Microscopic theory of the Curzon-Ahlborn heat engine based on a Brownian particle

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(Received 12 August 2021; revised 29 April 2022; accepted 6 July 2022; published 3 August 2022)

The Curzon-Ahlborn (CA) efficiency, as the efficiency at the maximum power (EMP) of the endoreversible Carnot engine, has significant impact on finite-time thermodynamics. However, the CA engine is based on many assumptions. In the past few decades, although a lot of efforts have been made, a microscopic theory of the CA engine is still lacking. By adopting the method of the stochastic differential equation of energy, we formulate a microscopic theory of the CA engine realized with a highly underdamped Brownian particle in a class of nonharmonic potentials. This theory gives microscopic interpretation of all assumptions made by Curzon and Ahlborn. In other words, we find a microscopic counterpart of the CA engine in stochastic thermodynamics. Also, based on this theory, we derive the explicit expression of the protocol associated with the maximum power for any given efficiency, and we obtain analytical results of the power and the efficiency statistics for the Brownian CA engine. Our research brings new perspectives to experimental studies of finite-time microscopic heat engines featured with fluctuations.

DOI: 10.1103/PhysRevE.106.024105

I. INTRODUCTION

For practical heat engines, not only the efficiency but also the power characterizes the performance. Optimizing the power and the efficiency of heat-engine cycles is one of the goals in the study of finite-time thermodynamics [1,2]. Compared to the Carnot efficiency achieved in infinite time [3] (but see Ref. [4]), the efficiency at the maximum power (EMP) attracts a lot of attention in the studies of finite-time heat engines. The early studies [5–12] of the endoreversible Carnot engine concluded that the EMP is the well-known Curzon-Ahlborn (CA) efficiency

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}}, \tag{1}$$

where T_C and T_H denote the temperatures of the cold and the hot heat baths. Like the Carnot efficiency, the CA efficiency depends only on the temperatures of the two heat reservoirs, independent of any other characteristics of the heat engine. The CA efficiency aroused a lot of attention and led to many following-up researches (see, for example, Refs. [9,13–15]). It has become a paradigmatic result in the study of thermodynamic optimization in the framework of finite-time and stochastic thermodynamics. In addition, the CA efficiency is relevant to many practical thermal machines [16].

The CA efficiency as the EMP has also been derived in some different setups [17]. For example, in Ref. [18] it is shown that CA efficiency is a result which can be obtained in the well-founded linear irreversible thermodynamics. In Ref. [16], the CA efficiency, as the EMP, is derived in the symmetric low-dissipation regime, where the irreversible entropy production is assumed inversely proportional to the period of a cycle. Such a $1/\tau$-scaling, as the first-order finite-time correction to the quasi-static driving [19], has been recently verified in the experiment of finite-time isothermal compression of dry air [20,21]. These studies confirm the validity of the CA efficiency as the EMP in many heat engine models. Meanwhile, in some other models, the EMP deviates from the CA efficiency [14,16,18,19,22–43].

The original derivation of the CA efficiency [5–8] is based on a lot of assumptions, such as the endoreversible assumption and the assumption of constant temperature difference. But how reliable are those assumptions made by Curzon and Ahlborn remains unclear due to the lack of a microscopic theory of the CA engine. Also, due to this lack, the control scheme of the work parameter which is essential to construct the optimal cycle cannot be determined (except for the harmonic potential [15]), neither can the work and heat statistics. In the past few decades, a lot of efforts have been made to seek a microscopic interpretation of the CA engine, but were unsuccessful.

In this article, we realize a microscopic CA engine with a highly underdamped Brownian particle [15,44–52] in a time-dependent potential as the working substance. By adopting the method of stochastic differential equation of energy [53,54], we give microscopic interpretation of all assumptions made by Curzon and Ahlborn, including the endoreversibility, Newton's cooling law and the constant temperature difference.

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Thus, we find a microscopic counterpart of the CA engine in stochastic thermodynamics. Furthermore, this microscopic theory allows us to determine the control scheme of the work parameter of the Brownian CA engine and study the fluctuations of the power and the efficiency of the Brownian CA engine. Our study demonstrates that when downsizing the working substance to a single Brownian particle, results about the average power and efficiency of the CA engine remain valid, but the fluctuations become prominent.

This article is organized as follows. In Sec. II, we introduce the stochastic differential equation of energy for a highly underdamped Brownian particle. In Sec. III, we realize the CA engine with a highly underdamped Brownian particle. All preconditions assumed by Curzon and Ahlborn are derived from microscopic dynamics. In Sec. IV, we derive the analytical expression of the joint generating function of work and heat in an isothermal process, and study the statistics of the power and the efficiency of the Brownian CA engine. Summary and discussions are given in Sec. V.

II. STOCHASTIC DIFFERENTIAL EQUATION OF ENERGY FOR A HIGHLY UNDERDAMPED BROWNIAN PARTICLE

The working substance of the engine is modeled as a Brownian particle [46,48,55–66] constrained in a controllable potential $\mathcal{U}(x,t) = k(t)x^{2n}/(2n)$ with the control parameter $k(t)$ and a positive integer n. In the isothermal expansion (compression) process, the control parameter $k(t)$ is varied when the engine is in contact with the heat bath at temperature T_b. The motion of the particle with mass m is governed by the complete Langevin equation

$$\ddot{x} + \gamma\dot{x} + \frac{k(t)}{m}x^{2n-1} = \frac{1}{m}\xi(t), \tag{2}$$

where the random force $\xi(t)$ represents a Gaussian white noise satisfying $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2m\gamma T_\mathrm{b}\delta(t-t')$ with the friction coefficient γ. Throughout the text the Boltzmann constant is set to be $k_B = 1$.

We consider the highly underdamped regime $\tau_p \ll \gamma^{-1}$ and slow external driving $\tau_p \ll k/\dot{k}$, where τ_p is the period of the unperturbed motion of the particle, e.g., $\tau_p = 2\pi\sqrt{m/k}$ for a harmonic oscillator ($n = 1$). Under these two conditions, the variation of the stochastic energy of the particle within a period is relatively small, which allows us to study the dynamics of the stochastic energy $E = m\dot{x}^2/2 + \mathcal{U}(x,t)$. Based on Ito's lemma and Virial theorem, the equation of motion is expressed as the stochastic differential equation of the energy E [53,54],

$$dE = \frac{\dot{\lambda}}{\lambda}Edt - \Gamma\left(E - \frac{f_n T_\mathrm{b}}{2}\right)dt + \sqrt{2\Gamma T_\mathrm{b}E}dB_t. \tag{3}$$

where the work parameter is rewritten into $\lambda(t) = k(t)^{1/(n+1)}$ with the increment dB_t of the Wiener process, the effective friction coefficient $\Gamma = 2n\gamma/(n+1)$, and the effective degrees of freedom $f_n = 1 + 1/n$. The kinetic energy and the potential energy of the particle obey a generalized equipartition theorem. The increment of trajectory work for this system is

$$dW = \frac{\dot{\lambda}}{\lambda}Edt. \tag{4}$$

The trajectory heat is obtained from the first law of thermodynamics as

$$dQ = -\Gamma\left(E - \frac{f_n T_\mathrm{b}}{2}\right)dt + \sqrt{2\Gamma T_\mathrm{b}E}dB_t. \tag{5}$$

The Fokker-Planck equation associated with Eq. (3) can be solved explicitly [54]. During the dynamical evolution process, the system remains in a Maxwell-Boltzmann distribution in the energy space and thus can be described by an effective temperature $\theta(t)$ as

$$P(E,t) = \frac{e^{-E/\theta(t)}}{\Gamma(f_n/2)}\frac{E^{f_n/2-1}}{\theta(t)^{f_n/2}}, \tag{6}$$

where $\Gamma(x) = \int_0^\infty e^{-y}y^{x-1}dy$ is the Euler gamma function. Notice that Eq. (6) leads to the endoreversibility, which is usually assumed in previous studies relevant to the CA engine, but is derived as a consequence of the equation of motion in our setup. The ensemble average of the energy is $\langle E(t)\rangle = f_n\theta(t)/2$ with the effective temperature $\theta(t)$ governed by

$$\dot{\theta}(t) = \frac{\dot{\lambda}}{\lambda}\theta(t) - \Gamma[\theta(t) - T_\mathrm{b}]. \tag{7}$$

We emphasize that the left-hand side of Eq. (7) corresponds to the time derivative of the average energy up to a factor $f_n/2$. The two terms on the right-hand side correspond to the average work flux and heat flux, respectively. The average heat flux satisfies Newton's cooling law, which is also derived as a consequence of the equation of motion. The effective friction coefficient Γ, as a cooling rate, is independent of the work parameter λ. We would like to point out that a similar equation of motion for the effective temperature $\theta(t)$ has been obtained previously for the ideal gas as the working substance [13]. However, their derivation relies on several assumptions, for example, the equation of state of ideal gas, the phenomenological Newton's cooling law and the endoreversible assumption. On the contrary, the results presented here are all derived from the microscopic dynamics, and are capable of describing microscopic systems featured with fluctuations.

III. REALIZATION OF CURZON-AHLBORN ENGINE BASED ON A BROWNIAN PARTICLE

With the model introduced above, we study the EMP of such a microscopic Brownian engine and formulate a microscopic theory of the CA engine. To construct a finite-time Carnot cycle, two heat baths at temperatures T_C and T_H are required in the isothermal compression and the isothermal expansion processes. The (effective) friction coefficients γ_C and γ_H (Γ_C and Γ_H) may be different in the two processes. Based on Curzon and Ahlborn's derivation [8], we summarize the preconditions of the CA engine as follows

(i) Endoreversibility [67]. The state of the working substance of the engine can be described by an effective temperature.

(ii) Newton's cooling law (or linear heat transfer law). The heat flux between the working substance and the heat bath is proportional to the temperature difference.

(iii) Constant temperature difference. During the isothermal expansion (compression) process, the effective

temperature of the working substance remains at a constant value θ_H (θ_C) different from that of the heat bath T_H (T_C).

(iv) Internal reversible cycle. All irreversibilities are associated with the heat exchange between the working substance and the heat baths while the adiabatic processes remain reversible [17].

(v) Constant heat capacity [68]and cooling rate [69]. Both the heat capacity of the working substance and the cooling rate are independent of the temperature and the work parameters.

According to Eqs. (6) and (7), our setup fulfills requirements (i) and (ii). When the precondition (v) is satisfied, the optimal protocol of finite-time isothermal processes becomes the exponential protocol [13,15,49,54]. Here the precondition (v) is guaranteed in the highly underdamped regime. With the exponential protocols, the optimal cycle corresponding to the maximum power is exactly the CA cycle satisfying preconditions (iii) and (iv). The heat engine operates like a reversible Carnot engine between two virtual heat baths at temperatures θ_C and θ_H, respectively.

We attempt to construct and optimize a finite-time Brownian Carnot engine with the microscopic model. A Carnot cycle consists of two isothermal processes and two adiabatic processes. According to Refs. [13,15,49,54], we know that for a given duration of time and the initial and the final values of the work parameter, the optimal protocol $\lambda(t)$ of an isothermal expansion (compression) process that maximizes the work output (minimizes the work input) in the highly underdamped regime is an exponential function of time. The adiabatic processes are the instantaneous jumps at the beginning and the end of the isothermal processes. The effective temperature of the working substance is kept at a constant except for sudden jump processes. The jump-exponential-jump form of the optimal work protocol $\lambda(t)$ and the constant effective temperature are general properties of the optimal control, and are independent of specific settings of the control time and the initial and final values of the work parameter, as long as we maximize the work output. Based on the equation of motion of the effective temperature [Eq. (7)] and the definition of trajectory work and heat, we can optimize the average power of the finite-time Brownian engine. In Fig. 1(a), we plot the cycle diagram of the Brownian CA engine. To construct a closed CA cycle, the values of work parameter at the end of each process in Fig. 1(a) satisfy

$$\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_4} = r, \tag{8}$$

with the compression ratio r. The four processes of the CA cycle are illustrated as follows:

(I) Isothermal compression. The working substance is in contact with the cold heat bath. Initiated from λ_1 at $t = 0$, the work parameter is varied exponentially with time $\lambda(t) = \lambda_1(\lambda_2/\lambda_1)^{t/\tau_C}$, where τ_C is the duration of the process. From the protocol, it can be found that the effective temperature of the working substance remains at a constant during the process:

$$\theta_C = \frac{\tau_C \Gamma_C}{\tau_C \Gamma_C - \ln r} T_C. \tag{9}$$

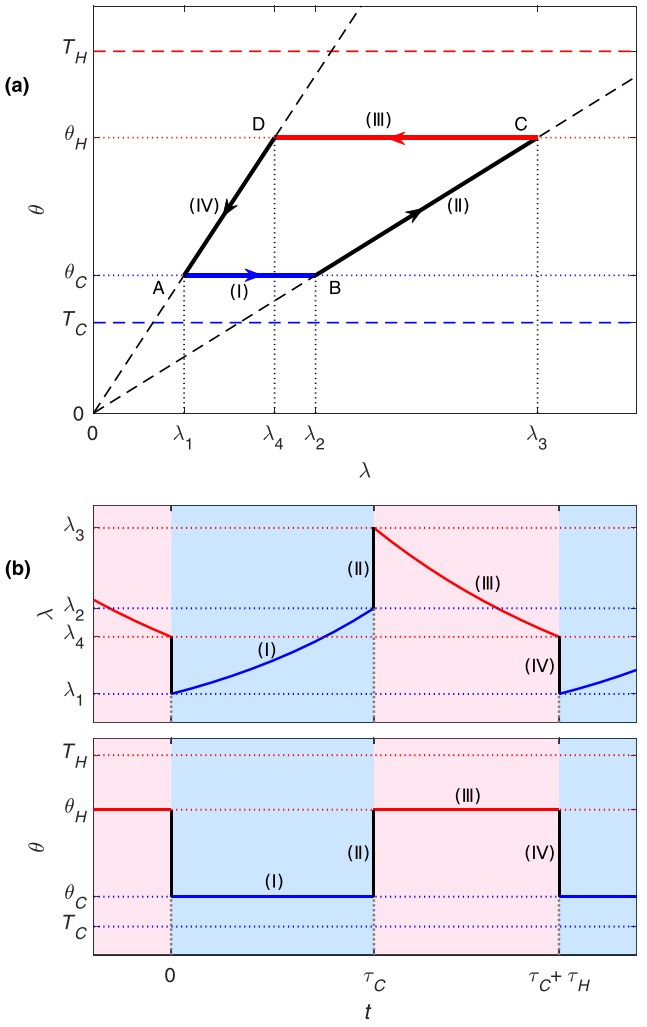

FIG. 1. CA cycle based on a Brownian particle. (a) The cycle diagram in the space of the work parameter λ and the effective temperature θ. In the isothermal expansion (compression) process, the working substance remains at a constant effective temperature $\theta_H(\theta_C)$, which is different from the temperature $T_H(T_C)$ of the hot (cold) heat bath. In the two adiabatic processes, the effective temperature θ of the working substance is proportional to the work parameter λ. (b) The control scheme of the work parameter $\lambda(t)$ and the evolution of the effective temperature $\theta(t)$ in a finite-time cycle. The work parameter λ is varied exponentially with time in an isothermal process, and is quenched abruptly in an adiabatic process.

The heat released to the cold heat bath during the isothermal compression process is

$$\begin{aligned} -\langle Q_C \rangle &= \frac{f_n}{2} \Gamma_C (\theta_C - T_C) \tau_C \\ &= \frac{f_n}{2} \ln r \frac{\tau_C \Gamma_C}{\tau_C \Gamma_C - \ln r} T_C. \end{aligned} \tag{10}$$

(II) Adiabatic compression. The work parameter $\lambda(t)$ is quenched instantaneously from λ_2 to λ_3 with the effective temperature $\theta(t)$ changing from θ_C to θ_H accordingly. When the timescale of varying the work parameter is much shorter

than that of the heat dissipation, Eq. (7) becomes $\dot{\theta}(t) = \dot{\lambda}\theta(t)/\lambda$, which leads to $\theta_H/\theta_C = \lambda_3/\lambda_2$.

(III) Isothermal expansion. The working substance is in contact with the hot heat bath. The work parameter is varied exponentially with time $\lambda(t) = \lambda_3(\lambda_4/\lambda_3)^{(t-\tau_C)/\tau_H}$, and the working substance remains at a constant effective temperature,

$$\theta_H = \frac{\tau_H\Gamma_H}{\tau_H\Gamma_H + \ln r}T_H, \tag{11}$$

where τ_H is the duration of the process. The heat absorbed from the hot heat bath during the isothermal expansion process is

$$\begin{aligned}\langle Q_H\rangle &= \frac{f_n}{2}\Gamma_H(T_H - \theta_H)\tau_H \\ &= \frac{f_n}{2}\ln r\frac{\tau_H\Gamma_H}{\tau_H\Gamma_H + \ln r}T_H.\end{aligned} \tag{12}$$

(IV) Adiabatic expansion. Finally, the work parameter is quenched from λ_4 to the initial value λ_1 instantaneously with the effective temperature changing from θ_H to θ_C accordingly, satisfying $\theta_H/\theta_C = \lambda_4/\lambda_1$.

The control scheme of the work parameter λ and the evolution of the effective temperature θ in a finite-time cycle are illustrated in Fig. 1(b). For long operation time $\tau_i \gg \ln r/\Gamma_i$, the system is in the low-dissipation regime, and the finite-time correction to the heat exchange is of the order τ_i^{-1}, as can be seen from Eqs. (10) and (12).

Combing Eqs. (9)–(12), it is straightforward to verify the precondition (iv) that the entropy change of the working substance after a cycle is zero $\Delta S = \langle Q_H\rangle/\theta_H + \langle Q_C\rangle/\theta_C = 0$. Therefore, the microscopic dynamics of the model, together with the explicit control scheme $\lambda(t)$, constitutes a microscopic theory of the CA engine based on a Brownian particle.

The net work of a full cycle is $-\langle W\rangle = \langle Q_H\rangle + \langle Q_C\rangle$, and the average power and the thermodynamic efficiency follow as $\overline{P} := -\langle W\rangle/(\tau_H + \tau_C)$ and $\eta := -\langle W\rangle/\langle Q_H\rangle$, which are explicitly

$$\overline{P} = \frac{f_n\ln r}{2(\tau_H + \tau_C)}\left(\frac{\tau_H\Gamma_H T_H}{\tau_H\Gamma_H + \ln r} - \frac{\tau_C\Gamma_C T_C}{\tau_C\Gamma_C - \ln r}\right), \tag{13}$$

and

$$\eta = 1 - \frac{1 + (\tau_H\Gamma_H)^{-1}\ln r}{1 - (\tau_C\Gamma_C)^{-1}\ln r}\frac{T_C}{T_H}. \tag{14}$$

To achieve the maximum power, we first fix r and optimize the power over τ_H and τ_C. The maximum power is obtained as

$$\overline{P}_{\max} = \frac{f_n\Gamma_C\Gamma_H(\sqrt{T_H} - \sqrt{T_C})^2}{2(\sqrt{\Gamma_H} + \sqrt{\Gamma_C})^2}, \tag{15}$$

with the corresponding optimal duration of the two isothermal processes

$$\begin{aligned}\tau_H^{\max} &= \frac{\ln r(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C})}{\Gamma_H\sqrt{\Gamma_C}(\sqrt{T_H} - \sqrt{T_C})}, \\ \tau_C^{\max} &= \frac{\ln r(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C})}{\Gamma_C\sqrt{\Gamma_H}(\sqrt{T_H} - \sqrt{T_C})}.\end{aligned} \tag{16}$$

It is straightforward to see that the EMP of the Brownian engine in the highly underdamped regime is the CA efficiency

$$\eta_{\text{EMP}} = 1 - \sqrt{\frac{T_C}{T_H}} = \eta_{CA}, \tag{17}$$

as we expect. Due to fact that the cooling rates are constant, both the maximum power and the EMP are independent of r. Hence $\overline{P}_{\max}$ is also the global maximum power.

Our optimization with respect to the durations of the isothermal processes τ_H and τ_C is mathematically equivalent to the optimization of the effective temperatures θ_H and θ_C as done in Ref. [8], since the effective temperatures and the durations of the isothermal processes are related by Eqs. (9) and (11) under fixed compression ratio r. Experimentally, the control of effective temperatures is realized by tuning the durations of the isothermal processes.

Based on Eqs. (14) and (15), we derive the trade-off relation between power and efficiency in Appendix A. Compared to previous studies [26,70–74], our trade-off relation is tight and is shown to be reachable with the explicit control scheme of the work parameter $\lambda(t)$. It is worth mentioning that the same trade-off relation was obtained in Ref. [75]. Also, the same trade-off relation as well as the control scheme for the harmonic potential was obtained in Ref. [15]. As a generalization of Ref. [15], our results are valid for a Brownian particle in a class of nonharmonic potentials.

IV. POWER AND EFFICIENCY STATISTICS OF THE BROWNIAN CURZON-AHLBORN ENGINE

For a microscopic Brownian engine, average values are insufficient to characterize the performance. Fluctuations are nonnegligible [76]. To evaluate the performance of a finite-time heat engine, we need to quantify the extracted work and heat absorbed from the hot bath in one heat-engine cycle. For the model introduced above, we can derive the analytical results of the joint generating function of work and heat $I(u,s) := \langle e^{uQ+sW}\rangle$ by generalizing the techniques in Ref. [77]. The final result is

$$I(u,s) = \left[\frac{1 + u\tilde{\psi}(\tau)}{1 + u\tilde{\psi}_0}\right]^{\frac{f_n}{2}} e^{\frac{f_n(s-u)}{2}\int_0^\tau \frac{\dot{\lambda}(t)}{\lambda(t)}\psi(t)dt}, \tag{18}$$

where τ is the time duration of the process, and the temperature-like variable $\psi(t)$ satisfies

$$\frac{d\psi}{dt} = \frac{\dot{\lambda}}{\lambda}\psi - \Gamma(\psi - T_{\text{b}}) + (s-u)\frac{\dot{\lambda}}{\lambda}\psi^2, \tag{19}$$

with the initial condition $\psi(0) = \tilde{\psi}_0/(1 + u\tilde{\psi}_0)$. The initial value $\tilde{\psi}_0$ is either set as the initial temperature θ_0 or obtained from the previous process. A shifted temperature-like variable is defined as $\tilde{\psi}(t) := \psi(t)/[1 - u\psi(t)]$, whose value $\tilde{\psi}(\tau)$ at the end of this process is used as the initial value $\tilde{\psi}_0$ of the subsequent process. Detailed derivations to Eqs. (18) and (19) are left in Appendix B.

Based on the microscopic theory, especially the joint generating function of work and heat $I(u,s)$ and the control scheme $\lambda(t)$ of the full cycle, we can further study the fluctuations of the power and the efficiency [76–82] together with the fluctuation theorems [83,84] of the finite-time Brownian

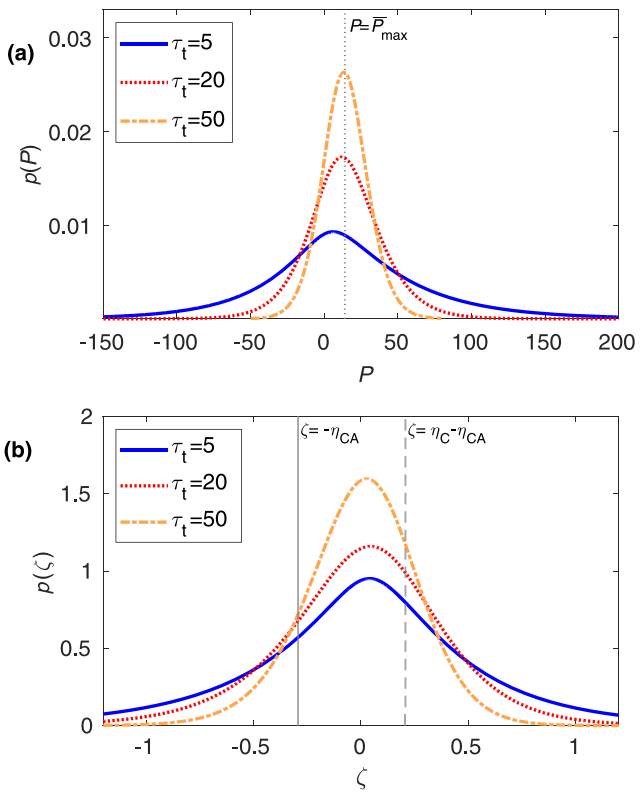

FIG. 2. Distribution of the power (a) and the fluctuating efficiency deviation (b) of a Brownian CA engine for three different periods $\tau_t = 5, 20, 50$. (a) The vertical dotted line indicates the average power. (b) The vertical solid and dashed lines correspond to efficiency $\eta = 0$ and $\eta = \eta_C$ (Carnot efficiency) respectively. The parameters are chosen as $T_C = 300$, $T_H = 600$, $\Gamma_C = 1$, $\Gamma_H = 1.2$.

Carnot engine. Specifically, we calculate the distribution $p(P)$ of the fluctuating power $P := -W/(\tau_C + \tau_H)$ and the distribution $p(\zeta)$ of the fluctuating efficiency deviation $\zeta := -(W + \eta Q_H)/\langle Q_H \rangle$ from the generating function $I_{\rm cycle}(u_H, u_C, s) = \langle e^{u_H Q_H + u_C Q_C + sW} \rangle$ (derivations are left in Appendix B) of a whole cycle. Please note that, instead of the stochastic efficiency $-W/Q_H$, we use the fluctuating efficiency deviation ζ to characterize the deviation from the thermodynamic efficiency η. It is straightforward to see that $\langle \zeta \rangle = 0$.

As a special case, we plot the distributions of the power and the efficiency of the Brownian CA engine in Fig. 2, where $\eta = \eta_{CA}$, $\tau_C = \tau_C^{\rm max}$, $\tau_H = \tau_H^{\rm max}$. Due to the fluctuation, the power can be negative or much larger than the average power. Similarly, the efficiency can be negative or larger than Carnot efficiency. From the analytical results of the joint generating function $I_{\rm cycle}(u_H, u_C, s)$, the leading-order contributions to the variance of the power and the fluctuating efficiency deviation are obtained as

$$\mathrm{Var}(P) \approx \frac{4\overline{P}_{\rm max}^2}{f_n \eta_{CA}^2} \frac{[(1-\eta_{CA})^2 + 1/\delta](1+\delta)}{\sqrt{\Gamma_C \Gamma_H}} \frac{1}{\tau_t}, \tag{20}$$

$$\mathrm{Var}(\zeta) \approx \frac{4(1-\eta_{CA})^2}{f_n \sqrt{\Gamma_C \Gamma_H}} \frac{(1+\delta)^2}{\delta} \frac{1}{\tau_t}, \tag{21}$$

where $\delta = \sqrt{\Gamma_H T_H/(\Gamma_C T_C)}$. The strict expressions of $\mathrm{Var}(P)$ and $\mathrm{Var}(\zeta)$ are given in Eqs. (B58) and (B59). For both the power and the fluctuating efficiency deviation, their variances decrease inversely with the duration of the cycle $\tau_t := \tau_C^{\rm max} + \tau_H^{\rm max}$.

V. SUMMARY AND DISCUSSIONS

We realize the Curzon-Ahlborn heat engine with a Brownian particle in the highly underdamped regime. By adopting the method of stochastic differential equation of energy, we formulate a microscopic theory of the CA engine based on this model. This theory gives microscopic interpretation of all assumptions of the CA engine including the endoreversibility, Newton's cooling law and the constant temperature difference. Hence, we find a microscopic counterpart of the CA engine in stochastic thermodynamics.

One well-known model of the microscopic heat engine is the overdamped Brownian engine [14,85]. Nevertheless, in the overdamped regime, one cannot reproduce the preconditions of the CA engine. The effective temperature is only defined for the potential energy when neglecting the kinetic energy. In fact, the kinetic energy remains in equilibrium with the heat bath. Thus, the whole system is not endoreversible. Besides, the cooling rate in the overdamped regime depends on the work parameter, and the work parameter-dependent cooling rate leads to varying effective temperature in isothermal processes of the optimal cycle. Therefore, the EMP of the overdamped Brownian engine is not the CA efficiency.

From this microscopic theory, we construct the optimal cycle of the maximum power of the Brownian engine, and the optimal cycle is exactly the CA cycle. The control scheme associated with the maximum power for any given efficiency can also be obtained based on the microscopic theory. In addition, we calculate the generating function of work and heat, and obtain the analytical results of statistics of the power and the efficiency together with the fluctuation theorems of the Brownian CA engine. These quantitative results about the Brownian CA engine bring important insights to the studies of finite-time thermodynamics beyond the low-dissipation regime [16,26,71,72,74,86,87]. For example, results about the average power and efficiency of the CA engine remain valid when downsizing the working substance to a single Brownian particle, but fluctuations become prominent. Our study will shed new light on the experimental explorations about finite-time Brownian engine, and may inspire future studies about the design of nanomachines with higher power and efficiency.

ACKNOWLEDGMENTS

H.T.Q. thanks Zhan-Chun Tu for valuable comments. We acknowledge support from the National Science Foundation of China under Grants No. 11775001, No. 11534002, No. 11825501, and No. 12147157.

APPENDIX A: TRADE-OFF RELATION BETWEEN POWER AND EFFICIENCY

The trade-off relation between power and efficiency, namely the maximum power for a given efficiency, has

recently attracted much attention in finite-time thermodynamics. Various trade-off relations have been found under different circumstances [26,70–74]. Nevertheless, these relations either are obtained based on the low-dissipation approximation, or are very loose constraints.

It is worth pointing out that in 1989, a tight trade-off relation between power and efficiency was obtained for the endoreversible Carnot cycle with the assumption of Newton's cooling law [75],

$$\overline{P}^* \propto \frac{\eta(\eta_C - \eta)}{1-\eta}. \tag{A1}$$

However, due to the lack of microscopic theory in their study, they were unable to determine the control scheme associated with the maximum power for a given efficiency.

Here we derive the control scheme $\lambda(t)$ associated with the above tight trade-off relation between power and efficiency for the finite-time Brownian Carnot engine. With the expressions of the power and the efficiency, we introduce the function

$$L = \frac{f_n}{2(\tau_H+\tau_C)}\left(\frac{\tau_H\Gamma_H \ln r}{\tau_H\Gamma_H + \ln r}T_H - \frac{\tau_C\Gamma_C \ln r}{\tau_C\Gamma_C - \ln r}T_C\right) + \mu\left(1 - \frac{1+(\tau_H\Gamma_H)^{-1}\ln r}{1-(\tau_C\Gamma_C)^{-1}\ln r}\frac{T_C}{T_H} - \eta\right), \tag{A2}$$

where μ is the Lagrange multiplier to include the given efficiency as the constraint. The maximum power for the given efficiency η is obtained from

$$\frac{\partial L}{\partial \tau_H} = 0, \tag{A3}$$

$$\frac{\partial L}{\partial \tau_C} = 0, \tag{A4}$$

$$\frac{\partial L}{\partial \mu} = 0. \tag{A5}$$

The solution to the above equations are

$$\tau_H^* = \frac{\ln r}{\sqrt{\Gamma_H}(\eta_C-\eta)}\left(\frac{1-\eta}{\sqrt{\Gamma_C}} + \frac{1-\eta_C}{\sqrt{\Gamma_H}}\right), \tag{A6}$$

$$\tau_C^* = \frac{\ln r}{\sqrt{\Gamma_C}(\eta_C-\eta)}\left(\frac{1-\eta}{\sqrt{\Gamma_C}} + \frac{1-\eta_C}{\sqrt{\Gamma_H}}\right), \tag{A7}$$

$$\mu^* = \frac{f_n\Gamma_C\Gamma_H T_H[(2-\eta)\eta-\eta_C]}{2(1-\eta)^2(\sqrt{\Gamma_H}+\sqrt{\Gamma_C})^2}. \tag{A8}$$

Since we have adopted the exponential protocol, the control scheme $\lambda(t)$ of the whole cycle for the Brownian CA engine is

$$\lambda(t) = \begin{cases} \lambda_1(\lambda_2/\lambda_1)^{t/\tau_C^*} & 0 < t < \tau_C^* \\ \lambda_3(\lambda_4/\lambda_3)^{(t-\tau_C^*)/\tau_H^*} & \tau_C^* < t < \tau_H^* \end{cases}. \tag{A9}$$

Substituting τ_H^* and τ_C^* into the expression of the power, we obtain the maximum power for a given efficiency η as

$$\overline{P}^* = \frac{f_n T_H \Gamma_C\Gamma_H}{2\left(\sqrt{\Gamma_C}+\sqrt{\Gamma_H}\right)^2}\frac{\eta(\eta_C-\eta)}{(1-\eta)}, \tag{A10}$$

which agrees with the trade-off relation obtained in Ref. [75]. Notice that the trade-off relation [Eq. (A10)] is independent of the compression ratio r.

We emphasize that the dependence of the maximum power on the efficiency is always in the factorized form, which is universal and independent of the cooling rate Γ_H (Γ_C) in the two isothermal processes. In comparison with Ref. [70], the trade-off relation (A10) is tighter and this upper bound of the power can be achieved with the explicit control scheme.

APPENDIX B: THE GENERATING FUNCTION AND THE STATISTICS OF THE FINITE-TIME CARNOT CYCLE

Based on the method proposed in Ref. [77], we derive the joint generating function of work and heat for both the finite-time driving process (finite-time isothermal process). The system is coupled to only one heat bath with the exchanged heat $Q[X]$, and the work parameter of the system is tuned with the performed work $W[X]$. In stochastic thermodynamics, both heat and work are defined on the trajectory $X = \{x(t)|0 \leqslant t \leqslant \tau\}$, where $x(t)$ denotes a phase-space point of the system with the energy $E(t)$ [77,88]. The joint generating function of work and heat is defined as the path integral in the trajectory space

$$I(u,s) := \langle e^{uQ+sW}\rangle \tag{B1}$$

$$= \int D[X]p[X]e^{uQ[X]+sW[X]}, \tag{B2}$$

where $p[X]$ denotes the probability of a given trajectory X.

The method [77] to be applied requires that

(1) The system always obeys a Maxwell-Boltzmann distribution in the energy space described by an effective temperature θ, whose evolution is governed by

$$\dot{\theta} = \Phi_t(\theta). \tag{B3}$$

Here, $\Phi_t(\theta)$ contains the contribution of the work. When work is performed on the system, the temperature of the system increases. If no modulation is performed, $\Phi_t(\theta) = -\Gamma(\theta - T_b)$ is Newton's law of cooling with a constant cooling rate Γ. T_b is the temperature of the bath.

(2) The increment in work is proportional to the stochastic energy of the system,

$$dW = \alpha_t E dt, \tag{B4}$$

where α_t is determined by the control protocol, and is chosen as $\alpha_t = \dot{\lambda}/\lambda$ relating to the work parameter $\lambda = \lambda(t)$; $E = E(t)$ is the stochastic energy of the system affected by both the heat exchange and the work performed. As a result, $\Phi_t(\theta)$ is explicitly

$$\Phi_t(\theta) = -\Gamma(\theta - T_b) + \alpha_t\theta. \tag{B5}$$

(3) The structure (shape) of the partition function does not depend on λ. Generally, the partition function $Z_\lambda(\beta)$ should depend on the work parameter λ. The inverse temperature is $\beta = 1/\theta$, where we set $k_B = 1$ for convenience. For example, for a quantum harmonic oscillator, the frequency determines energy-level spacing which affects the $Z_\lambda(\beta)$. But here, we require $Z_\lambda(\beta)$ to be in the same form of λ which means

$$Z_\lambda(\beta) = g(\lambda) \times Z(\beta), \tag{B6}$$

where $g(\lambda)$ is a function of the work parameter, and $Z(\beta)$ characterize the form of the energy density.

All conditions are satisfied in the current microscopic model. The probability distribution of the energy is

$$p(E|\theta) = \frac{E^{\frac{f_n}{2}-1}e^{-\beta E}}{Z(\beta)}. \tag{B7}$$

The partition function is $Z(\beta) = \int_0^\infty E^{f_n/2-1}e^{-\beta E}dE = \beta^{-f_n/2}\Gamma(f_n/2)$ [89], which is also the normalization factor in the energy space. The average energy is determined by

$$\langle E\rangle = -\frac{\partial \ln[Z(\beta)]}{\partial\beta} = \frac{f_n}{2}\theta. \tag{B8}$$

Namely, the heat capacity of the working substance is $f_n/2$, which remains a constant.

Along a given trajectory $X = \{E(t)|0 \leqslant t \leqslant \tau\}$, the increment of work at time t is

$$dW = \alpha_t E dt, \tag{B9}$$

and the increment of heat is obtained from the first law

$$dQ = dE - \alpha_t E dt. \tag{B10}$$

1. Discretization to calculate the joint generating function of work and heat

We discretize the dynamics with $t_j = j\epsilon$, $j = 0, 1, ..., N, N+1$ and $\epsilon = \tau/(N+1)$, the thermal distribution of the system is $p(E_j|\theta_j)$ with the inverse temperature $\beta_j = 1/\theta_j$ at t_j. The work parameter $\lambda(t)$ remains a constant during each time slice $t_j < t < t_{j+1}$ and is quenched at the moment t_j. As a result, the work $W_j = \ln[\lambda(t_j^+)/\lambda(t_j^-)]E_j \approx \alpha_j E_j\epsilon$ is performed at each moment t_j with the neglected exchanged heat, while the heat $Q_j = E_{j+1} - E_j - \alpha_j E_j\epsilon$ is generated during each time slice. The evolution between every adjacent moment t_j and t_{j+1} can be described by the transition probability $\mathcal{R}^j_{j+1} = \mathcal{R}(E_{j+1}, t_{j+1}; E_j, t_j)$ (also named as the propagator), which maps a local equilibrium state to another local equilibrium state

$$\int dE_j p(E_j|\theta_j)\mathcal{R}^j_{j+1} = p(E_{j+1}|\theta_{j+1}). \tag{B11}$$

The temperature of the next step is

$$\theta_{j+1} = \theta_j + \Phi_j(\theta_j)\epsilon, \tag{B12}$$

where we denote $\Phi_j(\cdot) = \Phi_{t_j}(\cdot)$ for simplicity.

We calculate the joint generating function $I(u, s)$ as follows. We rewrite the joint generating function with the $N+1$ discrete steps,

$$I(u,s) = \int \prod_{j=0}^{N+1} dE_j \prod_{j=0}^{N} (\mathcal{R}^j_{j+1}) p(E_0|\tilde\psi_0) \prod_{j=0}^{N} e^{uQ_j + sW_j}, \tag{B13}$$

where the initial value $\tilde\psi_0$ is a temperature-like quantity. We use a different notation ψ to distinguish with the temperature θ. The meaning of "tilde" will be clarified later [by Eq. (B23)]. If there is no previous process, $\tilde\psi_0 = \theta_0$ is the local temperature of the system. Otherwise, the initial value $\tilde\psi_0$ is obtained from the previous process.

From the first law, we can rewrite

$$uQ_j + sW_j = (s-u)\alpha_j E_j\epsilon + u(E_{j+1} - E_j). \tag{B14}$$

The result of the integral in the first step is

$$\int dE_0 \mathcal{R}^0_1 p(E_0|\tilde\psi_0)e^{uQ_0+sW_0} = \frac{Z(1/\psi_0)}{Z(1/\tilde\psi_0)}\frac{e^{E_1(u-1/\varphi_1)}}{Z(1/\varphi_1)}, \tag{B15}$$

where ψ_0 is

$$\psi_0 = 1/[1/\tilde\psi_0 - (s-u)\epsilon\alpha_0 + u], \tag{B16}$$

and the intermediate variable φ_1 is obtained as

$$\varphi_1 = \psi_0 - \Gamma(\psi_0 - T_b)\epsilon + \alpha_0\psi_0\epsilon. \tag{B17}$$

After doing the integral over E_j, $1 \leqslant j \leqslant N+1$, the result is

$$I(u,s) = \frac{Z(1/\psi_0)}{Z(1/\tilde\psi_0)} \cdot \prod_{n=1}^{N}\left[\frac{Z(1/\psi_n)}{Z(1/\varphi_n)}\right] \cdot \frac{Z(1/\tilde\psi_{N+1})}{Z(1/\varphi_{N+1})}, \tag{B18}$$

where the initial value ψ_0 is given by Eq. (B16), and the value at the final step is

$$\tilde\psi_{N+1} = (1/\varphi_{N+1} - u)^{-1}. \tag{B19}$$

The reduction formulas are $\varphi_n = \psi_{n-1} + \Phi_{n-1}(\psi_{n-1})\epsilon$ and $\psi_n = [1/\varphi_n - (s-u)\alpha_n\epsilon]^{-1}$, which lead to $(\psi_n - \psi_{n-1})/\epsilon = \Phi_{n-1}(\psi_{n-1}) + (s-u)\alpha_n\psi^2_{n-1}$. Its continuum limit is

$$\frac{d\psi}{dt} = \Phi_t(\psi) + (s-u)\alpha_t\psi^2. \tag{B20}$$

The initial condition is obtained from Eq. (B16) as

$$\psi(0) = \frac{\tilde\psi_0}{1+u\tilde\psi_0}. \tag{B21}$$

At the end $t = \tau$, Eq. (B19) leads to the final condition

$$\tilde\psi(\tau) = \frac{\psi(\tau)}{1-u\psi(\tau)}. \tag{B22}$$

We can define a shifted temperature-like variable $\tilde\psi(t)$ as

$$\tilde\psi(t) := \frac{\psi(t)}{1-u\psi(t)}. \tag{B23}$$

The corresponding differential equation is

$$\frac{d\tilde\psi}{dt} = (1+u\tilde\psi)^2\Phi_t\left(\frac{\tilde\psi}{1+u\tilde\psi}\right) + (s-u)\alpha_t\tilde\psi^2, \tag{B24}$$

and the initial value is $\tilde\psi(0) = \tilde\psi_0$.

In the continuum limit, the factors of the joint generating function (B18) are explicitly

$$\frac{Z(1/\psi_0)}{Z(1/\tilde\psi_0)} \to (1+u\tilde\psi_0)^{-\frac{f_n}{2}}, \tag{B25}$$

$$\frac{Z(1/\tilde\psi_{N+1})}{Z(1/\varphi_{N+1})} \to (1+u\tilde\psi(\tau))^{\frac{f_n}{2}}, \tag{B26}$$

$$\prod_{n=1}^{N}\frac{Z(1/\psi_n)}{Z(1/\varphi_n)} \to e^{(s-u)\frac{f_n}{2}\int_0^\tau \alpha_t\psi(t)dt}. \tag{B27}$$

We then obtain the final result (18) of the joint generating function. Notice that the temperature-like variable $\psi(t)$ is

solved from the differential Eq. (B20) with the initial condition (B21) and the final condition (B22).

As follows, we discuss the joint generating function for the sudden jump process and the exponential protocol process. These two processes are used to form the finite-time Carnot cycle.

2. Sudden jump process

In a sudden jump process, work is performed with the neglected heat exchange. The work parameter is quenched from λ^- to λ^+ with temperature-like quantity changed from $\tilde{\psi}^-$ to $\tilde{\psi}^+$. No exchanged heat is produced in such a sudden process, and we can set $u = 0$. Therefore, $\tilde{\psi}$ becomes the same as ψ in this process. The differential Eq. (B24) [or Eq. (19)] becomes

$$\frac{d\tilde{\psi}}{dt} = \alpha_t \tilde{\psi} + s\alpha_t \tilde{\psi}^2. \tag{B28}$$

The final value $\tilde{\psi}^+$ is associated with $\tilde{\psi}^-$ as

$$\frac{\tilde{\psi}^+}{1+\tilde{\psi}^+ s} = \frac{\lambda^+}{\lambda^-}\frac{\tilde{\psi}^-}{1+\tilde{\psi}^- s}. \tag{B29}$$

Notice that $\tilde{\psi}^+$ serves as the initial value $\tilde{\psi}_0$ of the next process. If there is a previous process, then $\tilde{\psi}^-$ is substituted with $\tilde{\psi}(\tau)$ of the previous process. The joint generating function of the sudden jump process is

$$I_{\text{jump}}(u,s) = \left[1 + \left(1 - \frac{\lambda^+}{\lambda^-}\right)\tilde{\psi}^- s\right]^{-\frac{f_n}{2}}, \tag{B30}$$

which is in fact independent of u since no heat is exchanged in this process.

3. Exponential protocol process

For a exponential protocol process $\alpha_t = \alpha = \text{const}$, Eq. (19) becomes a Ricatti equation (time-independent):

$$\dot{\psi} = -\Gamma(\psi - T_{\text{b}}) + \alpha\psi + (s-u)\alpha\psi^2. \tag{B31}$$

We solve the above differential equation with the initial condition $\psi(0) = \tilde{\psi}_0/(1+u\tilde{\psi}_0)$. The solution to Eq. (B31) is

$$\psi(t) = \frac{\Gamma - \alpha + \sqrt{D(s-u)}\tan\left[\frac{c_1+t}{2}\sqrt{D(s-u)}\right]}{2\alpha(s-u)}, \tag{B32}$$

where c_1 is determined by the initial condition $\psi(0)$ as

$$\tan\left[\frac{c_1}{2}\sqrt{D(s-u)}\right] = \frac{2\alpha(s-u)\psi(0) - (\Gamma-\alpha)}{\sqrt{D(s-u)}}, \tag{B33}$$

with the function $D(x) = 4\alpha\Gamma T_{\text{b}} x - (\Gamma-\alpha)^2$.

We obtain the joint generating function as

$$I_{\text{exp}}(u,s) = \left\{\frac{\exp\left[(\Gamma-\alpha)\tau/2\right][1+u\tilde{\psi}(\tau)]/[1+u\tilde{\psi}_0]}{\cos\left[\frac{t}{2}\sqrt{D(s-u)}\right] - g(u,s)\sin\left[\frac{t}{2}\sqrt{D(s-u)}\right]}\right\}^{\frac{f_n}{2}}, \tag{B34}$$

with $g(u,s) = [2\alpha(s-u)\psi(0) - (\Gamma-\alpha)]/\sqrt{D(s-u)}$. If there is a previous process, we need to substitute $\tilde{\psi}_0$ as the final value $\tilde{\psi}(\tau)$ of the previous process.

4. Joint generating function of a finite-time Carnot cycle

Now we calculate the joint generating function of the finite-time Carnot cycle. The heat is absorbed from the hot bath and released to the cold, namely, $Q_H > 0$ and $Q_C < 0$. The joint generating function $I(u_H, u_C, s)$ of work and heat for a whole finite-time cycle is expressed in the path integral form

$$I(u_H, u_C, s) = \int D[X]p[X]e^{u_H Q_H[X] + u_C Q_C[X] + sW[X]}. \tag{B35}$$

Notice that the system is in contact with only one heat bath at a time. We start from A point (Fig. 1 in main text) and calculate the joint generating function of a whole cycle.

a. Process I: Isothermal compression

We define the compression ratio as $r = \lambda_2/\lambda_1$. In this process, the work parameter is varied exponentially with the time $\lambda(t) = \lambda_1 r^{t/\tau_C}$. The quantity α of this process is explicitly $\alpha_C = \ln r/\tau_C$. The initial temperature of the working substance is (which is also the initial condition) $\tilde{\psi}_0 = \theta_C = \Gamma_C T_C/(\Gamma_C - \alpha_C)$. If we consider the heat engine runs for many cycles, $\tilde{\psi}_0$ takes the final value of $\tilde{\psi}(\tau_{\text{t}})$ of the previous cycle. According to Eq. (B34), the joint generating function of this process is

$$I_{\text{I}}(u_H, u_C, s) = \left\{\frac{e^{(\Gamma_C-\alpha_C)\tau_C/2}[1+u_C\tilde{\psi}(\tau_C^-)]/[1+u_C\tilde{\psi}_0]}{\cos(\Omega_C\tau_C) - g_{\text{I}}(u_C,s)\sin(\Omega_C\tau_C)}\right\}^{\frac{f_n}{2}}, \tag{B36}$$

where $g_{\text{I}}(u_C,s) = [\alpha_C(s-u_C)\psi(0) - (\Gamma_C-\alpha_C)/2]/\Omega_C$ with the initial value $\psi(0) = \tilde{\psi}_0/(1+u_C\tilde{\psi}_0)$, τ_C^- represents the moment of the end of process I, and the frequency Ω_C is $\sqrt{\alpha_C\Gamma_C T_C(s-u_C) - (\Gamma_C-\alpha_C)^2/4}$. Notice that the expression I_{I} does not contain u_H. But if there exist a previous cycle, $\tilde{\psi}_0$ also relies on u_H. We always write $I(u_H, u_C, s)$ for all the processes in a finite-time heat engine cycle.

b. Process II: Adiabatic compression

At the end of process I, the temperature-like quantity is $\tilde{\psi}(\tau_C)$, which is plugged as the initial value for the process II as a sudden quench of the work parameter λ from λ_2 to λ_3 at the moment $t = \tau_C$. Notice that the ratio of the work parameter in this process satisfy $\lambda_3/\lambda_2 = \theta_H/\theta_C$. According to Eq. (B29), the temperature-like quantity after the quench $\tilde{\psi}^+_{\tau_C}$ is associated with $\tilde{\psi}^-_{\tau_C} = \tilde{\psi}(\tau_C^-)$ as

$$\tilde{\psi}^+_{\tau_C} = \frac{\tilde{\psi}(\tau_C^-)}{1-\eta-s\eta\tilde{\psi}(\tau_C^-)}, \tag{B37}$$

where the efficiency of the finite-time Carnot cycle is $\eta = 1 - \theta_C/\theta_H$. According to Eq. (B30), the joint generating function of process II is

$$I_{\text{II}}(u_H, u_C, s) = \left[1 - \frac{s\eta}{1-\eta}\tilde{\psi}(\tau_C^-)\right]^{-\frac{f_n}{2}}. \tag{B38}$$

Notice that the value of the temperature-like quantity $\tilde{\psi}(\tau_C^-)$ depends on u_C and s.

c. Process III: Isothermal expansion

The work parameter is varied exponentially with the time $\lambda(t) = \lambda_3 r^{-(t-\tau_C)/\tau_H}$. The quantity α of this process is explicitly $\alpha_H = -\ln r/\tau_H$. The initial condition of this process is $\tilde{\psi}(\tau_C^+) = \tilde{\psi}_{\tau_C}^+$ or $\psi(\tau_C^+) = \tilde{\psi}_{\tau_C}^+/(1+u_H\tilde{\psi}_{\tau_C}^+)$.

The adiabatic process is completed suddenly at the moment $t = \tau_C$, and we use τ_C^+ to indicate the beginning of process III. Equation (B34) gives the joint generating function of this process

$$I_{\rm III}(u_H, u_C, s) = \left\{ \frac{e^{(\Gamma_H-\alpha_H)\tau_H/2}[1+u_H\tilde{\psi}(\tau_C+\tau_H^-)]/[1+u_H\tilde{\psi}_{\tau_C}^+]}{\cos(\Omega_H\tau_H) - g_{\rm III}(u_H,s)\sin(\Omega_H\tau_H)} \right\}^{\frac{f_n}{2}}, \tag{B39}$$

where $g_{\rm III}(u_H, s) = [\alpha_H(s-u_H)\psi(\tau_C^+) - (\Gamma_H-\alpha_H)/2]/\Omega_H$, $\tau_C+\tau_H^-$ represents the moment of the end of process III, and the frequency Ω_H is $\sqrt{\alpha_H\Gamma_H T_H(s-u_H) - (\Gamma_H-\alpha_H)^2/4}$.

d. Process IV: Adiabatic expansion

This process is a sudden quench with the work parameter λ changing from λ_4 to λ_1 at the moment $t = \tau_C + \tau_H$. The ratio of the work parameter satisfies $\lambda_1/\lambda_4 = \theta_C/\theta_H$. According to Eq. (B29), the temperature-like quantity after the quench $\tilde{\psi}_{\tau_C+\tau_H}^+$ is associated with $\tilde{\psi}_{\tau_C+\tau_H}^- = \tilde{\psi}(\tau_C+\tau_H^-)$ as

$$\tilde{\psi}_{\tau_C+\tau_H}^+ = \frac{(1-\eta)\tilde{\psi}(\tau_C+\tau_H^-)}{1+s\eta\tilde{\psi}(\tau_C+\tau_H^-)}. \tag{B40}$$

According to Eq. (B30), the joint generating function of process IV is

$$I_{\rm IV}(u_H, u_C, s) = \left[1+s\eta\tilde{\psi}(\tau_C+\tau_H^-)\right]^{-\frac{f_n}{2}}. \tag{B41}$$

e. Joint generating function of a whole cycle

The joint generating function $I_{\rm cycle}(u_H, u_C, s) = I_{\rm I}\times I_{\rm II}\times I_{\rm III}\times I_{\rm IV}$ of a whole cycle is

$$I_{\rm cycle} = \left\{ \frac{e^{(\Gamma_C\tau_C+\Gamma_H\tau_H)/2}\left[1-\frac{s\eta}{1-\eta}\tilde{\psi}(\tau_C^-)\right]^{-1}\left[1+s\eta\tilde{\psi}(\tau_C+\tau_H^-)\right]^{-1}\frac{1+u_C\tilde{\psi}(\tau_C^-)}{1+u_C\tilde{\psi}_0}\frac{1+u_H\tilde{\psi}(\tau_C+\tau_H^-)}{1+u_H\tilde{\psi}_{\tau_C}^+}}{[\cos(\Omega_C\tau_C) - g_{\rm I}(u_C,s)\sin(\Omega_C\tau_C)][\cos(\Omega_H\tau_H) - g_{\rm III}(u_H,s)\sin(\Omega_H\tau_H)]} \right\}^{\frac{f_n}{2}}. \tag{B42}$$

The evolution of $\psi(t)$ is governed by

$$\dot{\psi} = \begin{cases} -\Gamma_C(\psi-T_C) + \alpha_C\psi + (s-u_C)\alpha_C\psi^2, & 0 < t < \tau_C, \\ -\Gamma_H(\psi-T_H) + \alpha_H\psi + (s-u_H)\alpha_H\psi^2, & \tau_C < t < \tau_H. \end{cases} \tag{B43}$$

The initial and the connecting conditions are $\psi(0) = \tilde{\psi}_0/(1+u_C\tilde{\psi}_0)$ and $\psi(\tau_C^+) = \tilde{\psi}_{\tau_C}^+/(1+u_H\tilde{\psi}_{\tau_C}^+)$, where $\tilde{\psi}_{\tau_C}^+$ is associated with $\tilde{\psi}(\tau_C^-)$ through process II. As follows, we give the generating functions for the work and the fluctuating efficiency deviation, respectively.

f. Generating function for the work

By setting $u_C = 0$ and $u_H = 0$, the generating function for work is $I_{\rm cycle}^W(s) = I_{\rm cycle}(0, 0, s)$, explicitly as

$$I_{\rm cycle}^W(s) = \left\{ \frac{e^{(\Gamma_C\tau_C+\Gamma_H\tau_H)/2}\left[1-\frac{s\eta}{1-\eta}\psi(\tau_C^-)\right]^{-1}[1+s\eta\psi(\tau_C+\tau_H^-)]^{-1}}{[\cos(\Omega_C^W\tau_C) - g_{\rm I}^W\sin(\Omega_C^W\tau_C)][\cos(\Omega_H^W\tau_H) - g_{\rm III}^W\sin(\Omega_H^W\tau_H)]} \right\}^{\frac{f_n}{2}}, \tag{B44}$$

with $\Omega_C^W = \sqrt{\alpha_C\Gamma_C T_C s - (\Gamma_C-\alpha_C)^2/4}$, $\Omega_H^W = \sqrt{\alpha_H\Gamma_H T_H s - (\Gamma_H-\alpha_H)^2/4}$, $g_{\rm I}^W = [\alpha_C s\psi(0) - (\Gamma_C-\alpha_C)/2]/\Omega_C^W$ and $g_{\rm III}^W = [\alpha_H s\psi(\tau_C^+) - (\Gamma_H-\alpha_H)/2]/\Omega_H^W$. Notice that Eq. (B23) indicates $\tilde{\psi}(t) = \psi(t)$ in this situation.

g. Generating function for the fluctuating efficiency deviation

By setting $u_C = 0$ and $u_H = s\eta$, the generating function for the fluctuating efficiency deviation is $I_{\rm cycle}^\zeta(s) = I_{\rm cycle}(s\eta, 0, s)$, explicitly as

$$I_{\rm cycle}^\zeta(s) = \left\{ \frac{e^{(\Gamma_C\tau_C+\Gamma_H\tau_H)/2}}{\left[\cos\left(\Omega_C^\zeta\tau_C\right) - g_{\rm I}^\zeta\sin\left(\Omega_C^\zeta\tau_C\right)\right]\left[\cos\left(\Omega_H^\zeta\tau_H\right) - g_{\rm III}^\zeta\sin\left(\Omega_H^\zeta\tau_H\right)\right]} \right\}^{\frac{f_n}{2}}, \tag{B45}$$

with $\Omega_C^\zeta = \sqrt{\alpha_C\Gamma_C T_C s - (\Gamma_C-\alpha_C)^2/4}$, $\Omega_H^\zeta = \sqrt{\alpha_H\Gamma_H T_H s(1-\eta) - (\Gamma_H-\alpha_H)^2/4}$, $g_{\rm I}^\zeta = [\alpha_C s\psi(0) - (\Gamma_C-\alpha_C)/2]/\Omega_C^\zeta$ and $g_{\rm III}^\zeta = [\alpha_H s(1-\eta)\psi(\tau_C^+) - (\Gamma_H-\alpha_H)/2]/\Omega_H^\zeta$. The condition $u_C = 0$ gives $\tilde{\psi}(t) = \psi(t)$ during process I ($0 < t < \tau_C$).

5. Self-consistent check of the joint generating function from the fluctuation theorem

The fluctuation theorem for heat engines [83] provides a microscopic understanding of Carnot's theorem. For the model presented here which never reaches thermal equilibrium with the heat bath, the fluctuation theorem takes a different form. In the following, we derive the fluctuation theorem for our finite-time Brownian Carnot engine.

Suppose the engine operates between a hot bath at temperature T_H and a cold bath at temperature T_C. Moreover, the the working substance of the engine obeys the same Maxwell-Boltzmann distribution at an effective temperature θ_C at both the beginning and the end of the cycle [point A in Fig. 1(a)]. The ratio of the probability of a trajectory X to that of its time reversal $\tilde{X}$ under the time reversed protocol is [84]

$$\frac{p[X]}{\tilde{p}[\tilde{X}]} = \exp(\Delta s_{\rm sys} + \Delta s_{\rm bath}). \tag{B46}$$

The entropy change of the system is $\Delta s_{\rm sys} = -\ln \tilde{\pi}_i(\tilde{x}_i) + \ln \pi_i(x_i)$, where $\pi_i(x_i)$ and $\tilde{\pi}_i(\tilde{x}_i)$ are the initial equilibrium distributions of the forward and reverse processes at temperature θ_C. The work parameter returns to its initial value at the end of the cycle, and thus $\tilde{\pi}_i(\tilde{x}_i)/\pi_i(x_i) = \exp[-(E_f - E_i)/\theta_C]$ with the initial (final) stochastic energy $E_{i(f)}$ on the trajectory X. The entropy change of the heat bath is $\Delta s_{\rm bath} = -Q_H/T_H - Q_C/T_C$. According to the first law of thermodynamics along a single trajectory, the stochastic energy change comprises the trajectory work and heat,

$$E_f - E_i = W + Q_H + Q_C. \tag{B47}$$

Accordingly, we can rewrite Eq. (B46) into

$$p[X]e^{Q_H(\frac{1}{T_H}-\frac{1}{\theta_C})+Q_C(\frac{1}{T_C}-\frac{1}{\theta_C})-\frac{W}{\theta_C}} = \tilde{p}[\tilde{X}]. \tag{B48}$$

By taking ensemble average, we obtain the following fluctuation theorem

$$\left\langle e^{Q_H(\frac{1}{T_H}-\frac{1}{\theta_C})+Q_C(\frac{1}{T_C}-\frac{1}{\theta_C})-\frac{W}{\theta_C}} \right\rangle = 1. \tag{B49}$$

With Jensen's inequality $\exp\langle x\rangle \leqslant \langle \exp(x)\rangle$, Carnot's theorem can be derived as a corollary of both fluctuation theorems

$$\eta = -\frac{\langle W\rangle}{\langle Q_H\rangle} \leqslant 1 - \frac{T_C}{T_H} = \eta_C, \tag{B50}$$

where we have used the conservation of the energy for a periodic steady closed cycle $\langle W\rangle + \langle Q_H\rangle + \langle Q_C\rangle = \langle E_f\rangle - \langle E_i\rangle = 0$.

The fluctuation theorem (B49) can be rewritten as $I_{\rm cycle}(1/T_H - 1/\theta_C, 1/T_C - 1/\theta_C, -1/\theta_C) = 1$, which can be used to verify our results of the joint generating function $I_{\rm cycle}(u_H, u_C, s)$. By choosing $u_H = 1/T_H - 1/\theta_C$, $u_C = 1/T_C - 1/\theta_C$, $s = -1/\theta_C$, the auxiliary quantity $\psi(t)$ in the cycle is solved from Eq. (B43) as

$$\psi(t) = \begin{cases} T_C & 0 < t < \tau_C, \\ T_H & \tau_C < t < \tau_C + \tau_H, \end{cases} \tag{B51}$$

and the shifted temperature-like variable is $\tilde{\psi}(t) \equiv \theta_C$ throughout the whole cycle. Ω_C and Ω_H are explicitly $\Omega_C = i(\Gamma_C + \alpha_C)/2$ and $\Omega_H = i(\Gamma_H + \alpha_H)/2$. Therefore, the joint generating function of each process becomes

$$I_{\rm I} = e^{-\frac{f_n}{2}\alpha_C\tau_C}, \tag{B52}$$

$$I_{\rm II} = (1-\eta)^{\frac{f_n}{2}}, \tag{B53}$$

$$I_{\rm III} = e^{-\frac{f_n}{2}\alpha_H\tau_H}, \tag{B54}$$

$$I_{\rm IV} = (1-\eta)^{-\frac{f_n}{2}}. \tag{B55}$$

Thus, we verify the fluctuation theorem $I_{\rm cycle}(1/T_H - 1/\theta_C, 1/T_C - 1/\theta_C, -1/\theta_C) = 1$ for heat engines.

6. Calculating the variances of power and fluctuating efficiency deviation

The derivatives of the generating function give the moments of work distribution. The average work is

$$\langle W\rangle = \frac{\partial}{\partial s}\langle e^{sW}\rangle|_{s=0} = \frac{f_n}{2}(\theta_H - \theta_C)\ln r. \tag{B56}$$

The variance of the work output in a cycle $\langle \Delta W^2\rangle = \langle W^2\rangle - \langle W\rangle^2$ can also be calculated from the generating function

$$\langle \Delta W^2\rangle = \left[\frac{\partial^2}{\partial s^2}\langle e^{sW}\rangle - \left(\frac{\partial}{\partial s}\langle e^{sW}\rangle\right)^2\right]|_{s=0}. \tag{B57}$$

The variance of the power is explicitly obtained as

$$\begin{aligned}\frac{{\rm Var}(P)}{\overline{P}^2} =& \frac{\langle \Delta W^2\rangle}{\langle W\rangle^2} \\ =& \frac{4}{f_n}\left[\frac{1}{\kappa_C\tau_C}\left(\frac{1-\eta}{\eta}\right)^2 + \frac{1}{\kappa_H\tau_H}\frac{1}{\eta^2}\right] + \frac{4}{f_n}\left[\frac{1-e^{-\kappa_H\tau_H}}{\kappa_H^2\tau_H^2}\left(\frac{\kappa_H^2\tau_H^2}{(\ln r)^2} - \frac{1}{\eta^2}\right) - \frac{1-e^{-\kappa_C\tau_C}}{\kappa_C^2\tau_C^2}\left(\frac{1-\eta}{\eta}\right)^2\right] \\ &+ \frac{4}{f_n}\frac{(1-e^{-\kappa_C\tau_C})(1-e^{-\kappa_H\tau_H})}{\kappa_C\tau_C\kappa_H\tau_H}\left(\frac{\kappa_H\tau_H}{\ln r} - \frac{1}{\eta}\right)\frac{1-\eta}{\eta},\end{aligned} \tag{B58}$$

where $\kappa_C = \Gamma_C - \ln r/\tau_C$ and $\kappa_H = \Gamma_H - \ln r/\tau_H$. The variance of the fluctuating efficiency deviation is given by

$$\begin{aligned}\mathrm{Var}(\zeta) &:= \frac{\langle (W+\eta Q_H)^2\rangle}{\langle Q_H\rangle^2} \\ &= \frac{4(1-\eta)^2}{f_n}\left[\frac{1}{\kappa_C\tau_C}+\frac{1}{\kappa_H\tau_H}\right] - \frac{4(1-\eta)^2}{f_n}\left[\frac{1-e^{-\kappa_C\tau_C}}{\kappa_C^2\tau_C^2}+\frac{1-e^{-\kappa_H\tau_H}}{\kappa_H^2\tau_H^2}+-\frac{(1-e^{-\kappa_C\tau_C})(1-e^{-\kappa_H\tau_H})}{\kappa_C\tau_C\kappa_H\tau_H}\right]. \end{aligned} \tag{B59}$$

To achieve the maximum power, we should choose $\tau_C = \tau_C^{\max}$ and $\tau_H = \tau_H^{\max}$ according to Eq. (16) to realize the Brownian CA engine. As a result, the efficiency η should be replaced by η_{CA}.

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