Perturbative expansion of irreversible work in symmetric and asymmetric processes

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The systematic expansion method of the solution of the Fokker-Planck equation is developed by generalizing the formulation proposed in [J. Phys. A: Math. Theor. **50**, 325001 (2017)]. Using this method, we obtain an alternative formula to calculate the mean work perturbatively which is applicable to systems with degeneracy in the eigenvalues of the Fokker-Planck operator. This method enables us to study how the geometrical symmetry affects thermodynamic description of a Brownian particle. To illustrate the application of the derived theory, we consider the Fokker-Planck equation with a two-dimensional harmonic potential. To investigate the effect of symmetry of the potential, we study thermodynamic properties in symmetric and asymmetric deformation processes of the potential: the rotational symmetry of the harmonic potential is held in the former, but it is broken in the latter. Optimized deformations in these processes are defined by minimizing mean work. Comparing these optimized processes, we find that the difference between the symmetric and asymmetric processes is maximized when the deformation time of the potential is given by a critical time which is characterized by the relaxation time of the Fokker-Planck equation. This critical time in the mean work is smaller than that of the change of the mean energy because of the hysteresis effect in the irreversible processes.

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I. INTRODUCTION

There is no established framework to generalize the thermodynamic description to a small-fluctuating system, but such a system is often modeled using Brownian motion confined in an external potential [1–4]. We can then define mean work, heat, and entropy in the process induced by the deformation of the confinement potential, and show that these quantities satisfy the laws analogous to the first and second laws in thermodynamics. Differently from the idealized discussion in thermodynamics, the deformation of the potential is implemented within a finite time period, and thus any process is irreversible. In the construction of, for example, an efficient nanomachines, it is important to find the optimized process where the irreversible contribution in mean work is minimized [4–18].

The distribution function of a Brownian particle in the configuration space is known to be described by the Fokker-Planck equation. Therefore the thermodynamic quantities in a small-fluctuating system can be obtained by solving it. In order to find the optimized process, then, it is desirable to calculate the mean work in an analytic form [4,5] because it is difficult to calculate variation numerically. In fact, the optimization has been exclusively studied for exactly solvable models like the harmonic potential [15–28]. As applications to more general potentials, see, for example, [29,30]. In Ref. [18], the present author developed a perturbative expansion method to calculate the solution of the Fokker-Planck equation in the one-dimensional system. Using this theory, a formula to calculate the mean work perturbatively was obtained. Applying the derived perturbation theory to the one-dimensional harmonic potential, we confirmed that our perturbative calculations are consistent with the exact results.

In Ref. [18], however, the effect of degeneracy is not considered, and thus the theory is not applicable to study the systems in higher spatial dimensions. The purpose of this paper is to generalize the perturbation theory to arbitrary spatial dimensions. Such a generalization enables us to study how the geometrical symmetry of the external potential affects thermodynamic description of a Brownian particle. The external potential determines the shape of the system, and thus the generalized theory will be useful to study the surface effect in the thermodynamic behavior which vanishes in the thermodynamic limit. See also Ref. [31] as a related reference. To illustrate the application of the derived theory, we study thermodynamic description of a Brownian particle confined in a two-dimensional harmonic potential. We are interested in the effect of symmetry and hence investigate symmetric and asymmetric compression processes. The rotational symmetry of the harmonic potential is held in the former, but it is broken in the latter. Perturbative calculations are affected by these deformation processes because the degeneracy of the expansion basis (foot and head states) depends on the symmetry of the potentials. The optimized processes are found by minimizing the mean work. Comparing the optimized symmetric and asymmetric processes, we find that these behaviors are the same in the vanishing and infinity limits of the deformation time of the potential. The difference between these processes is maximized when the deformation time is given by a critical time which is characterized by the relaxation time of the Fokker-Planck equation. This critical time in the mean work is smaller than that in the change of the mean energy because of the hysteresis effect in irreversible processes.

This paper is organized as follows. In Sec. II the thermodynamic interpretation in the Fokker-Planck equation is

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summarized following stochastic energetics [4]. In Sec. III the eigenvalues and eigenfuctions of the time-dependent Fokker-Planck operator are introduced. Using these, we develop the perturbation theory in the D-dimensional Fokker-Planck equation in Sec. IV. This result is applied to find the perturbative formula to calculate the mean work in Sec. V. To illustrate the applications of the formula, we consider symmetric and asymmetric compression processes of the two dimensional harmonic potential in Secs. VI and VII, respectively. These results are numerically calculated and compared in Sec. VIII. Section IX is devoted to the concluding remarks.

II. THERMODYNAMIC INTERPRETATION IN FOKKER-PLANCK EQUATION

Before discussing the perturbation theory, we briefly summarize the thermodynamic description of the systems given by the Fokker-Planck equation. The discussion in this section is based on stochastic energetics [4].

We consider a D-dimensional Brownian particle which is confined in an external potential V and interacts with a thermal bath with temperature T. Let us define the distribution of the Brownian particle in the configuration space by.

$$\rho(\mathbf{x},t)=\int d\mathbf{R}\,\rho_0(\mathbf{R})\mathrm{E}\big[\delta^{(D)}(\mathbf{x}-\mathbf{x}_t)\big], \tag{1}$$

where D is the spatial dimension and $\mathbf{x}_t$ represents the position of the Brownian particle at t. The initial distribution of the Brownian particle is denoted by $\rho_0(\mathbf{R})$ with $\mathbf{R}$ being the position at an initial time t_i, $\mathbf{x}_{t_i}=\mathbf{R}$. The evolution of this distribution is described by the Fokker-Planck equation,

$$\begin{aligned}\partial_t\rho(\mathbf{x},t)&=\nabla\cdot\left\{\frac{1}{\nu\beta}\nabla+\frac{1}{\nu}(\nabla V(\mathbf{x},\mathbf{a}_t))\right\}\rho(\mathbf{x},t)\\&\equiv\mathcal{L}_t(\mathbf{x})\rho(\mathbf{x},t),\end{aligned} \tag{2}$$

where ν is a friction constant and $\beta=1/(k_BT)$ with k_B being the Boltzmann constant. In this work, the temperature is constant but the form of the potential V is deformed by changing the control parameters $\mathbf{a}_t=(a_t^{(1)},a_t^{(2)},\dots)$. Because of the confinement potential V, the particle distribution $\rho(\mathbf{x},t)$ vanishes quickly at an infinite distance, $\lim_{|\mathbf{x}|\to\infty}\rho(\mathbf{x},t)=0$.

As is well known, the Fokker-Planck equation (2) is reproduced from Brownian motion which is described by the stochastic differential equation [4,32],

$$d\mathbf{x}_t=-\frac{dt}{\nu}\nabla V(\mathbf{x}_t,\mathbf{a}_t)+\sqrt{\frac{2}{\beta\nu}}d\mathbf{B}_t, \tag{3}$$

where $d\mathbf{B}_t=\mathbf{B}_{t+dt}-\mathbf{B}_t$ is the inclination of the standard Wiener process $\mathbf{B}_t$ satisfying

$$\mathrm{E}\big[dB_t^i\big]=0, \tag{4}$$

$$\mathrm{E}\big[dB_t^i dB_{t'}^j\big]=dt\,\delta_{t,t'}\,\delta_{i,j}. \tag{5}$$

The ensemble average of the Wiener process is denoted by $\mathrm{E}[\cdot]$.

In this model, the Brownian particle interacts with the thermal bath through the dissipative force $(-\nu d\mathbf{x}_t/dt)$ and the random force $(\sqrt{2/\beta\nu}\,d\mathbf{B}_t/dt)$. In stochastic energetics, the heat dq_t associated with a single stochastic trajectory is interpreted as the work done by these interactions and thus given by

$$dq_t=\sum_{i=1}^{D}\left(-\nu\frac{dx_t^i}{dt}+\sqrt{\frac{2}{\beta\nu}}\frac{dB_t^i}{dt}\right)\circ dx_t^i, \tag{6}$$

where $\circ$ is the Stratonovich definition of the product,

$$A_t\circ dB_t=\frac{A_t+A_{t+dt}}{2}dB_t. \tag{7}$$

Applying Eq. (3) to the definition (6), the above heat is reexpressed as

$$dq_t=dV(\mathbf{x}_t,\mathbf{a}_t)-dw_t. \tag{8}$$

The first term on the right-hand side gives the change of the internal energy and the second term represents the work done by the deformation of the external potential, which is defined by

$$dw_t=\sum_{i=1}^{D}\frac{\partial V(\mathbf{x}_t,\mathbf{a}_t)}{\partial a_t^i}da_t^i. \tag{9}$$

Note that the relation (8) is satisfied for each stochastic trajectory

The expectation value of Eq. (8) leads to a law analogous to the first law of thermodynamics,

$$dQ_t=[E(t+dt)-E(t)]-dW_t, \tag{10}$$

where the expectation values are introduced by

$$dQ_t=\int d^D\mathbf{R}\,\rho_0(\mathbf{R})\mathrm{E}[dq_t], \tag{11}$$

$$E(t)=\int d^D\mathbf{R}\,\rho_0(\mathbf{R})\mathrm{E}[V(\mathbf{x}_t,\mathbf{a}_t)]=\int d^D\mathbf{x}\,\rho(\mathbf{x},t)V(\mathbf{x},\mathbf{a}_t), \tag{12}$$

$$\begin{aligned}dW_t&=\int d^D\mathbf{R}\,\rho_0(\mathbf{R})\mathrm{E}[dw_t]\\&=dt\int d^D\mathbf{x}\,\rho(\mathbf{x},t)\sum_{i=1}^{D}\frac{\partial V(\mathbf{x},\mathbf{a}_t)}{\partial a_t^i}\frac{da_t^i}{dt}.\end{aligned} \tag{13}$$

Using the Shannon entropy (multiplying the Boltzmann constant) defined by

$$S(t)=-k_B\int d^D\mathbf{x}\,\rho(\mathbf{x},t)\ln\rho(\mathbf{x},t), \tag{14}$$

we further find that there exists the following inequality [4]:

$$\frac{dS(t)}{dt}-\frac{1}{T}\frac{dQ_t}{dt}\geqslant 0. \tag{15}$$

The equality is satisfied for the stationary state of the Fokker-Planck equation. This inequality can be regarded as the second law of thermodynamics. Note, however, that the Shannon entropy is definable even in nonequilibrium states. Therefore the above inequality is not exactly the same as the the thermodynamic second law.

In this paper, we consider the irreversible isothermal processes where the system interacts with the thermal bath with

a fixed temperature T, and thus the mean work should satisfy the following inequality:

$$dW_t = [E(t+dt) - E(t)] - dQ_t \geqslant F_{\rm helm}(\mathbf{a}_{t+dt}) - F_{\rm helm}(\mathbf{a}_t). \tag{16}$$

This is obtained by using the inequality (15) and the Helmholtz free energy defined by

$$F_{\rm helm}(\mathbf{a}_t) = E(t) - TS(t). \tag{17}$$

Here, however, $S(t)$ is not the thermodynamic entropy but the Shannon entropy introduced above. Therefore this Helmholtz free energy is not the same as the corresponding quantity in thermodynamics.

III. EIGENVALUES AND EIGENFUNCTIONS OF TIME-DEPENDENT FOKKER-PLANCK OPERATOR

In this section, we define the expansion basis following Ref. [18]. A similar expansion basis is discussed, for example, in Ref. [32]. We generalize this method to the case of the time-dependent Fokker-Planck operator referring to the discussion in Ref. [33]. Note that the eigenvalue theory of the time-periodic Fokker-Planck operator (the Kolmogorov operator) is discussed in Ref. [34] and the properties found below are consistent with the result. See also Ref. [35].

The eigenvalues and eigenfunctions of the time-dependent Fokker-Planck operator $\mathcal{L}_t(\mathbf{x})$ are defined by

$$\mathcal{L}_t(\mathbf{x})\,\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = -\underline{\lambda}_n(t)\,\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t). \tag{18}$$

We call these eigenfunctions the *foot* states. Here the index n characterizes the eigenvalue $\underline{\lambda}_n(t)$ which is a function of time. The degeneracy associated with the index n is characterized by a set of nonnegative integers $\boldsymbol{\alpha}_n = (\alpha_{1,n}, \alpha_{2,n}, \dots)$, satisfying $0 \leqslant \alpha_{i,n} \leqslant d^{(i,n)}(\mathbf{a}_t)$. Thus the upper limit of the sums of the degeneracy is characterized by a set of integers $\mathbf{d}^{(n)}(\mathbf{a}_t) = (d^{(1,n)}(\mathbf{a}_t), d^{(2,n)}(\mathbf{a}_t), \dots)$. If there is no degeneracy, $\mathbf{d}^{(n)}(\mathbf{a}_t) = (0,0,\dots) = \mathbf{0}$ and thus $\alpha_{i,n}$ can take only zeros, $\boldsymbol{\alpha}_n = (0,0,\dots) = \mathbf{0}$.

Note that $\mathcal{L}_t(\mathbf{x})$ is not self-adjoint and the above eigenfunctions do not form a complete set. To find the complete set, we define the adjoint operator of $\mathcal{L}_t(\mathbf{x})$ by

$$\int d^D\mathbf{x}\, g(\mathbf{x})\mathcal{L}_t(\mathbf{x})f(\mathbf{x}) = \int d^D\mathbf{x}\,\big(\mathcal{L}_t^\dagger(\mathbf{x})g(\mathbf{x})\big)f(\mathbf{x}), \tag{19}$$

where $f(\mathbf{x})$ and $g(\mathbf{x})$ are smooth arbitrary functions and

$$\mathcal{L}_t^\dagger(\mathbf{x}) = \left[\frac{1}{\nu\beta}\nabla^2 - \frac{1}{\nu}[\nabla V(\mathbf{x},\mathbf{a}_t)]\cdot\nabla\right]. \tag{20}$$

The eigenfunctions of $\mathcal{L}_t^\dagger(\mathbf{x})$ are called the *head* states and defined by

$$\mathcal{L}_t^\dagger(\mathbf{x})\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = -\overline{\lambda}_n(t)\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t). \tag{21}$$

As shown soon later, the eigenvalue of the head state $\overline{\lambda}_n(t)$ is the same as that of the foot state $\underline{\lambda}_n(t)$.

Note that both of $\mathcal{L}_t(\mathbf{x})$ and $\mathcal{L}_t^\dagger(\mathbf{x})$ are characterized by a self-adjoint operator $\mathcal{H}_t(\mathbf{x})$ defined by [18,32,33]

$$e^{G(\mathbf{x},\mathbf{a}_t)/2}\mathcal{L}_t(\mathbf{x})e^{-G(\mathbf{x},\mathbf{a}_t)/2} = e^{-G(\mathbf{x},\mathbf{a}_t)/2}\mathcal{L}_t^\dagger(\mathbf{x})e^{G(\mathbf{x},\mathbf{a}_t)/2} = -\mathcal{H}_t(\mathbf{x}), \tag{22}$$

where

$$\mathcal{H}_t(\mathbf{x}) = \left[-\frac{1}{\nu\beta}\nabla^2 - \frac{1}{2\nu}[\nabla^2 V(\mathbf{x},\mathbf{a}_t)] + \frac{\beta}{4\nu}[\nabla V(\mathbf{x},\mathbf{a}_t)]^2\right], \tag{23}$$

$$G(\mathbf{x},\mathbf{a}_t) = \beta V(\mathbf{x},\mathbf{a}_t). \tag{24}$$

To give the representations of the foot and head states, we introduce the eigenfunctions of $\mathcal{H}_t(\mathbf{x})$,

$$\mathcal{H}_t(\mathbf{x})u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = \lambda_n(t)u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t). \tag{25}$$

As is well known in quantum mechanics, these eigenfunctions form a complete orthogonal set [18,36],

$$\int d^D\mathbf{x}\, u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)u_{m,\boldsymbol{\beta}_m}(\mathbf{x},\mathbf{a}_t) = \delta_{n,m}\delta_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n},$$
$$\sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)u_{n,\boldsymbol{\alpha}_n}(\mathbf{x}',\mathbf{a}_t) = \delta^{(D)}(\mathbf{x}-\mathbf{x}'), \tag{26}$$

where $\delta_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} = \delta_{\alpha_{1n},\beta_{1,n}}\,\delta_{\alpha_{2,n},\beta_{2,n}}\,\dots$. Note that the eigenvalues $\lambda_n(t)$ are nonnegative real numbers as shown later and thus $u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$ can be chosen to be real functions.

Using these results, we construct the foot and head states. The set of the foot and head states form a biorthogonal system and satisfies the following orthogonal and completeness conditions:

$$\int d^D\mathbf{x}\,\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)\underline{\rho}_{m,\boldsymbol{\beta}_m}(\mathbf{x},\mathbf{a}_t) = \delta_{n,m}\delta_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_m},$$
$$\sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} \underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x}',\mathbf{a}_t) = \delta^{(D)}(\mathbf{x}-\mathbf{x}'). \tag{27}$$

Because of these properties, the solution of the Fokker-Planck equation can be expanded using the foot and head states. The foot and head states are represented by

$$\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = \sqrt{\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)}e^{-G(\mathbf{x},\mathbf{a}_t)/2}u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t),$$
$$\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = \frac{1}{\sqrt{\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)}}e^{G(\mathbf{x},\mathbf{a}_t)/2}u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t). \tag{28}$$

Here we introduced the real factor $\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)$. It is because, differently from quantum mechanics, the normalization condition is applied to a pair of foot and head states, and thus, for example, a foot state itself is not necessarily normalized by one.

From Eqs. (28), we find that all eigenvalues of $\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$, $\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$ and $u_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$ are equal,

$$\lambda_n(t) = \overline{\lambda}_n(t) = \underline{\lambda}_n(t). \tag{29}$$

Moreover, the foot and head states satisfy the relation

$$\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t) = \frac{1}{\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)}e^{G(\mathbf{x},\mathbf{a}_t)}\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t). \tag{30}$$

Therefore it is easy to understand that the degeneracy of $\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$ is the same as that of $\overline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x},\mathbf{a}_t)$.

For an arbitrary external potential, we can show the following properties:

(1) The smallest eigenvalue is given by zero. Thus, without loss of generality, we can set the order of the eigenvalues as

$$0 = \lambda_0(t) < \lambda_1(t) < \lambda_2(t) < \cdots . \tag{31}$$

(2) The "ground" foot and head states $\underline{\rho}_0(\mathbf{x}, \mathbf{a}_t)$ and $\overline{\rho}_0(\mathbf{x}, \mathbf{a}_t)$ have the eigenvalue λ_0. These states are not degenerate and given by

$$\underline{\rho}_{0,\mathbf{0}}(\mathbf{x}, \mathbf{a}_t) = \sqrt{\mathcal{N}_{0,\mathbf{0}}(\mathbf{a}_t)}\frac{1}{\sqrt{\mathcal{Z}(\mathbf{a}_t)}}e^{-\beta V(\mathbf{x},\mathbf{a}_t)},$$
$$\overline{\rho}_{0,\mathbf{0}}(\mathbf{x}, \mathbf{a}_t) = \frac{1}{\sqrt{\mathcal{N}_{0,\mathbf{0}}(\mathbf{a}_t)}}\frac{1}{\sqrt{\mathcal{Z}(\mathbf{a}_t)}}, \tag{32}$$

respectively. Here we introduced

$$\mathcal{Z}(\mathbf{a}_t) = \int d^D\mathbf{x}\, e^{-\beta V(\mathbf{x},\mathbf{a}_t)}. \tag{33}$$

The derivations of these properties are the same as those in one dimension [18] and are summarized in Appendix A.

When we choose

$$\mathcal{N}_{0,\mathbf{0}}(\mathbf{a}_t) = \frac{1}{\mathcal{Z}(\mathbf{a}_t)}, \tag{34}$$

the ground head state becomes trivial,

$$\overline{\rho}_{0,\mathbf{0}}(\mathbf{x}, \mathbf{a}_t) = 1. \tag{35}$$

This choice is considered in Ref. [36], where it is shown that the time evolution of the Fokker-Planck, the Kramers, and the relativistic Kramers systems can be regarded as a special case of Schrödinger's reciprocal process, and then the Jarzynski relation is reproduced from the symmetry of the Fokker-Planck operator.

A. Bra-ket representation

For the sake of simplicity, we introduce the quantum-mechanical bra-ket notation. The foot and heat states are expressed in terms of the inner products of bra-ket vectors,

$$\underline{\rho}_{n,\boldsymbol{\alpha}_n}(\mathbf{x}, \mathbf{a}_t) = \langle \mathbf{x}|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle,$$
$$\overline{\rho}_{m,\boldsymbol{\beta}_m}(\mathbf{x}, \mathbf{a}_t) = \langle \mathbf{x}|\overline{m, \boldsymbol{\beta}_m, \mathbf{a}_t}\rangle. \tag{36}$$

Here we introduce the position basis $|\mathbf{x}\rangle$ which satisfies $(|\mathbf{x}\rangle)^\dagger = \langle \mathbf{x}|$ and

$$\int d^D\mathbf{x}\, |\mathbf{x}\rangle\langle \mathbf{x}| = 1, \quad \langle \mathbf{x}|\mathbf{x}'\rangle = \delta^{(D)}(\mathbf{x} - \mathbf{x}'). \tag{37}$$

Then the conditions for the complete orthogonal set, Eq. (27), are, respectively, represented by

$$\langle\overline{m, \boldsymbol{\beta}_m, \mathbf{a}_t}|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle = \delta_{m,n}\delta_{\boldsymbol{\beta}_m,\boldsymbol{\alpha}_n},$$
$$\sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} |\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle\langle\overline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}| = 1, \tag{38}$$

where

$$(|\overline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle)^\dagger = \langle\overline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}|, \quad (|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle)^\dagger = \langle\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}|. \tag{39}$$

From Eq. (30), we find

$$|\overline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle = \frac{1}{\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)}e^{\widehat{G}(\mathbf{a}_t)}|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle, \tag{40}$$

where the operator $\widehat{G}(\mathbf{a}_t)$ is defined by

$$\langle \mathbf{x}|\widehat{G}(\mathbf{a}_t)|\mathbf{x}'\rangle = G(\mathbf{x}, \mathbf{a}_t)\delta^{(D)}(\mathbf{x} - \mathbf{x}'). \tag{41}$$

In this notation, the Fokker-Planck equation is represented by

$$\partial_t|\rho(t)\rangle = \widehat{\mathcal{L}}_t|\rho(t)\rangle, \tag{42}$$

where

$$\langle \mathbf{x}|\widehat{\mathcal{L}}_t|\mathbf{x}'\rangle = \mathcal{L}_t(\mathbf{x})\delta^{(D)}(\mathbf{x} - \mathbf{x}'). \tag{43}$$

The eigenvalue equations are then expressed as

$$\widehat{\mathcal{L}}_t|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle = -\lambda_n(t)|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle,$$
$$\widehat{\mathcal{L}}_t^\dagger|\overline{m, \boldsymbol{\beta}_m, \mathbf{a}_t}\rangle = -\lambda_m(t)|\overline{m, \boldsymbol{\beta}_m, \mathbf{a}_t}\rangle. \tag{44}$$

Here we have used Eq. (29).

One may notice that these representations in the biorthogonal basis look very similar to the non-Hermitian generalization of quantum mechanics (which is, sometimes, called $\mathcal{PT}$-symmetric quantum mechanics) [37–39]. Indeed, the operator $e^{G(\mathbf{x},\mathbf{a}_t)}/\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)$ in Eq. (30) seems to be a kind of the metric operator. Thus a head state is obtained from a foot state by multiplying it and vice versa. See also the discussion in Ref. [36].

IV. PERTURBATIVE SOLUTION OF TIME-DEPENDENT FOKKER-PLANCK EQUATION

There are various expansions to express the solution of the Fokker-Planck equation. See Refs. [40–44] and references therein. In our method, we expand the solution of the Fokker-Planck equation in terms of the foot states introduced in the previous section,

$$|\rho(t)\rangle = \sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} A_{n,\boldsymbol{\alpha}_n}(t)e^{-\int_{t_i}^t ds\,\lambda_n(s)}|\underline{n, \boldsymbol{\alpha}_n, \mathbf{a}_t}\rangle. \tag{45}$$

Note that the sums in the above expansion can be time-dependent because of $\mathbf{d}^{(n)}(\mathbf{a}_t)$. In the following calculation, however, we consider the case where the sums associated with n and $\boldsymbol{\alpha}_n$ are independent of time,

$$\sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} = \sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_{t'})} \quad (t \neq t'). \tag{46}$$

For the sake of simplicity, we further decompose the coefficient $A_{n,\boldsymbol{\alpha}_n}(t)$ as

$$A_{n,\boldsymbol{\alpha}_n}(t) = \sum_{\boldsymbol{\beta}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} [\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}C_{n,\boldsymbol{\beta}_n}(t). \tag{47}$$

Here the matrix $[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}$ is obtained by the solution of the following differential equation:

$$\partial_t[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} = -\sum_{\boldsymbol{\gamma}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} \langle \overline{n,\boldsymbol{\alpha}_n,\mathbf{a}_t}|\partial_t|\underline{n,\boldsymbol{\gamma}_n,\mathbf{a}_t}\rangle[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\gamma}_n,\boldsymbol{\beta}_n}, \tag{48}$$

with the initial condition

$$[\mathbf{M}^{(n)}(t_i)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} = \delta_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}. \tag{49}$$

Substituting this expansion into the Fokker-Planck equation, we find that the expansion coefficients are given by the solutions of the following differential equation:

$$\partial_t C_{n,\boldsymbol{\alpha}_n}(t) + \sum_{\boldsymbol{\beta}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} \sum_{m\neq n\geqslant 0} \sum_{\boldsymbol{\gamma}_m,\boldsymbol{\delta}_m=\mathbf{0}}^{\mathbf{d}^{(m)}(\mathbf{a}_t)} e^{\int_{t_i}^t ds\,[\lambda_n(s)-\lambda_m(s)]} \times \frac{[\mathbf{M}^{(n)}(t)]^{-1}_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}\left[\dot{\mathbf{L}}^{(n,m)}(t)\right]_{\boldsymbol{\beta}_n,\boldsymbol{\gamma}_m}[\mathbf{M}^{(m)}(t)]_{\boldsymbol{\gamma}_m,\boldsymbol{\delta}_m}}{\lambda_n(t)-\lambda_m(t)} \times C_{m,\boldsymbol{\delta}_m}(t) = 0, \tag{50}$$

where

$$[\dot{\mathbf{L}}^{(n,m)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_m} = \langle \overline{n,\boldsymbol{\alpha}_n,\mathbf{a}_t}|(\partial_t\widehat{\mathcal{L}}_t)|\underline{m,\boldsymbol{\beta}_m,\mathbf{a}_t}\rangle, \tag{51}$$

and the inverse matrix is defined by

$$\sum_{\boldsymbol{\gamma}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} [\mathbf{M}^{(n)}(t)]^{-1}_{\boldsymbol{\alpha}_n,\boldsymbol{\gamma}_n}[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\gamma}_n,\boldsymbol{\beta}_n} = \delta_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}. \tag{52}$$

It should be noted that the coefficient $C_{0,\mathbf{0}}(t)$ is time-independent,

$$C_{0,\mathbf{0}}(t) = C_{0,\mathbf{0}}(t_i), \tag{53}$$

because

$$[\dot{\mathbf{L}}^{(0,m)}(t)]_{0,\boldsymbol{\beta}_m} \propto \int d^Dx\,\boldsymbol{\nabla}\cdot\{\underline{\rho_{m,\boldsymbol{\beta}_m}}(\mathbf{x},t)\boldsymbol{\nabla}[\partial_t V(\mathbf{x},\mathbf{a}_t)]\} = 0. \tag{54}$$

A. Evolution from equilibrium state

In the following, we focus on the time evolution from a thermal equilibrium state with the control parameters $\mathbf{a}_{t_i}$,

$$\rho_{eq}(\mathbf{x},t_i) = \frac{1}{\mathcal{Z}(\mathbf{a}_{t_i})}e^{-\beta V(\mathbf{x},\mathbf{a}_{t_i})}, \tag{55}$$

where $\mathcal{Z}(\mathbf{a}_{t_i})$ is defined by Eq. (33). The expansion in terms of the foot state leads to

$$\rho_{eq}(\mathbf{x},t_i) = \sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} C_{n,\boldsymbol{\alpha}_n}(t_i)\langle\mathbf{x}|\underline{n,\boldsymbol{\alpha}_n,\mathbf{a}_{t_i}}\rangle, \tag{56}$$

where

$$C_{0,\mathbf{0}}(t_i) = \frac{1}{\sqrt{\mathcal{N}_{0,\mathbf{0}}(t_i)}}\frac{1}{\sqrt{\mathcal{Z}(\mathbf{a}_{t_i})}} = \langle\overline{0,\mathbf{0},\mathbf{a}_{t_i}}|\mathbf{x}\rangle$$
$$C_{n,\boldsymbol{\alpha}_n}(t_i) = 0 (n>0). \tag{57}$$

For this initial condition, Eq. (50) is further simplified. Let us reexpress the expansion coefficient $C_{n,\boldsymbol{\alpha}_n}(t)$ as

$$C_{n,\boldsymbol{\alpha}_n}(t) = \langle\overline{0,\mathbf{0},\mathbf{a}_{t_i}}|\mathbf{x}\rangle D_{n,\boldsymbol{\alpha_n}}(t). \tag{58}$$

Then, by solving Eq. (50) formally, the expansion coefficient $D_{n,\boldsymbol{\alpha}_n}(t)$ is determined by solving the following equation iteratively:

$$D_{n,\boldsymbol{\alpha}_n}(t) = D_{n,\boldsymbol{\alpha}_n}(t_i) - \int_{t_i}^t ds \sum_{\boldsymbol{\beta}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_s)} \sum_{m\neq n\geqslant 0} \sum_{\boldsymbol{\gamma}_m,\boldsymbol{\delta}_m=\mathbf{0}}^{\mathbf{d}^{(m)}(\mathbf{a}_s)} \times e^{\int_{t_i}^s ds'\,[\lambda_n(s')-\lambda_m(s')]} \times \frac{[\mathbf{M}^{(n)}(s)]^{-1}_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}[\dot{\mathbf{L}}^{(n,m)}(s)]_{\boldsymbol{\beta}_n,\boldsymbol{\gamma}_m}[\mathbf{M}^{(m)}(s)]_{\boldsymbol{\gamma}_m,\boldsymbol{\delta}_m}}{\lambda_n(s)-\lambda_m(s)} \times D_{m,\boldsymbol{\delta}_m}(s), \tag{59}$$

with the initial conditions

$$D_{n,\boldsymbol{\alpha}_n}(t_i) = \delta_{n,0}\delta_{\boldsymbol{\alpha}_n,\mathbf{0}}. \tag{60}$$

Again, note that $D_{0,\mathbf{0}}(t)$ is independent of time, $D_{0,\mathbf{0}}(t) = D_{0,\mathbf{0}}(t_i)$, because of Eq. (53). In the following calculations, we use exclusively this expansion coefficient $D_{n,\boldsymbol{\alpha}_n}(t)$.

B. Pseudodensity matrix

Let us introduce the pseudodensity matrix defined by

$$\widehat{\rho}(t) = \sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(t)} e^{-\int_{t_i}^t ds\,\lambda_n(s)}|\underline{n,\boldsymbol{\alpha}_n,\mathbf{a}_t}\rangle \times [\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}D_{n,\boldsymbol{\beta}_n}(t)\langle\overline{0,\mathbf{0},\mathbf{a}_{t_i}}|. \tag{61}$$

This satisfies the following properties:

$$\mathrm{Tr}[\widehat{\rho}(t)] = 1, \tag{62}$$

$$\widehat{\rho}^2(t) = \widehat{\rho}(t). \tag{63}$$

To show the second equation, we have used

$$\langle\overline{0,\mathbf{0},\mathbf{a}_{t_i}}| = [\mathbf{M}^{(0)}(t)]^{-1}_{\mathbf{0},\mathbf{0}}\langle\overline{0,\mathbf{0},\mathbf{a}_t}|. \tag{64}$$

The expectation values are represented by the trace with the pseudo density matrix. For example, the mean work done by the deformation of the external potential is given by

$$W = \int dt\,\mathrm{Tr}[(\partial_t\widehat{V}_t)\widehat{\rho}(t)]. \tag{65}$$

V. FORMULA FOR MEAN WORK

Let us consider the time evolution from a thermal equilibrium state which is induced by changing the control parameters $\mathbf{a}_t$ in a finite time interval $t_i \leqslant t \leqslant t_f$ where t_f is a finial time. The mean work done by the deformation of the potential is given by the time integral of Eq. (13). Using

Eq. (61), we find

$$W = \int_{t_i}^{t_f} dW_t = \int_{t_i}^{t_f} dt \sum_{n\geqslant 0} \sum_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n=\mathbf{0}}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} e^{-\int_{t_i}^{t} d\,\lambda_n(s)} \times [\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} D_{n,\boldsymbol{\beta}_n}(t) \langle \overline{0,\mathbf{0},\mathbf{a}_{t_i}}|(\partial_t \widehat{V}_t)|\underline{n,\boldsymbol{\alpha}_n,\mathbf{a}_t}\rangle. \quad (66)$$

This formula is the generalization of Eq. (38) of Ref. [18] to the systems in higher spatial dimensions. Indeed, this is reduced to Eq. (38) by setting $[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} = e^{-\theta_n(t)}\delta_{\boldsymbol{\alpha}_n,\mathbf{0}}\delta_{\boldsymbol{\beta}_n,\mathbf{0}}$. The role of $\theta_n(t)$ is discussed in Sec. V C.

To expand the right-hand side in terms of the time derivative of the control parameters, $\dot{\mathbf{a}}_t$, we introduce the expansions of the mean work W and the expansion coefficient $D_{n,\boldsymbol{\beta}_n}(t)$,

$$W = W^{(1)} + W^{(2)} + \cdots, \quad (67)$$

$$D_{n,\boldsymbol{\beta}_n}(t) = D^{(0)}_{n,\boldsymbol{\beta}_n}(t) + D^{(1)}_{n,\boldsymbol{\beta}_n}(t) + \cdots, \quad (68)$$

where $W^{(m)}$ and $D^{(m)}_{n,\boldsymbol{\beta}_n}(t)$ indicate the mth order terms of $\dot{\mathbf{a}}_t$. From the initial condition (60), we find

$$D^{(0)}_{n,\boldsymbol{\beta}_n}(t) = \delta_{n,0}\delta_{\boldsymbol{\beta}_n,\mathbf{0}} \quad D^{(m)}_{0,\mathbf{0}}(t) = 0 \quad (m \geqslant 1). \quad (69)$$

A. Lowest order contribution $W^{(1)}$

The lowest order contribution in the mean work is given by $W^{(1)}$,

$$W^{(1)} = \int_{t_i}^{t_f} dt\, [\mathbf{M}^{(0)}(t)]_{\mathbf{0},\mathbf{0}} \langle \overline{0,\mathbf{0},\mathbf{a}_{t_i}}|(\partial_t \widehat{V}_t)|\underline{0,\mathbf{0},\mathbf{a}_t}\rangle. \quad (70)$$

This quantity is calculated exactly without referring to the detailed behavior of the control parameters $\mathbf{a}_t$, and given by the difference of the free energies,

$$W^{(1)} = F(\mathbf{a}_{t_f}) - F(\mathbf{a}_{t_i}), \quad (71)$$

where

$$F(\mathbf{a}_t) = -\frac{1}{\beta}\ln \mathcal{Z}(\mathbf{a}_t) = -\frac{1}{\beta}\ln\left(\int d^D x\, e^{-\beta V(\mathbf{x},\mathbf{a}_t)}\right). \quad (72)$$

It should be emphasized that this free energy is not the Helmholtz free energy defined by Eq. (17). We see that $W^{(1)}$ is determined only by the initial and final values of $\mathbf{a}_t$ and independent of the intermediate deformation processes of $V(\mathbf{x},\mathbf{a}_t)$.

By definition, the initial state is given by a thermal equilibrium state, and thus this free energy agrees with the Helmholtz free energy (17) at the initial time t_i, $F(\mathbf{a}_{t_i}) = F_{helm}(\mathbf{a}_{t_i})$. In general, $F(\mathbf{a}_t)$ for $t > t_i$ does not necessarily agree with $F_{helm}(\mathbf{a}_t)$. However, in the quasistatic process, the final state is approximately given by a thermal equilibrium state, and then the right-hand side of Eq. (71) is identified with the change of the Helmholtz free energy. This is the well-known behavior in the reversible process.

B. Higher order contributions

All higher order terms in the mean work $W^{(n)}$ $(n \geqslant 2)$ give only irreversible contributions. The lowest order term $W^{(2)}$ is obtained by using $D^{(1)}_{n,\boldsymbol{\beta}_n}(t)$,

$$W^{(2)} = \int_{t_i}^{t_f} dt \sum_{n\geqslant 1} \sum_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n=0}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} e^{-\int_{t_i}^{t} ds\,\lambda_n(s)} \times [\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n} D^{(1)}_{n,\boldsymbol{\beta}_n}(t) \langle \overline{0,\mathbf{0},\mathbf{a}_{t_i}}|(\partial_t \widehat{V}_t)|\underline{n,\boldsymbol{\alpha}_n,\mathbf{a}_t}\rangle, \quad (73)$$

where

$$D^{(1)}_{n,\boldsymbol{\beta}_n}(t) = -(1-\delta_{n,0}) \int_{t_i}^{t} ds \sum_{\boldsymbol{\gamma}_n=0}^{\mathbf{d}^{(n)}(\mathbf{a}_s)} e^{\int_{t_i}^{s} ds'\,\lambda_n(s')} \times \frac{[\mathbf{M}^{(n)}(s)]^{-1}_{\boldsymbol{\beta}_n,\boldsymbol{\gamma}_n}[\dot{\mathbf{L}}^{(n,0)}(s)]_{\boldsymbol{\gamma}_n,\mathbf{0}}[\mathbf{M}^{(0)}(s)]}{\lambda_n(s)}. \quad (74)$$

The factor $(1-\delta_{n,0})$ is due to the property (60). When $\widehat{V}_t$ is given by the harmonic potential, higher order terms $W^{(n)}$ $(n > 2)$ vanish and the exact mean work is given by

$$W = W^{(1)} + W^{(2)}. \quad (75)$$

This is discussed in detail later.

C. Relation to Berry's phase

In quantum mechanics, it is known that the wave function acquires a phase induced by a cyclic adiabatic motion of a system. This additional phase is affected by the geometrical properties of the parameter of the Hamiltonian and called Berry's geometric phase.

In the quasistatic (adiabatic) limit, the solution of the Fokker-Planck equation is obtained by substituting $D_{n,\boldsymbol{\alpha}_n}(t) = D^{(1)}_{n,\boldsymbol{\alpha}_n}(t)$,

$$|\rho(t)\rangle = \langle \overline{0,\mathbf{0},\mathbf{a}_{t_i}}|\mathbf{x}\rangle e^{-\theta_0(t)}|\underline{0,\mathbf{0},\mathbf{a}_t}\rangle, \quad (76)$$

where

$$\theta_0(t) = -\ln[\mathbf{M}^{(0)}(t)]_{\mathbf{0},\mathbf{0}} = \int_{t_i}^{t} ds \langle \overline{0,\mathbf{0},\mathbf{a}_s}|\partial_s|\underline{0,\mathbf{0},\mathbf{a}_s}\rangle. \quad (77)$$

We observe that a nontrivial time-dependent factor $e^{-\theta_0(t)}$ appears besides the time evolution of $|\underline{0,\mathbf{0},\mathbf{a}_t}\rangle$, and this factor corresponds to Berry's geometric phase. See also the discussion in Ref. [18].

Wilczek and Zee found that Berry's phase becomes a matrix in the system with degeneracy [45]. Indeed, $[\mathbf{M}^{(n)}(t)]_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}$ is generally given by a matrix for $n \neq 0$. In the present calculation, the ground foot state is always not degenerate. Therefore, to observe the Wilczek-Zee-type phase, we should consider the time evolutions in the "excited" state of the Fokker-Planck operator.

VI. APPLICATION I: SYMMETRIC PROCESS

To illustrate the applications of the derived formula, we consider the two-dimensional harmonic potential,

$$V(\mathbf{x},\mathbf{a}_t) = \tfrac{1}{2}a^{(1)}_t x_1^2 + \tfrac{1}{2}a^{(2)}_t x_2^2. \quad (78)$$

In such a system, the degeneracy related to the rotational symmetry of the potential is characterized by a single integer. Let us consider the symmetric deformation of the

harmonic potential where $a_t^{(1)} = a_t^{(2)} = a_t$ and thus the potential has the rotational symmetry around $(x_1, x_2) = (0, 0)$ in the two-dimensional plane during the deformation process. Then $d^{(n)}(a_t)$ are given by time-independent constants, $d^{(n)}$, and thus the condition (46) is satisfied. The foot and head states are defined by

$$\langle \mathbf{x}|\underline{n, \alpha_n, \mathbf{a}_t}\rangle = \underline{\rho}_{\alpha_n}(x_1, a_t)\underline{\rho}_{n-\alpha_n}(x_2, a_t), \tag{79}$$

$$\langle \mathbf{x}|\overline{n, \alpha_n, \mathbf{a}_t}\rangle = \overline{\rho}_{\alpha_n}(x_1, a_t)\overline{\rho}_{n-\alpha_n}(x_2, a_t), \tag{80}$$

where α_n take nonnegative integers satisfying $0 \leqslant \alpha_n \leqslant d^{(n)} = n$, and

$$\underline{\rho}_n(x, a_t) = \sqrt{N_n(a_t)}e^{-\beta a_t x^2/4}u_n(x, a_t), \tag{81}$$

$$\overline{\rho}_n(x, a_t) = \frac{1}{\sqrt{N_n(a_t)}}e^{\beta a_t x^2/4}u_n(x, a_t). \tag{82}$$

Here $N_n(a_t)$ is an arbitrary real function associated with the ambiguity of the normalization of the foot states discussed below Eq. (28). We introduced

$$u_n(x_i, a_t^{(i)}) = \frac{1}{\sqrt{Z(a_t^{(i)})}}\sqrt{\frac{1}{2^n n!}}e^{-\beta a_t^{(i)} x_i^2/4}H_n(\sqrt{\beta a_t^{(i)}/2}x_i), \tag{83}$$

where $H_n(x)$ denote the Hermite polynomials and

$$Z(a) = \sqrt{\frac{2\pi}{\beta a}}. \tag{84}$$

The eigenvalues of the foot and head states are the same and given by

$$\lambda_n(t) = \frac{a_t}{\nu}n \quad (n \geqslant 0). \tag{85}$$

The lowest order contribution in the mean work $W^{(1)}$ is already calculated. The next order contribution is given by Eq. (73), which is simplified as

$$W_{sym}^{(2)} = 2\int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \frac{da_s}{ds}\left(\frac{1}{2\beta a_s^2}e^{-\int_s^t ds' \frac{2a_{s'}}{\nu}}\right)\frac{da_t}{dt}. \tag{86}$$

See Appendix B for the detailed derivation.

In the harmonic potential, the exact mean work is given by

$$W_{\rm sym} = W^{(1)} + W_{\rm sym}^{(2)}. \tag{87}$$

To understand this, note that, as seen from Eqs. (59) and (69), the expansion coefficient $D_{2,\beta_2}^{(n)}(t)$ has the following product:

$$\begin{aligned}&\sum_{m_1\neq 2}\sum_{m_2\neq m_1}\cdots\sum_{m_{n-1}\neq 0}[\mathbf{M}^{(2)}(\tau_1)]^{-1}\left[\dot{\mathbf{L}}^{(2,m_1)}(\tau_1)\right][\mathbf{M}^{(m_1)}(\tau_1)]\\ &\quad\times[\mathbf{M}^{(m_1)}(\tau_2)]^{-1}\left[\dot{\mathbf{L}}^{(m_1,m_2)}(\tau_2)\right][\mathbf{M}^{(m_2)}(\tau_2)]\cdots\\ &\quad\times[\mathbf{M}^{(m_{n-1})}(\tau_n)]^{-1}\left[\dot{\mathbf{L}}^{(m_{n-1},0)}(\tau_n)\right][\mathbf{M}^{(0)}(\tau_n)].\end{aligned} \tag{88}$$

Note that $[\dot{\mathbf{L}}^{(n,m)}(\tau)]$ has a finite contribution only when $n = m + 2$ and $n = m$, but the latter is excluded in the sums [see Eq. (B5)]. Therefore all higher order terms $D_{2,\beta_2}^{(n)}(t)$ $(n \geqslant 2)$ disappear.

The mean work in the one-dimensional harmonic potential is obtained in Ref. [18]. Comparing the one- and two-dimensional results, we find that $W_{sym}^{(2)}$ in the two-dimensional calculation is twice as large as the one-dimensional result given by Eq. (50) in Ref. [18]. The same property is found in the three-dimensional symmetric deformation of the harmonic potential. Therefore the D-dimensional result is comprehensively summarized by

$$W_{\rm sym}^{(2)} = D\int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \frac{da_s}{ds}\left(\frac{1}{2\beta a_s^2}e^{-\int_s^t ds' \frac{2a_{s'}}{\nu}}\right)\frac{da_t}{dt}, \tag{89}$$

where D is the number of the spatial dimension and can take $D = 1, 2$, or 3. This simple D dependence is understood by the fact that the harmonic potential is given by the independent sum of the each spatial component and thus the Fokker-Planck operator is represented by the sum of the independent spatial components,

$$\mathcal{L}_t = \sum_{i=1}^{D}\left(\frac{1}{\nu\beta}\partial_i^2 - \partial_i\frac{a_t x_i}{\nu}\right). \tag{90}$$

Therefore the optimization with respect to the ith spatial component does not affect that to the $j(\neq i)$-th spatial components, and the mean work is given by a linear function of D in the present symmetric process.

A. Energy

Following the thermodynamic interpretation introduced in Sec. II, the mean energy of this system is defined by Eq. (12),

$$E(t) = \mathrm{Tr}[\widehat{V}_t\widehat{\rho}(t)]. \tag{91}$$

Applying the same procedure used in the calculation of the mean work, the exact mean energy is calculated by using $D_{2,\beta_2}^{(1)}$ in the pseudodensity matrix. The result is given by

$$E_{sym}(t) = \frac{1}{\beta} + \frac{a_t}{\beta}\int_{t_i}^{t} ds\, e^{-\int_s^t ds' \frac{2a_{s'}}{\nu}}\frac{1}{a_s^2}\frac{da_s}{ds}. \tag{92}$$

Because $E_{sym}(t_i) = 1/\beta$, the change of the mean energy in the symmetric process is given by the second term of the above equation,

$$\Delta E_{\rm sym} = E_{\rm sym}(t_f) - E_{\rm sym}(t_i) = \frac{a_{t_f}}{\beta}\int_{t_i}^{t_f} ds\, e^{-\int_s^{t_f} ds' \frac{2a_{s'}}{\nu}}\frac{1}{a_s^2}\frac{da_s}{ds}. \tag{93}$$

B. Optimization in symmetric process

We are in particular interested in the optimized process which minimizes the mean work. Because $W^{(1)}$ is determined only by the initial and final values of the parameters, $\mathbf{a}_{t_i}$ and $\mathbf{a}_{t_f}$, independently of processes, it is sufficient to consider the variation of $W_{sym}^{(2)}$ by fixing the operation time $\tau_{op} = t_f - t_i$. The variation of the control parameter is defined by

$$a_t \longrightarrow a_t + \delta a_t. \tag{94}$$

We fix the initial and final forms of the harmonic potential and hence this variation satisfies the boundary conditions,

$$\delta a_{t_i} = \delta a_{t_f} = 0. \tag{95}$$

The variation of $W^{(2)}_{sym}$ leads to

$$\int_{t_i}^{t} ds\,\frac{1}{a_s^2}\frac{da_s}{ds}\partial_t e^{-\frac{2}{\nu}\int_s^t d\tau\, a_\tau} + \frac{1}{a_t^2}\int_t^{t_f} ds\,\frac{da_s}{ds}\partial_t e^{-\frac{2}{\nu}\int_t^s d\tau\, a_\tau} + \frac{2}{\nu}\int_t^{t_f} ds_2\int_{t_i}^{t} ds_1\, e^{-\frac{2}{\nu}\int_{s_1}^{s_2} d\tau\, a_\tau}\frac{1}{a_{s_1}^2}\frac{da_{s_1}}{ds_1}\frac{da_{s_2}}{ds_2} = 0. \quad (96)$$

Because of the reason discussed below Eq. (89), this integro-differential equation is the same as Eq. (59) of Ref. [18] which determines the optimized process in the one-dimensional harmonic potential. See Appendix B in Ref. [18] for the detailed derivation.

The optimized protocol is determined by solving this equation, but, so far, the exact solution is not known. Therefore we consider an approximated process in the large τ_{op} limit as is discussed in Ref. [18]. For this purpose, we reexpress $W^{(2)}_{\rm sym}$ in the adimensional form,

$$W^{(2)}_{\rm sym} = 2\int_0^1 d\tau_1\int_0^{\tau_1} d\tau_2\,\frac{d\bar{a}_{\tau_2}}{d\tau_2} \times\left[e^{-\frac{2\tau_{op}a_{t_i}}{\nu}\int_{\tau_2}^{\tau_1} d\tau_3\,\bar{a}_{\tau_3}}\frac{1}{2\beta(\bar{a}_{\tau_2})^2}\right]\frac{d\bar{a}_{\tau_1}}{d\tau_1}, \quad (97)$$

where adimensional quantities are introduced by

$$\tau = (t-t_i)/\tau_{op}, \quad (98)$$

$$\bar{a}_\tau = \frac{a_{t_i+\tau\tau_{op}}}{a_{t_i}}. \quad (99)$$

In the large τ_{op} limit, we consider the following approximation:

$$\int_0^1 d\tau_1\int_0^{\tau_1} d\tau_2\, G(\tau_1,\tau_2)e^{-\frac{2\tau_{op}a_{t_i}}{\nu}\int_{\tau_2}^{\tau_1} d\tau_3\,\bar{a}_{\tau_3}} \approx \int_0^1 d\tau_1\int_0^{\tau_1} d\tau_2\, G(\tau_1,\tau_1)e^{-\frac{2\tau_{op}a_{t_i}}{\nu}\bar{a}_{\tau_1}(\tau_1-\tau_2)} \approx \int_0^1 d\tau_1\, G(\tau_1,\tau_1)\frac{\nu}{2\tau_{op}a_i\bar{a}_{\tau_1}}, \quad (100)$$

where $G(\tau_1,\tau_2)$ is a smooth function. Using this approximation, $W^{(2)}_{\rm sym}$ is calculated as

$$W^{(2)}_{\rm sym} = \frac{\nu}{2\beta a_{t_i}\tau_{op}}\int_0^1 d\tau\,\frac{1}{(\bar{a}_\tau)^3}\left(\frac{d\bar{a}_\tau}{d\tau}\right)^2. \quad (101)$$

We minimize this approximated mean work. Applying the variation defined above, we find

$$\frac{1}{\bar{a}_\tau^3}\frac{d^2\bar{a}_\tau}{d\tau^2} - \frac{3}{2}\frac{1}{\bar{a}_\tau^4}\left(\frac{d\bar{a}_\tau}{d\tau}\right)^2 = 0. \quad (102)$$

This equation is analytically solvable and the optimized external parameter in the large τ_{op} limit is given by

$$\bar{a}_\tau = \frac{\bar{a}_1}{[\sqrt{\bar{a}_1}-(\sqrt{\bar{a}_1}-1)\tau]^2} \longrightarrow a_t = \frac{\tau_{op}^2\, a_{t_i}a_{t_f}}{[(t-t_i)\sqrt{a_{t_i}}+(t_f-t)\sqrt{a_{t_f}}]^2}. \quad (103)$$

This is the same as those in Refs. [5,15,18].

Substituting this into Eq. (101), the optimized mean work in the large τ_{op} limit is expressed as

$$\lim_{\tau_{op}\to\infty}\frac{W_{\rm sym}}{W^{(1)}} = \lim_{\tau_{op}\to\infty}\frac{W^{(1)}+W^{(2)}_{\rm sym}}{W^{(1)}} = 1+\frac{2}{\bar{a}_1\ln\bar{a}_1}\left(\sqrt{\bar{a}_1}-1\right)^2\frac{\nu}{a_{t_i}\tau_{op}}, \quad (104)$$

where, from Eq. (71),

$$W^{(1)} = -\frac{1}{\beta}\ln\frac{\mathcal{Z}(a_{t_f})}{\mathcal{Z}(a_{t_i})} = \frac{1}{\beta}\ln\bar{a}_1. \quad (105)$$

The time evolution in the large τ_{op} limit is very slow and thus $\rho(\mathbf{x},t_f)$ is approximately given by the thermal equilibrium state with a_{t_f}, $\rho_{eq}(\mathbf{x},t_f)$. Then $W^{(1)}$ in Eq. (71) corresponds to the one in the reversible process and is represented by the change of the Helmholtz free energy. One can see that the irreversible contribution of the mean work asymptotically disappears as a function of τ_{op}^{-1} in Eq. (104). It is worth emphasizing that this τ_{op} dependence is experimentally verified [46].

In a similar fashion, the change of the mean energy (93) is calculated with this approximated optimized process (103). In the large τ_{op} limit, this quantity is simplified as

$$\Delta E_{\rm sym} \approx \frac{\nu}{2\tau_{op}a_{t_i}\beta}\frac{1}{\bar{a}_1^2}\left.\frac{d\bar{a}_\tau}{d\tau}\right|_{\tau=1}. \quad (106)$$

Substituting Eq. (103) into this, the change of the mean energy is calculated as

$$\lim_{\tau_{op}\to\infty}\frac{\Delta E_{\rm sym}}{E(t_i)} = \frac{1}{\bar{a}_1}(\sqrt{\bar{a}_1}-1)\frac{\nu}{a_{t_i}\tau_{op}}. \quad (107)$$

This equation implies that there is no change of the mean energy in the quasistatic isothermal process as is the case of the ideal gas.

Later, we will discuss the asymmetric deformation of the harmonic potential and compare the results with those in the symmetric process. In the asymmetric process, we choose the initial and final parameters of the harmonic potential as $(a_{t_i}^{(1)},a_{t_i}^{(2)}) = (a_0,a_0)$ and $(a_{t_f}^{(1)},a_{t_f}^{(2)}) = (2a_0,2a_0)$, respectively. The corresponding symmetric process is realized by choosing $a_{t_i} = a_0$ and $a_{t_f} = 2a_0$. Then, from Eqs. (104) and (107), the mean work and the change of the mean energy in the large τ_{op} limit are, respectively, given by

$$\lim_{\tau_{op}\to\infty}\frac{W_{\rm sym}}{W^{(1)}} = 1+\frac{(\sqrt{2}-1)^2}{\ln 2}\frac{\tau^*}{\tau_{op}}, \quad (108)$$

$$\lim_{\tau_{op}\to\infty}\frac{\Delta E_{\rm sym}}{E(t_i)} = \frac{(\sqrt{2}-1)}{2}\frac{\tau^*}{\tau_{op}}, \quad (109)$$

where $W^{(1)} = \beta^{-1}\ln 2$ and we introduced $\tau^* = \nu/a_0$.

VII. APPLICATION II: ASYMMETRIC PROCESS

In this section, we consider the asymmetric deformation of the harmonic potential where $a_t^{(1)} \neq a_t^{(2)}$ for $t_i < t < t_f$. The initial and final values of the control parameters of the harmonic potential are fixed by $(a_{t_i}^{(1)},a_{t_i}^{(2)}) = (a_0,a_0)$ and $(a_{t_f}^{(1)},a_{t_f}^{(2)}) = (2a_0,2a_0)$, respectively. We first increase $a_t^{(1)}$ fixing $a_t^{(2)} = a_0$ along path I as shown in Fig. 1. After the

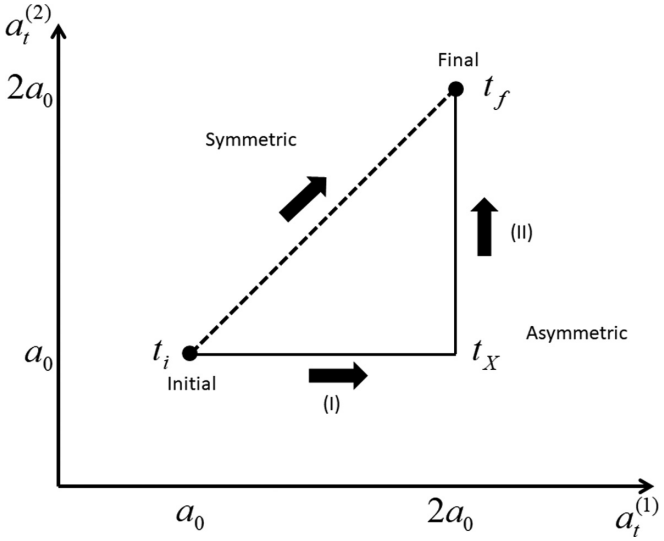


FIG. 1. Symmetric and asymmetric compression processes of the harmonic potential. The former is denoted by the dotted line and the latter by the solid line, respectively. The rotational symmetry of the harmonic potential is hold in the symmetry process where $a_t^{(1)} = a_t^{(2)}$. The asymmetric process is represented by two paths. The control parameter $a_t^{(2)}$ is fixed by a_0 on path I. At the switching time t_X, path I is switched to path II where $a_t^{(1)}$ is fixed by $2a_0$.

switching time t_X, the control parameters are changed along path II where $a_t^{(2)}$ is increased fixing $a_t^{(1)} = 2a_0$. That is, the corresponding time evolutions of the control parameters are represented by

$$\left(a_{t_f}^{(1)}, a_{t_f}^{(2)}\right) = \begin{cases} (f_t, a_0): & \text{path I} \quad t_i \leqslant t \leqslant t_X \\ (2a_0, g_t): & \text{path II} \quad t_X < t \leqslant t_f \end{cases}, \tag{110}$$

where f_t and g_t are monotonically increasing functions satisfying

$$f_{t_i} = g_{t_X} = a_0, \qquad f_{t_X} = g_{t_f} = 2a_0. \tag{111}$$

The switching time t_X and the functions f_t and g_t are determined by minimizing the mean work.

Because of the same reason discussed below Eq. (87), the exact mean work is given by the sum of $W^{(1)}$ and $W^{(2)}_{asym}$. As discussed, $W^{(1)}$ is independent of the evolution of the control parameters and given by Eq. (71), $W^{(1)} = \beta^{-1} \ln 2$. The second-order term $W^{(2)}_{asym}$ is given by the sum of the contributions from path I and path II,

$$W^{(2)}_{asym} = W^{(2)}_I + W^{(2)}_{II}, \tag{112}$$

where

$$W^{(2)}_I = \int_{t_i}^{t_X} dt \int_{t_i}^{t} ds \, \frac{df_s}{ds} \left[e^{-\int_s^t ds' \frac{2f_{s'}}{\nu}} \frac{1}{2\beta (f_s)^2} \right] \frac{df_t}{dt}, \tag{113}$$

$$W^{(2)}_{II} = \int_{t_X}^{t_f} dt \int_{t_X}^{t} ds \, \frac{dg_s}{ds} \left[e^{-\int_s^t ds' \frac{2g_{s'}}{\nu}} \frac{1}{2\beta (g_s)^2} \right] \frac{dg_t}{dt}. \tag{114}$$

See Appendix C for the detailed derivation.

A. Optimization in asymmetric process

As was done in Sec. VI B, we minimize of the mean work in the the large τ_{op} limit to find the optimized process. In this limit, $W^{(2)}_{asym}$ is given by

$$\lim_{\tau_{op}\to\infty} W^{(2)}_{\text{asym}} = \frac{\nu}{4\beta a_0 \tau_{op}} \int_0^{\tau_X} d\tau \, \frac{1}{(\bar{f}_\tau)^3} \left(\frac{d\bar{f}_\tau}{d\tau} \right)^2 + \frac{\nu}{4\beta a_0 \tau_{op}} \int_{\tau_X}^{1} d\tau \, \frac{1}{(\bar{g}_\tau)^3} \left(\frac{d\bar{g}_\tau}{d\tau} \right)^2, \tag{115}$$

where adimensional quantities are introduced by

$$\tau_X = (t_X - t_i)/\tau_{op}, \tag{116}$$

$$\bar{f}_\tau = \frac{f_{t_i + \tau\tau_{op}}}{a_0}, \tag{117}$$

$$\bar{g}_\tau = \frac{g_{t_i + \tau\tau_{op}}}{a_0}. \tag{118}$$

The optimized parameters $\bar{f}_\tau$ and $\bar{g}_\tau$ are obtained by minimizing $W^{(2)}_{\text{asym}}$. The variations of f_t and g_t are defined by

$$\bar{f}_\tau \longrightarrow \bar{f}_\tau + \delta\bar{f}_\tau, \tag{119}$$

$$\bar{g}_\tau \longrightarrow \bar{g}_\tau + \delta\bar{g}_\tau, \tag{120}$$

which satisfy $\delta\bar{f}_0 = \delta\bar{f}_{\tau_X} = \delta\bar{g}_{\tau_X} = \delta\bar{g}_1 = 0$. The optimized parameters are given by the solutions of the following differential equations:

$$\frac{1}{\bar{f}_\tau^3} \frac{d^2\bar{f}_\tau}{d\tau^2} - \frac{3}{2} \frac{1}{\bar{f}_\tau^4} \left(\frac{d\bar{f}_\tau}{d\tau} \right)^2 = 0, \tag{121}$$

$$\frac{1}{\bar{g}_\tau^3} \frac{d^2\bar{g}_\tau}{d\tau^2} - \frac{3}{2} \frac{1}{\bar{g}_\tau^4} \left(\frac{d\bar{g}_\tau}{d\tau} \right)^2 = 0. \tag{122}$$

These equations are solved using the following conditions:

$$\bar{f}_0 = \bar{g}_X = 1, \qquad \bar{f}_X = \bar{g}_1 = 2. \tag{123}$$

The solutions are given by

$$\begin{aligned} \bar{f}_\tau &= \frac{2\tau_X^2}{[\sqrt{2}\tau_X - (\sqrt{2}-1)\tau]^2} \longrightarrow f_t \\ &= \frac{2(t_X - t_i)^2}{[(t - t_i) + (t_X - t)\sqrt{2}]^2} a_0, \\ \bar{g}_\tau &= \frac{2(1-\tau_X)^2}{[\tau_X - \sqrt{2} + (\sqrt{2}-1)\tau]^2} \longrightarrow g_t \\ &= \frac{2(t_f - t_X)^2}{[(t - t_X) + (t_f - t)\sqrt{2}]^2} a_0. \end{aligned} \tag{124}$$

The optimized switching time t_X is determined by minimizing $W^{(2)}_{\text{asym}}$ which is obtained by substituting Eq. (124) into Eq. (115). Solving $\partial W^{(2)}_{\text{asym}}/\partial\tau_X = 0$, the optimized t_X is given by

$$\tau_X = \frac{1}{2} \longrightarrow t_X = \frac{t_f + t_i}{2}. \tag{125}$$

Using these results, the optimized mean work in the large τ_{op} limit is eventually given by

$$\lim_{\tau_{op}\to\infty} \frac{W_{\rm asym}}{W^{(1)}} = 1 + 2\frac{(\sqrt{2}-1)^2}{\ln 2}\frac{\tau^*}{\tau_{op}}, \tag{126}$$

where $\tau^* = \nu/a_0$ which is already defined below Eq. (109).

In a similar fashion, the change of the mean energy in the asymmetric process is calculated by applying the above results to the definition (12) as was done in Sec. VI A,

$$\Delta E^{\rm asym} = \frac{f_{t_X}}{2\beta}\int_{t_i}^{t_X} d\tau \left[e^{-\int_\tau^{t_X} ds\, \frac{2f_s}{\nu}}\frac{1}{(f_\tau)^2}\right]\frac{df_t}{dt} + \frac{g_{t_f}}{2\beta}\int_{t_X}^{t_f} d\tau \left[e^{-\int_\tau^{t_f} ds\, \frac{2g_s}{\nu}}\frac{1}{(g_\tau)^2}\right]\frac{dg_t}{dt}. \tag{127}$$

Using Eqs. (124) and (125) in the large τ_{op} limit, the optimized change of the mean energy is calculated by

$$\lim_{\tau_{op}\to\infty} \frac{\Delta E^{\rm asym}}{E(t_i)} = (\sqrt{2}-1)\frac{\tau^*}{\tau_{op}}. \tag{128}$$

VIII. COMPARISON OF SYMMETRIC AND ASYMMETRIC PROCESSES

We compare the mean work and the changes of the mean energy in the symmetric and asymmetric compression processes which are denoted by the dotted and solid lines in Fig. 1, respectively. The initial and final values of the control parameters are fixed in both processes, $(a_{t_i}^{(1)}, a_{t_i}^{(2)}) = (a_0, a_0)$ and $(a_{t_f}^{(1)}, a_{t_f}^{(2)}) = (2a_0, 2a_0)$. We use the optimized control parameters which are given by Eq. (103) in the symmetric process, and by Eqs. (124) and (125) in the asymmetric process.

A. Mean work

In Fig. 2 the mean work are plotted as functions of the operation time $\tau_{op} = t_f - t_i$. The black and gray solid lines represent the exact mean work in the symmetric and asymmetric processes, respectively. The former is calculated from Eqs. (86) and (105), and the latter from Eqs. (105) and (112). The black and gray dotted lines show the mean work in the large τ_{op} limit which are given by Eqs. (108) and (126), respectively. We find that the asymptotic behaviors of the solid lines are well reproduced by the corresponding dotted lines. In the large τ_{op} limit, these processes correspond to the quasistatic isothermal process. Then the black and gray lines converge to $W^{(1)}$, which corresponds to the mean work in the reversible process.

In the instantaneous jump limit $\tau_{op} \to 0$, the black and gray solid lines converge to the same value,

$$\lim_{\tau_{op}\to 0} W^{sym} = \lim_{\tau_{op}\to 0} W^{\rm asym} = \frac{1}{\beta}. \tag{129}$$

This behavior can be understood from the first law (10). In this limit, there is no enough time for a Brownian particle to interact with the thermal bath, and thus Eq. (10) leads to $dW_t = E(t+dt) - E(t)$ because $dQ_t = 0$. Moreover, in this limit, the final states are approximately given by the initial state, $\rho(\mathbf{x}, t_f) = \rho(\mathbf{x}, t_i)$. Therefore the changes of the mean energy are given by $1/\beta$ in both the symmetric and asymmetric processes,

$$\begin{aligned} E(t_f) - E(t_i) &= \int d^2\mathbf{x}\, \rho(\mathbf{x}, t_f) V(\mathbf{x}, a_f) - \int d^2\mathbf{x}\, \rho(\mathbf{x}, t_i) V(\mathbf{x}, a_i) \\ &= \int d^2\mathbf{x}\, \rho(\mathbf{x}, t_i)[V(\mathbf{x}, a_f) - V(\mathbf{x}, a_i)] \\ &= \frac{1}{\beta}. \end{aligned} \tag{130}$$

The mean work in the asymmetric process (gray solid line) is always larger than that in the symmetric process (black solid line). This indicates that irreversible contribution in the asymmetric process is always larger than that in the symmetric process. To understand this behavior, we should notice that Eq. (112) is reexpressed as

$$W_{\rm asym}^{(2)} = 2\int_0^{1/2} d\tau_1 \int_0^{\tau_1} d\tau_2\, \dot{\bar{f}}_{\tau_2} \times \left[e^{-\frac{2\tau_{op}a_0}{\nu}\int_{\tau_2}^{\tau_1} d\tau_3\, \bar{f}_{\tau_3}}\frac{1}{2\beta(\bar{f}_{\tau_2})^2}\right]\dot{\bar{f}}_{\tau_1}. \tag{131}$$

To obtain this expression, we have used that $\bar{f}_\tau$ and $\bar{g}_\tau$ defined by Eq. (124) satisfy

$$\bar{g}_{\tau+1/2} = \bar{f}_\tau. \tag{132}$$

when $\tau_X = 1/2$. Then one can see that $W_{\rm asym}^{(2)}$ (131) is reproduced from $W_{\rm sym}^{(2)}$ by replacing τ_{op} with $\tau_{op}/2$ in Eq. (97). That is, the deformation of the harmonic potential in the

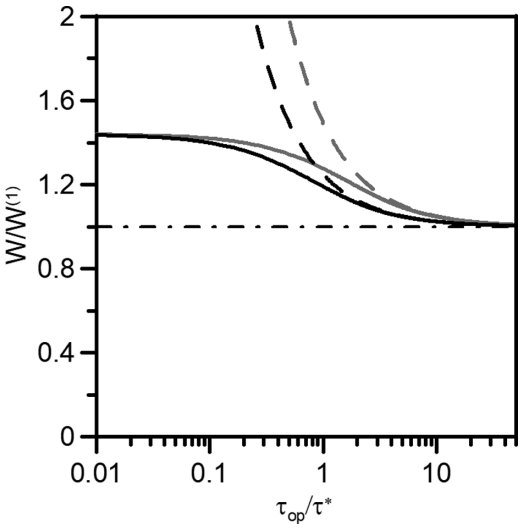


FIG. 2. Mean work plotted as functions of the operation time $\tau_{op} = t_f - t_i$. The black solid and dotted lines are results in the symmetric process and represent the exact mean work and the mean work in the large τ_{op} limit, respectively. The optimized control parameters are given by Eq. (103). The gray solid and dotted lines are results in the asymmetric process and denote the exact mean work and the mean work in the large τ_{op} limit, respectively. The optimized parameters are given by Eq. (124). The dotted-dashed line represents $W^{(1)}$, which is given by Eq. (105). We define $\tau^* = \nu/a_0$.

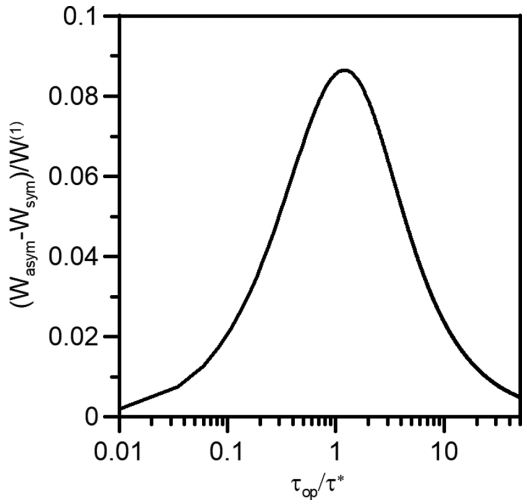


FIG. 3. Difference between $W_{\rm asym}$ and $W_{\rm sym}$ plotted as a function of the operation time $\tau_{op} = t_f - t_i$. The peak is located around $\tau_{op} \sim \tau^*$ We define $\tau^* = \nu/a_0$.

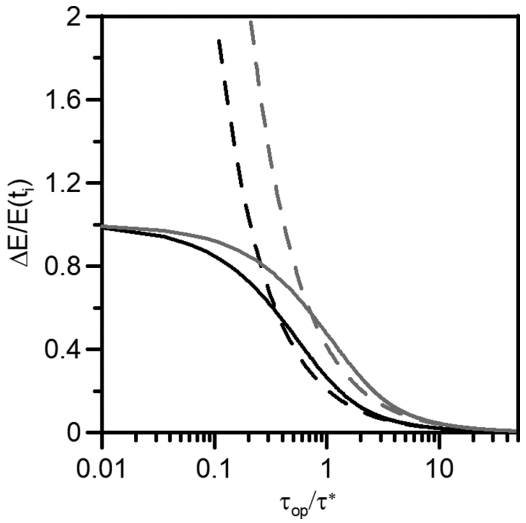


FIG. 4. Changes of the mean energy $\Delta E = E(t_f) - E(t_i)$ plotted as functions of the operation time $\tau_{op} = t_f - t_i$. The black solid and dotted lines are results in the symmetric process and represent the exact change of the mean energy and the change in the large τ_{op} limit, respectively. The optimized control parameters are given by Eq. (103). The gray solid and dotted lines are results in the asymmetric process and represent the exact change of the mean energy and the change in the large τ_{op} limit, respectively. The optimized parameters are given by Eq. (124). We define $\tau^* = \nu/a_0$.

asymmetric process is twice as fast as that in the symmetric process. Normally, the irreversible contribution is pronounced in rapid deformations and thus the mean work in the asymmetric process has more irreversible contributions than that in the symmetric process.

The difference of the mean work in the optimized symmetric and asymmetric processes disappears in the large τ_{op} and the instantaneous jump limits. Therefore, we have to choose an appropriate operation time to maximize the difference. As shown in Fig. 3, the difference is maximized when τ_{op} is chosen to be close to $\tau^* = \nu/a_0$, which characterizes the relaxation time of the Fokker-Planck equation. The exact position of the peak is located at a bit larger than τ^*. This small deviation from τ^* will be related to a hysteresis effect which is discussed in the next section.

B. Change of mean energy

The changes of the mean energy in the symmetric and asymmetric processes are shown as functions of the operation time τ_{op} in Fig. 4. The black and gray solid lines represent the exact changes of the mean energy in the symmetric and asymmetric processes, respectively. The former is calculated from Eqs. (93) and the latter from Eqs. (127). The black and gray dotted lines show the changes in the large τ_{op} limit which are given by Eqs. (109) and (128), respectively. We find that the asymptotic behaviors of the solid lines are approximately reproduced by the corresponding dotted lines.

In this system, the mean energy in equilibrium is given by $1/\beta$ independently of the value of $\mathbf{a}_t$. Therefore the deviation of ΔE from zero can be used to characterize the deviation of the final state from equilibrium. Indeed, the change of the mean energy is reexpressed as

$$\Delta E = \int d^2\mathbf{x}\, V(\mathbf{x}, \mathbf{a}_{t_f}) \left[\rho(\mathbf{x}, \mathbf{a}_{t_f}) - \frac{e^{-\beta V(\mathbf{x}, \mathbf{a}_{t_f})}}{\mathcal{Z}(\mathbf{a}_{t_f})} \right], \quad (133)$$

where $V(\mathbf{x}, \mathbf{a}_{t_f})$ is the harmonic potential (78). The quantity in the bracket represents the deviation of $\rho(\mathbf{x}, t_f)$ from the corresponding thermal equilibrium distribution. To obtain this, we have used

$$\int d^2\mathbf{x}\, V(\mathbf{x}, \mathbf{a}_{t_i}) \frac{e^{-\beta V(\mathbf{x}, \mathbf{a}_{t_i})}}{\mathcal{Z}(\mathbf{a}_{t_i})} = \int d^2\mathbf{x}\, V(\mathbf{x}, \mathbf{a}_{t_f}) \frac{e^{-\beta V(\mathbf{x}, \mathbf{a}_{t_f})}}{\mathcal{Z}(\mathbf{a}_{t_f})} = \frac{1}{\beta}. \quad (134)$$

From Fig. 4, one can see that ΔE's represented by the black and gray lines converge to zero in the large τ_{op} limit. That is, the processes behave as the reversible process in this limit.

In the instantaneous jump limit $\tau_{op} \to 0$, as discussed below Eq. (129), there is no enough time for the final state to evolve from the initial equilibrium state and thus

$$\lim_{\tau_{op}\to 0} \Delta E^{sym} = \lim_{\tau_{op}\to 0} \Delta E^{\rm asym} = \frac{1}{\beta}. \quad (135)$$

This is consistent with the behavior in Fig. 4. We further observe that the gray lines are always larger than the black lines. This is because, as was discussed in the mean work, the asymmetric process has larger irreversible contributions which give rise to the larger deviation of the final state from equilibrium.

The difference between $\Delta E_{\rm asym}$ and $\Delta E_{\rm sym}$ is shown in Fig. 5. For the sake of comparison, the result of Fig. 3 is shown by the dotted line. We find that it is maximized when τ_{op} is chosen to be close to τ^*. Exactly speaking, however, the exact position of the peak is a bit smaller than τ^* and this behavior is different from that in the mean work which is a bit larger. This difference will be related to a hysteresis effect: the change of the mean energy is obtained directly from the initial and final states while the mean work depends on the whole time

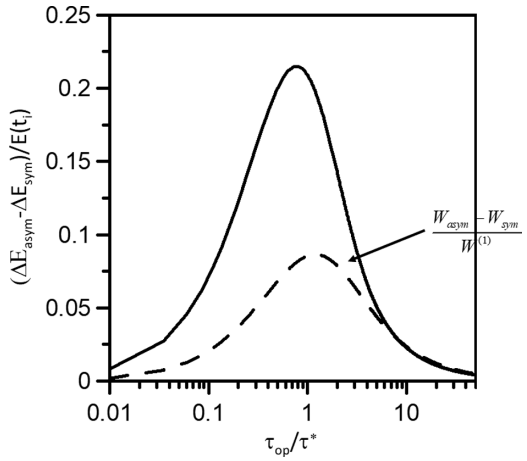


FIG. 5. Difference between $\Delta E_{\rm asym}$ and $\Delta E_{\rm sym}$ plotted as a function of the operation time $\tau_{op} = t_f - t_i$. For sake of comparison, the result of Fig. 3 is shown by the dotted line. The peak of the solid line is smaller than τ^* but that of the dotted line is larger. We define $\tau^* = \nu/a_0$.

evolution of the pseudo density matrix as seen from Eqs. (12) and (66). Because of this, the peak in the mean work appears in retard compared to that in the change of the mean energy.

As for the comparison of our perturbation theory with the numerical simulation of the stochastic differential equation, the time evolutions of the mean energy are shown in Appendix D.

IX. CONCLUDING REMARKS

In this paper, we developed the systematic expansion of the solution of the Fokker-Planck equation with degeneracy by generalizing the formulation developed in Ref. [18]. The Fokker-Planck equation describes the thermal relaxation of a Brownian particle confined in an external potential, which is deformed by changing control parameters. The time derivative of these parameters are utilized as the expansion parameter of the perturbation theory. Differently from the theory in Ref. [18], the present theory is applicable to systems in arbitrary spatial dimensions, and we obtained a formula to calculate the mean work perturbatively which is applicable to the systems with degeneracy in the eigenvalues of the Fokker-Planck operator. This formula enables us to study how the geometrical symmetry of the external potential affects thermodynamic description of a Brownian particle.

The application of the derived formula depends on the degeneracy of systems. To illustrate this, we considered the thermodynamic description of a Brownian particle confined in the two-dimensional harmonic potential. By changing the control parameters, the harmonic potential is monotonically compressed. Fixing the initial and final forms of the potential, we considered two irreversible processes: the symmetric and asymmetric processes. The rotational symmetry of the harmonic potential in the two-dimensional plan is held in the former, but it is broken in the latter. Perturbative calculations are affected by these deformation processes because the degeneracy of the expansion basis (foot and head states) depends on the symmetry of the potential.

The optimized deformation processes of the harmonic potential are obtained by minimizing the mean work. The exact solutions have, however, not yet been known and thus we considered the optimization in the limit of the large operation time $\tau_{op} = t_f - t_i$. Using these approximated optimized processes, we found that the mean work and the changes of the mean energy in the asymmetric process are always larger than those in the symmetric process for any τ_{op}, because the timescales of the deformation in the asymmetric process is shorter than that in the symmetric process.

When the operation time τ_{op} is very small, there is no enough time for the initial state to evolve. Moreover processes in the large τ_{op} limit converge to the quasistatic process. Therefore the symmetric and asymmetric processes show the same behaviors in these limits. To see the difference between the symmetric and asymmetric processes, we have to choose τ_{op} appropriately. We calculated the difference of the mean work in the optimized symmetric and asymmetric processes as a function of τ_{op} and found that the difference is maximized when $\tau_{op} \sim \tau^* = \nu/a_0$ where where ν is the friction constant in the Fokker-Planck equation and a_0 denotes the initial value of the spring constant of the harmonic potential.

The similar difference can be calculated for the changes of the mean energy. We then found that the difference between the symmetric and asymmetric processes is maximized when τ_{op} is close to τ^* but is smaller than the corresponding value in the mean work. This deviation may be related to a hysteresis effect: the change of the mean energy is obtained directly from the initial and final states while the mean work depends on the whole time evolution of the pseudodensity matrix. It is thus reasonable to consider that the maximum in the mean work appears in retard compared to that in the change of the mean energy.

In this theory, the solution of the Fokker-Planck equation is expanded with the foot and head states which are the eigenstates of the time-dependent Fokker-Planck operator. These states form the biorthogonal system and thus the foot or head state itself is not necessarily normalized by one. As a consequence, there is an ambiguity to multiply a real factor to the foot and head states. See the factor $\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)$ in Eq. (28). Our perturbation theory should be invariant for this ambiguity. Indeed, the mean work and the changes of the mean energy in the harmonic potential do not depend on this additional real factor $\mathcal{N}_{n,\boldsymbol{\alpha}_n}(\mathbf{a}_t)$. This invariance is, however, not yet shown for general external potential.

There are various applications which are not considered in this paper. To apply the present formulation to nonlinear potentials, we have to develop another perturbation theory to express the foot and head states in terms of those of an analytically solvable potential like the harmonic potential. The Floquet theory is known to be useful to solve periodic linear differential equations and provides a convenient method in quantum mechanics [47]. This approach is applied to the Fokker-Planck equation [34] but the corresponding optimization problems has not yet been investigated. We have considered transitions from an initial equilibrium state to a nonequilibrium state. To consider a transition starting from a nonequilibrium state, we should use Eq. (50) instead of Eq. (59). Our perturbation theory will be then applicable to study linear and nonlinear responses around equilibrium and

nonequilibrium steady states as is discussed in Ref. [48]. For this, an external field should be chosen as the expansion parameter instead of the time derivative of the control parameter. When the potential is deformed, we can consider the processes associated with the rotation of the potential. There is no thermodynamic counterpart of such a process and hence its behavior is worth investigating. In Ref. [49] the heat conduction process between two different heat baths is considered to study the protocol to realize the transition from an initial steady state to a final steady state. However, the relation between the derived protocol and the optimization of the mean work is not yet known. The framework developed in the present paper will provide a systematic approach to study this.

In the present paper, we assumed that the changes of the control parameters are given by smooth deterministic functions, but these changes can be stochastic in a microscopic timescale. For the optimization of stochastic control parameters, the minimization of the mean work will be replaced with the stochastic calculus of variation [50,51]. The applications to relativistic [36,52,53] and quantum [54,55] systems have not yet been investigated sufficiently. These generalizations are left as future tasks.

APPENDIX A: PROPERTIES OF FOOT AND HEAD STATES

We should notice that the operator $\mathcal{H}_t$ can be expressed as

$$\mathcal{H}_t = \frac{1}{\nu\beta}\sum_{i=1}^{D} B_i^\dagger B_i, \tag{A1}$$

where

$$B_i = \partial_i + \frac{\beta}{2}[\partial_i V(\mathbf{x}, \mathbf{a}_t)], \tag{A2}$$

$$B_i^\dagger = -\partial_i + \frac{\beta}{2}[\partial_i V(\mathbf{x}, \mathbf{a}_t)]. \tag{A3}$$

These operators satisfy the following commutation relations:

$$[B_i, B_j^\dagger] = \beta[\partial_i\partial_j V(\mathbf{x}, \mathbf{a}_t)], \tag{A4}$$

$$[B_i, B_i] = 0, \tag{A5}$$

$$[B_i^\dagger, B_j^\dagger] = 0. \tag{A6}$$

Then the eigenvalues are shown to be nonnegative,

$$\lambda_n(t) = \int d^D\mathbf{x}\, u_{n,\boldsymbol{\alpha}_n}(\mathbf{x}, t)\mathcal{H}_t u_{n,\boldsymbol{\alpha}_n}(\mathbf{x}, t)$$

$$= \frac{1}{\nu\beta}\sum_{i=1}^{D} |B_i u_{n,\boldsymbol{\alpha}_n}(\mathbf{x}, t)|^2 \geqslant 0. \tag{A7}$$

Thus, without loss of generality, the eigenvalues are ordered as

$$0 = \lambda_0(t) < \lambda_1(t) < \lambda_2(t)\cdots. \tag{A8}$$

The foot state with $\lambda_0 = 0$ is given by solving $B_i u_{0,\boldsymbol{\alpha}_0} = 0$, and we find

$$\underline{\rho}_{0,\mathbf{0}}(\mathbf{x}, \mathbf{a}_t) = \sqrt{\mathcal{N}_{0,\mathbf{0}}(\mathbf{a}_t)}e\frac{1}{\sqrt{\mathcal{Z}(\mathbf{a}_t)}}e^{-\beta V(\mathbf{x},\mathbf{a}_t)}, \tag{A9}$$

where

$$\mathcal{Z}(\mathbf{a}_t) = \int d^D\mathbf{x}\, e^{-\beta V(\mathbf{x},\mathbf{a}_t)}. \tag{A10}$$

One can see that this is the stationary solution of the Fokker-Planck equation when V is not time-dependent explicitly. There is no degeneracy in the stationary solution and thus we set $\boldsymbol{\alpha}_0 = \mathbf{0}$. The corresponding head state is easily found by using Eq. (30),

$$\overline{\rho}_{0,\mathbf{0}}(\mathbf{x}, \mathbf{a}_t) = \frac{1}{\sqrt{\mathcal{N}_{0,\mathbf{0}}(\mathbf{a}_t)}}\frac{1}{\sqrt{\mathcal{Z}(\mathbf{a}_t)}}. \tag{A11}$$

It should be noted that B_i and $B_i^\dagger$ are not the lowering and raising operators in general because the right-hand side of Eq. (A4) is not necessarily constant.

APPENDIX B: DERIVATION OF THE MEAN WORK IN THE SYMMETRIC PROCESS

The lowest order contribution in the mean work $W^{(1)}$ is already calculated. The next order contribution is given by Eq. (73), which is simplified as

$$W_{\mathrm{sym}}^{(2)} = \int_{t_i}^{t_f} dt \sum_{\alpha_2,\beta_2=0}^{d^{(2)}} e^{-\int_{t_i}^{t} d\,\lambda_2(s)}[\mathbf{M}^{(2)}(t)]_{\alpha_2,\beta_2} D_{2,\beta_2}^{(1)}(t)\langle\overline{0, 0, \mathbf{a}_{t_i}}|(\partial_t\widehat{V}_t)|\underline{2, \alpha_2, \mathbf{a}_t}\rangle, \tag{B1}$$

because, in the harmonic potential, we find

$$\langle\overline{0, 0, \mathbf{a}_\tau}|(\partial_t\widehat{V})|\underline{n, \alpha_n, \mathbf{a}_t}\rangle = \frac{1}{2}\frac{da_t}{dt}\sqrt{\frac{N_0^2(a_t)}{N_0^2(a_\tau)}}\left(\frac{a_\tau}{a_t}\right)^{1/2}\frac{1}{\beta a_t}\left[\delta_{n,0}e^{-i[\delta_{n+2}(a_t)-\delta_n(a_t)]} + \sqrt{\frac{N_2(a_t)}{N_0(a_t)}}e^{-i[\delta_{n-2}(a_t)-\delta_n(a_t)]}\sqrt{2}\delta_{n,2}\right][\delta_{0,\alpha_n} + \delta_{n,\alpha_n}]. \tag{B2}$$

The expansion coefficient $D_{2,\beta_2}^{(1)}(t)$ is given by Eq. (74). Substituting it into Eq. (B1), we obtain

$$W_{\mathrm{sym}}^{(2)} = 2\int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \frac{da_s}{ds}\left[\frac{1}{2\beta a_s^2}e^{-\int_s^t ds'\frac{2a_{s'}}{\nu}}\right]\frac{da_t}{dt}, \tag{B3}$$

where we have used Eq. (64) and the following results:

$$[\mathbf{M}^{(n)}(t)]_{\alpha_n,\beta_n} = \delta_{\alpha_n,\beta_n}\sqrt{\frac{N_{\alpha_n}(a_{t_i})N_{n-\alpha_n}(a_{t_i})}{N_{\alpha_n}(a_t)N_{n-\alpha_n}(a_t)}}\left(\frac{a_t}{a_{t_i}}\right)^{(n+1)/2}, \tag{B4}$$

$$\left[\dot{\mathbf{L}}^{(n,m)}(\tau)\right]_{\alpha_n,\beta_m} = -\frac{1}{\nu}\frac{da_\tau}{d\tau}\left[m\delta_{n,m}\delta_{\alpha_n,\beta_m} + \sqrt{\frac{N_{\beta_m}(a_\tau)}{N_{\beta_m+2}(a_\tau)}}\sqrt{(\beta_m+2)(\beta_m+1)}\delta_{n,m+2}\delta_{\alpha_n,\beta_m+2}\right.$$
$$\left. + \sqrt{\frac{N_{m-\beta_m}(a_\tau)}{N_{m-\beta_m+2}(a_\tau)}}\sqrt{(m-\beta_m+2)(m-\beta_m+1)}\delta_{n,m+2}\delta_{\alpha_n,\beta_m}\right]. \tag{B5}$$

APPENDIX C: DERIVATION OF THE MEAN WORK IN THE ASYMMETRIC PROCESS

To apply the perturbation theory to this case, it is convenient to introduce the states which are parameterized by the eigenvalues of each spatial component,

$$\langle\mathbf{x}|\underline{\text{“}\mathbf{l}'', \mathbf{a}_t}\rangle = \langle\mathbf{x}|\underline{\text{“}l_1, l_2'', \mathbf{a}_t}\rangle = \underline{\rho_{l_1}}(x_1, a_t^{(1)})\underline{\rho_{l_2}}(x_2, a_t^{(2)})$$
$$\langle\mathbf{x}|\overline{\text{“}\mathbf{l}'', \mathbf{a}_t}\rangle = \langle\mathbf{x}|\overline{\text{“}l_1, l_2'', \mathbf{a}_t}\rangle = \overline{\rho}_{l_1}(x_1, a_t^{(1)})\overline{\rho}_{l_2}(x_2, a_t^{(2)}), \tag{C1}$$

where we have used the functions defined in Eqs. (81) and (82). The corresponding eigenvalues of the Fokker-Planck operator are characterized by the pair of the two integers (l_1, l_2),

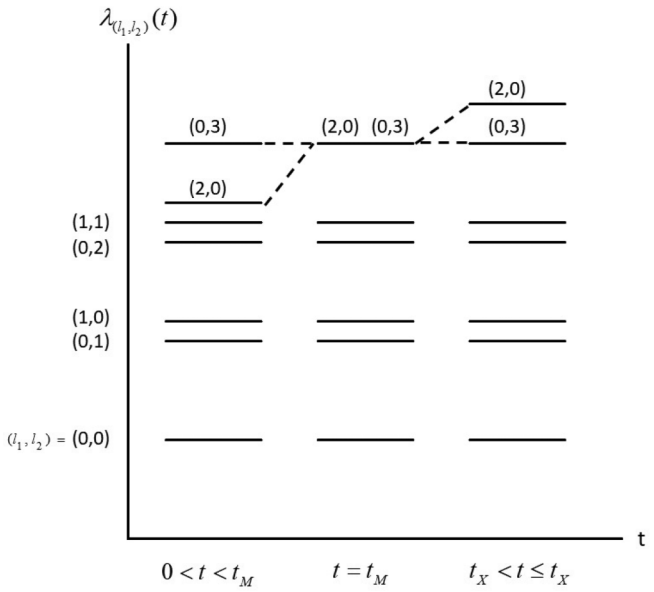
$$\lambda_{(l_1,l_2)}(t) = \frac{a_t^{(1)}}{\nu}l_1 + \frac{a_t^{(2)}}{\nu}l_2. \tag{C2}$$

These states satisfy the orthonormal condition,

$$\langle\overline{\text{“}\mathbf{m}'', \mathbf{a}_t}|\underline{\text{“}\mathbf{n}'', \mathbf{a}_t}\rangle = \delta_{n_1,m_1}\delta_{n_2,m_2} = \delta_{\mathbf{n},\mathbf{m}}. \tag{C3}$$

We call these the fictitious states. The sums associated with n and α_n, are shown to be equivalent to the sums of l_1 and l_2,

$$\sum_{n\geqslant 0}\sum_{\boldsymbol{\alpha}_n=0}^{\mathbf{d}^{(n)}(\mathbf{a}_t)} = \sum_{l_1\geqslant 0}\sum_{l_2\geqslant 0}, \tag{C4}$$



$\lvert n, \alpha_n, \mathbf{a}_t\rangle$	$\lambda_n(t)$	$\lvert\text{“}l_1, l_2\text{”}\mathbf{a}_t\rangle$	
$\lvert 0, 0, \mathbf{a}_t\rangle$	0	$\lvert\text{“}0, 0, \text{”}\mathbf{a}_t\rangle$	
$\lvert 1, 0, \mathbf{a}_t\rangle$	$\frac{a_0}{\nu}$	$\lvert\text{“}0, 1, \text{”}\mathbf{a}_t\rangle$	
$\lvert 2, 0, \mathbf{a}_t\rangle$	$\frac{a_t}{\nu}$	$\lvert\text{“}1, 0, \text{”}\mathbf{a}_t\rangle$	
$\lvert 3, 0, \mathbf{a}_t\rangle$	$\frac{2a_0}{\nu}$	$\lvert\text{“}0, 2, \text{”}\mathbf{a}_t\rangle$	
$\lvert 4, 0, \mathbf{a}_t\rangle$	$\frac{a_t}{\nu} + \frac{a_0}{\nu}$	$\lvert\text{“}1, 1, \text{”}\mathbf{a}_t\rangle$	
$\lvert 5, 0, \mathbf{a}_t\rangle$	$\frac{2a_t}{\nu}$	$\lvert\text{“}2, 0, \text{”}\mathbf{a}_t\rangle$	$t_i < t < t_M$
$\lvert 5, 0, \mathbf{a}_t\rangle$ $\lvert 5, 1, \mathbf{a}_t\rangle$	$\frac{3a_0}{\nu}$	$\lvert\text{“}2, 0, \text{”}\mathbf{a}_t\rangle$ $\lvert\text{“}0, 3, \text{”}\mathbf{a}_t\rangle$	$t = t_M$
$\lvert 5, 0, \mathbf{a}_t\rangle$	$\frac{3a_0}{\nu}$	$\lvert\text{“}0, 3, \text{”}\mathbf{a}_t\rangle$	$t_M < t \leq t_X$
$\lvert 6, 0, \mathbf{a}_t\rangle$	$\frac{3a_0}{\nu}$	$\lvert\text{“}0, 3, \text{”}\mathbf{a}_t\rangle$	$t_i < t < t_M$
$\lvert 6, 0, \mathbf{a}_t\rangle$ $\lvert 6, 1, \mathbf{a}_t\rangle$	$\frac{4a_0}{\nu}$	$\lvert\text{“}2, 1, \text{”}\mathbf{a}_t\rangle$ $\lvert\text{“}0, 4, \text{”}\mathbf{a}_t\rangle$	$t = t_M$
$\lvert 6, 0, \mathbf{a}_t\rangle$	$\frac{2a_t}{\nu}$	$\lvert\text{“}2, 0, \text{”}\mathbf{a}_t\rangle$	$t_M < t \leq t_X$

FIG. 6. Distribution of the eigenvalues of the fictitious states (C1) for $t_i < t \leqslant t_X$. The left panel shows the schematic figure of the change of the eigenvalues. The right panel represents the correspondence between the foot state $|\underline{n, \alpha_n, \mathbf{a}_t}\rangle$ and the fictitious state $|\underline{\text{“}l_1, l_2'', \mathbf{a}_t}\rangle$.

and hence the condition (46) is satisfied. It is however noted that the degeneracy of the fictitious state is not yet identified.

The second-order term $W^{(2)}_{\rm asym}$ is given by the sum of the contributions from path I and path II,

$$W^{(2)}_{\rm asym} = W^{(2)}_I + W^{(2)}_{II}, \tag{C5}$$

where

$$W^{(2)}_I = -\sum_{n\geqslant 1}\int_{t_i}^{t_X} dt \int_{t_i}^{t} ds \sum_{\alpha_n,\beta_n=0}^{d^{(n)}(\mathbf{a}_t)} \sum_{\gamma_n=0}^{d^{(n)}(\mathbf{a}_s)} e^{-\int_s^t ds'\,\lambda_2(s')} \langle\overline{0,0,\mathbf{a}_i}|(\partial_t \widehat{V})|\underline{n,\alpha_n,\mathbf{a}_t}\rangle \times \frac{[\mathbf{M}^{(n)}(t)]_{\alpha_n,\beta_n}[\mathbf{M}^{(n)}(s)]^{-1}{}_{\beta_n,\gamma_n}\left[\dot{\mathbf{L}}^{(n,0)}(s)\right]_{\gamma_n,0}[\mathbf{M}^{(0)}(s)]_{0,0}}{\lambda_n(s)}, \tag{C6}$$

$$W^{(2)}_{II} = -\sum_{n\geqslant 1}\int_{t_X}^{t_f} dt \int_{t_i}^{t} ds \sum_{\alpha_n,\beta_n=0}^{d^{(n)}(\mathbf{a}_t)} \sum_{\gamma_n=0}^{d^{(n)}(\mathbf{a}_s)} e^{-\int_s^t ds'\,\lambda_2(s')} \langle\overline{0,0,\mathbf{a}_i}|(\partial_t \widehat{V})|\underline{n,\alpha_n,\mathbf{a}_t}\rangle \times \frac{[\mathbf{M}^{(n)}(t)]_{\alpha_n,\beta_n}[\mathbf{M}^{(n)}(s)]^{-1}{}_{\beta_n,\gamma_n}\left[\dot{\mathbf{L}}^{(n,0)}(s)\right]_{\gamma_n,0}[\mathbf{M}^{(0)}(s)]_{0,0}}{\lambda_n(s)}. \tag{C7}$$

We see that the integration of t in $W^{(2)}_I$ is on path I while that in $W^{(2)}_{II}$ is on path II.

1. Calculation of $W^{(2)}_I$

To calculate these equations, we have to identify a fictitious state $|\underline{\text{“}l_1, l_2'', \mathbf{a}_t}\rangle$ with a foot state $|\underline{n,\alpha_n,\mathbf{a}_t}\rangle$. We first consider the contribution on path I. On this path, the correspondence between the fictitious states and the foot states are summarized in Fig. 6. We find that, for example, the sixth foot state $|5,0,\mathbf{a}_t\rangle$ corresponds to $|\text{“}2,0''\mathbf{a}_t\rangle$ at $t_i < t < t_M$ and to $|\text{“}3,0''\mathbf{a}_t\rangle$ at $t_M < t < t_f$. Here the time t_M is defined by

$$a_{t_M} = \tfrac{3}{2}a_0. \tag{C8}$$

This foot state is degenerate instantaneously at $t = t_M$ and hence

$$d^{(5)}(\mathbf{a}_t) = \begin{cases} 0 & t \neq t_M, \\ 1 & t = t_M. \end{cases} \tag{C9}$$

In the calculation of the mean work, the integrand is not singular and thus the contribution at the moment $t = t_M$ is negligible.

From the correspondence in Fig. 6, we observe, for example, that the fictitious state $(l_1, l_2) = (2, 0)$ corresponds to the sixth foot state $(n, \alpha_n) = (5, 0)$ for $t_i < t < t_M$, but it is changed to the seventh foot state $(n, \alpha_n) = (6, 0)$ for $t_M < t \leqslant t_X$. Because of this, $W^{(2)}_I$ is given by the sum of three contributions,

$$W^{(2)}_I = W^A_I + W^B_I + W^C_I = \int_{t_i}^{t_X} dt \int_{t_i}^{t} ds \frac{df_s}{ds}\left[e^{-\int_s^t ds' \frac{2f_{s'}}{\nu}} \frac{1}{2\beta(f_s)^2}\right]\frac{df_t}{dt}, \tag{C10}$$

where

$$W^A_I = -\int_{t_i}^{t_M} dt \int_{t_i}^{t} ds\, e^{-\int_s^t ds'\,\lambda_{(2,0)}(s')} \frac{[\mathbf{M}^{(5)}(t)]_{0,0}[\mathbf{M}^{(5)}(s)]^{-1}_{0,0}[\dot{\mathbf{L}}^{(5,0)}(s)]_{0,0}[\mathbf{M}^{(0)}(s)]_{0,0}}{\lambda_{(2,0)}(s)} \langle\overline{0,0,\mathbf{a}_{t_i}}|(\partial_t \widehat{V})|\underline{5,0,\mathbf{a}_t}\rangle = \int_{t_i}^{t_M} dt \int_{t_i}^{t} ds \frac{df_s}{ds}\left[e^{-\int_s^t ds' \frac{2f_{s'}}{\nu}} \frac{1}{2\beta(f_s)^2}\right]\frac{df_t}{dt}, \tag{C11}$$

$$W^B_I = -\int_{t_M}^{t_X} dt \int_{t_i}^{t_M} ds\, e^{-\int_s^t ds'\,\lambda_{(2,0)}(s')} \frac{[\mathbf{M}^{(6)}(t)]_{0,0}[\mathbf{M}^{(5)}(s)]^{-1}_{0,0}[\dot{\mathbf{L}}^{(5,0)}(s)]_{0,0}[\mathbf{M}^{(0)}(s)]_{0,0}}{\lambda_{(2,0)}(s)} \langle\overline{0,0,\mathbf{a}_{t_i}}|(\partial_t \widehat{V})|\underline{6,0,\mathbf{a}_t}\rangle = \int_{t_M}^{t_X} dt \int_{t_i}^{t_M} ds \frac{df_s}{ds}\left[e^{-\int_s^t ds' \frac{2f_{s'}}{\nu}} \frac{1}{2\beta(f_s)^2}\right]\frac{df_t}{dt}, \tag{C12}$$

$$W^C_I = -\int_{t_M}^{t_X} dt \int_{t_M}^{t} ds\, e^{-\int_s^t ds'\,\lambda_{(2,0)}(s')} \frac{[\mathbf{M}^{(6)}(t)]_{0,0}[\mathbf{M}^{(6)}(s)]^{-1}_{0,0}[\dot{\mathbf{L}}^{(6,0)}(s)]_{0,0}[\mathbf{M}^{(0)}(s)]_{0,0}}{\lambda_{(2,0)}(s)} \langle\overline{0,0,\mathbf{a}_{t_i}}|(\partial_t \widehat{V})|\underline{6,0,\mathbf{a}_t}\rangle = \int_{t_M}^{t_X} dt \int_{t_M}^{t} ds \frac{df_s}{ds}\left[e^{-\int_s^t ds' \frac{2f_{s'}}{\nu}} \frac{1}{2\beta(f_s)^2}\right]\frac{df_t}{dt}. \tag{C13}$$

In the calculations of W_I^A and W_I^C, there is no effect of the change of the eigenvalues at $t = t_M$. We, however, require attention to the calculation of W_I^B. The fictitious state $(l_1, l_2) = (2, 0)$ in $W_I^{(B)}$ corresponds to the foot state $(n, \alpha_n) = (5, 0)$ for the integral of s but the same state represents $(n, \alpha_n) = (6, 0)$ for the integral of t. This is the reason why we observe $[\mathbf{M}^{(6)}(t)]_{0,0}$ and $[\mathbf{M}^{(5)}(s)]^{-1}_{0,0}$ in Eq. (C12) at the same time,

$$\sum_{n\geqslant 1}\int_{t_M}^{t_X} dt \int_{t_i}^{t_M} ds\, [\mathbf{M}^{(n)}(t)]_{0,0}[\mathbf{M}^{(n)}(s)]^{-1}_{0,0}\langle\overline{0,0,\mathbf{a}_s}|(\partial_t\widehat{V})|\underline{n,\alpha_n,\mathbf{a}_t}\rangle$$
$$= \int_{t_M}^{t_X} dt \int_{t_i}^{t_M} ds\, [\mathbf{M}^{(6)}(t)]_{0,0}[\mathbf{M}^{(5)}(s)]^{-1}_{0,0}\langle\overline{0,0,\mathbf{a}_s}|(\partial_t\widehat{V})|\underline{6,0,\mathbf{a}_t}\rangle. \tag{C14}$$

2. Calculation of $W_{II}^{(2)}$

We consider the contribution from path II in this subsection. The distribution of the eigenvalues on path II is summarized in Fig. 7, which is much simpler than that of path I. In this case, Eq. (C7) is calculated as

$$W_{II}^{(2)} = -\int_{t_X}^{t_f} dt \int_{t_i}^{t} ds\, e^{-\int_s^t ds'\,\lambda_3(s')} \frac{[\mathbf{M}^{(3)}(t)]_{0,0}[\mathbf{M}^{(3)}(s)]^{-1}{}_{0,0}\left[\dot{\mathbf{L}}^{(3,0)}(s)\right]_{0,0}}{\lambda_n(s)}\langle\overline{0,0,\mathbf{a}_s}|(\partial_t\widehat{V})|\underline{3,0,\mathbf{a}_t}\rangle$$
$$= -\int_{t_X}^{t_f} dt \int_{t_X}^{t} ds\, e^{-\int_s^t ds'\,\lambda_3(s')} \frac{[\mathbf{M}^{(3)}(t)]_{0,0}[\mathbf{M}^{(3)}(s)]^{-1}{}_{0,0}\left[\dot{\mathbf{L}}^{(3,0)}(s)\right]_{0,0}}{\lambda_n(s)}\langle\overline{0,0,\mathbf{a}_s}|(\partial_t\widehat{V})|\underline{3,0,\mathbf{a}_t}\rangle$$
$$= \int_{t_X}^{t_f} dt \int_{t_X}^{t} ds\, \frac{dg_s}{ds}\left[e^{-\int_s^t ds'\,\frac{2g_{s'}}{\nu}}\frac{1}{2\beta(g_s)^2}\right]\frac{dg_t}{dt}. \tag{C15}$$

In the second equality, the lower limit of the integral of s is changed from t_i to t_X because

$$\dot{\mathbf{L}}^{(n,0)}(s) = -\frac{\sqrt{2}}{\nu}\begin{cases} \frac{df_s}{ds}\sqrt{\frac{N_0^{(1)}(s)}{N_2^{(1)}(s)}}\delta_{n,5}\delta_{\alpha_n,0} & t_i < s < t_M, \\ \frac{df_s}{ds}\sqrt{\frac{N_0^{(1)}(s)}{N_2^{(1)}(s)}}\delta_{n,6}\delta_{\alpha_n,0} & t_M < s < t_X, \\ \frac{dg_s}{ds}\sqrt{\frac{N_0^{(2)}(s)}{N_2^{(2)}(s)}}\delta_{n,3}\delta_{\alpha_n,0} & t_X < s < t_f. \end{cases} \tag{C16}$$

$\lvert n,\alpha_n,\mathbf{a}_t\rangle$	$\lambda_n(t)$	$\lvert$"l_1,l_2,"$\mathbf{a}_t\rangle$
$\lvert 0,0,\mathbf{a}_t\rangle$	0	$\lvert$"$0,0$,"$\mathbf{a}_t\rangle$
$\lvert 1,0,\mathbf{a}_t\rangle$	$\frac{a_t}{\nu}$	$\lvert$"$0,1$,"$\mathbf{a}_t\rangle$
$\lvert 2,0,\mathbf{a}_t\rangle$	$\frac{2a_0}{\nu}$	$\lvert$"$1,0$,"$\mathbf{a}_t\rangle$
$\lvert 3,0,\mathbf{a}_t\rangle$	$\frac{2a_t}{\nu}$	$\lvert$"$0,2$,"$\mathbf{a}_t\rangle$
$\lvert 4,0,\mathbf{a}_t\rangle$	$\frac{2a_0}{\nu}+\frac{a_t}{\nu}$	$\lvert$"$1,1$,"$\mathbf{a}_t\rangle$
$\lvert 5,0,\mathbf{a}_t\rangle$	$\frac{4a_0}{\nu}$	$\lvert$"$2,0$,"$\mathbf{a}_t\rangle$
$\lvert 6,0,\mathbf{a}_t\rangle$	$\frac{2a_0}{\nu}+\frac{2a_t}{\nu}$	$\lvert$"$1,2$,"$\mathbf{a}_t\rangle$

FIG. 7. Distribution of the eigenvalues of the fictitious state (C1) for $t_X < t < t_f$.

APPENDIX D: COMPARISON WITH STOCHASTIC DIFFERENTIAL EQUATION

To confirm that our perturbation theory is consistent with the numerical simulation of the stochastic differential equation, we consider the symmetric process and calculate the time evolution of the mean energy. To calculate this, we need the equations for x_{it}^2 $(i = 1, 2)$ which are obtained by applying

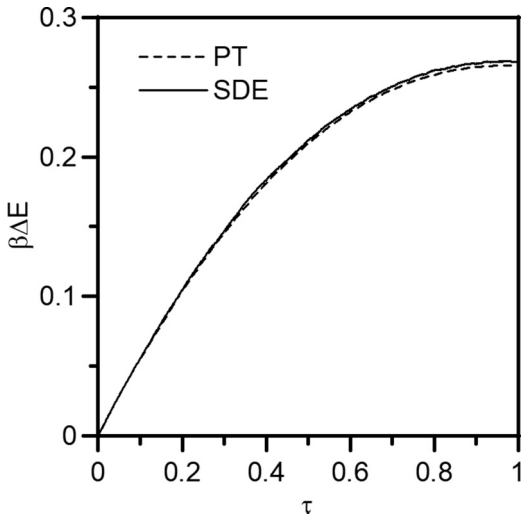


FIG. 8. Changes of the mean energy plotted as functions of $\tau = (t - t_i)/\tau_{op}$. The solid line is obtained by the numerical simulation of the stochastic differential equation. The dotted line is given by Eq. (93). The optimized protocol (103) is utilized to both calculations.

Itô's lemma to Eq. (3),

$$d(x_t^i)^2 = \frac{2}{\nu}\left[-a_t(x_t^i)^2 + \frac{1}{\beta}\right]dt + 2\sqrt{\frac{2}{\beta\nu}}x_t^i\, dB_t^i. \quad \text{(D1)}$$

The product of the last term on the right-hand side is given by the Itô definition [32]. We solved this equation using the Euler-Maruyama method using $d\tau = 10^{-3}$, $\tau_{op}a_{t_i}/\nu = 1$ and the optimized protocol (103).

In Fig. 8 the changes of the mean energy are plotted as functions of $\tau = (t - t_i)/\tau_{op}$. The solid line is obtained by the numerical simulation of the stochastic differential equation by taking the average of 200 000 events. The dotted line is the exact calculation in the perturbation theory, given by Eq. (93). Our analytical result agrees with the corresponding numerical simulation.

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