Absorbing phase transition in a unidirectionally coupled layered network

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We study the contact process on layered networks in which each layer is unidirectionally coupled to the next layer. Each layer has elements sitting on (i) an Erdös-Réyni network, and (ii) a d-dimensional lattice. The top layer is not connected to any layer and undergoes an absorbing transition in the directed percolation class for the corresponding topology. The critical infection probability p_c for the transition is the same for all layers. For an Erdös-Réyni network the order parameter decays as $t^{-\delta_l}$ at p_c for the lth layer with $\delta_l \sim 2^{1-l}$. This can be explained with a hierarchy of differential equations in the mean-field approximation. The dynamic exponent $z = 0.5$ for all layers and $\nu_\parallel \to 2$ for larger l. For a d-dimensional lattice, we observe a stretched exponential decay of the order parameter for all but the top layer at p_c.

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I. INTRODUCTION

The identification of the underlying topological structure for complex systems [1] has led to a new branch of "network science" [2]. Several researchers have studied different properties of real-life networks and proposed models. The most popular models are scale-free [3] and small-world networks [4]. These studies helped to better understand phenomena as diverse as the spreading of diseases in the population [5], information processing in gene circuits [6], and biological pathways [7]. It has also helped in understanding the transport properties of several man-made systems [8].

Another model which has attracted attention recently has been the multiplex network. It models multiple levels of interaction in a given network. One example is a social media network [9,10] where individuals are connected by Twitter, Facebook, WhatsApp, etc., and there is a certain information flow in the layers. Another example is the traffic network [11] where people travel using various modes of travel such as trams, buses, etc. In the spread of diseases [12,13], empirical studies on different strains of disease or different diseases have shown the necessity of modeling the underlying network as a multiplex network. The interaction between the nodes is described by a single-layer network and the different layers of networks describe the different modes of interaction. Various properties such as random walks [14] on these networks, eigenvalues [15] and the eigenvector structure of these networks, the spread of infection on such networks, etc., have been investigated.

In this paper, we study a simplified model of multilayer networks where all layers have the same type of connectivity within a given layer. Every site is connected to the sites in the next layer in a unidirectional manner. We study the contact process on this network. For low infection probability p, the infection dies down and the fraction of infected individuals goes to zero. For higher p, this fraction tends to be a constant. Usually, this is an absorbing transition in the universality class of directed percolation (DP). We find that the nature of decay of the order parameter at the p_c changes from layer to layer. Interestingly, for an Erdös-Réyni network, we observe a power-law decay of the order parameter with different exponents for different layers. On the other hand, for one- or two-dimensional (1D or 2D) basic networks, we find that the decay is well described by a stretched exponential for all but the top layer at the p_c.

We note that the DP universality class is extremely robust against perturbations. The Janssen-Grassberger conjecture stated the conditions for a DP transition to a nondegenerate absorbing state with a single scalar order parameter. The prerequisites are short-range interactions in space and time, and the absence of frozen randomness or multicritical points. It has been found that the DP class is very robust and observed even in the presence of memory, quenched disorder, infinitely many absorbing states, or unidirectional coupling [16–18]. There are very few universality classes known for transitions to an absorbing state such as a directed Ising class or voter class, dynamical percolation, etc. [19–21]. Thus it is quite surprising that this perturbation leads to a different universality class in the Erdös-Réyni network.

II. MODEL

First, we consider a multiplex network with L layers each having N sites. Each layer has an Erdös-Réyni network, i.e., each site is coupled to k randomly chosen sites in the same layer for the top layer and the same connectivity is repeated for all L layers. Each site is connected to the previous layer unidirectionally. Each mth site in the lth layer is connected to the mth site in the $(l-1)$th layer of the lattices in a unidirectional way for $l > 1$. The top layer $(l = 1)$ is not connected to the layer. The representative picture of the Erdös-Réyni network topology for $L = 2$ and $k = 2$ is shown in Fig. 1(a). We have also studied the case of a Cartesian lattice as a network for the top layer in later sections. Figures 1(b) and 1(c) show a representative multiplex structure for a 1D network $(L = 4)$ and a 2D network with $(L = 2)$. We have carried out

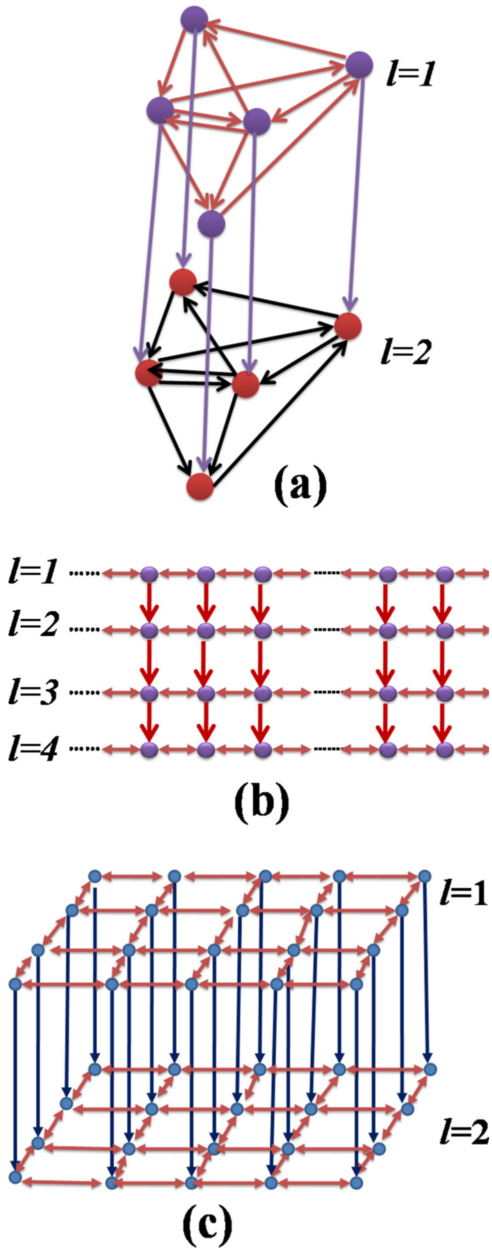

FIG. 1. Topological representation of a multiplex structure with (a) an identical Erdös-Réyni network on each layer ($k = 2$, two layers), (b) a 1D network at each layer (four layers), and (c) a 2D network for each layer (two layers).

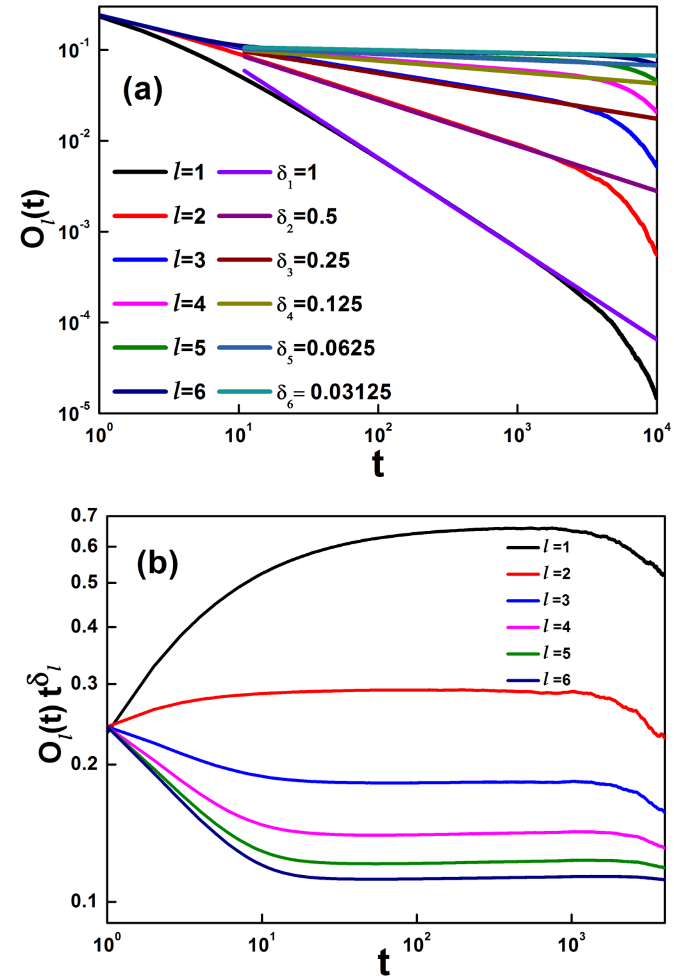

FIG. 2. (a) Plot of order parameter $O_l(t)$ vs t for various layers (from bottom to top) of an Erdös-Réyni network at $p = p_c = 0.25$ for $N = 8 \times 10^6$. The decay exponent is given by $\delta_l = 2^{1-l}$ and the fit is shown. (b) Plot of $O_l(t)t^{\delta_l}$ vs t (from top to bottom) for the same system. This quantity is a constant in time, confirming δ_l.

extensive numerical simulations for the contact process on the above Erdös-Réyni multiplex network for $k = 4$. We associate the variable $x_m^l(t)$ to the mth site on the lth layer of this NL-dimensional multiplex where L is the number of layers, each of which has N sites. Initially, we assign $x_m^l(0) = 0$ or $x_m^l(0) = 1$ with equal probability. Let $s_m^l(t)$ be the sum of $x_{m'}^l(t)$ where m' is connected to m. The evolution proceeds in a synchronous manner as $x_m^l(t+1) = 1$ with probability p if $s_m^l(t) \neq 0$ and 0 otherwise. Thus the sites become active with probability p if any of its neighbors is active. Being a contact process, this model shows the absorbing state transition. If all sites in the multiplex become inactive, they remain so forever. Furthermore, we observe another feature due to the unidirectional connection between layers. If the top layer becomes inactive, it remains so forever because it is not connected to any other layer. Similarly, if all sites in the top two layers become inactive, they stay inactive regardless of the presence of active sites in the next layers. On the other hand, the lth inactive layer can become active if there are active sites in any kth layer such that $k < l$. The activity persists only for $p > p_c$, which is a well-defined quantity in the thermodynamic limit. We expect the value of p_c to be the same for the entire lattice as it is for the top layer. The reason is simple. Below p_c, the top layer will become inactive. Now the second layer is the top layer for all practical purposes and will become inactive. When it becomes inactive, the third layer is the top layer which will be inactive for $p < p_c$. Thus the entire multiplex is expected to become inactive for $p < p_c$.

For an Erdös-Réyni network with k neighbors, we expect the absorbing state for $kp < 1$ in the mean-field limit. Thus we estimate $p_c = 1/k$. For $k = 4$, we numerically obtain $p_c = 0.250\,00 \pm 0.000\,15$ which is close to this approximation [22–24]. The dynamic phase transition for such

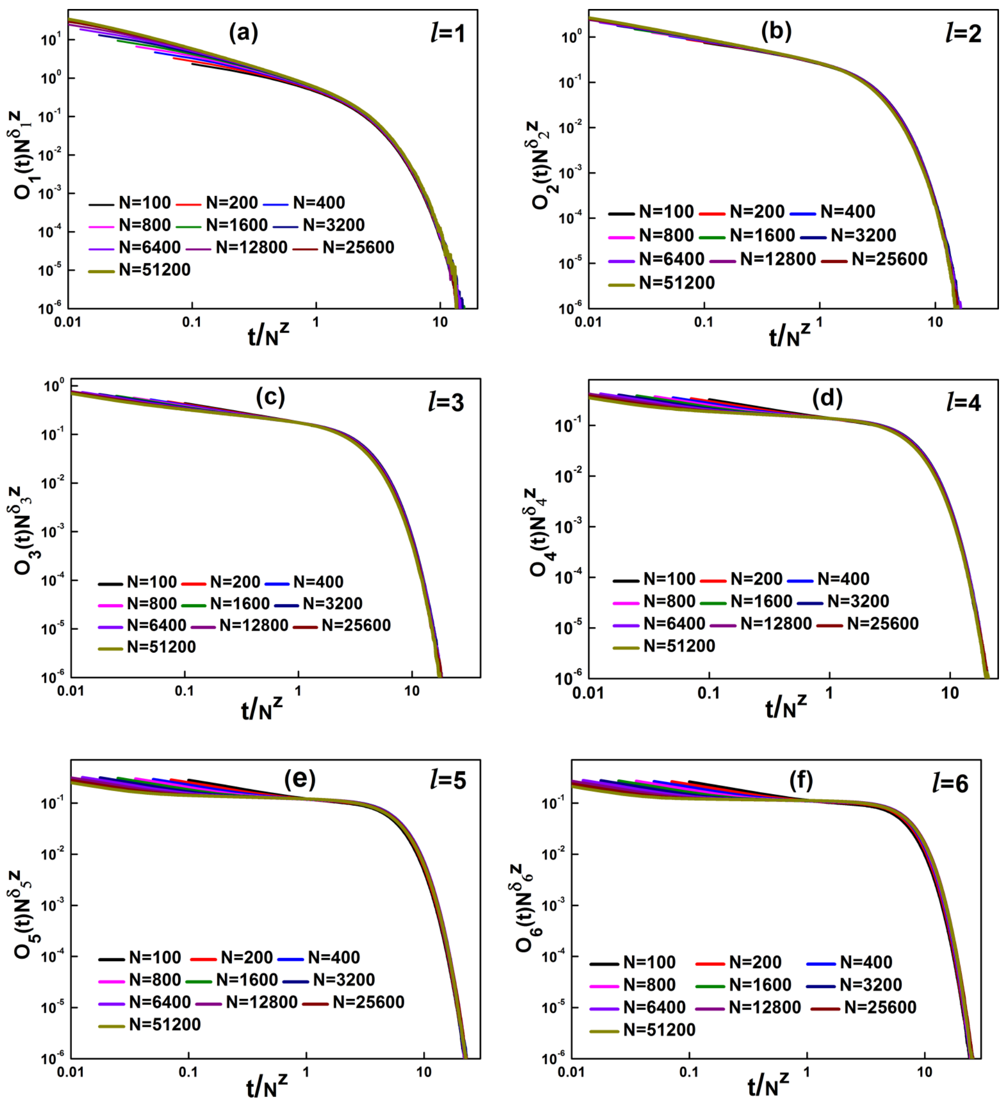

FIG. 3. For an Erdös-Réyni network, finite-size scaling is obtained by plotting $O_l(t)N^{z\delta_l}$ as a function of t/N^z for different system sizes N at $p = p_c = 0.25$ where $\delta_l = 2^{1-l}$ and $\delta_1 = 1$. The value of the dynamic exponent $z = 0.5$ is the same for all layers. (a) $l = 1$, (b) $l = 2$, (c) $l = 3$, (d) $l = 4$, (e) $l = 5$, and (f) $l = 6$.

connectivity is expected to be in the mean-field universality class. For nonequilibrium phase transitions, this expectation is not always fulfilled [25,26].

The p_c as well as the critical exponents for the absorbing phase transition in the top layer must be in the same universality class as the absorbing phase transition for that connectivity. As mentioned above, this is also the p_c for the entire multiplex structure. However, we may question how the critical exponents (if any) change for $l > 1$.

A. Erdös-Réyni network

We study the six-layer Erdös-Réyni network in which we study the absorbing phase transition using the order parameter $O_l(t)$ which is a fraction of active sites in the lth layer as a quantifier. We simulate the network for $N = 8 \times 10^6$ and average over more than 400 configurations. We indeed observe a power-law decay of the order parameter at $p = p_c$ for all l. The order parameter $O_l(t) \sim t^{-\delta_l}$ for each layer $p = p_c$. The power-law exponent value for the top layer is close to $\delta_1 = 1$, which is a mean-field value. For $l = 2$, $\delta_2 = 0.5$. We observe that the exponent for the lth layer is half of the exponent for the $(l-1)$th layer, i.e., $\delta_l = \delta_{l-1}/2$ $(l > 1)$. Due to a continuous infusion of infection from the layers above, the inactivation rate becomes slower for larger l. This is shown in Fig. 2(a). An excellent power law is obtained with $\delta_l = 2^{1-l}$. This behavior is confirmed by plotting $O_l(t)t^{\delta_l}$ as a function of time t and independent fits [see Fig. 2(b)]. These values are confirmed within 1%. We note that $O_l(t)t^{\delta_l}$ is constant in time over a few decades. While the exponent in the $l = 1$ is in the mean-field class, we report on the other exponents here. We study the finite-size scaling at the critical point for different layers. We simulate for $N = 2^m \times 100$ for $m = 0$–9. We average over at least 2×10^6 for $N \leqslant 12\,800$ and over 2×10^5 configurations for $N \geqslant 25\,600$. We obtain finite-size scaling for every layer in the network. The dynamic exponent z for all layers is the same and has the value $z = 0.5$. This scaling is shown in Fig. 3.

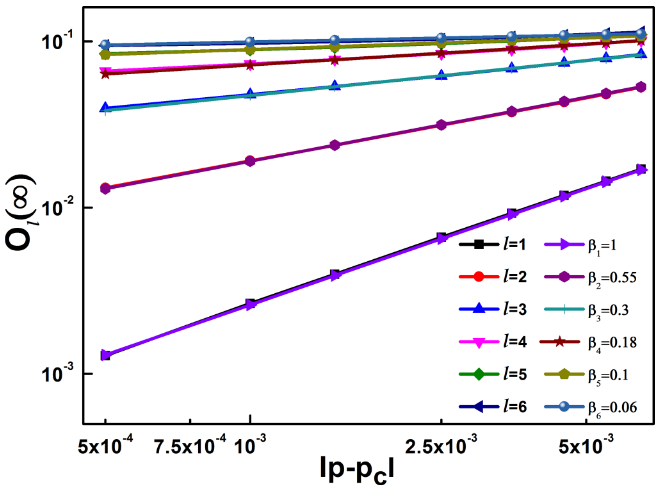

FIG. 4. $O_l(\infty)$ is plotted for various values of Δ ranging from 0.0005 to 0.0065 for various layers (from bottom to top). The behavior can be approximated as $O_l(\infty) \propto \Delta^{\beta_l}$ with $\beta_l = \nu_{\parallel,l}\delta_l$, and $\nu_{\parallel,1} = 1$, $\nu_{\parallel,2} = 1.1$, $\nu_{\parallel,3} = 1.2$, $\nu_{\parallel,4} = 1.44$, $\nu_{\parallel,5} = 1.6$, $\nu_{\parallel,6} = 1.92$.

We expect the asymptotic value of the order parameter to scale as $O_l(\infty) \propto \Delta^{\beta_l}$ where $\Delta = |p - p_c|$ and $O_l(\infty)$ is the fraction of active sites in the lth layer. We note that $\beta_l = \nu_{\parallel,l}\delta_l$ where $\nu_{\parallel,l}$ is related to divergence of the correlation time $\xi_\parallel$ close to criticality as $\xi_\parallel \sim (p - p_c)^{-\nu_\parallel}$. We carry out simulations for $N = 8 \times 10^6$ and average over more than 80 configurations (see Fig. 4). [We fit the function $O_l(\infty) \propto a\Delta^b$ using the fit function in GNUPLOT and the values of b_l obtained are 0.99 ± 0.002, 0.54 ± 0.002, 0.29 ± 0.003, 0.17 ± 0.005, 0.10 ± 0.005, and 0.07 ± 0.005 for $l = 1$–6, respectively.] We find that $\nu_{\parallel,1} = \delta_1 = 1$ for the first layer, which is a mean-field value. However, for $l \neq 1$, $\nu_{\parallel,l} > 1$ and $\beta_l \neq \delta_l$. In fact, $\nu_{\parallel,l} \to 2$ for higher layers.

To understand this behavior, we write mean-field equations for various layers. The mean-field equation for directed percolation can be derived as follows. Let the rate of offspring production reaction $A \to 2A$ be μ_p, annihilation reaction $A \to 0$ occur with rate μ_r, and coalescence reaction $2A \to A$ occur with rate μ_c. Thus there an effective birth rate $\tau = \mu_p - \mu_r$ which is linear in nature and a quadratic loss term with rate $g = \mu_c$. The mean-field equation for directed percolation is given by Eq. (3.6) in Ref. [27] as $\partial_t\rho_1(t) = \tau\rho_1(t) - g\rho_1(t)^2$.

For the critical point $\tau = 0$, $\rho_1(t) = \frac{1}{c+gt}$, where $c = [\rho_1(0)]^{-1}$. Thus $\delta_1 = 1$. For $\tau > 0$, $\rho_1(t) \sim \frac{\tau}{g}$ as $t \to \infty$, implying $\beta_1 = 1$ and hence $\nu_{\parallel,1} = 1$. These are exponents in the mean-field limit. We heuristically write equations for different layers as

$$\begin{aligned}
\partial_t\rho_1(t) &= \tau\rho_1(t) - g\rho_1(t)^2,\\
\partial_t\rho_2(t) &= \tau\rho_2(t) - g\rho_2(t)^2 + \rho_1(t)\\
&\vdots\\
\partial_t\rho_l(t) &= \tau\rho_l(t) - g\rho_l(t)^2 + \rho_{l-1}(t)\\
&\vdots\\
\partial_t\rho_L(t) &= \tau\rho_L(t) - g\rho_L(t)^2 + \rho_{L-1}(t).
\end{aligned} \tag{1}$$

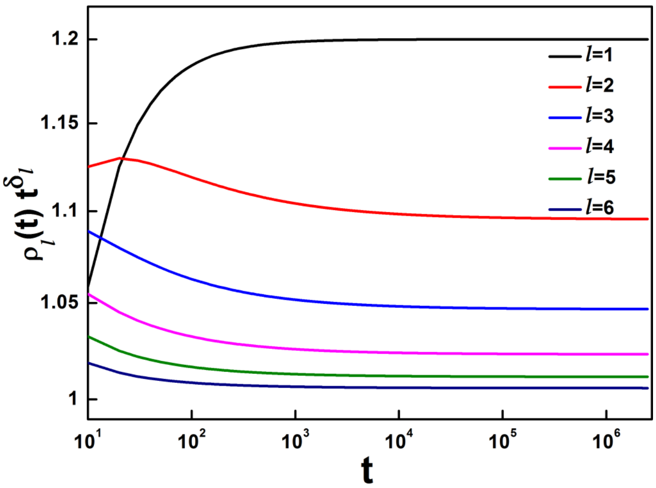

FIG. 5. $t^{\delta_l}\rho_l(t)$ is plotted as a function of time t for various layers (from top to bottom).

We simulate these equations at the critical point $\tau = 0$ using the fourth-order Runge-Kutta method with $h = 0.01$ with $\rho_i(0) = 0.9$ for $1 \leqslant i \leqslant L$. Asymptotically, we observe a power-law decay of the order parameter as $\rho_l(t) \sim t^{-\delta_l}$ with $\delta_l = 2^{1-l}$. The plots are shown in Fig. 5. Thus the hierarchy of mean-field equations explains the order density decay exponent at $p = p_c$ very well.

However, for $\tau > 0$, the behavior does not match with the Erdös-Réyni multiplex described above. We propose $\rho_l(\infty) \propto \tau^{\beta_l}$ and obtain $\beta_l = \delta_l$. The above equations yield $\nu_{\parallel,l} = 1$ for all layers which is the mean-field value. But the Erdös-Rényi network shows larger values of β_l for $l > 1$. The reason may be long crossover times or the mean-field limit may be approached for very large values of k. It has been shown that the nonequilibrium system networks with random nonlocal connectivity do not necessarily show a transition in the mean-field class [25,26].

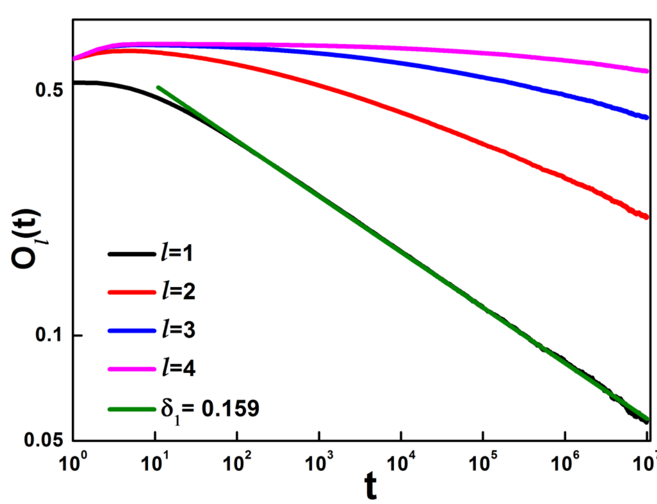

FIG. 6. Plot of $O_l(t)$ as function of time t for a 1D network (from bottom to top) for $N = 5 \times 10^5$ and $p = p_c = 0.705\,485\,15$. The order parameter decay exponent for the top layer is $\delta_1 = 0.159$.

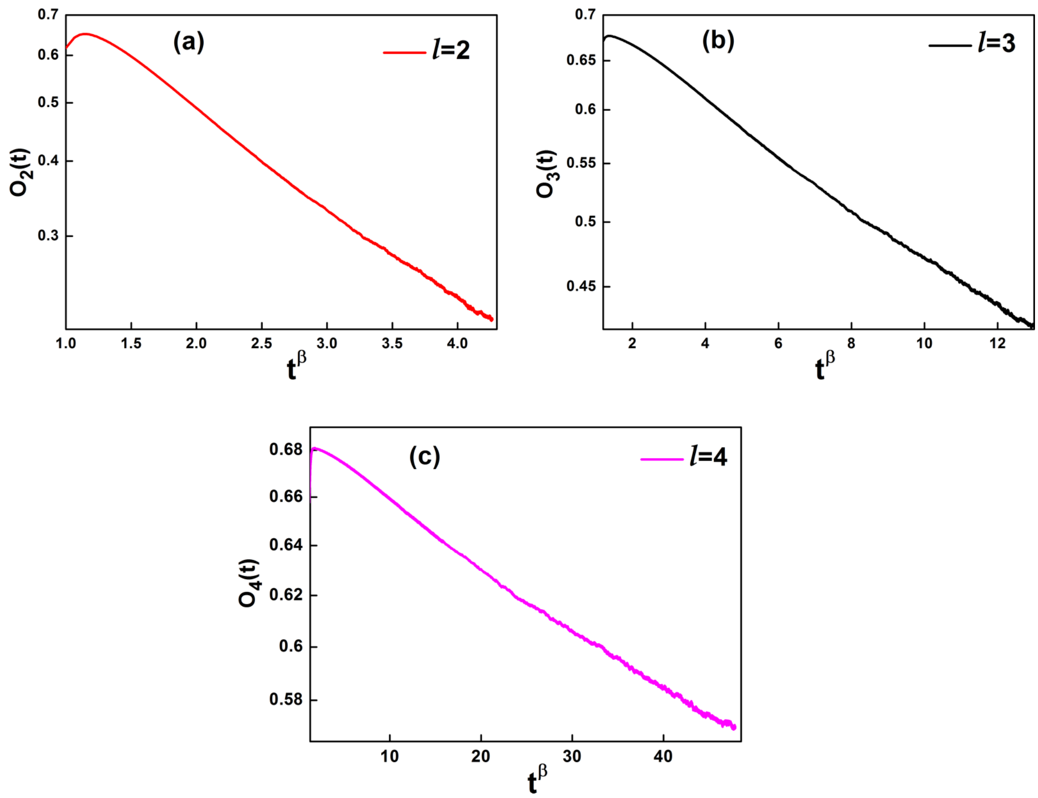

FIG. 7. For a 1D network, we plot $O_l(t)$ vs t^β on a semilogarithmic scale for $l \neq 1$ at $p = p_c$. A clear straight line shows that the decay is well described by a stretched exponential. (a) $l = 2$, $\beta = 0.09$, (b) $l = 3$, $\beta = 0.16$, and (c) $l = 4$, $\beta = 0.24$.

B. One-dimensional network

Now we consider the case in which each layer has internal connections such as a d-dimensional Cartesian lattice. Consider the case of a 1D lattice and $L = 4$ layers. We simulate for $N = 5 \times 10^5$ and averaged over 80 configurations. The top layer which is a 1D lattice undergoes a DP transition at $p = p_c = 0.705\,485\,15$ [27] and shows a power-law decay of $O_1(t)$ with a critical exponent $\delta_1 = 0.159$ (see Fig. 6) which is in the 1D DP class. The value of p_c is the same for all layers. However, $O_l(t)$ for $l > 1$ is not a power-law decay. It is better fitted with a stretched exponential as $O_l(t) \propto \exp(-B_l x^{c_l})$. The value of c_l increases with l (see Fig. 7) and $c_2 = 0.09$, $c_3 = 0.16$, and $c_4 = 0.24$ within 3%. This behavior is confirmed by fitting using standard software such as ORIGIN [28] and using a fit function in GNUPLOT which uses an implementation of the nonlinear least-squares (NLLS) Marquardt-Levenberg algorithm [29].

C. Two-dimensional network

In 2D, we simulate an $N \times N$ lattice in a given layer with $N = 3 \times 10^3$ at $p = p_c = 0.344\,57$ [27]. We averaged over 105 configurations and consider four layers. The $O_1(t)$ decays with exponent $\delta_1 = 0.45$ which is in a two-dimensional DP class (see Fig. 8). However, for $l > 1$, $O_l(t)$ is better described by a stretched exponential decay $\exp(-B_l x^{c_l})$ as in the 1D case. We obtain $c_2 = 0.1$, $c_3 = 0.19$, and $c_4 = 0.32$ within 1% (see Fig. 9). For $l = 4$, the curvature indicates the possible presence of nonlinear corrections to the stretched exponential fit.

III. SUMMARY

In this paper, we discussed three systems, i.e., the Erdös-Réyni, 1D, and 2D systems. In these systems, each layer is connected to the layer above it in a unidirectional manner. The top layer has no connection to any other layer. The contact process in this system is defined in the following manner. Any site becomes active with probability p if any of the connected sites is active. The critical point for the top layer is well known

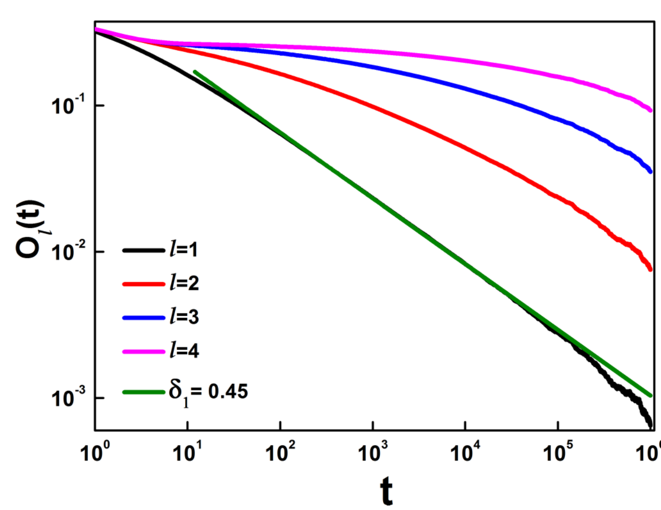

FIG. 8. Plot of $O_l(t)$ as a function of time t for a 2D network (from bottom to top) of size $N = 3 \times 10^3$ at $p = p_c = 0.344\,57$. For the first layer, the exponent $\delta_1 = 0.45$.

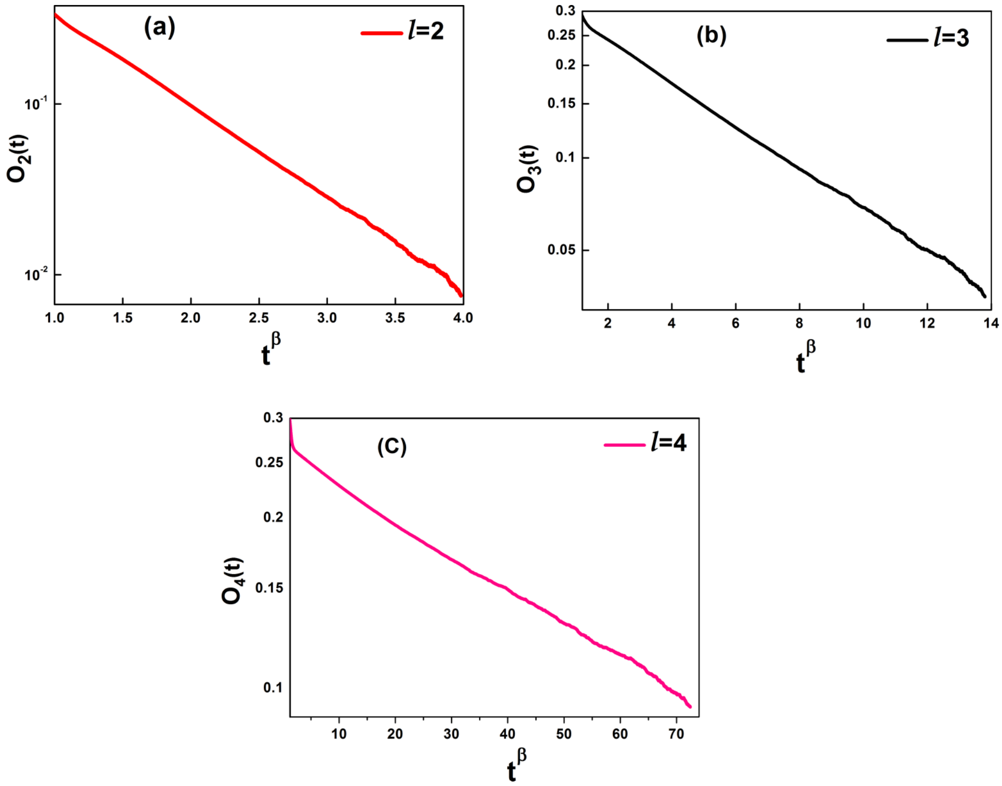

FIG. 9. Plot of $O_l(t)$ vs t^β on a semilog scale for $l \neq 1$ at $p = p_c$. Data are well fitted by a stretched exponential. (a) $l = 2$, $\beta = 0.10$, (b) $l = 3$, $\beta = 0.19$, and (c) $l = 4$, $\beta = 0.32$.

and the critical point is expected to be the same for the entire network. We compute the fraction of active sites $O_l(t)$ in a given layer l as an order parameter.

(a) In an Erdös-Réyni network, we find that there is a power-law decay of the order parameter at each layer for $p = p_c$ and the decay exponent is half of the previous layer. Since a well-defined order parameter decay exponent is observed, we compute other exponents such as finite-size scaling and off-critical scaling. We find that the dynamic exponent $z = 0.5$ for all layers is not the mean-field exponent. The saturation value of the order parameter for various layers scales as Δ^{β_l} where $\beta_l = \delta_l \nu_{\parallel,l}$, and even the value of $\nu_{\parallel,l} \neq 1$, except for the first layer which is a departure from the mean field. We propose a system of hierarchy of differential equations that correctly reproduces the behavior at a critical point for all layers, but not the behavior in a fluctuating phase.

(b) In 1D and 2D networks, the absorbing phase transition in the first layer leads to a power-law decay of the order parameter only in the top layer with the same exponent as DP. However, the decay is not described by the power law for other layers. It is better fitted by the stretched exponential.

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