Switching from a continuous to a discontinuous phase transition under quenched disorder

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Discontinuous phase transitions are particularly interesting from a social point of view because of their relationship to social hysteresis and critical mass. In this paper, we show that the replacement of a time-varying (annealed, situation-based) disorder by a static (quenched, personality-based) one can lead to a change from a continuous to a discontinuous phase transition. This is a result beyond the state of the art, because so far numerous studies on various complex systems (physical, biological, and social) have indicated that the quenched disorder can round or destroy the existence of a discontinuous phase transition. To show the possibility of the opposite behavior, we study a multistate q-voter model, with two types of disorder related to random competing interactions (conformity and anticonformity). We confirm, both analytically and through Monte Carlo simulations, that indeed discontinuous phase transitions can be induced by a static disorder.

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I. INTRODUCTION

The study of complex systems, for which the 2021 Nobel Prize was awarded, is arguably the most interdisciplinary field of science, influencing many seemingly unrelated disciplines. For example, it may seem that physics and the social sciences have little in common, yet a huge number of papers have been published in recent decades using statistical physics methods to model various social systems [1–4].

Someone might ask why physicists are concerned with social systems. Probably the first answer that comes to mind is that the methods and concepts of statistical physics can also be useful in the social sciences. But is feedback possible? Can problems and concepts from the social sciences trigger the development of physics itself? The very birth of statistical physics shows that this is what can happen [5]. However today, as sometimes claimed, physicists are completing the circle by applying physical methods (oftentimes those of statistical physics) to quantify social phenomena [4]. This is undeniably true, but does this mean that nowadays research at the intersection of physics and social science no longer contributes anything new to physics? There are several examples, which suggest the opposite [6,7].

In this paper, we also show that the result obtained within the model originally proposed to describe social opinion dynamics can go beyond the state of the art in physics. This result will address the effects of two types of approaches, so-called quenched and annealed, on phase transitions. Before we get to the point, let us clarify the terms *quenched* and *annealed* in the context of disorder in complex systems, because, in our experience, they are not widely known to the general audience. Within the quenched approach various randomness (heterogeneity) associated with interactions, topology, etc., are fixed in time. In the context of opinion dynamics, a good example of such an approach are inflexible agents [8] or zealots [9]. On the other hand, within the annealed approach all these randomnesses are changed at each time step [10]. A good example of this type of approach is a contrarian behavior introduced by Galam [11]. In the context of social systems, inspired by the long-standing person-situation debate [12], we related quenched to the personality-oriented, while annealed to the situation-oriented, approach [13,14]. Therefore, the analysis of differences resulting from the given approach (quenched versus annealed) is interesting in the context of social systems. But is this topic of interest in the context of physics itself? Looking at the literature it definitely does.

The role of the quenched disorder in shaping the type of the phase transitions (PTs) have been intensively studied from experimental and theoretical points of view, and applied to understand the behavior of various complex systems [15]. In particular, it has been found that in low dimensions quenched randomness results in rounding, smearing, or completely destroying discontinuous PTs [16–21]. The early prediction of this effect was given heuristically by Imry and Ma [16], and later proven by Aizenman and Wehr [17]. More recently, it was found to be true also in genuinely nonequilibrium systems [22,23]. Another possibility is that discontinuous (mixed order) PT remains discontinuous and the heterogeneity adds a Griffiths phase subcritically [21]. Moreover, for higher-dimensional systems (three-dimensional or in the mean-field limit), it has been shown that a discontinuous phase transition can simply remain discontinuous in the presence of quenched disorder [14,24,25]. However, the complementary effect, i.e., change from continuous to discontinuous PT under the quenched disorder, has yet to be observed, even in the mean-field limit.

In this paper, we will show, both analytically and by Monte Carlo simulations, that such an effect is possible, at least on a complete graph (CG), which corresponds to the mean-field approach (MFA). We will show this within the framework

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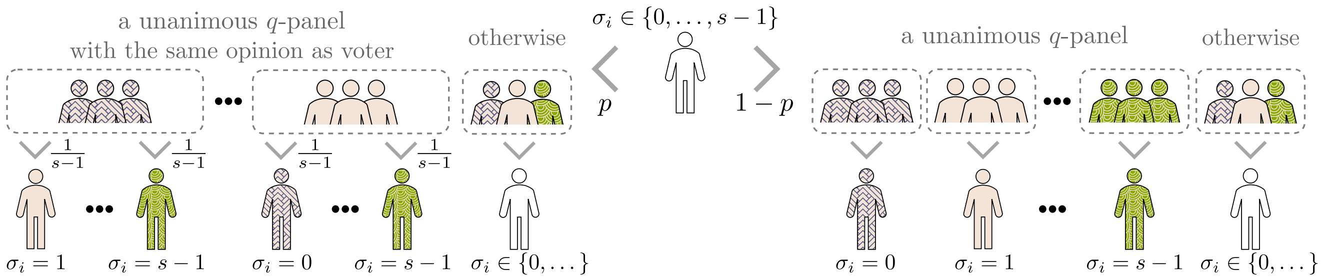

FIG. 1. Visualization of the elementary update for the multistate q-voter model with anticonformity. Within the annealed approach, a voter can anticonform or conform to the group of influence with complementary probabilities p and $1-p$. Within the quenched approach, a fraction p of all voters are permanently anticonformists, whereas others are always conformists.

of a model for which so far only the typical destruction (or softening) of discontinuous PT under quenched disorder has been observed, namely the q-voter model [26]. For example, it was shown that the quenched disorder rounds discontinuous PT in the multistate q-voter model with independence [25] or completely kills these transitions in the two-state version of this model [14]. Additionally, it was shown that in the two-state q-voter model with random competing interactions (conformity and anticonformity), both quenched and annealed disorders give exactly the same continuous PTs [14]. The multistate version of such a model will be the subject of this paper.

II. METHODS

The model, which we refer to as the multistate q-voter model with anticonformity, is defined as follows. There is a system of N voters, placed in the nodes of an arbitrary graph (here, we focus on CG). Each voter $i = 1, \dots, N$ can be in one of s possible states $\sigma_i = \alpha \in \{0, 1, 2, 3, \dots, s-1\}$. As in the original q-voter model, a voter can be influenced by q neighboring agents only if they are unanimous [26]. As usually, we use a random sequential updating and a unit of time ($t \to t+1$) is defined as N elementary updates of duration Δt, i.e., $N\Delta t = 1$, which corresponds to one Monte Carlo step (MCS). An elementary update (schematically presented in Fig. 1) consists of (1) choosing randomly voter i, (2) choosing randomly a group of q neighbors of i, (3) checking if all q neighbors are in the same state to form a group of influence, and (4) updating the state of voter i. The last step of an update depends on the considered approach, annealed or quenched.

Within the annealed approach, all agents are identical: With probability p an active voter acts as an anticonformist, and with complementary probability $1-p$ as a conformist. Within the quenched approach, the system consists of two types of agents: Each voter is set to be permanently anticonformist with probability p or conformist with complementary probability $1-p$. If the active voter is an anticonformist and all q neighbors are in the same state as the state of an active voter, it changes its state to any other, randomly chosen from the remaining equally probable $s-1$ states. On the other hand, if the active voter is a conformist and all q neighbors are in the same state, the active voter copies their state.

To describe the system on the macroscopic scale, we introduce a random variable c_α describing the concentration of agents having opinion α,

$$c_\alpha = \frac{N_\alpha}{N}, \quad \text{and} \quad \sum_{\alpha=0}^{s-1} c_\alpha = 1, \tag{1}$$

where N_α is the number of voters in a given state. Because we use the random sequential updating, c_α can change only by $\pm 1/N$ with the respective transition probabilities:

$$\gamma^+(c_\alpha) = \Pr\left\{c_\alpha(t+\Delta t) = c_\alpha(t) + \frac{1}{N}\right\},$$
$$\gamma^-(c_\alpha) = \Pr\left\{c_\alpha(t+\Delta t) = c_\alpha(t) - \frac{1}{N}\right\}. \tag{2}$$

The specific form of γ^+, γ^- can be easily calculated within MFA for the annealed as well as the quenched approach. Detailed calculations for the transition probabilities Eq. (2), as well as other detailed calculations, to which we will refer later in this paper, can be found in the Supplemental Material (SM) [27]. Although c_α is a random variable, we can write the evolution equation of the corresponding expected value:

$$\langle c_\alpha(t+\Delta t)\rangle = \langle c_\alpha(t)\rangle + \frac{1}{N}\gamma^+(c_\alpha) - \frac{1}{N}\gamma^-(c_\alpha). \tag{3}$$

Since $\Delta t = 1/N$, we obtain

$$\frac{\langle c_\alpha(t+\Delta t)\rangle - \langle c_\alpha(t)\rangle}{\Delta t} = \gamma^+(c_\alpha) - \gamma^-(c_\alpha). \tag{4}$$

Under the realistic assumption that for $N \to \infty$ random variable c_α localizes to the expectation value, thus

$$\frac{dc_\alpha}{dt} = \gamma^+(c_\alpha) - \gamma^-(c_\alpha) = F(c_\alpha), \tag{5}$$

where $F(c_\alpha)$ can be interpreted as the effective force acting on the system [28]. Because in this paper we focus on PTs, we are not interested in the temporal evolution of the system, but only in the stationary states:

$$\frac{dc_\alpha}{dt} = 0. \tag{6}$$

Due to the equivalence of all opinions, one can claim that the only possible symmetry-breaking schemes are those with at most two distinct stationary values [25,29,30]. If initially several (one, two, or more) opinions are equinumerous and dominate over all the others, the system reaches an absorbing state in which these opinions still dominate and are equinumerous. At the same time, all remaining opinions become equinumerous. Based on this observation, confirmed

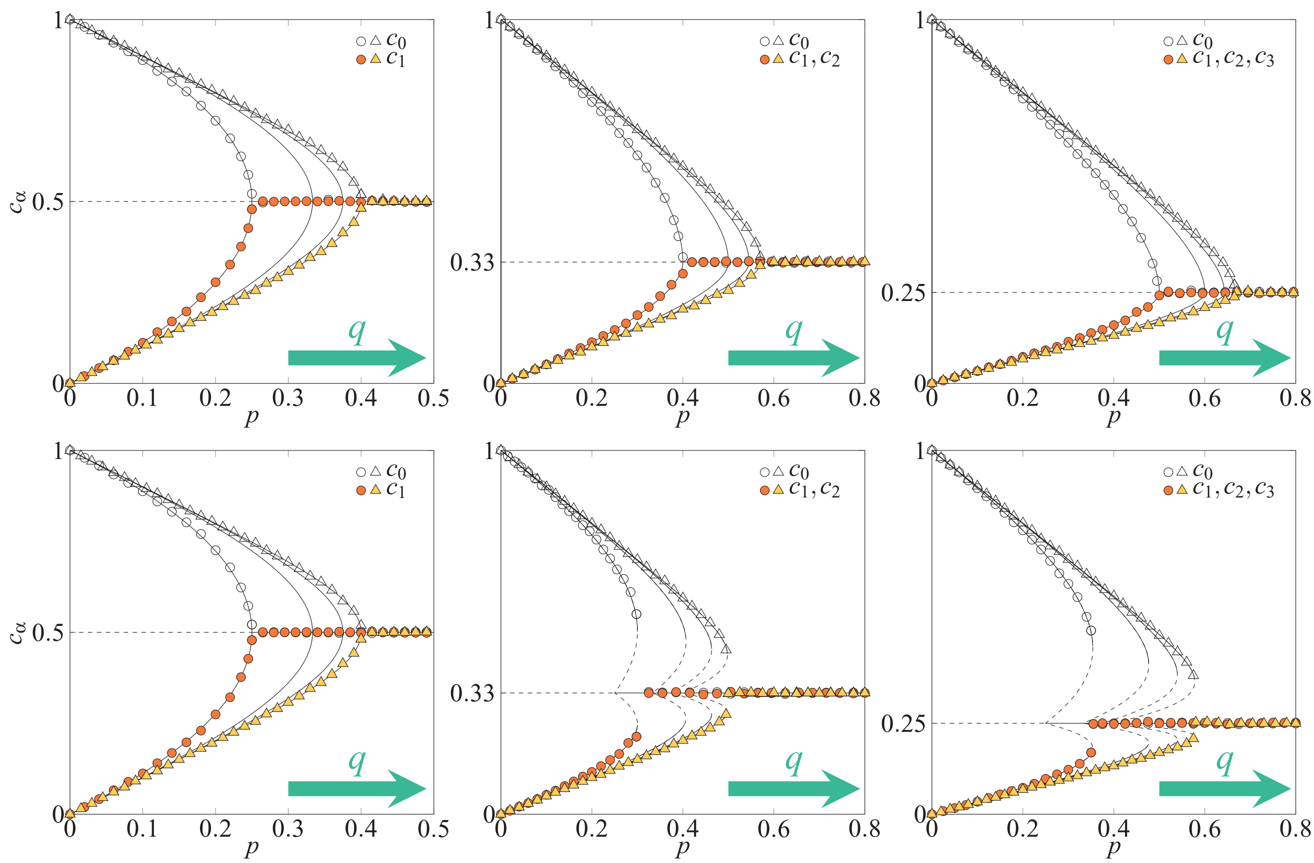

FIG. 2. Stationary concentration of agents in a given state as a function of the probability of anticonformity p within the annealed (upper panels) and quenched (bottom panels) approach for different values of the influence group size $q = \{2, 3, 4, 5\}$ (changing from left to right as indicated by arrows). The number of states $s = 2$ (left column), $s = 3$ (middle column), and $s = 4$ (right column). Lines represent analytical results: Solid and dashed lines correspond to stable and unstable steady states, respectively. Symbols represent the outcome of MC simulations for $q = \{2, 5\}$ and the system size $N = 5 \times 10^5$ performed from initial condition $c_0 = 1$. The results are averaged over ten runs and collected after $t = 5 \times 10^4$ MCS.

by Monte Carlo simulations, and the normalization condition Eq. (1) we are able to write down all solutions in terms of a single arbitrarily chosen state, denoted by c,

$$\begin{aligned} c_0 &= \cdots = c_{s-(\xi+1)} = c, \\ c_{s-\xi} &= \cdots = c_{s-1} = \frac{1-(s-\xi)c}{\xi}, \end{aligned} \tag{7}$$

where $\xi = 1, 2, 3, \ldots, s-1$. By $\xi = 0$ we denote the solution, where all states are equinumerous $c_0 = c_1 = \cdots = c_{s-1} = 1/s$. Knowing the above, we can determine the stationary states of the annealed and the quenched version of the model.

Under the annealed approach on the complete graph, Eq. (5) takes the form (see SM [27])

$$\frac{dc_\alpha}{dt} = -pc_\alpha^{q+1} + p\sum_{i\neq\alpha}\left[\frac{c_i^{q+1}}{s-1}\right] + (1-p)\sum_{i\neq\alpha}\left[c_i c_\alpha^q - c_\alpha c_i^q\right]. \tag{8}$$

Combining Eq. (7) with Eqs. (6) and (8), we obtain

$$p = \frac{c^q - (s-\xi)c^{q+1} - \xi c\left(\frac{1-(s-\xi)c}{\xi}\right)^q}{c^q - (s-\xi)c^{q+1} - \xi c\left(\frac{1-(s-\xi)c}{\xi}\right)^q - \frac{\xi}{s-1}\left[\left(\frac{1-(s-\xi)c}{\xi}\right)^{q+1} - c^{q+1}\right]}. \tag{9}$$

Based on the value of ξ we are able to recover s stationary solutions. Some of them are stable and some unstable. We can determine the stability by looking at the sign of the derivative of the effective force $\frac{dF(c_\alpha)}{dc}$ (see SM [27] for details). It turns out that only two types of stable stationary states of the system are possible: (*Disordered*) concentrations of all opinions are identical $c_\alpha = 1/s$ for all α and (*ordered*) in which the symmetry between opinions is broken and one opinion dominates over the others. As seen in Fig. 2, in the annealed case this order-disorder PT is continuous for all values of the model's parameters q and s.

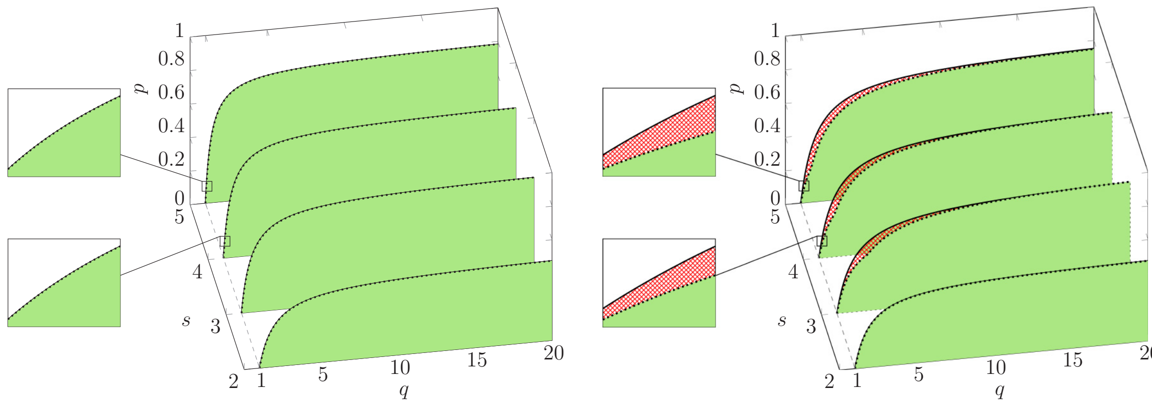

FIG. 3. Phase diagrams obtained within MFA for the multistate q-voter model under the annealed (left panels) and the quenched (right panels) approaches. The ordered phases are marked by solid color (green). The coexistence regions are marked by a crosshatched pattern (red). The disordered phases are shown as open regions (white). Lower and upper spinodals are marked by dotted and solid lines. respectively.

Under the quenched approach the system consists of two types of agents, who respond differently to group influence. Therefore, we must consider these two groups separately and write one evolution equation for the concentration $c_{(\mathbf{A},\alpha)}$ of anticonformists in state α, and the second one for concentration $c_{(\mathbf{C},\alpha)}$ of conformists in this state [14,31]. This will ultimately allow us to obtain the total concentration of voters in state α,

$$c_\alpha = pc_{(\mathbf{A},\alpha)} + (1-p)c_{(\mathbf{C},\alpha)}, \tag{10}$$

and the evolution of the system is given by two equations

$$\frac{dc_{(\mathbf{A},\alpha)}}{dt} = -c_{(\mathbf{A},\alpha)}c_\alpha^q + \sum_{i\neq\alpha}\left[\frac{c_{(\mathbf{A},i)}c_i^q}{s-1}\right],$$
$$\frac{dc_{(\mathbf{C},\alpha)}}{dt} = \sum_{i\neq\alpha}\left[c_{(\mathbf{C},i)}c_\alpha^q - c_{(\mathbf{C},\alpha)}c_i^q\right]. \tag{11}$$

By performing analogous reasoning to the annealed approach, we compute the stationary states,

$$p = \frac{\left(\frac{1-(s-\xi)c}{\xi}\right)^q c^q[cs^2-(1+2c\xi)(s-\xi)] + \xi c^{2q}[c(s-\xi)-1] + c\xi(s-\xi)\left(\frac{1-(s-\xi)c}{\xi}\right)^{2q}}{\xi\left(\frac{1-(s-\xi)c}{\xi}\right)^{2q} - \xi c^{2q}}, \tag{12}$$

and determine their stability (see the SM [27] for details). We again obtain a phase transition between the disordered state, in which concentrations of all opinions are identical $c_\alpha = 1/s$ and the ordered state, in which one opinion dominates. However, in contrast to the annealed case, this time for $s > 2$ this transition is discontinuous, as shown in the bottom panels of Fig. 2. For $s = 2$, the results for the annealed and quenched approach are identical, as already shown in Ref. [14].

III. DISCUSSION

For the binary q-voter model with anticonformity, that is, $s = 2$, the quenched approach gives the same result as the annealed one, and the phase transitions are continuous regardless of q, as shown in the left panels of Fig. 2. This result has already been obtained in the previous paper [14] and appears here only as a special case of the general multistate model. The new results refer to $s > 2$, for which the quenched model unexpectedly induces discontinuous phase transitions. While in the annealed version the phase transitions are still continuous, the quenched model displays discontinuous transitions already for $q > 1$ (see Figs. 2 and 3). For all values of the model parameters, the Monte Carlo results overlap analytical ones, as shown in Fig. 2, which was expected due to the structure of the complete graph.

As seen in Fig. 3 for the fixed value of $s > 2$, the size of the hysteresis, that is, the area in which ordered phase coexists with disordered one, and thus the final state depends on the initial one, depends nonmonotonically on the size of the influence group q. Initially, it increases with q and reaches the maximum value at

$$q = \frac{s}{s-2}, \tag{13}$$

which can be calculated analytically, as shown in SM [27]. At this point, we would like to draw attention to a fact that was already mentioned in the original paper on the q-voter model [1]. Although in the definition of the model both q and s are integer numbers, all analytical calculations make sense for any positive real values of q, s. Therefore, in Fig. 3 the phase boundaries appear as continuous lines as a function of q.

IV. CONCLUSIONS

The initial inspiration for this research came from social science and was specifically related to the question of factors that influence the emergence of discontinuous phase transitions in social systems. This question with respect to models of social dynamics has been asked before in several papers [25,32–35]. One might wonder why discontinuous phase transitions are relevant to social systems at all. In fact, they are important because it turns out that hysteresis and critical mass, which are indicators of discontinuous phase transitions, are empirically observed in real social systems [36–38]. The importance of discontinuous phase transitions was one of the reasons why, for example, the q-voter model with independence was studied more intensively than the q-voter model with anticonformity [39–44].

Within the annealed approach, the q-voter model with anticonformity displays only continuous PT, independently of the number of states s and the size of the influence source q. On the contrary, the q-voter model with independence shows discontinuous PTs under the annealed approach above the tricritical point $q^*(s)$, where $q^*(2) = 5$ [14,28] and $q^*(s > 2) = 1$ [25]. Moreover, it was shown that for the q-voter model with independence replacing the annealed disorder by the quenched one kills discontinuous phase transitions for $s = 2$ [14] or rounds them for $s > 2$ [25]. These previous results were in agreement with the state of the art [16,17,19,20,22,23]. On the contrary, in this paper, we have shown that the opposite phenomenon can also be observed.

We are aware that obtaining the same results independently within the two methods (analytical and Monte Carlo simulations) does not mean that we understand the observed phenomenon. Unfortunately, heuristic understanding is still lacking. Nevertheless, we have decided to present these results, hoping for the help of the readers. Admittedly, nowhere in the literature have we found a proof that the phenomenon we observe is impossible for a complete graph. On the other hand, we have not found any paper in which anyone has observed such a phenomenon. From this perspective, the model studied here should be treated as an example that shows that a quenched disorder can support discontinuous phase transitions in some cases.

We realize that a complete graph is very different from finite-dimensional systems, because in the latter case the nodes are not equivalent. Moreover, here, we consider only a bimodal disorder (conformist or anticonformist). This means that there are only two classes of sites in the complete graph, and the sites within either class are completely equivalent to each other. It would be desirable to consider the same model on top of other graphs to check what is responsible for the observed phenomena, the structure of the system, or maybe just a bimodal disorder. The consideration of the model on other structures would also allow us to analyze the behavior of the correlation length. This will help us to determine whether we are actually dealing with a binary division into continuous/discontinuous phase transitions or whether hybrid transitions, as reported for the asymmetric q-voter model on multiplex networks [42], also appear. Although many questions remain open, we believe that our finding goes far beyond social physics and will be interesting to a broad audience.

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