Exactly solvable one-dimensional quantum models with gamma matrices

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In this paper we write exactly solvable generalizations of one-dimensional quantum XY and Ising-like models by using 2^d-dimensional gamma matrices as the degrees of freedom on each site. We show that these models result in quadratic Fermionic Hamiltonians with Jordan-Wigner-like transformations. We illustrate the techniques using a specific case of four-dimensional gamma matrices and explore the quantum phase transitions present in the model.

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I. INTRODUCTION

Investigations of exactly solvable quantum many-body models are important due to their many applications in understanding a plethora of physical phenomena in statistical and condensed-matter physics, such as quantum phase transitions [1,2] and thermodynamic properties of many-body systems [3]. They not only provide platforms for testing new approximation schemes, but also serve as test beds for numerical techniques developed to tackle large many-body systems, in particular, higher-dimensional systems or nonintegrable systems [4–8].

Despite their enormous importance in various fields, only a handful of exactly solvable quantum many-body models are known to date, mostly in one dimension [9–13]. Among such models, perhaps the most celebrated ones are the Ising model [14–17] and the XY model [18–20] in a transverse field (see also [21]), consisting of a number of spin-$\frac{1}{2}$ particles arranged on a one-dimensional lattice. These models have a rich history of aiding research in various directions over the years, including understanding order-disorder quantum phase transitions [1,2], quantum information science and technology [22], and materials research in condensed-matter physics [23]. Moreover, realization of these models through currently available techniques using different substrates such as trapped ions [24], nuclear magnetic resonance systems [25], solid-state systems [26], and optical lattices [27] has made the verification of theoretical results possible.

While the simplest variants of the Ising and XY models deal with the spin-$\frac{1}{2}$ particles arranged on a one-dimensional lattice where a spin only interacts with its nearest neighbors, these models have been extended further in various directions. For example, one could have asymmetric Dzyaloshinskii-Moriya-type interactions [28], with staggered magnetic field [29], and multiple spin-exchange interactions [30–33]. The Ising and XY models in a transverse field, along with their generalizations mentioned above, can be solved by transforming the spin variables to spinless fermions via a Jordan-Wigner (JW) transformation [34], followed by a Bogoliubov–de Gennes transformation.

In this paper we explore one such exactly solvable generalization with higher-dimensional Hilbert space associated with each lattice site (for a different approach to exactly solvable one-dimensional generalization involving multiple spins at each site, see [35,36]). Noting that the anticommutation relations of Pauli operators, i.e., $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, $i, j \in (1, 2, 3)$, play a crucial role in the JW approach, earlier works [37] proposed replacing Pauli matrices by higher-dimensional gamma matrices Γ^i, $i \in (1, 2, 3, \dots)$, satisfying similar anticommutation relations, i.e.,

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \rightarrow \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}. \quad (1)$$

It was shown in [37] that the generalized versions of XXZ models can be fermionized. However, XXZ models are considerably harder to solve than the XY models and in this context it is natural to ask if there is a generalization of the XY models using gamma matrices. In this work we construct explicit models with 2^d-dimensional gamma matrices (for any d), which when fermionized give Hamiltonians which are quadratic in fermions and hence solvable, just like the XY models. We also discuss the $d = 2$ case in more detail and explore an Ising-like quantum critical point. While in this paper we focus on the models in one dimension, we point out here that several interesting gamma matrix generalizations have also been considered for quantum spin models on higher-dimensional lattices [38–43].

The rest of the paper is organized as follows. In Sec. II, after reviewing the one-dimensional solvable XY and related models, we define our model using the gamma matrices. We then rewrite the model in terms of fermions, employing the JW transformation, and solve it. In Sec. III we illustrate our results via solving the model explicitly for a special case. We also demonstrate the quantum phase transitions occurring in the system and comment on the calculation of critical exponents. We summarize in Sec. IV, as well as pointing out possible future directions.

II. MODEL

In this section we construct the model, perform the fermionization, and write the Hamiltonian in Fourier space.

We start with a quick recap of XY and related models in one dimension.

A. The XY model in one dimension

Consider a class of quantum spin models consisting of a lattice of N sites with two degrees of freedom at each site. We write the Hamiltonian representing such models in a compact form as

$$H = -i \sum_a \sum_{\mu,\nu=1}^{2} J_{\mu\nu} \sigma_a^\mu \sigma_a^3 \sigma_{a+1}^\nu - h \sum_a \sigma_a^3, \tag{2}$$

where σ^1, σ^2, and σ^3 are Pauli matrices and $a = 1, \dots N$ is the lattice index such that on each site

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}. \tag{3}$$

The spin-exchange couplings are represented by $J_{\mu\nu}$, while the strength of the external magnetic field in the z direction is denoted by h. Note that both $J_{\mu\nu}$ and h are real for the Hamiltonian to be Hermitian. It is worth mentioning that the above Hamiltonian can be easily rewritten in a more familiar form that is quadratic in Pauli matrices using $\sigma^1\sigma^2 = i\sigma^3$ and so on [see Eqs. (4)–(6) for details]. However, we will work with (2), since it is better suited to the generalizations that we will define later.

A number of well-known quantum spin models with nearest-neighbor spin-exchange interactions can be identified as particular cases of the Hamiltonian in (2) as follows: (i) the Ising model in a transverse field $J_{21} = J_1$, with the rest of $J_{\mu\nu}$ vanishing

$$H = \sum_a \left(J_1 \sigma_a^1 \sigma_{a+1}^1 - h\sigma_a^3\right); \tag{4}$$

(ii) the XY model in a transverse field $J_{12} = -J_2, J_{21} = J_1$, with the rest of $J_{\mu\nu}$ vanishing

$$H = \sum_a \left(J_1 \sigma_a^1 \sigma_{a+1}^1 + J_2 \sigma_a^2 \sigma_{a+1}^2 - h\sigma_a^3\right); \tag{5}$$

and (iii) the XY model in a transverse field with asymmetric Dzyaloshinskii-Moriya interaction $J_{12} = -J_2$, $J_{21} = J_1$, and $J_{11} = J_{22} = D$, with the rest of $J_{\mu\nu}$ vanishing

$$H = \sum_a \left(J_1 \sigma_a^1 \sigma_{a+1}^1 + J_2 \sigma_a^2 \sigma_{a+1}^2 - h\sigma_a^3\right) + \sum_a D\left(\sigma_a^1 \sigma_{a+1}^2 - \sigma_a^2 \sigma_{a+1}^1\right). \tag{6}$$

For convenience, we refer to the class of quantum spin models represented by (2) as the generalized XY models. As mentioned earlier, these models are solvable using the JW transformations, which rewrite Pauli matrices in terms of fermionic creation and annihilation operators obeying canonical anticommutation relations, as we will see below. The Hamiltonian (2) is quadratic in terms of these fermionic operators and hence solvable.

In what follows we generalize the generalized XY model to allow for more degrees of freedom per lattice site in such a way that the JW transformations remain applicable and the resulting Hamiltonian remains quadratic in terms of the fermionic operators. As noted in [37], such a generalization is possible by replacing the Pauli matrices on each lattice site with appropriate Γ matrices,[1] which have the following algebra at each site:

$$\{\Gamma_a^\mu, \Gamma_a^\nu\} = 2\delta^{\mu\nu}. \tag{7}$$

Here $\mu, \nu \in \{1, 2, \dots, 2d\}$, while Γ matrices at different sites commute. For the specific representation of the Γ matrices in terms of Pauli matrices see Sec. II B 1; the Pauli matrices σ_a^μ on the lattice site a correspond to the special case of $d = 1$. In the next few sections we work out this generalization in detail and demonstrate the solvability of the generalized model [see (18) for the Hamiltonian of the model]. We will call this class of models generalized XY models with gamma matrices (generalized $XY\Gamma$), parametrized (apart from the different interaction parameters appearing in the Hamiltonian) by the parameter d.

B. Review of gamma matrices

We begin by reviewing a number of features of the gamma matrices which will be important in the rest of the paper. For brevity, we suppress the lattice index and write the anticommutation relation of the gamma matrices as

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \tag{8}$$

where Γ^μ is Hermitian and $\mu = \{1, 2, \dots, 2d\}$. As we will see later in Sec. II B 1, these Γ^μ are $2^d \times 2^d$ matrices and can be thought of as operators acting on a Hilbert space of d spin-$\frac{1}{2}$ degrees of freedom. For $d = 1$, it is clear that (8) simply reduces to Pauli matrices σ^1 and σ^2. The matrix Γ^{2d+1}, which is the analog of the σ^3 matrix for the $d = 1$ case and plays an important role in defining the Hamiltonian for the generalized $XY\Gamma$ models (see Sec. II D), is defined as

$$\Gamma^{2d+1} \equiv (-i)^d \prod_{\mu=1}^{2d} \Gamma^\mu \tag{9}$$

and obeys the anticommutation relation

$$\{\Gamma^{2d+1}, \Gamma^\mu\} = 0 \,\forall \mu, \tag{10}$$

with $(\Gamma^{2d+1})^2 = 1$. Additionally, we define a set of d mutually commuting operators S_i such that $[S_i, S_j] = 0$, where

$$S_i \equiv (-i)\Gamma^{2i-1}\Gamma^{2i}, \quad i \in \{1, 2, \dots, d\}. \tag{11}$$

These operators will facilitate the field term in the generalized $XY\Gamma$ model (see Sec. II D).

1. Specific representation of the Γ *matrices*

While defining and solving our model can be done purely algebraically [i.e., using the algebra defined in (7)], it is sometimes useful to have explicit realization for the Γ_a^μ (for each lattice site a) operators as $2^d \times 2^d$ matrices on the Hilbert space. One such realization of the Γ matrices is in terms of the tensor products of Pauli matrices given as (again suppressing

[1] See [44] and references therein for a recent exposition on the general conditions when such rewriting is possible.

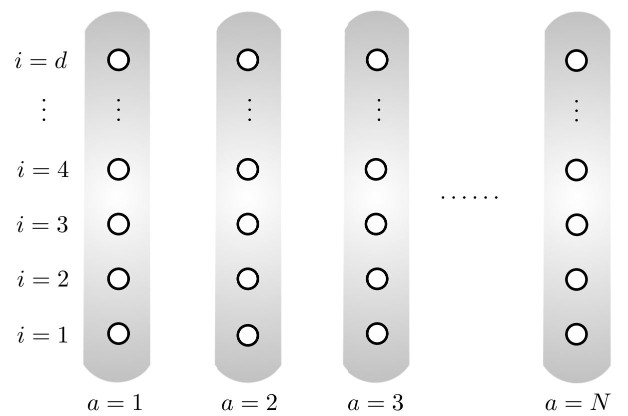

FIG. 1. Sublattice structure for generalized $XY\Gamma$ models. Each lattice site (gray block), denoted by the index a, consists of d sublattice points (white circles), marked by the index i.

the lattice index a)

$$\begin{aligned}
\Gamma^1 &= \sigma_1^1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_{d-1} \otimes \mathbb{1}_d,\\
\Gamma^2 &= \sigma_1^2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_{d-1} \otimes \mathbb{1}_d,\\
\Gamma^3 &= \sigma_1^3 \otimes \sigma_2^1 \otimes \mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_{d-1} \otimes \mathbb{1}_d,\\
\Gamma^4 &= \sigma_1^3 \otimes \sigma_2^2 \otimes \mathbb{1}_3 \otimes \cdots \otimes \mathbb{1}_{d-1} \otimes \mathbb{1}_d,\\
&\vdots\\
\Gamma^{2d-1} &= \sigma_1^3 \otimes \sigma_2^3 \otimes \sigma_3^3 \otimes \cdots \otimes \sigma_{d-1}^3 \otimes \sigma_d^1,\\
\Gamma^{2d} &= \sigma_1^3 \otimes \sigma_2^3 \otimes \sigma_3^3 \otimes \cdots \otimes \sigma_{d-1}^3 \otimes \sigma_d^2,
\end{aligned} \tag{12}$$

where it can be easily verified that the above matrices satisfy (8) (see [45] for a related realization). One can interpret it as d sublattice sites for each of the lattice sites a having a spin-$\frac{1}{2}$ degree of freedom on each of these sublattice sites or as d spin-$\frac{1}{2}$ pseudo-spin degrees of freedom at each lattice site. The subscripts on the Pauli matrices and the identity operators in (12) represents the sublattice points or pseudospin, which we denote by the index i $(i = 1, 2, \ldots, d)$, as mentioned before (see Sec. II B). A pictorial representation of the sublattice structure of the model can be found in Fig. 1. The above representation corresponds to the choice.

$$\Gamma^{2d+1} = \bigotimes_{i=1}^{d} \sigma_i^3 = \prod_{i=1}^{d} S_i, \tag{13}$$

where the commuting operators S_i (see Sec. II B) can be constructed as

$$S_i = \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \cdots \mathbb{1}_{i-1} \otimes \sigma_i^3 \otimes \mathbb{1}_{i+1} \otimes \cdots \otimes \mathbb{1}_d. \tag{14}$$

C. Jordan-Wigner transformation

The model we build below consists of Γ_a^μ, i.e., gamma matrices defined at each site a. We define the fermion operator χ_a^μ as

$$\chi_a^\mu \equiv \left(\prod_{b<a} \Gamma_b^{2d+1}\right) \Gamma_a^\mu \tag{15}$$

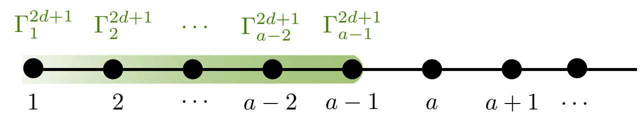

FIG. 2. Jordan string of Γ^{2d+1} operators corresponding to the fermionic operator χ_a^μ, spanning the lattice sites $1, 2, \ldots, a-1$.

such that $\chi_a^{\mu\dagger} = \chi_a^\mu$. It is easy to verify (also see [37]) that the operators $\{\chi_a^\mu\}$ satisfy the fermionic algebra

$$\{\chi_a^\mu, \chi_b^\nu\} = 2\delta^{\mu\nu}\delta_{ab} \tag{16}$$

and are called Majorana fermions. Note that this is a straightforward Γ-matrix generalization of the well-known JW transformations usually defined in terms of Pauli matrices, with the Γ^{2d+1} playing the role of σ^3. The so-called Jordan string now consists of a string of Γ^{2d+1} operators to the left of the site of interest (see Fig. 2). Although the Γ matrices at different sites commute, the presence of Γ^{2d+1} in the Jordan string of fermions along with the property $\{\Gamma_a^\mu, \Gamma_a^{2d+1}\} = 0$ makes the fermions at different sites anticommute with each other. An analogous treatment using complex fermions can also be done and some details are given in Appendix B.

D. Hamiltonian

With all the ingredients in place, we now write the generalized $XY\Gamma$ models as a generalization of the generalized XY models in terms of Γ matrices. Let us consider the Hamiltonian

$$H_G = -i \sum_a \sum_{\mu,\nu} J_{\mu\nu} \Gamma_a^\mu \Gamma_a^{2d+1} \Gamma_{a+1}^\nu - \sum_a \sum_i h_i S_a^i, \tag{17}$$

where $\mu, \nu = 1, 2, \ldots, 2d$, $i = 1, 2, \ldots, d$, and $J_{\mu\nu}$ and h_i are sets of $4d^2$ and d coupling constants, respectively.[2] The Hermiticity condition of the Hamiltonian implies that the coupling constants $J_{\mu\nu}$ and h_i must be real. Note that since for $d = 1$, Γ matrices reduce to Pauli σ matrices, the generalized $XY\Gamma$ Hamiltonian given above reduces to the generalized XY Hamiltonian (2) for $d = 1$. A pictorial representation of the $J_{\mu\nu}$ couplings for the $d = 2$ case can be found in Fig. 3. We sometimes write the coupling constants $J_{\mu\nu}$ as a sum of a symmetric and an antisymmetric part, as $J_{\mu\nu} = (\mathcal{S}_{\mu\nu} - \mathcal{A}_{\mu\nu})/2$, where we take $\mathcal{S}_{\mu\nu} = \mathcal{S}_{\nu\mu}$ and $\mathcal{A}_{\mu\nu} = -\mathcal{A}_{\nu\mu}$.

As described in Appendix A, the generalized $XY\Gamma$ models given by the Hamiltonian in (17) are quadratic in terms of the fermions defined by the JW transformations given in (15),

$$H_G = -i \sum_a \sum_{\mu,\nu} J_{\mu\nu} \chi_a^\mu \chi_{a+1}^\nu + i \sum_a \sum_i h_i \chi_a^{2i-1} \chi_a^{2i}. \tag{18}$$

Such quadratic fermionic Hamiltonians are often encountered in the literature [46]. In the next section we diagonalize the Hamiltonian by exploiting the translational symmetry. In the case of systems with finite N, the transformation of the Hamiltonian (17) to the Hamiltonian (18) requires a careful analysis of the boundary terms, which is given in Appendix A.

[2]As we will show in Appendix C, the number of independent couplings can be shown to be $4d^2$ rather than $4d^2 + d$.

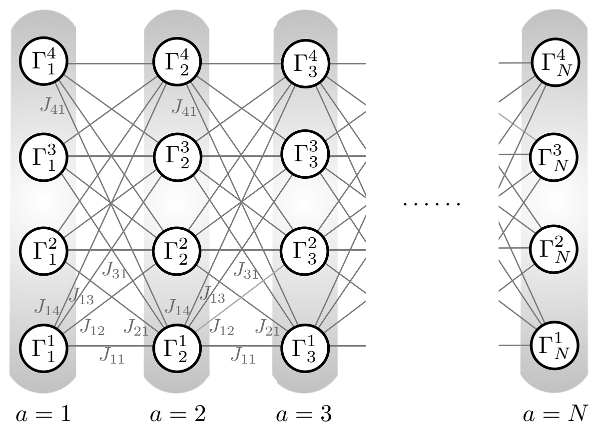

FIG. 3. Interactions in the Hamiltonian for $d = 2$. Each lattice site a (gray block) can house four Γ matrices $\{\Gamma_a^1, \Gamma_a^2, \Gamma_a^3, \Gamma_a^4\}$, $a = 1, 2, \ldots, N$. The lines in the diagram represent the couplings $J_{\mu\nu}$ involving Γ_a^μ and Γ_{a+1}^ν, $\mu, \nu \in \{1, 2, 3, 4\}$ [see (17)].

E. Hamiltonian in *k* space

To diagonalize the quadratic fermionic Hamiltonian in (18), we exploit the fact that the Hamiltonian is translationally invariant and go to the momentum space via defining the momentum modes χ_k^μ as

$$\chi_a^\mu \equiv \frac{1}{\sqrt{N}} \sum_k e^{ika} \chi_k^\mu, \tag{19}$$

where the sum over k runs symmetrically over both positive and negative values (see Appendix A for a detailed treatment of the finite-N scenario). Note that χ_k^μ are complex fermions and satisfy the following algebra:

$$\chi_k^{\mu\dagger} = \chi_{-k}^\mu, \tag{20}$$

$$\{\chi_k^{\mu\dagger}, \chi_{k'}^\nu\} = 2\delta^{\mu\nu}\delta_{k,k'}. \tag{21}$$

Under this transformation, the fermionic Hamiltonian of the generalized $XY\Gamma$ models (18) becomes (see Appendix A for the detailed calculation)

$$H_G = \sum_{k>0} \big[\{i\mathcal{A}_{\mu\nu}\cos k + \mathcal{S}_{\mu\nu}\sin k\}\chi_{-k}^\mu \chi_k^\nu + ih_i\{\chi_{-k}^{2i-1}\chi_k^{2i} - \chi_{-k}^{2i}\chi_k^{2i-1}\}\big]. \tag{22}$$

The Hamiltonian in (22) can also be written as

$$H_G = \sum_{k>0} V^\dagger H V, \quad \text{with } V \equiv \begin{pmatrix} \chi_k^1 \\ \vdots \\ \chi_k^{2d} \end{pmatrix}, \; k > 0, \tag{23}$$

where H is a $2d \times 2d$ matrix given by

$$H = \begin{pmatrix} \mathcal{S}_{11}\sin k & i(h_1 + \mathcal{A}_{12}\cos k) & i\mathcal{A}_{13}\cos k + \mathcal{S}_{13}\sin k & \ldots \\ -i(h_1 + \mathcal{A}_{12}\cos k)\cos k & \mathcal{S}_{22}\sin k & i\mathcal{A}_{23}\cos k + \mathcal{S}_{23}\sin k & \ldots \\ -i\mathcal{A}_{13}\cos k + \mathcal{S}_{13}\sin k & -i\mathcal{A}_{32}\cos k + \mathcal{S}_{23}\sin k & \mathcal{S}_{33}\sin k & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{24}$$

Notice that V and H have a k dependence which we have omitted in the notation for simplicity. As mentioned in Appendix C, with no loss of generality, we can choose $\mathcal{S}_{12} = \mathcal{S}_{34} = \cdots = 0$. Since these manipulations have rendered the problem of diagonalizing the Hamiltonian (17) (acting on a 2^{Nd}-dimensional Hilbert space) to simply diagonalizing the $2d \times 2d$ Hermitian matrix H, we term the model solvable. We will explicitly solve for the $d = 2$ case in the next section. We mention here that all of these can be repeated with complex fermions instead of Majorana fermions and some details are given in Appendix B.

F. Symmetries

The Hamiltonian in (17) has no symmetries for general values of the couplings $J_{\mu\nu}$ and h_i other than the discrete $\mathbb{Z}_2$ symmetry, which is given by

$$\Gamma_a^\mu \to -\Gamma_a^\mu. \tag{25}$$

However, the Hamiltonian may enjoy certain additional symmetries for special values of these couplings. One such symmetry is the reflection symmetry, which allows the exchange of the ath site with the $(N+1-a)$th site. For example, in the XY model of (5), this would amount to the Hamiltonian being invariant under the transformation

$$\sigma_a^\mu \leftrightarrow \sigma_{N+1-a}^\mu, \quad \mu = 1, 2, 3\, \forall a. \tag{26}$$

This symmetry is broken once we allow for the Dzyaloshinskii-Moriya interaction term (6). In our model given in (17), this symmetry can be incorporated as

$$\Gamma_a^{2i-1} \to -i\Gamma_{N+1-a}^{2i}\Gamma_{N+1-a}^{2d+1}, \quad \Gamma_a^{2i} \to i\Gamma_{N+1-a}^{2i-1}\Gamma_{N+1-a}^{2d+1}, \tag{27}$$

which for $d = 1$ translates to (26). Moreover, under the transformations (27),

$$S_a^i \to S_{N+1-a}^i, \quad \Gamma_a^{2d+1} \to \Gamma_{N+1-a}^{2d+1}. \tag{28}$$

The Hamiltonian in (17) is invariant under the reflection symmetry only if the couplings satisfy

$$J_{2i,2j} = -J_{2j-1,2i-1}, \quad J_{2i-1,2j} = J_{2j-1,2i}, \quad J_{2i,2j-1} = J_{2j,2i-1}. \tag{29}$$

Equivalently, in terms of couplings $\mathcal{S}_{\mu\nu}$ and $\mathcal{A}_{\mu,\nu}$,

$$\mathcal{S}_{2i,2j} = -\mathcal{S}_{2i-1,2j-1}, \quad \mathcal{S}_{2i,2j-1} = \mathcal{S}_{2i-1,2j}, \quad \mathcal{A}_{2i,2j} = \mathcal{A}_{2i-1,2j-1}, \quad \mathcal{A}_{2i,2j-1} = -\mathcal{A}_{2i-1,2j}. \tag{30}$$

We will work with reflection symmetric Hamiltonians when we solve the $d = 2$ model explicitly in the next section.

III. GENERALIZED $XY\Gamma$ MODELS FOR $d = 2$

In the preceding section we reduced the problem of solving for the spectrum of the Hamiltonian H_G in (17), a $2^{Nd} \times 2^{Nd}$ matrix, to solving for the eigenvalues of a $2d \times 2d$ matrix H given in (24). For $d = 1$, solving for the eigenvalues of the 2×2 matrix can be done analytically, which results in the known spectrum of generalized XY models. In this section we focus on the more complicated case of $d = 2$.

Before we go about solving the Hamiltonian, we comment on the physical interpretation of the $d = 2$ model. Recall that for $d = 2$ the Hilbert space can be taken to consist of two spin-$\frac{1}{2}$ degrees of freedom (say, σ and $\tilde{\sigma}$) per site. The most general nearest-neighbor Hamiltonian that can be written on this Hilbert space is

$$\tilde{H} = \sum_{i,j,k,l,a} J_{ijkl}\left(\sigma_a^i \sigma_{a+1}^k\right) \otimes \left(\tilde{\sigma}_a^j \tilde{\sigma}_{a+1}^l\right) + \sum_{i,j,a} h_{ij}\sigma_a^i \otimes \tilde{\sigma}_a^j, \quad (31)$$

where $i, j, \ldots$ are indices $1, \ldots, 4$ and $\sigma^0 \equiv \mathbb{I}_{2\times 2}$. There are $4^4 + 4^2 = 272$ coupling constants in this Hamiltonian.[3] As mentioned before, the generalized $XY\Gamma$ Hamiltonian spans a $4d^2 + d = 18$ parameter subspace of this general Hamiltonian. To get a sense of what the generalized $XY\Gamma$ interactions look like in the above conventions, consider the J_{12} term. The contribution to the generalized $XY\Gamma$ Hamiltonian (17) after substituting the representation (12) is given by

$$H_G \supset -iJ_{13}\Gamma_a^1\Gamma_a^5\Gamma_{a+1}^3 = -J_{13}\left(\sigma_a^2\sigma_{a+1}^2\right) \otimes \left(\tilde{\sigma}_a^3\tilde{\sigma}_{a+1}^1\right). \quad (32)$$

What we show below is that this 18-parameter subspace is exactly solvable. We also mention here that the four-dimensional Hilbert space can be equivalently thought of as a spin-$\frac{3}{2}$ system since the gamma matrices $\Gamma^1, \ldots, \Gamma^5$ can be represented by bilinear combinations of spin-$\frac{3}{2}$ operators (see [47]).

A. Hamiltonian

In this section we focus on the $d = 2$ generalized $XY\Gamma$ model. Before applying the techniques introduced in this paper, we can try to get some intuition by looking at the ground state and nature of excitations. While it is hard to do this for general values of the couplings, there exist choices of the couplings for which the analysis is much simpler. For example, there exist choices of coupling for which the above model reduces to the known XY and Ising models. To see this, consider the case when only $J_{12}, J_{21}, h_1, h_2 \neq 0$. Using the representation (12) the Hamiltonian is

$$\begin{aligned} H_G = \sum_a \Big[&-J_{12}\left(\sigma_a^2 \otimes \sigma_a^3\right)\left(\sigma_{a+1}^2 \otimes \mathbb{I}_{a+1}\right) \\ &+ J_{21}\left(\sigma_a^1 \otimes \sigma_a^3\right)\left(\sigma_{a+1}^1 \otimes \mathbb{I}_{a+1}\right) + h_1\left(\sigma_a^3 \otimes \mathbb{I}_a\right) \\ &+ h_2\left(\mathbb{I}_a \otimes \sigma_a^3\right)\Big]. \end{aligned} \quad (33)$$

In the first two terms above, one can think of the second sublattice as a control bit for the first sublattice, which controls whether the spin chain on the first sublattice has ferromagnetic or antiferromagnetic interactions. It is easy to see that in the $h_2 \to \infty$ limit, the dynamics of the second sublattice is frozen and the chain becomes the one-dimensional (1D) spin-$\frac{1}{2}$ XY model, for which the existing knowledge on the ground states and excitations would follow.

We now turn to solving the $d = 2$ generalized $XY\Gamma$ models assuming the reflection symmetry. Imposing the symmetry constraints given in (30), we obtain from (24) for H,[4]

$$H = \begin{pmatrix} \mathcal{S}_{11}\sin k & i(h_1 + \mathcal{A}_{12}\cos k) & i\mathcal{A}_{13}\cos k + \mathcal{S}_{13}\sin k & i\mathcal{A}_{14}\cos k + \mathcal{S}_{14}\sin k \\ -i(h_1 + A_{12}\cos k) & -\mathcal{S}_{11}\sin k & -i\mathcal{A}_{14}\cos k + \mathcal{S}_{14}\sin k & i\mathcal{A}_{13}\cos k - \mathcal{S}_{13}\sin k \\ -i\mathcal{A}_{13}\cos k + \mathcal{S}_{13}\sin k & i\mathcal{A}_{14}\cos k + \mathcal{S}_{14}\sin k & \mathcal{S}_{33}\sin k & i(h_2 + \mathcal{A}_{34}\cos k) \\ -i\mathcal{A}_{14}\cos k + \mathcal{S}_{14}\sin k & -i\mathcal{A}_{13}\cos k - \mathcal{S}_{13}\sin k & -i(h_2 + \mathcal{A}_{34}\cos k) & -\mathcal{S}_{33}\sin k \end{pmatrix}. \quad (34)$$

Before we discuss the spectrum of this matrix and find the energy gaps, we make some general comments about the spectrum. Note that H satisfies

$$S^{-1}HS = -H^*, \quad \text{with } S \equiv \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \quad (35)$$

This guarantees that for every eigenvector U with eigenvalue ϵ, there is a corresponding eigenvector $V \equiv SU^*$ with eigenvalue $-\epsilon$, namely,

$$HU = \epsilon U \Rightarrow HV = -\epsilon V. \quad (36)$$

Now let us define an antiunitary operator $\mathcal{P} \equiv \mathcal{K}S$, where $\mathcal{K}$ is the complex conjugation operator and S is defined via

$$S^{-1}\chi_k^{2i-1}S = -i\chi_k^{2i}, \quad S^{-1}\chi_k^{2i}S = i\chi_k^{2i-1}, \quad (37)$$

implying

$$\mathcal{P}^{-1}H_G\mathcal{P} = -H_G \quad (38)$$

for H_G defined in Eq. (22), upon using the conditions (30). This indicates that the property that the eigenvalues always

[3]The number of independent coupling constants can be reduced by rotating the Pauli matrices, etc.

[4]Via the choice of rotations as in Appendix C, we can set $\mathcal{S}_{12} = \mathcal{S}_{34} = 0$, $\mathcal{S}_{22} = -\mathcal{S}_{11}$, $\mathcal{S}_{44} = -\mathcal{S}_{33}$, $\mathcal{S}_{24} = -\mathcal{S}_{13}$, $\mathcal{S}_{23} = \mathcal{S}_{14}$, $\mathcal{A}_{24} = \mathcal{A}_{13}$, and $\mathcal{A}_{23} = -\mathcal{A}_{14}$.

come in pairs is actually more general and follows from the particle-hole symmetry, represented by $\mathcal{P}$. To see that $\mathcal{P}$ indeed acts like particle-hole symmetry, it is convenient to define complex fermions (more details about the complex fermions are given in Appendix B)

$$c_k^i \equiv \frac{\chi_k^{2i-1} + i\chi_k^{2i}}{\sqrt{2}}, \quad c^{i\dagger}_{-k} \equiv \frac{\chi_k^{2i-1} - i\chi_k^{2i}}{\sqrt{2}}, \tag{39}$$

which, under the action of $\mathcal{P}$, transforms as

$$\mathcal{P}^{-1} c_k^i \mathcal{P} = c^{i\dagger}_{-k}, \quad \mathcal{P}^{-1} c^{i\dagger}_{-k} \mathcal{P} = -c_k^i. \tag{40}$$

We now turn to computing the spectrum of Eq. (34), i.e., the energy gaps. It is convenient to define the quantities

$$\begin{aligned} F_k = \tfrac{1}{2}\big[&(h_1 + \mathcal{A}_{12}\cos k)^2 + (h_2 + \mathcal{A}_{34}\cos k)^2 \\ &+2\big(\mathcal{A}_{13}^2 + \mathcal{A}_{14}^2\big)\cos^2 k + \big(\mathcal{S}_{11}^2 + \mathcal{S}_{33}^2 + 2\mathcal{S}_{13}^2 \\ &+2\mathcal{S}_{14}^2\big)\sin^2 k\big] \end{aligned} \tag{41}$$

and

$$G_k = \det H, \tag{42}$$

with $F_k \geqslant 0$. By computing the characteristic polynomial for H, we can obtain the eigenvalues to be $\{\epsilon_\pm(k), -\epsilon_\pm(k)\}$, where

$$\epsilon_\pm(k) = \sqrt{F_k \pm \sqrt{F_k^2 - G_k}}. \tag{43}$$

Note that for

$$G_k = 0, \tag{44}$$

the eigenvalue ϵ_-, which is also the energy gap between the ground and the first excited state, vanishes, which corresponds to a quantum phase transition. From the eigenvectors, we can find the corresponding new quasiparticles, say, $b_k^\pm$ and $c_k^\pm$ for the positive-energy modes and negative-energy modes, respectively, in terms of which we can express the Hamiltonian H_G as [32]

$$H_G = \sum_{\substack{k>0 \\ s=\pm}} \epsilon_s(k)\big(b^\dagger_{s,k} b_{s,k} - c^\dagger_{s,k} c_{s,k}\big). \tag{45}$$

It is now obvious that the ground state is a filled Fermi sea with particlelike and holelike excitations. Various thermodynamic properties can now be extracted in a straightforward fashion from this expression. For instance, the ground-state energy of the system can be computed to be

$$E_g = -\sum_{k>0}[\epsilon_+(k) + \epsilon_-(k)] = -\sqrt{2}\sum_{k>0}\sqrt{F_k + \sqrt{G_k}}. \tag{46}$$

B. Quantum phase transitions

As mentioned before, the quantum phase transitions can be diagnosed using the gap closing condition given in (44). There may exist a number of conditions over the values of the system parameters involved in (41) for which this condition can be satisfied and each of these conditions will in principle provide a quantum phase transition occurring in the generalized $XY\Gamma$ models for $d = 2$. For the purpose of demonstration, we consider the simplest critical point of the generalized $XY\Gamma$ model, which is the analog of the order-disorder transition in the transverse-field Ising model, which takes place at the vanishing momentum, i.e., $k = 0$. The gap closing condition $G_0 = 0$ can then be solved to get the critical value of the system parameter h as

$$\begin{aligned} h_c = \frac{1}{2\mu}\big[&-\mathcal{A}_{12} - \mu^2\mathcal{A}_{34} \pm \big\{4\mu^2\big(\mathcal{A}_{13}^2 + \mathcal{A}_{14}^2\big) \\ &+ (\mathcal{A}_{12} - \mu^2\mathcal{A}_{34})^2\big\}^{1/2}\big], \end{aligned} \tag{47}$$

where we have defined μ and h via $h_1 \equiv \mu h$ and $h_2 \equiv \frac{h}{\mu}$. Note that h_c is *always* real, regardless of the choice of the values of the other coupling constants.

Motivated by the fact that the derivatives of two-point correlation functions and single-site magnetizations provide signatures of quantum phase transitions via nonanalytic behaviors, we probe the analogous quantities in the generalized $XY\Gamma$ models. The expectation value of $\langle S_a^i\rangle$, which is the analog of $\langle\sigma_a^3\rangle$ of the generalized XY model, is obtained from $\partial E_g/\partial h$ and is plotted as a function of h in Fig. 4(a), where we set $\mu = 2$, and the values of the rest of the coupling constants are set to 1, which leads to $h_c = 0.350\,781$ [from (47)]. At $h = h_c$, variation of $\partial E_g/\partial h$ as a function of h changes from convex to concave, thereby indicating a nonanalytic behavior of $\partial^2 E_g/\partial h^2$, as shown in Fig. 4(b). This susceptibility shows approximately $-\ln|h - h_c|$ divergence near $h = h_c$.

Critical exponents

The analysis above clearly indicates the presence of a quantum critical point in the generalized $XY\Gamma$ model. The next natural question is to extract various critical exponents. For this, let us analyze the behavior of the gap (equal to $2\epsilon_-$) in various limits. At the critical point $h = h_c$, it can be easily checked that ϵ_- vanishes linearly with k when expanded around the critical mode $k = 0$, i.e., $\epsilon_- \propto k$ up to first order in k. This gives the value of the dynamical critical exponent z to be equal to unity. On the other hand, for the critical mode $k = 0$, the gap vanishes as $h - h_c$ so that $\nu z = 1$, which implies that the correlation length exponent ν is also equal to unity. Since the critical exponents match that of the Ising model, this transition is of the Ising universality class, i.e., in the conformal field theory. it is $c = \frac{1}{2}$. Note that the continuum limit of our model is expected to be the free-field theory of d complex fermions (or equivalently $2d$ Majorana fermions) with mass terms controlled by the magnetic field. Consequently, the possible transitions can be understood in this context. As an example, the quantum phase transition that we found in Sec. III B corresponds to the field theory where all fermions except one have been given mass and gapped out.

We reiterate that we have considered the simplest quantum phase transition above. For example, in the analysis of the critical exponents, we could further fine-tune the couplings to obtain different critical exponents. We could also consider a different class of quantum phase transitions where the gaps (43) vanish at finite values of k, which would be the analog of the anisotropy transition occurring in the XY Ising model. We leave the detailed understanding of all the quantum phase transitions to future work.

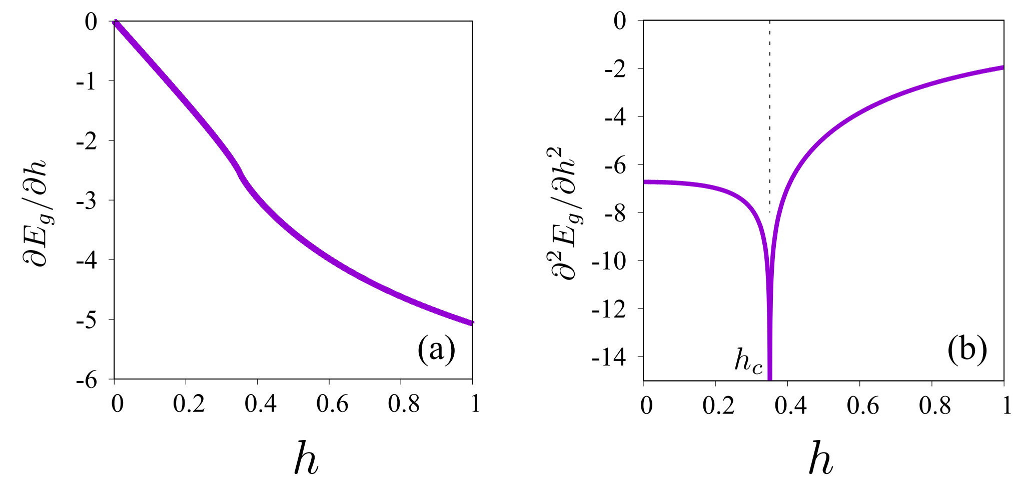

FIG. 4. Variation of (a) $\partial E_g/\partial h$ and (b) $\partial^2 E_g/\partial h^2$ as functions of h, setting $\mu = 2$, $\mathcal{A}_{12} = 1$, $\mathcal{A}_{13} = 1$, $\mathcal{A}_{14} = 1$, $\mathcal{A}_{34} = 1$, $\mathcal{S}_{11} = 1$, $\mathcal{S}_{33} = 1$, $\mathcal{S}_{13} = 1$, and $\mathcal{S}_{14} = 1$. The quantity $\partial^2 E_g/\partial h^2$ exhibits a divergence at $h_c = 0.350\,781$, indicating a quantum phase transition.

IV. CONCLUSION

In this work we have presented an exactly solvable generalization of the class of 1D quantum XY and Ising-like models by associating higher-dimensional Hilbert spaces with each lattice site, via replacing the Pauli matrices with gamma matrices. Using the Jordan-Wigner transformation, we fermionized the model and subsequently solved it. We illustrated an Ising-like quantum phase transition in the model for $d = 2$ with a specific set of system parameters.

We end with a discussion of possible future directions. Within the class of models explored in this paper, the $d = 2$ case provides a set of exactly solvable models with a 16-dimensional parameter space. A thorough exploration of this space may reveal more quantum phase transitions different from the one reported in this work. In addition, note that the 1D Jordan-Wigner transformations are useful in higher-dimensional models too, for example, the 2D Kitaev model on a honeycomb lattice [48,49] (see also [50]), which can be solved via a 1D Jordan-Wigner transformation on a special path. Since Γ-matrix generalizations have been proposed also for higher-dimensional models and more general lattices (see, for example, [38–41], and also [42,43] for a more general approach), it would be interesting to explore whether the generalizations we have described using the Jordan-Wigner transformation are applicable in such higher-dimensional contexts.

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APPENDIX A: FERMIONIZATION OF HAMILTONIAN

In this Appendix we give more details on the fermionization of the Hamiltonian. Though we are interested in the thermodynamic limit, we will work at finite N in this Appendix. Recall that the Γ matrices constructing the generalized $XY\Gamma$ models satisfy the periodicity condition

$$\Gamma^{\mu}_{N+1} = \Gamma^{\mu}_{1}\ \forall \mu = 1, 2, \dots, 2d. \tag{A1}$$

For the Jordan-Wigner fermions defined in (15), this translates to

$$\chi^{\mu}_{N+1} = W\chi^{\mu}_{1}, \tag{A2}$$

where $W \equiv \prod_{a=1}^{N}\Gamma_a^{2d+1}$. It is easy to see that $W^2 = 1$, and hence its eigenvalues (say, w) are ± 1. Consequently, the generalized $XY\Gamma$ model, when written in terms of fermions, consists of two sectors

$$\begin{aligned} \chi^{\mu}_{N+1} &= -\chi_1 \quad \text{for}\ \ w = -1, \\ \chi^{\mu}_{N+1} &= \chi_1 \quad \text{for}\ \ w = +1, \end{aligned} \tag{A3}$$

i.e., a sector with periodic fermions and another with antiperiodic fermions. To see the sectors more clearly, let us define $H_G^{\pm}$ via

$$H_G = \frac{1+W}{2}H_G^{+} + \frac{1-W}{2}H_G^{-}. \tag{A4}$$

The Hamiltonians $H_G^{\pm}$ written in terms of fermions become

$$H_G^{\pm} = -i\sum_{\mu,\nu,a} J_{\mu\nu}\chi_a^{\mu}\chi_{a+1}^{\nu} + i\sum_{i,a} h_i\chi_a^{2i-1}\chi_a^{2i}. \tag{A5}$$

The Hamiltonian in each sector is thus quadratic in terms of fermions and hence simple to solve; we just need to move to the Fourier basis. However, the periodicity condition enforces that the momentum modes are either integers or half integers. More precisely, we have[5]

$$\chi_a^{\mu} = \begin{cases} \frac{1}{\sqrt{N}}\sum_{q=-(N-1)/2}^{(N-1)/2} e^{i2\pi qa/N}\chi_q^{\mu} & \text{for}\ w = 1 \\ \frac{1}{\sqrt{N}}\sum_{q=-(N-2)/2}^{N/2} e^{i2\pi qa/N}\chi_q^{\mu} & \text{for}\ w = -1. \end{cases} \tag{A6}$$

Here we assume that N is odd, although the case of even N can also be done analogously and does not change the

[5]Note that in the main text we label the momentum modes by k, which is continuous, whereas here the label is q, which is an integer.

conclusions. The modes satisfy $\chi_q^\mu = \chi_{q+N}^\mu$, i.e., they sit on a periodic lattice. Let us denote the momentum lattice of the w sector by $\mathbb{Z}^{(w)}$. The modes which satisfy $\chi_q^\mu = \chi_{-q}^\mu$ play a special role in what follows and we denote this mode by q_0, i.e., $q_0 = 0$ for the $w = 1$ sector and $q_0 = \frac{N}{2}$ for the $w = -1$ sector.

It is easy to check that

$$\chi_q^{\mu\dagger} = \chi_{-q}^\mu, \quad \{\chi_q^{\mu\dagger}, \chi_{q'}^\nu\} = 2\delta^{\mu\nu}\delta_{q,q'}. \tag{A7}$$

Hence χ_q^μ can be treated as a complex fermion. Note, however, that the $\chi_{q_0}^\mu$ fermion is still a Majorana fermion. To solve the fermionic Hamiltonian, it is useful to note that for any b (the sector is denoted by $\pm$ below),

$$\begin{aligned}\sum_a \chi_a^\mu \chi_{a+b}^\nu &= \sum_{q\in\mathbb{Z}_+^{(w)}} \left[\left(e^{2\pi iqb/N}\chi_{-q}^\mu\chi_q^\nu - e^{-2\pi iqb/N}\chi_{-q}^\nu\chi_q^\mu\right)\right.\\ &\quad \left.+ \frac{1}{2}\left(e^{2\pi iq_0b/N}\chi_{q_0}^\mu\chi_{q_0}^\nu - e^{-2\pi iq_0b/N}\chi_{q_0}^\nu\chi_{q_0}^\mu\right)\right] + \kappa\delta^{\mu\nu}.\end{aligned} \tag{A8}$$

The constant κ is not relevant in the discussion below, since it just shifts the Hamiltonian by a constant. Thus, the Hamiltonian given in (A5) becomes

$$\begin{aligned}H_G^w &= \sum_{q\in\mathbb{Z}_+^{(w)}} \left(i\cos\frac{2\pi q}{N}\mathcal{A}_{\mu\nu} + \sin\frac{2\pi q}{N}\mathcal{S}_{\mu\nu}\right)\chi_{-q}^\mu\chi_q^\nu\\ &\quad + \frac{i}{2}\cos\frac{2\pi q_0}{N}\mathcal{A}_{\mu\nu}\chi_{q_0}^\mu\chi_{q_0}^\nu\\ &\quad + ih_i\sum_{q\in\mathbb{Z}_+^{(w)}}\left(\chi_{-q}^{2i-1}\chi_q^{2i} - \chi_{-q}^{2i}\chi_q^{2i-1}\right)\\ &\quad + \frac{ih_i}{2}\left(\chi_{q_0}^{2i-1}\chi_{q_0}^{2i} - \chi_{q_0}^{2i}\chi_{q_0}^{2i-1}\right).\end{aligned} \tag{A9}$$

Here we have used $J_{\mu\nu} = \frac{\mathcal{S}_{\mu\nu} - \mathcal{A}_{\mu\nu}}{2}$. The thermodynamic limit is $N \to \infty$, with $k = \frac{2\pi q}{N}$ kept fixed, and hence we can work with the Hamiltonian

$$\begin{aligned}H_G^w &= \sum_{\mu,\nu,k>0} [i\cos(k)\mathcal{A}_{\mu\nu} + \sin(k)\mathcal{S}_{\mu\nu}]\chi_{-k}^\mu\chi_k^\nu\\ &\quad + i\sum_{i,k>0} h_i\left(\chi_{-k}^{2i-1}\chi_k^{2i} - \chi_{-k}^{2i}\chi_k^{2i-1}\right),\end{aligned} \tag{A10}$$

where we have let $\chi_q^\mu \to \chi_k^\mu$. Also, the distinction between $w = \pm 1$ sectors goes away in this limit.

APPENDIX B: MAJORANA TO COMPLEX FERMIONS

In this Appendix we rewrite the equations we obtained in terms of complex fermions, instead of Majorana fermions. Let us define the complex fermions c_a^i via

$$c_a^i = \frac{\chi_a^{2i-1} + i\chi_a^{2i}}{\sqrt{2}}, \quad c_a^{i\,\dagger} = \frac{\chi_a^{2i-1} - i\chi_a^{2i}}{\sqrt{2}}. \tag{B1}$$

The relation between complex fermions and the matrices Γ_a^μ can be read off from the Jordan-Wigner transformation given in (15). We can also write the Fourier modes of complex fermions in terms of Fourier modes of Majorana fermions as

$$c_k^i = \frac{\chi_k^{2i-1} + i\chi_k^{2i}}{\sqrt{2}}, \quad c_k^{i\,\dagger} = \frac{\chi_{-k}^{2i-1} - i\chi_{-k}^{2i}}{\sqrt{2}}. \tag{B2}$$

Defining $c_k^{i+} = c_k^{i\,\dagger}$ and $c_k^{i-} = c_k^i$, we can rewrite the Hamiltonian in (22) in terms of the complex fermion modes as

$$H_G = \frac{1}{2}\sum_{\substack{i,j=1\\k>0\\s=\pm\\\tilde{s}=\pm}}^{d} \kappa_{ij}^{s\tilde{s}} c_{sk}^{is} c_{-\tilde{s}k}^{j\tilde{s}} + \sum_{i,k>0}^{d} h_i\left\{c_k^{i\dagger}c_k^i - c_{-k}^i c_{-k}^{i\dagger}\right\}, \tag{B3}$$

where

$$\kappa_{ij}^{s\tilde{s}} = \kappa_{2i-1,2j-1} + i\tilde{s}\kappa_{2i-1,2j} + is\kappa_{2i,2j-1} - s\tilde{s}\kappa_{2i,2j}, \tag{B4}$$

with

$$\kappa_{\mu\nu} = i\mathcal{A}_{\mu\nu}\cos k + \mathcal{S}_{\mu\nu}\sin k. \tag{B5}$$

APPENDIX C: A MORE GENERIC HAMILTONIAN

We can actually work with the more general Hamiltonian

$$H_G = -i\sum_a\sum_{\mu,\nu} J_{\mu\nu}\Gamma_a^\mu\Gamma_a^{2d+1}\Gamma_{a+1}^\nu + i\sum_a\sum_{\mu\nu} h_{\mu\nu}\Gamma_a^\mu\Gamma_a^\nu. \tag{C1}$$

Without loss of generality, we can choose $h_{\mu\nu} = -h_{\nu\mu}$ since $\{\Gamma_a^\mu, \Gamma_a^\nu\} = 2\delta^{\mu\nu}$. We can easily follow through the steps given in Sec. II and see that this gives rise to a Hamiltonian which is quadratic in fermions. We will see below that the number of independent coupling constants is smaller than what is expected by looking at (C1).

Working with a rotated set of Γ matrices given by

$$\tilde{\Gamma}_a^\mu \equiv \sum_\nu R_{\mu\nu}\Gamma_a^\nu, \tag{C2}$$

where $R_{\mu\nu}$ are matrix elements of a real $2d \times 2d$ rotation matrix R satisfying $R^T R = I$, it is easy to verify that they satisfy the same algebra as in (7). The Hamiltonian (C1) written in terms of the rotated $\tilde{\Gamma}$ matrices still retains its form, but with the coupling constants $\tilde{J}_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$. Labeling the matrix formed by $J_{\mu\nu}$ as J and so on, it is easy to see that

$$\tilde{J} = R^T J R, \quad \tilde{h} = R^T h R. \tag{C3}$$

We can always choose R such that $h_{\mu\nu}$ can be put in a block-diagonal form [51]

$$\begin{pmatrix} 0 & h_1 & 0 & 0 & 0 & \cdots\\ -h_1 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & h_2 & 0 & \cdots\\ 0 & 0 & -h_2 & 0 & 0 & \cdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{C4}$$

for some constants $h_1, h_2, \ldots, h_d$. This is the form that we used in the main text (17). However, the analysis above shows that we have not exhausted all the redefinitions yet; a further transformation by an R matrix of the form

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & 0 & \cdots \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 & 0 & \cdots \\ 0 & 0 & -\sin\theta_2 & \cos\theta_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{C5}$$

keeps the form of h in (C4) invariant. This freedom can be used to further simplify $J_{\mu\nu}$. Defining the symmetric and antisymmetric combinations of $J_{\mu\nu}$ via

$$\mathcal{S}_{\mu\nu} = J_{\mu\nu} + J_{\nu\mu}, \quad \mathcal{A}_{\mu\nu} = J_{\nu\mu} - J_{\mu\nu}, \tag{C6}$$

we can use the above-mentioned freedom to set $\mathcal{S}_{2i-1,2i} = 0$. This reduces the number of independent couplings to $4d^2$.

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