0}^{1} dz'\,(1 - z'^2)^{\alpha-1} e^{-\frac{(z'-z)^2}{2B}}[z'e^{-zz'/B}\mathrm{I}_0(\frac{zz'}{B})],$

The fact that all equations are third-order appears to be a hallmark of active motion, without a counterpart in passive Brownian dynamics. Without thermal fluctuations, or $B = 0$, all equations reduce to first-order, for which a solution is readily available as indicated in Table III. The onset of thermal fluctuations gives rise to second- and third-order terms.

The solutions in Table III are beta distributions with different α scaling. In each case, the crossover α_c, the point where the distribution changes from concave to convex, is different. The cases 1D and 2DL are "effectively" one-dimensional. The difference between these two cases comes from the distribution of swimming velocities. For the 1D case, the swimming velocities are discrete, $v = \pm v_0$, and for the 2DL case they are continuously distributed in the interval $v \in [-v_0, v_0]$ (only the x-projection of the swimming velocities is relevant). As a result, the swimming velocities at any time are smaller than v_0, and particles push less against the harmonic potential. This should result in a less concave distribution for the same value of α compared to the 1D case. This is what is observed. This also explains why the crossover for 2DL occurs at $\alpha_c = 1/2$ while that for 1D occurs at $\alpha_c = 2$.

We next compare the 1D and 2D cases. First we note that for $\alpha = 0$ the distributions are the same. This happens because for $\alpha = 0$ swimming orientations do not change in time and so $\theta_r = \theta$, where θ_r is the angular position of a particle in the trap, and θ is the swimming orientation. For finite α, however, it is expected that $\theta_r \neq \theta$. This in turn results in the velocity component projected onto the direction θ_r being less than v_0, or $v_0\cos(\theta - \theta_r) \leqslant v_0$.

As the higher-order terms do not vanish in the limit $\alpha = 0$, see Table IV, these terms need not necessarily be attributed to some property of nonequilibrium. Since a system at $\alpha = 0$ represents equilibrium with quenched disorder, higher-order terms can be interpreted as arising from disorder due to different swimming orientations. All equations can be written as a combination of two equations representing different limits. Taking the 1D case as an example, we have

$$0 = \{2zp - [1 - z^2 - 3B]p' + 2Bzp'' + B^2p'''\} + \alpha\{-zp - Bp'\}. \tag{51}$$

TABLE III. As in Table I, but for zero temperature.

RTP, $B = 0$		
1D	$j = (2-\alpha)zp - (1 - z^2)p'$	$p \propto (1 - z^2)^{\frac{\alpha}{2}-1}$
2DL	$j = (1-2\alpha)zp - (1 - z^2)p'$	$p \propto (1 - z^2)^{\alpha-\frac{1}{2}}$
2D	$j = (2-2\alpha)zp - (1 - z^2)p'$	$p \propto (1 - z^2)^{\alpha-1}$

TABLE IV. As in Table I but for $\alpha = 0$. This limit represents a system in equilibrium with quenched disorder. Equations for the three-dimensional case are included since they can be obtained using the procedure in Appendix E.

RTP, $\alpha = 0$	
1D	$j = 2zp - [1 - z^2 - 3B]p' + 2Bzp'' + B^2p'''$
2DL	$j = zp - [1 - z^2 - 2B]p' + 2Bzp'' + B^2p'''$
2D	$j = 2zp - [1 - z^2 - 4B + \frac{B^2}{z^2}]p' + [2Bz + \frac{B^2}{z}]p'' + B^2p'''$
3DL	$j = -[1 - z^2 - B]p' + 2Bzp'' + B^2p'''$
3D	$j = 2zp - [1 - z^2 - 5B + \frac{2B^2}{z^2}]p' + [2Bz + \frac{2B^2}{z}]p'' + B^2p'''$

Both parts of the equation taken separately represent equilibrium: equilibrium for quenched disorder and equilibrium of passive Brownian particles. Parameter α determines the balance between the two equilibria.

VI. CONCLUSION

In this work, we posit the problem of stationary distributions of active particles in a harmonic trap as a third-order linear homogenous differential equation. Using the procedure developed in this work, we were able to determine such an equation for RTP particles in 2D for different trap symmetries. We did not obtain an analogous differential equations for RTP particles in higher dimensions. Similar difficulties were encountered for the case of ABP particles in any dimension and for any symmetry of a trap. In none of those cases was it possible to infer a differential equation from even moments $\langle z^2\rangle$. More fundamental causes of why a simple differential equation is possible for RTP particles in 2D and not for other cases remains to be better understood.

In addition to formulating the problem as a third-order equation, we obtain solutions in the form of a convolution of two probability distributions. This is a reflection of the fact that the two random processes, namely thermal fluctuations and active motion, are independent and the total distribution is for a sum of those two processes. Retrospectively, such an explanation is intuitively correct. On the other hand, we should not expect that it applies to other types of external potentials.

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ACKNOWLEDGMENTS

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APPENDIX A: INTEGRAL SOLUTIONS TO THIRD-ORDER DIFFERENTIAL EQUATIONS

In this Appendix, we derive a solution to Eq. (10) given in the form of convolution, introduced in the main text in Eq. (15). For convenience, we provide Eq. (10) below:

$$0 = (2-\alpha)zp - (1 - z^2 - 3B + B\alpha)p' + 2Bzp'' + B^2p'''.$$

The above equation is transformed by the application of a two-sided Laplace transform [25]:

$$\hat{p} = \int_{-\infty}^{\infty} dz\, e^{-sz}p, \tag{A1}$$

leading to

$$0 = (1 + B + B\alpha - B^2s^2)s\hat{p} + (2Bs^2 - \alpha)\hat{p}' - s\hat{p}''. \tag{A2}$$

If the transformed solution can be represented as $\hat{p}(s) = K_\alpha(s)K_B(s)$, where $K_\alpha(s)$ depends only on the parameter α, and $K_B(s)$ depends only on the parameter B, then

$$K_\alpha(s)K_B(s) = \int_{-\infty}^{\infty} dz\, e^{-sz}p(z), \tag{A3}$$

which implies that p is the convolution of the inverse transforms of $K_\alpha(s)$ and $K_B(s)$:

$$p(z) \propto \int_{-1}^{1} dz'\, K_\alpha^{-1}(z')K_B^{-1}(z - z'). \tag{A4}$$

Our next task is to determine the functions of K_α and K_B.

To find K_α, we set $B = 0$ so that Eq. (A2) reduces to

$$0 = s\hat{p} - \alpha\hat{p}' - s\hat{p}''. \tag{A5}$$

But since we know that the distribution $p(z)$ in this case is $p = (1 - z^2)^{\frac{\alpha}{2}-1}$, the solution to Eq. (A5) must be

$$\hat{p} \propto \int_{-1}^{1} dz\, e^{-sz}(1 - z^2)^{\frac{\alpha}{2}-1}. \tag{A6}$$

We are next going to claim that for $B = 0$, $K_B = 1$ (or some other constant) so that

$$K_\alpha(s) \propto \int_{-1}^{1} dz\, e^{-sz}(1 - z^2)^{\frac{\alpha}{2}-1}. \tag{A7}$$

We will be able to verify this assumption later when we have an expression for K_B.

To determine K_B, we want to identify the limit where K_α becomes independent of s, namely $K_\alpha = 1$ (or some other constant), so that $\hat{p} = K_B$. Examining Eq. (A7), we identify this to be $\alpha \to \infty$, where $(1 - z^2)^{\frac{\alpha}{2}-1}$ becomes a δ function and K_α reduces to a constant. Equation (A2) for this limiting situation reduces to

$$0 = Bs\hat{p} - \hat{p}', \tag{A8}$$

where the solution is $\hat{p} = e^{Bs^2/2}$. Based on this result, we get

$$K_B(s) \propto e^{\frac{Bs^2}{2}} \propto \int_{-\infty}^{\infty} dz\, e^{-sz}e^{-z^2/2B}. \tag{A9}$$

Going back to Eq. (A4), we can write

$$p(z) \propto \int_{-1}^{1} dz'\, (1 - z'^2)^{\frac{\alpha}{2}-1}e^{-(z'-z)^2/2B}. \tag{A10}$$

This is the solution first given in Eq. (15).

To obtain the solution in Eq. (41) for the harmonic potential in 2D, we assume that the solution has a similar form to that in Eq. (A6), except the convolution is over the 2D space:

$$p \propto e^{-\frac{z^2}{2B}} \int d\mathbf{z}' \, (1 - z'^2)^{\alpha-1} e^{-\frac{(\mathbf{z}'-\mathbf{z})^2}{2B}}, \quad \text{(A11)}$$

where $\mathbf{z}$ and $\mathbf{z}'$ are the vectors in 2D. Using polar coordinates, this becomes

$$p \propto e^{-\frac{z^2}{2B}} \int_0^1 dz' \, z'(1 - z'^2)^{\alpha-1} e^{-\frac{z'^2}{2B}} \int_0^{2\pi} d\theta \, e^{\frac{zz'\cos\theta}{B}},$$

which after integration over θ and some rearrangement becomes

$$p \propto \int_0^1 dz' \, (1 - z'^2)^{\alpha-1} e^{-\frac{(z'-z)^2}{2B}} \left[z' e^{-zz'/B} \mathrm{I}_0\left(\frac{zz'}{B}\right)\right]. \quad \text{(A12)}$$

This can be verified to be a solution to Eq. (40).

APPENDIX B: PROCEDURE FOR REDUCING EQ. (1)

This Appendix presents the procedure for reducing two coupled equations in (1) into Eq. (2) for the distribution $p = p_+ + p_-$. We begin by adding and subtracting the two equations in (1). This procedure transforms the two equations into

$$\dot{p} = -v_0\sigma' + Dp'',$$
$$\dot{\sigma} = -v_0 p' + D\sigma'' - \frac{\sigma}{\tau}, \quad \text{(B1)}$$

where $\sigma = p_+ - p_-$. The first equation in (B1) is used to obtain a number of expressions for derivatives of σ in terms of p and its derivatives:

$$v_0\sigma' = Dp'' - \dot{p},$$
$$v_0\sigma''' = Dp'''' - \dot{p}'',$$
$$v_0\dot{\sigma}' = D\dot{p}'' - \ddot{p}. \quad \text{(B2)}$$

By differentiating the second equation in (B1) with respect to x, we get

$$\dot{\sigma}' = -v_0 p'' + D\sigma''' - \frac{\sigma'}{\tau}. \quad \text{(B3)}$$

The expressions in (B2) are then used to eliminate all terms with σ:

$$\dot{p} - 2\tau D\dot{p}'' + \tau\ddot{p} = (D + \tau v_0^2)p'' - \tau D^2 p''''. \quad \text{(B4)}$$

The result agrees with Eqs. (2) and (3).

APPENDIX C: PROCEDURE FOR REDUCING EQ. (7)

This Appendix presents the procedure for reducing the two coupled equations in (7) into a single equation in (8). We start by transforming those equations into

$$\dot{p} = zp' - \sigma' + Bp'' + p,$$
$$\dot{\sigma} = z\sigma' - p' + B\sigma'' + (1-\alpha)\sigma, \quad \text{(C1)}$$

where $p = p_+ + p_-$ and $\sigma = p_+ - p_-$. The first equation in (C1) is used to obtain a number of expressions for different derivatives of σ in terms of p and its derivatives:

$$\sigma' = zp' + Bp'' + p - \dot{p},$$
$$\sigma'' = zp'' + Bp''' + 2p' - \dot{p}',$$
$$\sigma''' = p'' + zp''' + Bp'''' + 2p'' - \dot{p}'',$$
$$\dot{\sigma}' = z\dot{p}' + B\dot{p}'' + \dot{p} - \ddot{p}. \quad \text{(C2)}$$

Next, we differentiate the second equation in (C1) with respect to z:

$$\dot{\sigma}' = \sigma' + z\sigma'' - p'' + B\sigma''' + (1-\alpha)\sigma'. \quad \text{(C3)}$$

The terms involving σ are next eliminated using the expressions in (C2):

$$(\alpha - 3)\dot{p} - 2z\dot{p}' - 2B\dot{p}'' + \ddot{p} = (\alpha - 2)p - (4-\alpha)zp' - (z^2 - 1 + 5B - B\alpha)p'' - 2Bzp''' - B^2 p''''.$$

To convert the right-hand side (rhs) of the equation to conform with the expression $-j'$, we rewrite it as

$$\text{rhs} = -(2-\alpha)p - (2-\alpha)zp' - 2zp' - (-1 + 3B - B\alpha)p'' - z^2p'' - 2Bp'' - 2Bzp''' - B^2p''''. \quad \text{(C4)}$$

This can be rearranged into

$$\text{rhs} = -[(2-\alpha)zp + (z^2 - 1 + 3B - B\alpha)p' + 2Bzp'' + B^2p''']'. \quad \text{(C5)}$$

Equations (C4) and (C5) yield the results in Eqs. (8) and (9).

APPENDIX D: FOKKER-PLANCK EQUATIONS FOR RTP PARTICLES IN A HARMONIC TRAP: DERIVATION OF EQS. (19) AND (32)

The FP equation for the RTP model in 2D subject to an external force $\mathbf{F}$ is

$$\dot{\rho} = -\nabla \cdot [(\mu\mathbf{F} + v_0\mathbf{n})\rho] + D\nabla^2\rho - \frac{1}{\tau}\left(\rho - \int_0^{2\pi} \frac{d\theta}{2\pi}\rho\right). \quad \text{(D1)}$$

For $\mathbf{F} = -\mathbf{e}_x Kx$ and at steady-state $\rho \equiv \rho(x,\theta)$, and the FP equation becomes

$$0 = \frac{\partial}{\partial x}[(\mu Kx - v_0\cos\theta)\rho] + D\frac{\partial^2\rho}{\partial x^2} - \frac{1}{\tau}\left(\rho - \int_0^{2\pi} \frac{d\theta}{2\pi}\rho\right). \quad \text{(D2)}$$

Using dimensionless units, this recovers Eq. (19).

In the case of an isotropic force $\mathbf{F} = -K\mathbf{r}$, at steady-state the FP equation becomes

$$0 = \mu K\mathbf{r}\cdot\nabla\rho + \mu K\rho(\nabla\cdot\mathbf{r}) - v_0\nabla\cdot\mathbf{n}\rho + D\nabla^2\rho - \frac{1}{\tau}\left(\rho - \int_0^{2\pi}\frac{d\theta}{2\pi}\rho\right). \quad \text{(D3)}$$

The stationary ρ should be a function of the position of a particle in a trap. In polar coordinates this includes r and θ_r, and the swimming orientation θ. Actually, the only relevant angular dependence is the difference $\theta_r - \theta$. This allows us to represent a stationary ρ as a function of two variables. From now on we use θ to designate the difference $\theta_r - \theta$.

For convenience, we fix the swimming orientation $\mathbf{n}$ along the x-axis, $\mathbf{n} = \mathbf{e}_x$. The resulting equation becomes

$$0 = \mu K\mathbf{r}\cdot\nabla\rho + \mu K\rho(\nabla\cdot\mathbf{r}) - v_0\frac{\partial\rho}{\partial x} + D\nabla^2\rho - \frac{1}{\tau}\left(\rho - \int_0^{2\pi}\frac{d\theta}{2\pi}\rho\right). \quad \text{(D4)}$$

In polar coordinates, various terms are expressed as $\nabla^2\rho = \frac{\partial^2\rho}{\partial r^2} + \frac{1}{r}\frac{\partial\rho}{\partial r} + \frac{1}{r^2}\frac{\partial^2\rho}{\partial\theta^2}$, $\frac{\partial\rho}{\partial x} = \cos\theta\frac{\partial\rho}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\rho}{\partial\theta}$, $\mathbf{r}\cdot\nabla\rho =$

$r\frac{\partial\rho}{\partial r}$, $\nabla\cdot\mathbf{r}=2$. Substituting those expressions in the above equation leads to

$$0=\mu K r\frac{\partial\rho}{\partial r}+2\mu K\rho-v_0\cos\theta\frac{\partial\rho}{\partial r}+v_0\frac{\sin\theta}{r}\frac{\partial\rho}{\partial\theta}+D\left[\frac{\partial^2\rho}{\partial r^2}+\frac{1}{r}\frac{\partial\rho}{\partial r}+\frac{1}{r^2}\frac{\partial^2\rho}{\partial\theta^2}\right]-\frac{1}{\tau}\left(\rho-\int_0^{2\pi}\frac{d\theta}{2\pi}\rho\right). \quad \text{(D5)}$$

Using dimensionless units, this recovers Eq. (32).

APPENDIX E: REVERSE PROCEDURE OF INFERRING EQUATION FROM SOLUTION

In this Appendix, we outline a procedure for inferring an equation from a solution. The procedure is used to obtain Eqs. (27) and (39) from the solution

$$p(z)\propto e^{-\frac{z^2}{2B}}\int_0^{2\pi}d\theta\, e^{-\frac{\cos^2\theta}{2B}}e^{\frac{z\cos\theta}{B}}, \quad \text{(E1)}$$

which first appears in Eq. (26), and

$$p(z)\propto e^{-z^2/2B}\,\mathrm{I}_0(z/B), \quad \text{(E2)}$$

which appears in Eq. (38). As both functions are even, their Taylor expansion

$$p(z)=p^{(0)}(0)+\frac{z^2}{2}p^{(2)}(0)+\frac{z^4}{4!}p^{(4)}(0)+\cdots.$$

We next propose the following ansatz for an unknown third-order equation:

$$\begin{aligned}0&=(c_{11}+c_{21}z^2+c_{31}z^4)zp\\&+(c_{12}+c_{22}z^2+c_{32}z^4)p'\\&+(c_{13}+c_{23}z^2+c_{33}z^4)zp''\\&+(c_{14}+c_{24}z^2+c_{34}z^4)p'''.\end{aligned} \quad \text{(E3)}$$

Note that the ansatz does not introduce mixed even and odd terms. This is to ensure that the symmetry of p is preserved.

The coefficients c_{ij} are determined from a truncated expansion of p, for example

$$p_{\mathrm{tr}}=p^{(0)}+\frac{z^2}{2}p^{(2)}+\frac{z^4}{4!}p^{(4)}+\frac{z^6}{6!}p^{(6)},$$

where $p^{(0)}=p(0)$, $p^{(1)}=p'(0)$, etc. The number of terms in p_{tr} depends on the number of parameters c_{ij}. For example, for nine c_{ij}, we should include at least nine terms in the truncated series (actually, eight is enough since we assume $c_{11}=1$).

If the coefficients do not become altered by using p_{tr} for a larger number of terms, it is assumed that the equation is complete. For the distribution in (E1), the nonzero coefficients are

$$\begin{aligned}&c_{11}=1,\\&c_{12}=2B-1,\ c_{22}=1,\\&c_{13}=2B,\\&c_{14}=B^2.\end{aligned} \quad \text{(E4)}$$

This suggests the following equation:

$$0=zp-(1-z^2-2B)p'+2Bzp''+B^2p''', \quad \text{(E5)}$$

which can be verified to yield the solution in Eq. (E1).

For the distribution in (E2), the nonzero coefficients are

$$\begin{aligned}&c_{21}=2,\\&c_{12}=-B^2,\ c_{22}=4B-1,\ c_{32}=1,\\&c_{13}=B^2,\ c_{23}=2B,\\&c_{24}=B^2,\end{aligned} \quad \text{(E6)}$$

and after manipulation, the ansatz becomes

$$0=2zp+\left(z^2-1+4B-\frac{B^2}{z^2}\right)p'+\left(\frac{B^2}{z^2}+2B\right)zp''+B^2p'''. \quad \text{(E7)}$$

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