Brownian motion theory of the two-dimensional quantum vortex gas

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A theory of Brownian motion is presented for an assembly of vortices. The attempt is motivated by a realization of Dyson's Coulomb gas in the context of quantum condensates. By starting with the time-dependent Landau-Ginzburg (LG) theory, the dynamics of the vortex gas is constructed, which is governed by the canonical equation of motion. The dynamics of point vortices is converted to the Langevin equation, which results in the generalized Fokker-Planck (GFP) (or Smolkovski) equation using the functional integral on the ansatz of the Gaussian white noise. The GFP, which possesses a non-Hermitian property, is characterized by two regimes called the *overdamping* and the *underdamping* regimes. In the overdamping regime, where the dissipation is much larger that the vortex strength, the GFP becomes the standard Fokker-Planck equation, which is transformed into the two-dimensional many-particle system. Several specific applications are given of the Fokker-Planck equation. An asymptotic limit of small diffusion is also discussed for the two-vortices system. The underdamping limit, for which the vortex charge is much larger than the dissipation, is briefly discussed.

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I. INTRODUCTION

The study of vortex gas is one of the main topics in physics. The original idea dates back to the work of Kirchhoff (see [1]), which describes the dynamics of an assembly of point vortices as an aggregate of particles interacting with the logarithmic potential. This idea was transferred to the Onsager's theory of vortex gas [2] aiming at exploring the turbulence. Since then, many investigations began to flourish in the study of a quantum vortex; see, e.g., [3–5]. The topics are still in the limelight in condensed matter physics.

The modern quantum condensed matter physics is based on the concept of the order parameter or, more specifically, the macrowave function [6]. The macrowave function provides a useful device to incorporate the vortex degree of freedom in theory. In this way it is natural to teat the quantized vortex in the framework of the Landau-Ginzburg theory for the order parameter of the quantum fluids such as superfluids, superconductivity, and Bose-Einstein condensates [7].

In treating condensed matter systems, it is inevitable to deal with the effect of "noise" or "fluctuations." The systematic study of the fluctuation has been one of the central subjects in nonequilibrium statistical physics [8–11]. The recent trend of studying random fluctuations has been focused on Brownian motion, which is facilitated by various physical situations (see, e.g., [12]). As for the quantum condensates, the most ready case is caused by the fluctuation of temperature as well as the effect arising from existence of impurities. The effect of random noise has been investigated for the superconductivity vortices (see, e.g., [13–15]) and in a classic article on two-dimensional superfluid vortices [16].

In the present article, we address a Brownian motion model for a quantum vortex gas. The attempt is inspired by the Dyson theory of Coulomb gas [17], which was studied in connection with the random matrix theory. We take it up from a renewed viewpoint as a problem of real vortices in quantum condensates. The starting point is the canonical equation of motion (Kirchhoff equation) for the vortex center, which is facilitated by the time-dependent LG theory: namely, we use the complex order parameter such that it incorporates the coordinate vortex center. The resultant Hamiltonian is given by a sum of the logarithmic potential and the harmonic confinement potential. The basic idea is to construct the Langevin equation by modifying the canonical equation of motion for a vortex so as to include the dissipation and random force. This is converted to the Fokker-Planck (FP) equation by adopting the functional integral based on the Gaussian white noise for the random force. The procedure follows a previous article [18], which developed a general formulation of the stochastic theory of the Schrödinger equation by refining the formalism so as to adapt the present purpose. The FP theory indeed provides a standard framework of the study of random systems (see, e.g., [19,20]).

The crux of the present attempt is as follows: The resultant FP equation is regarded as a generalization of the conventional FP equation [hence we call it a generalized FP equation (GFP)]. The GFP is characterized by three parameters; the diffusion constant, the dissipation constant, and the vortex charge. Keeping the diffusion constant fixed, there are two cases, which depend on the choice of two constants: the dissipation (say, μ) and the vortex strength (κ), namely, (1) $\mu \gg \kappa$ and (2) $\mu \ll \kappa$, which is called the "overdamping" and "underdamping," respectively. In the latter case the GFP is written as a form of the Liouville equation, which can be connected to a transport equation.

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Our interest is mainly focused on the case of the overdamping, for which the GFP is described by the non-Hermitian Schrödinger equation resulting in the standard FP (or Smolkowski) equation using the "change of phase." On the basis of the general formulation, we address several aspects of the FP equation as well as an asymptotic behavior for the functional integral.

The content of the paper is as follows: The next section gives a brief sketch of the dynamics of the quantum vortex. In Sec. III a general framework of the Brownian motion theory is developed. In Sec. IV specific problems are discussed for the FP equation. Section V is devoted to the "semiclassical analysis" for the functional integral in the small diffusion limit. In the last section we give a brief discussion of the aspect of the underdamping aspect of the generalized FP equation.

II. DYNAMICS OF THE QUANTUM VORTICES

A. Preliminaries

We start with a brief sketch for the dynamics of the assembly of vortex gas occurring in the quantum boson fluids and spin system [21,22]. Here we have in mind mainly the Bose-Einstein condensate of atomic gas [7].

To begin, let us consider the complex order parameter written in a polar form, that is, expressed in terms of the density ρ and the phase ϕ:

$$\psi = \sqrt{\rho}\exp(-i\phi). \tag{1}$$

The LG Lagrangian density is written as the sum of two terms:

$$\begin{aligned} l &= l_C - l_H \\ &\equiv i\hbar(\psi^*\dot{\psi} - \text{c.c.}) - H(\psi, \psi^*). \end{aligned} \tag{2}$$

The first term is called *canonical term* for convention, whereas the second term represents the Hamiltonian that consists of the kinetic energy as well as the interaction terms:

$$H = \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi + V \tag{3}$$

with m being the mass of constituent particle of condensate. Thus the Lagrangian becomes

$$l = \rho\dot{\phi} - H,\ H = \frac{1}{2}m\rho\mathbf{v}^2 + V(\rho), \tag{4}$$

where $\mathbf{v} = \frac{\hbar}{m}\nabla\phi$ gives the velocity field. We restrict the argument to the case where the potential term is expressed as a function of the density.

B. Hamiltonian dynamics of planer vortices

The first problem is to construct a dynamical equation for the vortex center [21]. In the following argument, we consider the vortex in a two-dimensional plane (x, y). We derive the equation of motion for the center of vortices which are parametrized by $\mathbf{R}_i(t) = (X_i(t), Y_i(t))\ (i = 1, \ldots, n)$. To carry out this we prepare a profile for the density ρ such that the vortex configuration is incorporated. The field argument $\mathbf{x}$ is thus replaced by $\mathbf{x} \to \mathbf{x} - \mathbf{R}_i(t)$. The phase angle $\phi = \tan^{-1}(y/x)$ is written in an extended form:

$$\phi = \sum_i \kappa_i \tan^{-1}\left[\frac{y - Y_i(t)}{x - X_i(t)}\right] \equiv \sum_i \kappa_i\phi_i, \tag{5}$$

where the winding number is chosen to be $n = 1$. The attached parameter κ_i represents the *vortex charge* with $|\kappa_i| = \kappa \equiv \frac{\hbar}{m}$ (namely, the same absolute charge for all vortices). Using the chain rule $\frac{\partial\phi}{\partial t} = \dot{X}\cdot\frac{\partial\phi}{\partial X}$ it follows that

$$L_C = m\sum_i \int \rho\kappa_i\mathbf{v}_i\cdot\dot{\mathbf{R}}_i\, d^2x \tag{6}$$

with $\mathbf{v}_i = \kappa_i\nabla\phi_i$, the velocity from the ith vortex; in other words, $m\rho\mathbf{v}_i \equiv \mathbf{p}_i$ defines the canonical conjugate to $\mathbf{R}_i$. In what follows we write the effective Lagrangian for the assembly of vortices in terms of the vortex center coordinates $(X_i(t), Y_i(t))$.

1. Intervortex interaction

Noting the relation $\nabla\phi_i = \kappa_i\mathbf{k}\times\nabla\log|\mathbf{x} - \mathbf{R}_i(t)|$ [$\mathbf{k}$ denotes the unit vector in the z direction that is perpendicular to the (x, y) plane], H turns out to be

$$H = \sum_{ij}\kappa_i\kappa_j\int \rho\nabla\log|\mathbf{x} - \mathbf{R}_i(t)|\cdot\nabla\log|\mathbf{x} - \mathbf{R}_j(t)|\, d^2x. \tag{7}$$

The profile of density near a vortex is assumed to have a form $\rho(r) = \rho_0 f(r)$, where $f(r)$ is a monotonically increasing function such that $f(0) = 0,\ f(\infty) = \rho_0$. The integral in (7) is carried out for a pair of vortices, the centers of which are $\mathbf{R}_i$ and $\mathbf{R}_j$. Let us write $\mathbf{y} = \mathbf{x} - \mathbf{R}_{ij}$ with $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$. By partial integration we have

$$-\int \nabla(\rho\nabla\log|\mathbf{y}|)\cdot\log|\mathbf{y} - \mathbf{R}_{ij}|\, d^2y,$$

and noting $\log|\mathbf{y} - \mathbf{R}_{ij}| \simeq \log|\mathbf{R}_{ij}|$ [23], the integral turns out to be $-\log|\mathbf{R}_{ij}|\int_o^\infty \frac{d\rho}{dr}\, dr = \rho_0$. Thus we arrive at the familiar form of the effective Hamiltonian for the assembly of vortices:

$$H_{int} = -\frac{1}{2}m\rho_0\sum_{ij}\kappa_i\kappa_j\log|\mathbf{R}_i - \mathbf{R}_j|. \tag{8}$$

2. Confinement potential

Besides this well-known log potential, we have another term coming from the common confinement potential [7], which is written as

$$H_u \sim \int \Psi(\mathbf{x}, \{\mathbf{R}_i\})^*\Psi(\mathbf{x}, \{\mathbf{R}_i\})U(\mathbf{x})\, d^2x.$$

Taking account of the vortex profile, one sees that $\Psi^*(\mathbf{x})\Psi(\mathbf{x}) \sim \sum_i \rho(\mathbf{x} - \mathbf{R}_i)$. Then the contribution from each vortex can be approximated by the well-localized function. To obtain the actual value, we need to subtract the contribution from the uniform background $\rho = \rho_0$. The profile of the function $f(r)$ can be chosen as the Gaussian form $f(r) = \exp[-ar^2]$; therefore we get the contribution from the

confinement potential

$$H_u \sim \int \rho_0 U(\mathbf{x})\, d^2x - \int \rho_0 (1 - f(\mathbf{x}-\mathbf{R})) \times U(\mathbf{x})\, d^2x = U(\mathbf{R}). \tag{9}$$

As the most typical case, we adopt the harmonic oscillator form, $U = k\mathbf{x}^2$. So the contribution from n vortices is given by $k\sum_{i=1}^{n}\mathbf{R}_i^2$. By summarizing the log potential and the confinement potential term, we have the effective Hamiltonian

$$H_{\rm eff} = -\frac{1}{2}m\rho_0 \sum_{ij}\kappa_i\kappa_j \log|\mathbf{R}_i - \mathbf{R}_j| + \sum_i k\mathbf{R}_i^2. \tag{10}$$

Here we note that there are additional contributions arising from the *pinning* effect caused by the presence of impurities, which we do not use explicitly in the following argument.

3. Canonical equation of motion

We turn to the canonical term L_C (the first term of the Lagrangian). Here instead of calculating L_C directly in terms of the vortex center coordinate $\mathbf{R}$, we derive the equation of motion by using the differentiation under the integral symbol. After some manipulation, we obtain

$$\mathbf{F}_{ix}^C = \frac{d}{dt}\frac{\partial L_C}{\partial \dot{X}_i} - \frac{\partial L_C}{\partial X_i} = m\int\left[\frac{\partial(\rho v_{iy})}{\partial x} - \frac{\partial(\rho v_{ix})}{\partial y}\right]d^2x. \tag{11}$$

Here the integral becomes the line integral $\int \rho\mathbf{v}\cdot d\mathbf{s}$, which yields $2\pi\rho_0$ by taking account of the boundary condition $\rho(\infty) = \rho_0$, hence we get $F_{ix}^C = 2\pi m\rho_0\dot{Y}_i$. In a similar way, it follows that $F_{iy}^C = -m\rho_0\dot{X}_i$. Writing these in terms of the vector notation, we have

$$\mathbf{F}_i^C = \frac{d}{dt}\frac{\partial L_C}{\partial \dot{\mathbf{R}}_i} - \frac{\partial L_C}{\partial \mathbf{R}_i} = 2\pi m\rho_0(\kappa_i\mathbf{k}\times\dot{\mathbf{R}}_i). \tag{12}$$

From the above result, one can guess the form of the Lagrangian, which is simply given as

$$L_C = 2m\rho_0\sum_i \kappa_i(Y_i\dot{X}_i - X_i\dot{Y}_i). \tag{13}$$

Thus the effective Lagrangian including the effective Hamiltonian is given by $L_{\rm eff} = L_C - H_{\rm eff}$. In the following argument we focus the argument on the special case that $\kappa_i = \kappa$ for all indices i; namely, all vortices have the same charge. In this way the equation of motion for the assembly of vortices becomes

$$\bar{\kappa}\frac{dX_i}{dt} = \frac{\partial H_{\rm eff}}{\partial Y_i},\ \bar{\kappa}\frac{dY_i}{dt} = -\frac{\partial H_{\rm eff}}{\partial X_i}, \tag{14}$$

where we use an abbreviated symbol, $\bar{\kappa} \equiv 2m\rho_0\kappa$. This set of equations of motion implies that the pairs (X_i, Y_i) form a canonical variable with each other. Alternatively, this is written in terms of vector notation:

$$\bar{\kappa}\mathbf{k}\times\frac{d\mathbf{R}_i}{dt} = \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i}. \tag{15}$$

This vector equation represents the balance between two types of forces: the left-hand side represents the "Magnus force," and the right-hand side is the potential force.

Thus the present vortex Hamiltonian has just the same form as the one for the charged particles interacting through the Coulomb repulsion confined in the harmonic potential.

4. An example of a two-point vortex

For this simplest problem the Hamiltonian becomes

$$H_{\rm eff} = -\frac{1}{2}m\rho_0\kappa^2\log|\mathbf{R}_1 - \mathbf{R}_2| + \frac{1}{2}k\left(\mathbf{R}_1^2 + \mathbf{R}_2^2\right). \tag{16}$$

This can be separated into the relative and center-of-mass coordinates,

$$2\mathbf{r} = \mathbf{R}_1 - \mathbf{R}_2,\ 2\mathbf{R}_G = \mathbf{R}_1 + \mathbf{R}_2,$$

by which the $H_{\rm eff}$ is given by the sum

$$H_r = -\frac{1}{2}m\rho_0\kappa^2\log|\mathbf{r}| + k\mathbf{r}^2,\ H_G = k\mathbf{R}_G^2. \tag{17}$$

Thus the equation of motion turns out to be

$$\frac{d}{dt}(\mathbf{r}^2) = 0,\ \frac{d}{dt}\left(\mathbf{R}_G^2\right) = 0, \tag{18}$$

which results in the energy surfaces, $H_r = E_r$, $H_G = E_G$. This case of two vortices will be used for the solution for the FP equation in a later section.

III. THE BROWNIAN MOTION

A. The Langevin equation

Now we discuss the stochastic aspect on the basis of the vortex dynamics given above. The effect of fluctuations arises from a variety of origins mainly caused by impurities. If these impurities are distributed in an irregular way, the vortex centers will acquire randomness as a result of, for example, the effect of scattering. Apart from such a scattering effect among vortices, the thermal effect may provide the more direct effect. Then it is natural to expect that the Brownian motion of the vortex centers occurs, which can be described by a random fluctuation denoted as ξ.

Besides the random fluctuation, we have to take account of the effect coming from dissipation, which is required from the *fluctuation-dissipation theorem* (see, e.g., [10]). Indeed, the dissipation can be caused by the inevitable effect of absorption of the vortex energy caused by an interaction with environment; more specifically the dissipation arises from the interaction with the normal fluid component (see, e.g., [16]).

Here we mention an early attempt of the Langevin and FP approach of the quantum vortex that was worked out in [16]. This paper gave a dynamical theory of the vortex pairs of opposite charge aiming at a description of dissociation of a bound pair of vortices. On the other hand, in the present attempt we are concerned with a quite different aspect of a two-dimensional vortex gas, namely, the repulsive Coulomb gas in the confinement potential.

To be noted here is that the random force ξ_i acting on the ith vortex is independent of ξ_j, that is, these have no correlation with each other. Furthermore, the dissipative force is common to all the vortices, which is denoted by μ. Indeed, this assumption on the fluctuation and dissipation is simplest and and most reasonable.

Now taking account both dissipation and random noise, the gradient force in Eq. (15) can be simply modified such that

$$\frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} \to \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} + \mu \frac{d\mathbf{R}_i}{dt} + \xi_i.$$

This modification is somehow a well-known procedure in general condensed matter physics. We here borrow the procedure used for ferromagnetic particles [24], which deals with the thermal fluctuation of the magnetic moment. In the present case the magnetic moment is replaced by the velocity of the vortex center $\dot{\mathbf{R}}$. By taking account of this prescription, the equation of motion is given as follows:

$$\bar{\kappa}\left(\mathbf{k} \times \frac{d\mathbf{R}_i}{dt}\right) = \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} + \mu \frac{d\mathbf{R}_i}{dt} + \xi_i. \tag{19}$$

Multiplying $\mathbf{k}$ by (19) one obtains

$$\bar{\kappa} \frac{d\mathbf{R}_i}{dt} = -\mathbf{R}_i \times \left(\frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} + \mu \frac{d\mathbf{R}_i}{dt} + \xi_i\right). \tag{20}$$

Using these two equations (19) and (20), the equation for $\dot{\mathbf{R}}_i$ can be derived [25]:

$$\frac{d\mathbf{R}_i}{dt} = -\mathbf{A}_i + \zeta_i, \tag{21}$$

where we adopt the following scaling of time variable $t \to (\mu^2 + \bar{\kappa}^2)t$ [26] and $\mathbf{A}_i$ and ζ_i are given as

$$\mathbf{A}_i = \left[\mu \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} + \left(\bar{\kappa}\mathbf{k} \times \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i}\right)\right], \tag{22}$$

$$\zeta_i = -(\mu \xi_i + \bar{\kappa}\mathbf{k} \times \xi_i). \tag{23}$$

The expression (23) can be written in terms of (x, y) components:

$$\zeta_i^x = \mu \xi_i^x - \bar{\kappa}\xi_i^y, \ \zeta_i^y = \bar{\kappa}\xi_i^x + \mu \xi_i^y, \tag{24}$$

which means the orthogonal transformation in a two-dimensional plane with the rotational angle, $\tan\Theta = \frac{\bar{\kappa}}{\mu}$; namely, (24) represents *just* a rotation of the original fluctuation ξ, so that the "additive nature" is safely kept.

Without ζ, Eq. (21) is an analogy of the "Landau-Lifschitz equation," which is well known in ferromagnetic theory [24,27]. We note that the vector $\mathbf{A}_i$ consists of two terms: the first term can be called the *gradient term*, and the second term the *gyration term*, which can be a counterpart of the "Magnus force." The mutual interplay between these two terms characterizes the stochastic process of the vortex motion, which we discuss below.

Now we put an Ansatz of the Gaussian white noise for ξ_i, which satisfies the correlation

$$\left\langle \xi_i^\alpha(t) \right\rangle = 0,$$
$$\left\langle \xi_i^\alpha(t)\xi_j^\beta(t+u) \right\rangle = h\delta_{\alpha\beta}\delta_{ij}\delta(u), \tag{25}$$

where the suffix (α, β) represents (x, y) and $\delta(u)$ means the delta function. The diffusion constant h is assumed to take a common value for all components i. The correlation (25) is transferred to the resultant random force (23) $\zeta = (\zeta^x, \zeta^y)$; we have that the same form of the correlation

$$\left\langle \zeta_i^\alpha(t) \right\rangle = 0,$$
$$\left\langle \zeta_i^\alpha(t)\zeta_j^\beta(t+u) \right\rangle = (\bar{\kappa}^2 + \mu^2)h\delta_{\alpha\beta}\delta_{ij}\delta(u). \tag{26}$$

Taking into account this feature, it is possible to replace $(\bar{\kappa}^2 + \mu^2)h \to h$ in the following argument. Thus the above set of the Langevin equations obeys a set of the random forces $\zeta_i{:}\{i = 1 \sim n\}$ which satisfy the common correlation relation independent of the vortex indices.

B. The functional integral and non-Hermitian Schrödinger equation

If we recall that constituent vortices are independent of each other, the probability distribution for an assembly of vortices is given by the product of the Gaussian noise for $(\zeta_i{:}i = 1 \sim n)$, which becomes the standard Gaussian functional form [28]:

$$P\{[\zeta_i(t)]\} = \prod_{i=1}^{n} \exp\left[-\frac{1}{2h}\int_0^t \zeta_i^2(t)\,dt\right]. \tag{27}$$

Using this distribution, the transition probability from $\mathbf{R}(0)$ to $\mathbf{R}(t)$ is given by the path integral

$$K[\mathbf{R}(t)|\mathbf{R}(0)] = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \prod_i \exp\left[-\int_0^t \frac{\zeta_i^2(t)}{2h}\,dt\right]\mathcal{D}[\zeta_i(t)], \tag{28}$$

where the notation for a set of vortex centers $\mathbf{R} \equiv (\mathbf{R}_1, \ldots, \mathbf{R}_n)$ is used. With this expression, the process of transition caused by the random noise can be built in an implicit way.

In order to explicate the process of building up the path integral over the orbits in the space of vortex center $\mathbf{R}$, we adopt the following steps: To ensure the Langevin equation, we insert the expression of the δ-functional integral

$$\int \prod_{i=1}^{n} \prod_t \delta[\mathbf{F}_i(t) - \zeta_i(t)]\mathcal{D}\mathbf{F}_i(t) = 1, \tag{29}$$

where we use the notation

$$\mathbf{F}_i = \frac{d\mathbf{R}_i}{dt} + \mathbf{A}_i \tag{30}$$

and $\prod_t$ means the *continuous product* over time interval $[0, t]$. Using the above delta functional, the transition rate (28) can be brought to the path integral

$$K[\mathbf{R}(t)|\mathbf{R}(0)] = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \prod_i \exp\left[-\int_0^t \frac{\zeta_i^2(t)}{2h}\,dt\right] \times \prod_{i=1}^{n} \prod_t \delta[\mathbf{F}_i(t) - \zeta_i(t)]\mathcal{D}\mathbf{F}_i(t)\mathcal{D}[\zeta_i(t)], \tag{31}$$

which enables us to derive the FP equation in most direct way. The intermediate step will be given in Appendix A. Noting this result, we use an "imaginary time trick," that is, we define $\tau = -it$, and then the propagator is written in the *quantum mechanical* path integral form

$$K[\mathbf{R}(\tau)|\mathbf{R}(0)] = \int \exp\left(\frac{i}{h}\int \mathcal{L}\,d\tau\right)\mathcal{D}[\mathbf{R}]. \tag{32}$$

Here the *Lagrangian* becomes

$$\mathcal{L} = \sum_i \left[\frac{1}{2}\left(\frac{d\mathbf{R}_i}{d\tau}\right)^2 + i\mathbf{A}_i \cdot \frac{d\mathbf{R}_i}{d\tau} \right] - W \tag{33}$$

with the potential function

$$W = \sum_i \left(\frac{\mathbf{A}_i^2}{2} - M_i h \right), \quad M_i = \frac{1}{2}\frac{\partial}{\partial \mathbf{R}_i} \cdot \mathbf{A}_i, \tag{34}$$

where the second term in W comes from the Jacobian written in an imaginary time form (see Appendix A):

$$J = \exp\left(\frac{i}{h} \int_0^\tau \sum_i M_i h \, d\tau \right). \tag{35}$$

Now by introducing the "wave function" $\Psi(\mathbf{R}, \tau)$, we write the integral equation:

$$\Psi(\mathbf{R}, \tau) = \int K[\mathbf{R}(\tau)|\mathbf{R}(0)]\Psi(\mathbf{R}, 0)\, d\mathbf{R}(0). \tag{36}$$

Following the standard procedure of a Feynman path integral, we obtain the Schrödinger equation [29]:

$$ih\frac{\partial \Psi}{\partial \tau} = \frac{1}{2}\sum_i (\mathbf{p}_i - i\mathbf{A}_i)^2 \Psi + W\Psi, \tag{37}$$

where $\mathbf{p} = -ih\frac{\partial}{\partial \mathbf{R}}$. This form of the wave equation has the same form as the particle in the presence of the *vector potential* $\mathbf{A}$. We note that (37) is apparently non-Hermitian in general (see, e.g., [30]); more details on this point will be discussed later).

By replacing the imaginary time τ with the original (genuine) time t, namely, $\tau \to -it$, and rewriting the wave function Ψ by the distribution function P, then one arrives at

$$\frac{\partial P}{\partial t} = \frac{h}{2}\sum_i \frac{\partial^2 P}{\partial \mathbf{R}_i^2}, + \sum_i \frac{\partial}{\partial \mathbf{R}_i} \cdot \left\{ \left[\mu \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} + \left(\bar{\kappa}\mathbf{k} \times \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} \right) \right] P \right\}. \tag{38}$$

Here use is made of the relation $(\nabla \cdot \mathbf{A})P + \mathbf{A} \cdot \nabla P = \nabla \cdot (\mathbf{A}P)$. Equation (38) is the main consequence of the present paper, though it looks simple enough. This can be regarded as a two-dimensional generalization of the FP (Smolkowski) equation used in Dyson's theory [17], where the second term in $\mathbf{A}$, namely, the gyration term is missing. We rewrite this generalized FP equation in the form of the conservation of current:

$$\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{39}$$

with $\nabla \equiv \sum_i \frac{\partial}{\partial \mathbf{R}_i}$. The probability current is defined as the sum

$$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2,$$
$$\mathbf{j}_1 = -\frac{h}{2}\nabla P - \mu \sum_i \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} P, \tag{40}$$
$$\mathbf{j}_2 = \mathbf{k} \times \left(-\frac{h}{2}\nabla P - \bar{\kappa} \sum_i \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} P \right). \tag{41}$$

These expressions are significant; the first term of the respective terms stands for a diffusion effect, and the second term represents the "transport of probability mass," namely, $\sum_i \frac{d\mathbf{R}_i}{dt} P$. This feature suggests that the FP equation thus obtained describes a diffusive behavior of the vortex gas.

Now we recall that the vector $\mathbf{A}$ consists of two terms: the gradient and the gyration terms. These two terms are controlled by a competition between two parameters, $\bar{\kappa}$ and μ, the vortex charge and the magnitude of dissipation, respectively. We have two *extreme* cases. (1) The relation $\mu \gg \kappa$ holds, that is, the gradient term becomes dominant, and the gyration term is discarded in (38). This case is called "overdamping." (2) On the other hand, if $\mu \ll \bar{\kappa}$, this corresponds to "underdamping," for which the gyration is dominant. The original idea to separate these cases dates back to the articles [31,32].

In what follows, we restrict the argument to the overdamping approximation, and the underdamping case will be briefly sketched in the last section. Thus we obtain

$$\frac{\partial P}{\partial t} = \frac{h}{2}\sum_i \frac{\partial^2 P}{\partial \mathbf{R}_i^2} + \sum_i \mu \frac{\partial}{\partial \mathbf{R}_i} \cdot \left[\left(\frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} \right) P \right] \equiv \mathcal{L}_{FP} P. \tag{42}$$

If (42) is regarded as a nonperturbed term, the term coming from the gyration $\mathbf{j}_2$ can be treated as a perturbation.

IV. SPECIFIC ASPECTS OF THE FP EQUATION

A. Statistical mechanical consequences

1. Statistical average using the FP equation

We first examine the general properties concerning the FP equation. Here we take up a typical example, the evolution of the statistical average for the function $\langle K\mathbf{R}_1, \ldots, \mathbf{R}_n)\rangle$. Using the FP equation this satisfies the equation

$$\mu \frac{d\langle K\rangle}{dt} = -\sum_i \left\langle \frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i}\frac{\partial K}{\partial \mathbf{R}_i} \right\rangle + \frac{h}{2}\sum_i \left\langle \frac{\partial^2 K}{\partial \mathbf{R}_i^2} \right\rangle. \tag{43}$$

As a special case, we consider the moment of the vortex center: $K = \sum_i \mathbf{R}_i^2$, for which we have

$$\mu \frac{d\langle K\rangle}{dt} = \left(\frac{hN}{2} - k^2 \right) - 2k\langle K\rangle, \tag{44}$$

which leads to

$$\langle K\rangle = K_0 \left[1 - \exp\left(-\frac{2k}{\mu} t \right) \right] \tag{45}$$

with $K_0 = \frac{1}{\mu}(\frac{hN}{2} - k^2)$.

2. Stationary distribution: Two-dimensional Coulomb gas

Let us consider the stationary state, namely, distribution function satisfies $\frac{\partial P}{\partial t} = 0$. If we put an Ansatz $P = \exp[-\beta H_{\rm eff}]$ and substitute this into (42), we get the relation

$$\left(\frac{h\beta}{2} - \mu \right) \left\{ \sum_i \left[\frac{\partial^2 H_{\rm eff}}{\partial \mathbf{R}_i^2} - \beta \left(\frac{\partial H_{\rm eff}}{\partial \mathbf{R}_i} \right)^2 \right] \right\} = 0. \tag{46}$$

From the requirement that this relation should hold for arbitrary Hamiltonian, we get

$$\frac{h\beta}{2} = \mu, \tag{47}$$

which is nothing other than the content of the fluctuation-dissipation relation [10]. In this way the equilibrium state of the assembly of vortices is realized as the two-dimensional Coulomb gas [17], for which $P = \exp[-\beta H_{\rm eff}]$ becomes the Boltzman factor

$$P = \exp\left[\kappa^2\beta\sum_{ij}\log|\mathbf{R}_i - \mathbf{R}_j|\right] \times \exp\left(-\frac{1}{2}k\beta\sum_i|\mathbf{R}_i|^2\right) \tag{48}$$

with $\beta \equiv 1/k_BT$ being the inverse temperature. It is to be mentioned that $\exp[-\beta H_{\rm eff}]$ is the eigenfunction of $\mathcal{L}_{FP}$ with the *zero eigenvalue*, namely, $\mathcal{L}_{FP}[\exp(-\beta H_{\rm eff})] = 0$ (see the argument below).

B. Reduction to a quantum mechanical many-particle system

As has been noted in the previous section, the "wave equation" (37) is a non-Hermitian Schrödinger equation. In the overdamping case, for which the gyration term is discarded, one can eliminate the *vector potential* $\mathbf{A}$ by adopting the change of phase [33], $\Psi = \exp(-if)\tilde{\Psi}$, where f is chosen such that the following equation holds:

$$\nabla f = \frac{1}{h}\sum_i \mathbf{A}_i \tag{49}$$

from which one sees that f is proportional to $H_{\rm eff}$; hence noting the relation (47), then we write the transformation like

$$P = \exp\left(-\frac{\beta H_{\rm eff}}{2}\right)\tilde{P}. \tag{50}$$

Thus one can reduce the FP equation to the familiar form of the Schrödinger-type equation for $\tilde{P}$:

$$\frac{\partial\tilde{P}}{\partial t} = \frac{h}{2}\sum_i\nabla_i^2\tilde{P} + \mu\left[\frac{1}{2}\sum_i\nabla_i^2 H_{\rm eff} - \frac{\beta}{4}\left(\sum_i\nabla_i H_{\rm eff}\right)^2\right]\tilde{P} \equiv \hat{\mathcal{H}}_{FP}\tilde{P}, \tag{51}$$

$$\hat{\mathcal{H}}_{FP} = \exp\left(\frac{\beta H_{\rm eff}}{2}\right)\mathcal{L}_{FP}\exp\left(-\frac{\beta H_{\rm eff}}{2}\right), \tag{52}$$

which is explicitly written as

$$\frac{\partial\tilde{P}}{\partial t} = \left[\frac{h}{2}\sum_i\nabla_i^2 - V(\mathbf{R}_1,\ldots,\mathbf{R}_N)\right]\tilde{P} \equiv \hat{\mathcal{H}}_{FP}\tilde{P},$$

$$V = \mu\left(B\sum_i\mathbf{R}_i^2 + A\sum_{i,j}\frac{1}{|\mathbf{R}_i - \mathbf{R}_j|^2}\right),$$

$$A = \frac{1}{4}(m\rho_0)^2\kappa^4,\ B = 4k^2 \tag{53}$$

up to some additional constant. We see that the following equation holds:

$$\hat{\mathcal{H}}_{FP}\left[\exp\left(-\frac{\beta H_{\rm eff}}{2}\right)\right] = 0,$$

which indicates that $\exp[-\frac{\beta H_{\rm eff}}{2}] \equiv \tilde{P}_0$ is an eigenfunction of $\hat{\mathcal{H}}_{FP}$ corresponding to zero eigenvalue. Here (53) is a typical two-dimensional many-body problem, which had once been studied as a major topic (e.g., [34,35]), for which a brief sketch will be given in Appendix B.

C. Special case of two vortices

For this case, as is seen from (17), the relative and the center-of-mass coordinates are separated, and one can put aside the center-of-mass degree. Hence we can treat the problem in such a way that one vortex is fixed at the origin, say, $\mathbf{R}_2$. So let us write the "wave function" in the form $\tilde{P} = \exp[-\epsilon t]G$, then it turns out to be

$$\left[\frac{h}{2}\nabla^2 + (\epsilon - V(r))\right]G = 0, \tag{54}$$

where $r \equiv |\mathbf{R}_1|$, and hence the potential becomes

$$V(r) = Br^2 + \frac{A}{r^2}. \tag{55}$$

Let us put $G(r,\theta) = v(r)u(\theta)$ and choose the angular part to be constant.

The time-dependent distribution function (the solution for the reduced FP equation) is written in the form

$$\tilde{P}(r,t) = \tilde{P}_0(r) + \sum_{n=0}c_n(t)\exp(-\epsilon_n t)G_n(r). \tag{56}$$

Here we need to separate the zero energy solution from the other parts and the coefficients $c_n(t)$ are determined by the initial condition $\tilde{P}(r,0)$, which will be given below. We first solve the eigenvalue equation (54), for which we change the eigenfunction: $v(r) = \sqrt{r}u(r)$, then (54) turns out to be

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \left[\epsilon - \left(Br^2 + \frac{A}{r^2}\right)\right]u = 0, \tag{57}$$

where A shifts by an amount $\frac{1}{4}$, namely, $A \to A - \frac{1}{4}$: The eigenvalue equation is known to be analytically solved [36], from which we simply borrow the result: namely, by putting $\xi = \sqrt{2B/h}r^2$, $u(r)$ satisfies

$$\xi\frac{d^2u}{d\xi^2} + \frac{3}{2}\frac{du}{d\xi} + \left[n + s + \frac{3}{4} - \frac{\xi}{4} - \frac{s(s+1/2)}{\xi}\right]u = 0, \tag{58}$$

where s, n is settled such that $2s(s+1) = \sqrt{B/h}$ and $\sqrt{B/h}\epsilon = 4(n+s)+3$. Hence the eigenvalue ϵ is given by

$$\epsilon_n = \sqrt{\frac{hB}{2}}\left[4n + \sqrt{\frac{4(A-1/8)}{h}}\right] \tag{59}$$

with n being the nonnegative integer. From this expression $A \geqslant 1/8$ should hold. The corresponding eigenstate is given by the hypergeometric function

$$u_n(\xi) = F\left(-n, 2s + \frac{3}{2}, \xi\right). \tag{60}$$

Hence we obtain the solution up to the lowest "excited state," namely, by choosing only the case $n = 0, 1$:

$$\tilde{P}(r,t) = \exp\left(-\frac{\beta H_{\rm eff}}{2}\right)[\tilde{P}_0 + c_0 u_0(r)\exp(-\epsilon_0 t) + c_1 u_1(r)\exp(-\epsilon_1 t)]. \quad (61)$$

In this way, we see that in the limit of $t \to \infty$, the equilibrium state is recovered: $P \simeq \exp(-\beta H_{\rm eff})$. The above result can be used for a basis of the more complicated system.

Here a remark is in order concerning the above solution. The problem is connected with the confinement harmonic potential that is reflected in the coefficient B. About this point, we note that it is possible to choose the more general case that the confinement potential is anisotropic and even letting it to allow time varying. By extending the problem to such a general case, there may appear a variety of problems; e.g., if this anisotropy of the harmonic oscillator changes adiabatically, we expect adiabatic control of the stochastic behavior of the vortex system.

V. SMALL DIFFUSION LIMIT

If we get back to the starting functional integral, it is intriguing to examine an asymptotic limit in which the diffusion constant h is regarded as small. We here look for an alternative way to obtain an approximate scheme so as to approach to random behavior for the vortex gas apart from the FP equation.

We consider a system of two vortices for which one vortex is pinned at the origin as before; hence the functional integral is written in the form

$$K = \int \exp\left[\left(-\frac{1}{h}\int \mathcal{L}dt\right)\right]\mathcal{D}(r).$$

Here the "Lagrangian" is given as

$$\mathcal{L} = \left(\frac{d\mathbf{r}}{dt} + \mu\frac{\partial H_r}{\partial \mathbf{r}}\right)^2. \quad (62)$$

Using the polar coordinate, one write $\frac{d\mathbf{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ ($\hat{r}$ and $\hat{\theta}$ stand for the unit vector of radial and its perpendicular direction). In the limit of $h \simeq 0$, the functional integral is treated by the stationary phase approximation, which is written in a form $K_{cl} = \exp(-\frac{1}{h}S_{cl})$. Here S_{cl} denotes the *classical action* that satisfies the extreme condition $\delta S = 0$. The extreme condition leads to the Euler-Lagrange equation. We have the contribution from the deviation of an extreme path that is written in terms of the Gaussian functional integral with respect to the deviation from the extreme path. However, we discard this for the sake of simplicity. We note a peculiar feature of the Lagrangian: The variable θ does not appear in the Lagrangian, namely, θ is cyclic coordinate, so the "momentum" conjugate to θ is the constant of motion, which is given by $r^2\dot{\theta} = C$. Thus, following the well-known procedure in analytical dynamics, we construct the Rouse function [37], in which the θ variable is eliminated to be

$$R = C\dot{\theta} - \mathcal{L} = -\left(\dot{r} + \mu\frac{\partial H_r}{\partial r}\right)^2 + \frac{C^2}{r^2}. \quad (63)$$

Thus the equation of motion is derived using the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0.$$

By substituting the solution (classical orbit) of this equation into the expression $\mathcal{L}$:

$$K_{cl} = \exp\left\{-\frac{1}{h}\int_{t_i}^{t_f}\left[\left(\dot{r} + \mu\frac{dH_r}{dr}\right)^2 + \frac{C^2}{r^2}\right]dt\right\}. \quad (64)$$

We look for a further reduced form of K_{cl}; that is, we consider the case that C is regarded as small enough such that it is treated as perturbation parameter. Hence we can omit the last term in (63); so the equation of motion becomes in a simple form

$$\dot{r} + \mu\frac{dH_r}{dr} = 0, \quad (65)$$

which is solved to lead to the orbit, (r,θ):

$$r^2(t) = \alpha + (r_i^2 - \alpha)\exp\left(-\frac{2\mu}{k}t\right),\ \theta = C\int_{t_i}^{t}\frac{dt}{r^2} \quad (66)$$

putting $\alpha = \frac{m\rho_0\kappa^2}{2k}$. The orbit describes the spiral which starts with the initial point $r = r_i$ and converges to the limiting radius, $r = \sqrt{\alpha}$. Then noting that the first integral in K_{cl} vanishes as a result of (65), we have

$$K_{cl} = \exp\left[-\frac{C^2\mu}{kh}\int_{r_i}^{r_f}\frac{dr}{r(r^2-\alpha)}\right]. \quad (67)$$

Then we get $K_{cl}(r_f, r_i) = \exp(X)$ with

$$X = \beta\left[\log\left(\frac{r_f^2-\alpha}{r_i^2-\alpha}\right) - \log\left(\frac{r_f}{r_i}\right)\right], \quad (68)$$

where we put $\beta = \frac{C^2\mu}{2kh\alpha}$. Using the transition amplitude thus calculated, the probability is calculated to be

$$P(r_f) = \int_0^\infty K_{cl}(r_f, r_i)P(r_i)\,dr_i, \quad (69)$$

and choosing the initial distribution $P(r_i) = \delta(r_i - r_0)$, one gets

$$P(r_f) = \left[\frac{r_0(r_f^2-\alpha)}{r_f(r_0^2-\alpha)}\right]^\beta. \quad (70)$$

This result indicates that the probability beyond the limiting radius vanishes.

VI. AN ASPECT FROM THE TRANSPORT THEORY

Up to now our consideration of the generalized FP equation has been restricted to the overdamping case, that is, $\mu \gg \bar{\kappa}$. Owing to this restriction, the standard FP equation is converted to the Schrödinger equation, by which we can use the resources of quantum mechanics. Now there arises a problem: What about the opposite case, namely, the current $\mathbf{j}_2$ is dominant. That is, the extremely opposite limit $\mu \ll \bar{\kappa}$ holds together with the simultaneous restriction that h is enough

small to be discarded, then (38) turns out to be

$$\frac{\partial P}{\partial t} = \bar{\kappa} \sum_i \frac{\partial}{\partial \mathbf{R}_i} \cdot \left[\left(\mathbf{k} \times \frac{\partial H_{\text{eff}}}{\partial \mathbf{R}_i} \right) P \right], \tag{71}$$

which is alternatively written as the Liouville equation

$$\frac{\partial P}{\partial t} = -\{P, \bar{\kappa} H_{\text{eff}}\}, \tag{72}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial X_i} \frac{\partial B}{\partial Y_i} - \frac{\partial A}{\partial Y_i} \frac{\partial B}{\partial X_i} \right).$$

The right-hand side of (72) is proportional to $\nabla \cdot (\sum_i \mathbf{v}_i P)$, which can be the divergence of the probability flow with the "phase space velocity" for each vortex:

$$\mathbf{v}_i \left(= \frac{d\mathbf{R}_i}{dt} \right) = \mathbf{k} \times \frac{\partial H_{\text{eff}}}{\partial \mathbf{R}_i}.$$

Thus (72) apparently suggests that there does not occur the equilibrium state in the vortex motion, whereas the gradient term in (38) drives the equilibrium state. In other words, the underdamping regime means that the dissipative as well as fluctuation force is rather weaker than the repulsive forces acting between vortices. As a result of this, there might not maintain the stable equilibrium state among the vortex gas. As such, if the remaining term of (42) coming from $\mathbf{j}_1$, written as G, namely,

$$G = \sum_i \left(\frac{h}{2} \frac{\partial^2 P}{\partial \mathbf{R}_i^2} + \mu \frac{\partial}{\partial \mathbf{R}_i} \cdot \frac{\partial H_{\text{eff}}}{\partial \mathbf{R}_i} P \right), \tag{73}$$

which can be treated as if the "collision term," (72) can connect with the transport equation [38]:

$$\frac{\partial P}{\partial t} - \{\bar{\kappa} H_{\text{eff}}, P\} = G. \tag{74}$$

This can be treated by applying the perturbation scheme using an iteration procedure. Let us write $P = P_0 + P_1$, where P_0 means the unperturbed term, which satisfies the stationary equation $\{H_{\text{eff}}, P\} = 0$. By introducing the linear operator defined as

$$\{H_{\text{eff}}, P\} \equiv \mathcal{L}_H P, \tag{75}$$

then we have the equation for the perturbed term P_1,

$$\frac{\partial P_1}{\partial t} - \mathcal{L}_H P_1 = G(P_0), \tag{76}$$

and $G(P_0)$ is the one for which the unperturbed solution P_0 is substituted in (73). The formal solution for (76) can be obtained with the aid of the method of variation of parameters. First, the homogenous equation is formally solved as

$$P_1(t) = \exp(\mathcal{L}_H t) P_1(0). \tag{77}$$

To look for a special solution of the inhomogeneous equation, we put $P(t) = \exp(\mathcal{L}t) Q(t)$. By substituting this into (76) the equation for Q is derived as

$$\frac{dQ}{dt} = \exp(-\mathcal{L}_H t) G(P_0), \tag{78}$$

from which we obtain a special solution

$$P_s(t) = \exp(\mathcal{L}_H t) \int \exp(-\mathcal{L}_H t') G(t')\, dt'. \tag{79}$$

This serves as a formal perturbation solution for the transport equation. The more detailed analysis will be left for a future study.

VII. CONCLUDING REMARKS

The stochastic approach to the quantum vortex gas in two dimensions has been investigated. The starting point is the Hamiltonian dynamics for the vortex gas in which the coordinates (X_i, Y_i) form a canonical pair each other. This can be transcribed to the Langevin equation with the Gaussian white noise. Owing to the white noise, the Langevin equation is converted to the functional integral, which results in the generalized Fokker-Planck (FP) equation. In particular we have examined the overdamping limit yielding the standard FP equation, for which we have examined several aspects in detail. As for the underdamping case, we have discussed it briefly, but there may still remain a variety of problems to be explored. The study will be left for future research. As a final remark, it would be interesting to address the problem to extend the present Langevin and FP formalism to the three-dimensional dynamics that shows an intricate process of entanglement of vortex curves [39,40].

APPENDIX A: REDUCTION TO THE PATH INTEGRAL

Here a prescription is given for the some step leading to the path integral form (32). Using the Fourier transform of the delta functional in (31), it follows that

$$\begin{aligned} K[\mathbf{R}(t)|\mathbf{R}(0)] = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \prod_i \exp\left[-\int_0^t \frac{\zeta_i^2(t)}{2h}\, dt \right] \\ \times \prod_{i=1}^n \prod_t \exp\{2\pi i \int \lambda_i(t)[\mathbf{F}_i(t) - \zeta_i(t)]\, dt\} \\ \times \mathcal{D}\mathbf{F}_i(t) \mathcal{D}[\zeta_i(t)] \mathcal{D}[\lambda_i(t)]; \end{aligned} \tag{A1}$$

then by carrying out the Gaussian functional integral over $\zeta(t)$ and $\lambda(t)$, one gets

$$K[\mathbf{R}(t)|\mathbf{R}(0)] = \int \exp\left[-\frac{1}{2h} \int_0^t \sum_i \mathbf{F}_i^2(t)\, dt \right] \prod_{i=1}^n \mathcal{D}\mathbf{F}_i. \tag{A2}$$

This is converted to the functional integral over the vortex centers:

$$\begin{aligned} K[\mathbf{R}(t)|\mathbf{R}(0)] = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \exp\left[-\frac{1}{2h} \int_0^t \sum_i \left(\frac{d\mathbf{R}_i}{dt} + \mathbf{A}_i \right)^2 dt \right] \\ \times \prod_i J(\mathbf{R}_i) \mathcal{D}[\mathbf{R}_i]. \end{aligned} \tag{A3}$$

Here $J(\mathbf{R})$ is the functional Jacobian given by

$$J(\mathbf{R}) = \det\left(\frac{\delta \mathbf{F}(\mathbf{R}(t))}{\delta \mathbf{R}(t')} \right), \tag{A4}$$

and after some steps of calculating the functional determinant [18], this leads to

$$J = \exp\left(\int_0^t \frac{1}{2}\frac{\partial}{\partial \mathbf{R}} \cdot \mathbf{A}\, dt\right). \tag{A5}$$

Noting the exponential form, this factor can be incorporated into the action function, and hence it plays a crucial role in determining the form of the FP equation given in the main text.

APPENDIX B: A BRIEF SKETCH FOR THE *N*-PARTICLE PROBLEM

We give an outline for treating the reduced quantum many-particle system described by the short-range repulsive force coming from the inverse square interaction that balances with the attractive force coming from the harmonic potential. We adopt the procedure of the method of the collective coordinate [34]. The central idea is to separate the original particle degree of freedom into the collective degree and the internal one. In what follows we borrow it aiming at an application to the present problem. According to Tomonaga, a natural candidate of the collective coordinate for the present case can be chosen as $\mathcal{Q}_c = \frac{1}{2}\sum_i (X_i^2 - Y_i^2)$ together with the conjugate momentum

$$\mathcal{P}_c = \frac{1}{2}\sum_i \left(X_i \frac{\partial}{\partial X_i} - Y_i \frac{\partial}{\partial Y_i}\right). \tag{B1}$$

The commutation relation for these becomes

$$[\mathcal{Q}_c, \mathcal{P}_c] = \frac{1}{NR_0^2}\left\langle \sum_i \left(X_i^2 + Y_i^2\right)\right\rangle, \tag{B2}$$

where $\langle\cdot\rangle$ is an average appropriately defined and R_0, which is a circle radius, chosen such that (B2) satisfies the canonical commutation relation. The *quantum mechanical Hamiltonian* $\mathcal{H}_{\rm eff}$ is thus written as a form of the coupling between the collective coordinate $\mathcal{Q}_c$, $\mathcal{P}_c$ and the internal coordinate, say, $(\mathcal{Q}_{in}, \mathcal{P}_{in})$:

$$\hat{\mathcal{H}}_{FP} = \mathcal{H}_0(\mathcal{Q}_{in}, \mathcal{P}_{in}) + \mathcal{H}_1(\mathcal{Q}_{in}, \mathcal{P}_{in})\mathcal{Q}_c + \mathcal{H}_2(\mathcal{Q}_{in}, \mathcal{P}_{in})\mathcal{Q}_c^2 + \frac{1}{2I}\mathcal{P}_c^2. \tag{B3}$$

Having accomplished the separation of variables, the wave function can be expressed as the direct product:

$$\tilde{P} = \psi_c(\mathcal{Q}_c)\psi_{in}(\mathcal{Q}_c, \{\mathcal{Q}_{in}, \mathcal{P}_{in}\}). \tag{B4}$$

When the adiabatic separation is assumed, the internal coordinates are fixed and eliminated by integrating over them, namely, $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are replaced by the expectation value with respect to the internal state ψ_{in} which results in the wave function, which is written in terms of the collective coordinate $\mathcal{Q}_c$.

As a special case, we consider three vortices of special configuration. Namely, we suppose that the third and second vortices are pinned at the origin, $\mathbf{R}_3 = (0, 0)$ and $\mathbf{R}_2 = (1, 0)$, respectively, and hence the stationary counterpart of the eigenvalue equation becomes $H_{\rm eff}P = \epsilon P$ with the "Hamiltonian" (53):

$$\begin{aligned} H_{\rm eff} &= H^0 + V, \\ H^0 &= -\frac{h}{2}\nabla_1^2 + \frac{B\mu}{\mu^2+\bar{\kappa}^2}R_1^2 + \frac{A\mu}{\mu^2+\bar{\kappa}^2}\frac{1}{R_1^2}, \\ V &= \frac{A\mu}{\mu^2+\bar{\kappa}^2}\frac{1}{(X-1)^2+Y^2}. \end{aligned} \tag{B5}$$

The crude estimate for this can be carried out by applying the perturbation procedure if H^0 is regarded as nonperturbative term and treating V as the perturbation: The solution for H^0 has been obtained for the two vortices case, and the lowest energy change can be simply obtained by taking the expectation value with respect to the lowest eigenstate state of H^0.

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