Letter

Three-loop order approach to flat polymerized membranes

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We derive the three-loop order renormalization group equations that describe the flat phase of polymerized membranes within the modified minimal subtraction scheme, following the pioneering one-loop order computation of Aronovitz and Lubensky [Phys. Rev. Lett. **60**, 2634 (1988)] and the recent two-loop order one of Coquand, Mouhanna, and Teber [Phys. Rev. E **101**, 062104 (2020)]. We analyze the fixed points of these equations and compute the associated field anomalous dimension η at three-loop order. Our results display a marked proximity with those obtained using nonperturbative techniques and reexpanded in powers of $\epsilon = 4 - D$. Moreover, the three-loop order value that we get for η at the stable fixed point, $\eta = 0.8872$, in $D = 2$, is compatible with known theoretical results and within the range of accepted numerical values.

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Introduction. The flat phase of polymerized membranes has been recently the subject of intense investigations mainly motivated by the fact that it seems to encode in a satisfying way the elastic degrees of freedom of materials such as graphene [1,2] and, more generally, graphene-like systems (see, e.g., [3]). Early, first-order, perturbative computations [4,5] have revealed the stability of such a phase, ensured by a mechanism of coupling of the capillary (flexural), **h**, modes with the elastic (phonons), **u**, modes, allowing one to circumvent the Mermin-Wagner theorem; see, e.g., [6] for an explanation. This flat phase is controlled by a fully stable infrared fixed point, characterized by power-law behaviors for the phonon-phonon and flexural-flexural correlation functions [5,7–9]:

$$G_{uu}(q) \sim q^{-(2+\eta_u)} \quad \text{and} \quad G_{hh}(q) \sim q^{-(4-\eta)}, \qquad (1)$$

where η_u and η are nontrivial anomalous dimensions related by a Ward identity: $\eta_u = 4 - D - 2\eta$ [5,7–9]. A major challenge in this context is an accurate determination of the exponent η at the stable fixed point. Due to the distance between the upper critical dimension, $D_{\rm uc} = 4$, and the physical dimension, $D = 2$, as well as of the intricacy of the diagrammatic analysis involved in the perturbative approach of membranes, the pioneering works have been followed by various nonperturbative approaches able to tackle the physics directly in dimension $D = 2$: $1/d$ expansion [7–12], self-consistent screening approximation (SCSA) [13–17], and the so-called nonperturbative renormalization group (NPRG) [18–25]. The two last ones have produced roughly compatible results: $\eta^{\rm l}_{\rm scsa} \simeq 0.821$ [13,17] at leading order ($\eta^{\rm nl}_{\rm scsa} \simeq 0.789$ at controversial next-to-leading order [14]) and $\eta_{\rm nprg} = 0.849$ [18]. As for Monte Carlo simulations of membranes, they have also led to scattered values $\eta = 0.81(3)$ [26], $\eta = 0.750(5)$ [27], and $\eta = 0.795(10)$ [28] and Monte Carlo simulations of graphene to $\eta \simeq 0.85$ [29]. In order to get a better understanding of the structure of the underlying field theory, several groups have, very recently, engaged in perturbative studies of both pure [30,31] and disordered membranes [32] going beyond leading order. The two-loop order approach performed in particular in [31] has revealed an intriguing agreement between the perturbative and nonperturbative approaches in the vicinity of the upper critical dimension. Moreover, the value of the two-loop order anomalous dimension in $D = 2$, $\eta^{2l} = 0.9139$ [31], when compared to the one-loop order one, $\eta^{1l} = 0.96$ [5,7–9], has been found to move in the right direction when referring to the generally accepted values that lie in the range [0.72,0.88].

We extend here the work done in [31] by means of a *three-loop* order, weak-coupling, perturbative approach performed near the upper critical dimension $D_{\rm uc} = 4$ within the modified minimal subtraction ($\overline{\rm MS}$) scheme. We compute the renormalization group (RG) equations at this order for both the flexuron-phonon *two-field* model as well as for the flexural *effective* model, which are both defined below. We determine the fixed points and the corresponding field anomalous dimensions at order ϵ^3. We finally compare our results to those obtained within the nonperturbative context either reexpanded in powers of ϵ or directly in the physical dimensions $D = 2$.

As will be seen in the following, our analysis confirms unambiguously the order-by-order agreement between perturbative and nonperturbative approaches, already identified in our previous work [31]. Moreover, the value that we get for the three-loop order anomalous dimension in $D = 2$, $\eta^{3l} = 0.8872$, is compatible with the analytical and numerical nonperturbative results. Such a fast convergence raises the issue of the unusual nature of the series in ϵ obtained in this context.

The models. We now present the two models studied here. One describes a membrane as a D-dimensional manifold embedded in a d-dimensional Euclidean space. The parametrization of a point $\mathbf{x} \in \mathbb{R}^D$ in the membrane is realized through the mapping $\mathbf{x} \to \mathbf{R}(\mathbf{x})$ with $\mathbf{R} \in \mathbb{R}^d$. The flat

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configuration of a membrane is given by $\mathbf{R}^0(\mathbf{x}) = (\mathbf{x}, \mathbf{0}_{d_c})$ where $\mathbf{0}_{d_c}$ is the null vector of codimension $d_c = d - D$. To parametrize the fluctuations around this configuration one decomposes the field $\mathbf{R}$ into $\mathbf{R}(x) = [\mathbf{x} + \mathbf{u}(\mathbf{x}), \mathbf{h}(\mathbf{x})]$ where $\mathbf{u}$ and $\mathbf{h}$ represent D longitudinal (phonon) and $d - D$ transverse (flexural) modes, respectively. The action in the flat phase is given by [4,5,7–10,31]

$$S[\mathbf{h}, \mathbf{u}] = \int \mathrm{d}^D x \left\{ \frac{\kappa}{2} (\Delta \mathbf{h})^2 + \frac{\lambda}{2} u_{ii}^2 + \mu \, u_{ij}^2 \right\}, \qquad (2)$$

where, as usual, one neglects a term $(\Delta \mathbf{u})^2$ in the curvature energy contribution $(\Delta \mathbf{R})^2$. In Eq. (2), u_{ij} is the strain tensor that encodes the elastic fluctuations around the flat phase configuration $\mathbf{R}^0(\mathbf{x})$: $u_{ij} = \frac{1}{2}(\partial_i \mathbf{R}.\partial_j \mathbf{R} - \partial_i \mathbf{R}^0.\partial_j \mathbf{R}^0) = \frac{1}{2}(\partial_i \mathbf{R}.\partial_j \mathbf{R} - \delta_{ij})$. It is given by neglecting nonlinearities in the phonon field $\mathbf{u}$:

$$u_{ij} \simeq \tfrac{1}{2}[\partial_i u_j + \partial_j u_i + \partial_i \mathbf{h}.\partial_j \mathbf{h}]. \qquad (3)$$

In Eq. (2), κ is the bending rigidity constant, whereas λ and μ are the Lamé (elasticity) coefficients; stability considerations require κ, μ, and the bulk modulus $B = \lambda + 2\mu/D$ to be all *positive*. The action (2) together with Eq. (3) defines the two-field model.

Now one can take advantage of the fact that the phonon field $\mathbf{u}$ appears quadratically in the action (2) to integrate over it exactly. In this way one gets an effective action depending only on the flexural field $\mathbf{h}$. It reads, in Fourier space [13,17,31],

$$S_{\mathrm{eff}}[\mathbf{h}\,] = \frac{\kappa}{2} \int_{\mathbf{k}} k^4 \, |\mathbf{h}(\mathbf{k})|^2 + \frac{1}{4} \int_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4} \mathbf{h}(\mathbf{k}_1) \cdot \mathbf{h}(\mathbf{k}_2) \, R_{ab,cd}(\mathbf{q}) \, k_1^a \, k_2^b \, k_3^c \, k_4^d \; \mathbf{h}(\mathbf{k}_3) \cdot \mathbf{h}(\mathbf{k}_4), \qquad (4)$$

where $\int_{\mathbf{k}} = \int d^D k/(2\pi)^D$ and $\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{k}_3 - \mathbf{k}_4$. The $\mathbf{q}$-transverse tensor $R_{ab,cd}(\mathbf{q})$ reads [13,17,31]

$$R_{ab,cd}(\mathbf{q}) = b \, N_{ab,cd}(\mathbf{q}) + \mu \, M_{ab,cd}(\mathbf{q}) \qquad (5)$$

with $N_{ab,cd}$ and $M_{ab,cd}$ given by

$$N_{ab,cd}(\mathbf{q}) = \frac{1}{D-1} \, P^T_{ab}(\mathbf{q}) \, P^T_{cd}(\mathbf{q}),$$
$$M_{ab,cd}(\mathbf{q}) = \frac{D-1}{2} [N_{ac,bd}(\mathbf{q}) + N_{ad,bc}(\mathbf{q})] - N_{ab,cd}(\mathbf{q}), \quad (6)$$

where $P^T_{ab}(\mathbf{q}) = \delta_{ab} - q_a q_b/\mathbf{q}^2$ is the transverse projector. In Eq. (5) we have defined the coupling constant $b = \mu\,(D\lambda + 2\mu)/(\lambda + 2\mu)$, which is proportional to the bulk modulus B and has a nontrivial D-dependence. The action (4), together with Eqs. (5) and (6), defines the flexural effective model. Working with this model allows a strong check of the computations performed with the action (2). However, we would like to emphasize that, in order to have a complete correspondence between the physical quantities computed with both the two-field and the flexural effective models, one has to consider b as a *D-independent* coupling constant when computing within the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme; see [31,33]. Moreover, b is also treated as a coupling constant *independent* of λ and μ.

The renormalization group equations at three-loop order. We have derived the RG equations at three-loop order for the flexural-phonon, two-field, model (2) and then for the flexural effective model (4), both within the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme and in the massless case. As discussed in [9,31], the renormalizability of the models relies on Ward identities following the (partially broken) rotation invariance in the flat phase [7,9]. Our computations have been performed using techniques of massless Feynman diagram calculations; see, e.g., the review [34]. Their automation has been implemented using Qgraf [35] for the generation of the diagrams, as well as Mathematica to perform the numerator algebra. LiteRed [36,37] has also been used to reduce the loop integrals to a finite set of master integrals.

In the case of the two-field model, we had to evaluate 32 distinct three-loop diagrams for the flexuron self-energy (note that there were only five distinct diagrams at two-loop order and one diagram at one-loop order) and 19 diagrams for the three-loop phonon self-energy (there were only three distinct diagrams at two-loop order and one at one-loop order). In the case of the flexural effective model, we had to evaluate 15 distinct three-loop diagrams for the flexuron self-energy (there were only three distinct diagrams at two-loop order and one diagram at one-loop order) and 11 diagrams for the effective three-loop polarization (there were only two distinct diagrams at two-loop order and one at one-loop order). As for the masters, the analytic result of [38–40] was used in order to compute complicated primitively two-loop master diagrams with a noninteger index on the central line. More details will be given in [33].

The two-field model. For the two-field model one introduces the renormalized fields $\mathbf{h}_R$ and $\mathbf{u}_R$ through $\mathbf{h} = Z^{1/2}\kappa^{-1/2}\mathbf{h}_R$ and $\mathbf{u} = Z\kappa^{-1}\mathbf{u}_R$ and the renormalized coupling constants λ_R and μ_R through $\lambda = k^\epsilon Z^{-2}\kappa^2 Z_\lambda \lambda_R$ and $\mu = k^\epsilon Z^{-2}\kappa^2 Z_\mu \mu_R$, where k is the renormalization momentum scale and $\epsilon = 4 - D$. Within the $\overline{\mathrm{MS}}$ scheme, one moreover introduces the scale $\overline{k}^2 = 4\pi e^{-\gamma_E} k^2$ where γ_E is the Euler constant. One then defines the RG β-functions $\beta_{\lambda_R} = \partial_t \lambda_R$ and $\beta_{\mu_R} = \partial_t \mu_R$, with $t = \ln \overline{k}$ as well as the field anomalous dimension:

$$\eta = \beta_{\lambda_R} \frac{\partial \ln Z}{\partial \lambda_R} + \beta_{\mu_R} \frac{\partial \ln Z}{\partial \mu_R},$$

where, for simplicity, we have omitted all explicit references to k in our notations of the renormalized coupling constants.

The RG functions are too long to be displayed in the main text; they are given in Sec. I of the Supplemental Material [41]. Here we discuss only the fixed points and the corresponding field anomalous dimensions, whose expressions are explicitly provided in Sec. II of [41]. Similarly to what happens at one [5,7–9] and two-loop [31] orders, the three-loop order RG equations display four fixed points (note that, for simplicity, we omit the R indices on the renormalized coupling constants):

(1) The Gaussian fixed point P_1 with $\mu_1^* = 0$, $\lambda_1^* = 0$ and $\eta_1 = 0$; it is twice unstable.

(2) The shearless fixed point P_2 with $\mu_2^* = 0$, $\lambda_2^* = 16\pi^2\,\epsilon/d_c$ and $\eta_2 = 0$ whose coordinates are the same as those obtained at one- and two-loop orders. It lies on the stability line $\mu = 0$; it is once unstable.

(3) The fixed point P_3 whose coordinates μ_3^*, λ_3^* and anomalous dimension η_3 are given in Table I of [41]. It is once unstable. For this fixed point, whose bulk modulus B vanishes at one-loop order, $B_3^* = \lambda_3^* + 2\mu_3^*/(4-\epsilon)$ receives negative contributions of order ϵ^2 at two-loop order and of order ϵ^3 at three-loop order. As observed in [31], P_3 is thus apparently located out of the stability region of the model (2) that requires $B \geqslant 0$. However, as emphasized in [31], this could be an artifact of the perturbative computation; see below. It is instructive to consider the physical case $d_c = 1$ for the anomalous dimension. One gets from Table I of [41]

$$\eta_3 = 0.4762\,\epsilon - 0.01776\,\epsilon^2 - 0.00872\,\epsilon^3 + O(\epsilon^4). \qquad (7)$$

This series shows, up to, and including, the order ϵ^3, a strong decrease of its coefficients.

(4) The flat phase fixed point P_4 whose coordinates μ_4^*, λ_4^* and anomalous dimension η_4 are given in Table II of [41]. It is fully stable and thus controls the asymptotic behavior of the flat phase. Note that, at one-loop order, this fixed point is located on the line $3\lambda + \mu = 0$, which is stable, in the RG sense, at one-loop order. In D dimensions, it corresponds to the line $(D+2)\lambda + 2\mu = 0$. However, at two- [31] and three-loop orders, the coordinates of the fixed point P_4 no longer obey the relation $(D+2)\lambda_4^* + 2\mu_4^* = (6-\epsilon)\lambda_4^* + 2\mu_4^* = 0$. This results in the following nonvanishing anomalous ratio:

$$\delta_1 = \frac{\lambda_4^*}{\mu_4^*} + \frac{1}{3} = 0.00889\,\epsilon + 0.02434\,\epsilon^2 + O(\epsilon^3); \qquad (8)$$

see discussion below Eq. (11) for further details.

Finally, considering the physical case $d_c = 1$ one gets from Table II of [41]

$$\eta_4 = 0.4800\,\epsilon - 0.01152\,\epsilon^2 - 0.00334\,\epsilon^3 + O(\epsilon^4), \qquad (9)$$

which was referred to simply as η in the Introduction. This series in ϵ behaves in a way similar to Eq. (7) with even a stronger decrease of the coefficients than at the fixed point P_3.

The flexural effective model. As said in the Introduction, we have also considered, as in [31], the flexural effective model. This model provides a field theory structurally very different from that provided by the two-field model. From a practical point of view, the task seemed to be less imposing as it involves only a total of 26 distinct diagrams. However, the computational time remains the same as for the 51 diagrams of the two-field model. From a more conceptual point of view, analyzing this model allows one to get valuable insights about phenomena observed in the context of the two-field model that could correspond to artifacts of the corresponding perturbative approach. Indeed, the two models, although nonperturbatively equivalent, are parametrized by different sets of coupling constants, (λ, μ) and (b, μ), respectively.

TABLE I. Anomalous η_2' at order ϵ^3 at the fixed point P_2' obtained from the three-loop order (this work), SCSA, and NPRG approaches.

Approach	P_2'
Three-loop	$\eta_2' = 0.4000\,\epsilon - 0.00133\,\epsilon^2 + 0.00138\,\epsilon^3$
SCSA	$\eta_2' = 0.4000\,\epsilon - 0.00133\,\epsilon^2 + 0.00310\,\epsilon^3$
NPRG	$\eta_2' = 0.4000\,\epsilon + 0.00867\,\epsilon^2 + 0.00123\,\epsilon^3$

As for the two-field model, one introduces the renormalized field $\mathbf{h}_R$ through $\mathbf{h} = Z^{1/2}\kappa^{-1/2}\mathbf{h}_R$, the renormalized coupling constants b_R and μ_R through $b = k^\epsilon Z^{-2}\kappa^2 Z_b\, b_R$ and $\mu = k^\epsilon Z^{-2}\kappa^2 Z_\mu\, \mu_R$. One then defines the RG β-functions $\beta_{b_R} = \partial_t b_R$ and $\beta_{\mu_R} = \partial_t \mu_R$ as well as the field anomalous dimension:

$$\eta = \beta_{b_R}\frac{\partial \ln Z}{\partial b_R} + \beta_{\mu_R}\frac{\partial \ln Z}{\partial \mu_R}.$$

Again, the RG functions are too long to be given in this paper; they are given in Sec. III of [41], while the fixed points and the corresponding field anomalous dimensions are given in Sec. IV of [41]. We now discuss the fixed points of these equations. At three-loop order one finds four fixed points (once again, we omit the R indices on the renormalized coupling constants):

(1) The Gaussian one P_1 with $\mu_1^* = 0$, $b_1^* = 0$ and $\eta_1 = 0$, which is twice unstable.

(2) A fixed point P_2' with $\mu_2'^* = 0$ and nontrivial values for both $b_2'^*$ and η_2'; see Table III of [41]. It is once unstable. The series in ϵ for η_2', in the physical $d_c = 1$ case, is given in Table I. Note that this fixed point has no counterpart within the two-field model where b, which is proportional to μ, vanishes at P_2'. One recalls that the two-loop order correction to this fixed point has first been computed by Mauri and Katsnelson [30]. Considering the physical case $d_c = 1$ one gets from Table III of [41]

$$\eta_2' = 0.4000\,\epsilon - 0.00133\,\epsilon^2 + 0.00138\,\epsilon^3 + O(\epsilon^4), \qquad (10)$$

where the coefficients of the ϵ-expansion are still small but, contrary to what is observed in Eqs. (7) and (9), the three-loop order coefficient is now slightly higher than the two-loop order one. This may reveal the asymptotic nature of the expansion. That this manifests at P_2' rather than at P_3 and P_4 seems to be due to the structure of the perturbative series involving denominators that are odd powers of $n + d_c$ with $n = 4$ at P_2' while $n = 20$ at P_3 and $n = 24$ at P_4; see Tables VI, VII, and VIII in [41], as well as the discussion below. We do not exclude that, at higher orders, the coefficients of the ϵ-expansions in Eqs. (7) and (9) may increase as well.

(3) The infinitely compressible fixed point P_3 which is characterized by $b_3^* = 0$, thus for which the bulk modulus B vanishes, and by nontrivial values for μ_3^* and η_3; see Table IV of [41]. It is once unstable. This fixed point identifies with the fixed point P_3 of the two-field model. However, as discussed above, the condition $B_3^* = 0$ is violated at both two- and three-loop orders for the two-field model. Therefore, the result $B_3^* = 0$ obtained within the flexural effective model seems to indicate that this should be, in fact, an artifact of the perturbative (ϵ expansion) approach performed on the two-field model and, more precisely, of the handling of D-dependent

TABLE II. Anomalous η_3 at order ϵ^3 at the fixed point P_3 obtained from the three-loop order (this work), SCSA, and NPRG approaches.

Approach	P_3
Three-loop	$\eta_3 = 0.4762\,\epsilon - 0.01776\,\epsilon^2 - 0.00872\,\epsilon^3$
SCSA	$\eta_3 = 0.4762\,\epsilon - 0.01668\,\epsilon^2 - 0.00700\,\epsilon^3$
NPRG	$\eta_3 = 0.4762\,\epsilon - 0.01349\,\epsilon^2 - 0.00649\,\epsilon^3$

relations as $\lambda + 2\mu/D$ that governs the value taken by the bulk modulus at a given fixed point. Despite this, the series in ϵ for the anomalous dimension η_3 (see Table IV of [41]) coincides exactly with that obtained within the two-field model; see Table I of [41]. This quantity, in the physical $d_c = 1$ case, is given in Table II and obviously coincides exactly with Eq. (7).

(4) The fixed point P_4, whose coordinates μ_4^*, b_4^* and anomalous dimension η_4 are given in Table V of [41]. It is fully stable and therefore controls the flat phase. It identifies with the fixed point P_4 of the two-field model. At two- and three-loop orders, the coordinates of P_4 differ from those obtained from the two-field model; see Table II of [41]. Also, these coordinates do not meet the condition $(D+1)b_4^* - 2\mu_4^* = (5-\epsilon)b_4^* - 2\mu_4^* = 0$ corresponding to the one-loop order stable line, a result already true at two-loop order [31]. The corresponding anomalous ratio reads, using $\lambda = 2\mu\,(\mu - b)/(b - D\,\mu)$:

$$\delta_2 = \frac{\lambda_4^*}{\mu_4^*} + \frac{1}{3} = 0.00519\,\epsilon + 0.02122\,\epsilon^2 + O(\epsilon^3). \quad (11)$$

One should notice that this ratio is different from the one found via the two-field approach, Eq. (8), i.e., $\delta_2 \neq \delta_1$, therefore implying that corrections to λ/μ are very likely scheme-dependent. This should also be the case for the Poisson ratio that is given by $\nu = \lambda_4^*/[2\mu_4^* + (D-1)\lambda_4^*]$ since one has $\nu = -1/3 + \delta\nu$ with $\delta\nu(\delta_1) \neq \delta\nu(\delta_2)$. This contrasts with the anomalous dimension η_4 (see Table V of [41]), which coincides exactly with the one obtained within the two-field model; see Table II of [41]. The corresponding value in the physical $d_c = 1$ case is given in Table III and coincides with Eq. (9). Finally it is worth noticing that, contrary to the case of the anomalous dimensions at the fixed points, the coefficients of the series giving δ_1 [Eq. (8)] and δ_2 [Eq. (11)] increase with the order of the expansion so that Eqs. (8) and (11) seem to deserve resummations. Performing a simple symmetric Padé approximant with $\epsilon = 2$ reveals that both δ are very small ($\sim 10^{-3}$), implying small deviations from the line $3\lambda + \mu = 0$, and thus a Poisson ratio close to $-1/3$.

Discussion: comparison with nonperturbative approaches. We further discuss our results in particular in comparison with those obtained using alternative methods: the SCSA and the NPRG approaches that are known to have produced numerical results in $D = 2$ rather close to the accepted—although dispersed—values obtained by means of numerical computations. Also, these methods offer explicit expressions of the various anomalous dimensions as functions of D that can be expanded in powers of ϵ and compared to those obtained in this work. The anomalous dimensions obtained within the SCSA approach [13,17] are given in Table VII of [41] and those obtained within the NPRG approach [18] in Table VIII of [41]. One first has to note that the basic structure of the series, where denominators are odd powers of $n + d_c$ with $n = 4, 20, 24$, is recovered within all approaches. Also, as already noted in [31], the agreement between the perturbative approach and the SCSA is particularly good for all anomalous dimensions up to two-loop order; see Table VI of [41], where we have gathered all perturbative values of η, and Table VII of [41]. The agreement is less pronounced with the NPRG approach except for η_4; see Table VIII of [41].

TABLE III. Anomalous η_4 at order ϵ^3 at the fixed point P_4 obtained from the three-loop order (this work), SCSA, and NPRG approaches.

Approach	P_4
Three-loop	$\eta_4 = 0.4800\,\epsilon - 0.01152\,\epsilon^2 - 0.00334\,\epsilon^3$
SCSA	$\eta_4 = 0.4800\,\epsilon - 0.01190\,\epsilon^2 - 0.00349\,\epsilon^3$
NPRG	$\eta_4 = 0.4800\,\epsilon - 0.00918\,\epsilon^2 - 0.00333\,\epsilon^3$

Due to the complexity of the numerators at three-loop order, the comparison between these three approaches is not obvious for an arbitrary codimension d_c. We thus consider the series in the physical case $d_c = 1$. In this case one recovers (see [31]) at the fixed point P_2' the exact agreement between the perturbative computation and the SCSA up to order ϵ^2; see Table I. The comparison, at three-loop order, with the SCSA is less satisfying as with this last method one observes a strong increase of the ϵ^3 coefficient with respect to the ϵ^2 one. Conversely the agreement with the NPRG approach is not very good at order ϵ^2—for unclear reasons one finds the wrong sign at this order (see [31])—but satisfying at three-loop order. Nevertheless at the fixed point P_3 (see Table II) and especially at P_4 (see Table III), the agreement between *all* approaches is particularly good—up to three significant digits with the SCSA, a little bit less with the NPRG—at two-loop but also at three-loop orders.

We now discuss our result for η_4 associated to the fully stable fixed point P_4. The series giving η_4 is meant to be asymptotic, and resummation techniques should be used in principle. However, as far as the three-loop order is considered, the series appears to be convergent due to the large regular denominator structure of the form $\epsilon^l/(n+d_c)^{2l-1}$, with l the loop order and $n = 4, 20, 24$. We expect the singular nature of the series to show up at higher orders when the numerators become large enough to overcome the denominators. Since, up to three-loop order, the coefficients are small and decreasing, asymptotic analysis allows us to truncate the series (up to the smallest coefficient) with minimum error and without any resummation. This is the so-called "optimal truncation rule" (see, e.g., [42]) and is supposed to give a good approximation to an asymptotic series. At some higher loop order, one should reach some increasing coefficients and need performing resummations to get better approximations. At $\epsilon = 2$ one gets successively at one-, two-, and three-loop orders:

$$\eta_4^{1l} = \frac{24}{25} = 0.96, \quad \eta_4^{2l} = \frac{2856}{3125} \simeq 0.9139,$$
$$\eta_4^{3l} = \frac{2856}{3125} + \frac{4[568\,241 - 1\,286\,928\,\zeta(3)]}{146\,484\,375} \simeq 0.8872.$$

Clearly, η_4 gets closer and closer, order by order, to the values obtained directly in $D = 2$ from the SCSA, where $\eta_4 = 0.821$ and the NPRG where $\eta_4 = 0.849$. Moreover, the three-loop order value we find is already compatible with the value obtained by these techniques and within the range [0.72,0.88] where many numerical values lie [26–29,43–49].

Let us finally discuss our predictions about the anomalous and Poisson ratios δ and ν. Our results, as well as those already obtained at two-loop order [31], display (small) corrections with respect to the leading order values. They contrast with those obtained within both the SCSA and NPRG approaches that lead to $\delta\nu = 0$ and, thus, $\nu = -1/3$. Several comments are required. First, we have indicated that our results for δ and ν are model-, and thus very likely, scheme-dependent. It is very desirable to get scheme-independent values for these quantities in order to compare our results with those obtained from other methods in a relevant way. Then, if these order-dependent corrections persist, one should inquire about higher-order contributions of the series providing ν. Finally, one should also inquire about the contributions that have been neglected within the SCSA and NPRG approaches, which could affect the leading order results. More generally, this raises the question of the deviation of the Poisson ratio with respect to the value $-1/3$, a fact that has been recently proposed, notably in [50].

Conclusion. We have investigated the flat phase of polymerized membranes at three-loop order by means of a weak-coupling, perturbative approach of two complementary models. We have determined the RG equations, their fixed points, and the associated anomalous dimensions. The agreement between the results obtained from the two models shows that we have obtained unambiguous control of the renormalization procedure in both models. The details of the—involved—computations will be given in a forthcoming publication [33].

From our results, the order-by-order agreement found between the perturbative approach and the nonperturbative ones when the later are reexpanded in powers of ϵ, which has already been observed at two-loop order, is confirmed at three-loop order. Let us add that, though our perturbative results are supposed to be valid at weak coupling, they are nevertheless exact order by order in the coupling constants. Hence, they serve as a benchmark for nonperturbative and numerical techniques that also rely on their own sets of approximations. In this context, a noticeable feature of our results with respect to the one- and two-loop orders calculations is that the value found for the three-loop order critical exponent η_4 in $D = 2$ (without any resummation of the ϵ-series) is compatible with the usually accepted ones from (all orders) nonperturbative methods.

Such an agreement raises the question of its very origin and, thus, of the very nature of the field theory describing the flat phase of membranes. This salient feature can be explained by the smallness of the coefficients found in the ϵ-series; see Eqs. (7) and (9). As can be seen from these equations, the coefficients even get smaller with increasing the loop order, thus seemingly alleviating the asymptotic nature of the series, at least up to three loops. A higher order examination of these coefficients would be very interesting but is beyond the scope of this paper.

Finally, one can note that recent attempts have been made to probe more deeply the nonperturbative structure of the theory, notably concerning the relation between scale invariance and conformal symmetry; see [51]. The result could be considered as deceptive: the scale invariance at the infrared fixed point not being promoted to conformal invariance at the fixed point, the use of methods such as conformal bootstrap techniques seems to be excluded. The crux of the matter still lies ahead of us.

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