Parrondo's paradox in quantum walks with three coins

Zbigniew Walczak [*] and Jarosław H. Bauer [†]
Department of Theoretical Physics, Faculty of Physics and Applied Informatics, University of Lodz, Pomorska 149/153, 90-236 Lodz, Poland

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Parrondo's paradox refers to the apparently paradoxical effect whereby a certain combination of biased random walks displays a counterintuitive reversal of the bias direction. We show that Parrondo's paradox can occur not only in the case of one-dimensional discrete-time quantum walks with a deterministic sequence of two quantum coins but also in the case of one-dimensional discrete-time quantum walks with a deterministic sequence of three quantum coins. Moreover, we show how Parrondo's paradox affects the time evolution of quantum entanglement for such quantum walks.

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I. INTRODUCTION

In 1996, Parrondo [1] invented a game-theoretic model of the flashing Brownian ratchet introduced by Ajdari and Prost [2]. The original Parrondo game consists of two games A and B, in which a player's capital can be increased or decreased by one at each game run. In game A, the capital is increased by one with probability $1/2 - \varepsilon$ and decreased by one with probability $1/2 + \varepsilon$, that is, the player wins with probability $1/2 - \varepsilon$ and loses with probability $1/2 + \varepsilon$, where ε is a small positive number. In game B, if the player's capital is a multiple of three, then the player wins with probability $1/10 - \varepsilon$ and loses with probability $9/10 + \varepsilon$, but if the player's capital is not a multiple of three, then the player wins with probability $3/4 - \varepsilon$ and loses with probability $1/4 + \varepsilon$. It turns out that, for $\varepsilon = 0.005$, if each of games A and B is played individually (i.e., $AAAA\ldots$ or $BBBB\ldots$), the Parrondo game results in losing, that is, the average capital is a decreasing function of the number of runs, however, if they are played randomly or in a certain deterministic sequence (e.g., $ABBABB\ldots$, $ABBAABBA\ldots$, etc.), the Parrondo game results in winning, that is, the average capital is an increasing function of the number of runs (a detailed explanation of the Parrondo game, in terms of Markov chains, can be found in Refs. [3–5]). This counterintuitive phenomenon was called Parrondo's paradox [6,7]. In general, the term Parrondo's paradox refers to the situation where two or more dynamics in which a given variable decreases are combined in such a way that the same variable increases in the resulting dynamics [3]. In the last two decades, Parrondo's paradox (called also the Parrondo's effect) has received attention not only in game theory where different variants of Parrondo's games were studied [3–5,8–17], but also in various areas of research, ranging from chaos and nonlinear dynamics [18–25], ecology and evolutionary biology [26–37], economics and social dynamics [38–42], to quantum information [43–62]. Comprehensive reviews on Parrondo's paradox can be found in Refs. [63–67].

In quantum information science, Parrondo's paradox has been studied mainly in the quantum walk approach [46,48,49,51–53,55–62]. A discrete-time coined quantum walk on an infinite one-dimensional lattice (DTQW) is a bipartite quantum system consisting of a coin and a walker moving along the lattice [68–70]. The coin states play a nontrivial role in determining the dynamics of the walker because a unitary operator driving the walker-coin system dynamics performs a transformation of the coin states and then, depending on the resulting state, shifts the walker from one position on the lattice to another. When this unitary operator is applied at each time step, a coherent superposition between all possible states of the walker-coin system emerges.

The DTQWs with a deterministic periodic sequence of two two-state quantum coins (representing games A and B) that exhibit Parrondo's effect have been presented in Refs. [51,52]. However, in both cases this effect disappears after a certain relatively small number of time steps. Interestingly, the long-lasting Parrondo's effect (called also the genuine Parrondo's effect) occurs in the case of some DTQWs with a deterministic periodic sequence of two three- and four-state quantum coins [56,57] as well as in the case of some DTQWs with a deterministic periodic sequence of two entangled two-state quantum coins [55]. Until recently, it was believed that DTQWs with a deterministic periodic sequence of two two-state quantum coins exhibiting the genuine Parrondo's effect do not exist. But it has been shown both theoretically and experimentally that this is not the case [58–61]. More recently, it has been shown that the genuine Parrondo's effect can occur also in the case of some DTQWs with a deterministic aperiodic sequence of two two-state quantum coins [62].

Although it has been already known that there exist DTQWs with a deterministic aperiodic as well as a periodic sequence of two two-state quantum coins exhibiting the genuine Parrondo's effect, the occurrence of this effect in DTQWs with a deterministic aperiodic as well as a periodic sequence of three two-state quantum coins has remained an open problem so far. In this paper, we provide examples of

[*]zbigniew.walczak@uni.lodz.pl
[†]jaroslaw.bauer@uni.lodz.pl

such DTQWs and show how, in these cases, the genuine Parrondo's effect affects the time evolution of the walker-coin quantum entanglement.

II. DTQWs

A DTQW is defined on the Hilbert space $\mathcal{H}=\mathcal{H}_w\otimes\mathcal{H}_c$, where $\mathcal{H}_w$ is the walker Hilbert space spanned by the orthonormal states $\{|x\rangle : x\in\mathbb{Z}\}$ representing the walker's position on the lattice while $\mathcal{H}_c$ is the coin Hilbert space spanned by the orthonormal states $\{|0\rangle, |1\rangle\}$. At time step t, the walker-coin system is in the following state [70]:

$$|\psi(t)\rangle=\sum_x |x\rangle[a_x(t)|0\rangle+b_x(t)|1\rangle], \tag{1}$$

where the probability amplitudes $a_x(t)$ and $b_x(t)$ satisfy the normalization condition $\sum_x[|a_x(t)|^2+|b_x(t)|^2]=1$. The one-step time evolution of the walker-coin system

$$|\psi(t+1)\rangle=U|\psi(t)\rangle \tag{2}$$

is governed by the unitary operator

$$U=S[I\otimes C(q,\theta,\phi)], \tag{3}$$

where

$$S=\sum_x |x-1\rangle\langle x|\otimes|0\rangle\langle 0|+\sum_x |x+1\rangle\langle x|\otimes|1\rangle\langle 1| \tag{4}$$

is the conditional shift operator acting on $\mathcal{H}$, I is the identity operator acting on $\mathcal{H}_w$, and

$$C(q,\theta,\phi)=\begin{pmatrix}\sqrt{q} & \sqrt{1-q}e^{i\theta}\\ \sqrt{1-q}e^{i\phi} & -\sqrt{q}e^{i(\theta+\phi)}\end{pmatrix} \tag{5}$$

with $0\leqslant q\leqslant 1$, $\theta\geqslant 0$, and $\phi\leqslant\pi$ is the coin operator acting on $\mathcal{H}_c$ [70].

At time step t, the probability distribution of the position of the walker has the form

$$p(x,t)=|a_x(t)|^2+|b_x(t)|^2, \tag{6}$$

while the average position of the walker is given by

$$\langle x\rangle(t)=\sum_x xp(x,t). \tag{7}$$

Interestingly, if the walker-coin initial state has the form $|\psi(0)\rangle=|0\rangle[a_0(0)|0\rangle+b_0(0)|1\rangle]$, then the average position of the walker at time step t does not depend on θ and ϕ but only on q, $a_0(0)$, and $b_0(0)$ [71,72]. Moreover, if $1/2\leqslant q\leqslant 1$, $a_0(0)=1$, and $b_0(0)=0$, then the average position of the walker at time step t is nonpositive and reads [71,72]

$$\langle x\rangle(t)=-q^{t-1}\left[(2q-1)t+\sum_{k=1}^{\left[\frac{t-1}{2}\right]}\sum_{\gamma=1}^{k}\sum_{\delta=1}^{k}\left(\frac{q-1}{q}\right)^{\gamma+\delta}\frac{(t-2k)^2\kappa_{\gamma,\delta,t,k}}{\gamma\delta}[(2q-1)t+\gamma+\delta]\right], \tag{8}$$

where $\kappa_{\gamma,\delta,t,k}=\binom{k-1}{\gamma-1}\binom{k-1}{\delta-1}\binom{t-k-1}{\gamma-1}\binom{t-k-1}{\delta-1}$ while $[\frac{t-1}{2}]$ denotes the maximal integer smaller than or equal to $\frac{t-1}{2}$.

At time step t, the walker-coin quantum entanglement, that is, the quantum entanglement present in the state (1), can be quantified by entropy of entanglement [73]

$$S_E(t)=-\mathrm{Tr}[\rho_c(t)\log_2\rho_c(t)], \tag{9}$$

where $\rho_c(t)=\mathrm{Tr}_w[|\psi(t)\rangle\langle\psi(t)|]$ is the reduced density matrix describing the coin state at time step t and Tr_w denotes the partial trace over the walker's degree of freedom. The entropy of entanglement ranges from 0 for the walker-coin separable states to 1 for the walker-coin maximally entangled states. At time step t, the coin state is given by [73]

$$\rho_c(t)=\begin{pmatrix}\rho_{11}(t) & \rho_{12}(t)\\ \rho_{12}^*(t) & 1-\rho_{11}(t)\end{pmatrix}, \tag{10}$$

where

$$\rho_{11}(t)=\sum_x |a_x(t)|^2, \tag{11}$$

$$\rho_{12}(t)=\sum_x a_x(t)b_x^*(t). \tag{12}$$

The entropy of entanglement (9) can be written in the following form [73]:

$$S_E(t)=-\lambda_+\log_2\lambda_+-\lambda_-\log_2\lambda_-, \tag{13}$$

where

$$\lambda_\pm=\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}-\rho_{11}(t)[1-\rho_{11}(t)]+|\rho_{12}(t)|^2} \tag{14}$$

are the eigenvalues of $\rho_c(t)$, and the convention that $0\log_2 0=0$ is adopted.

In order to show that the genuine Parrondo's effect can occur in the case of DTQWs with a deterministic aperiodic as well as a periodic sequence of three two-state quantum coins, let us consider three DTQWs with the coin operators respectively given by

$$C_A\equiv C(1/2,0,0), \tag{15}$$

$$C_B\equiv C(1/2,\pi,\pi), \tag{16}$$

$$C_C\equiv C(1/2,\pi,\pi/2), \tag{17}$$

where in each of the above DTQWs a separable state of the form

$$|\psi(0)\rangle=|0\rangle\otimes|0\rangle \tag{18}$$

is the walker-coin initial state. Since q, $\rho_{11}(t)$, and $|\rho_{12}(t)|^2$ have the same form for each of the above DTQWs, understood respectively as DTQWs with the period-one sequence of the coin operators C_A, C_B, and C_C, therefore at a given time step t, the probability distribution of the position of the

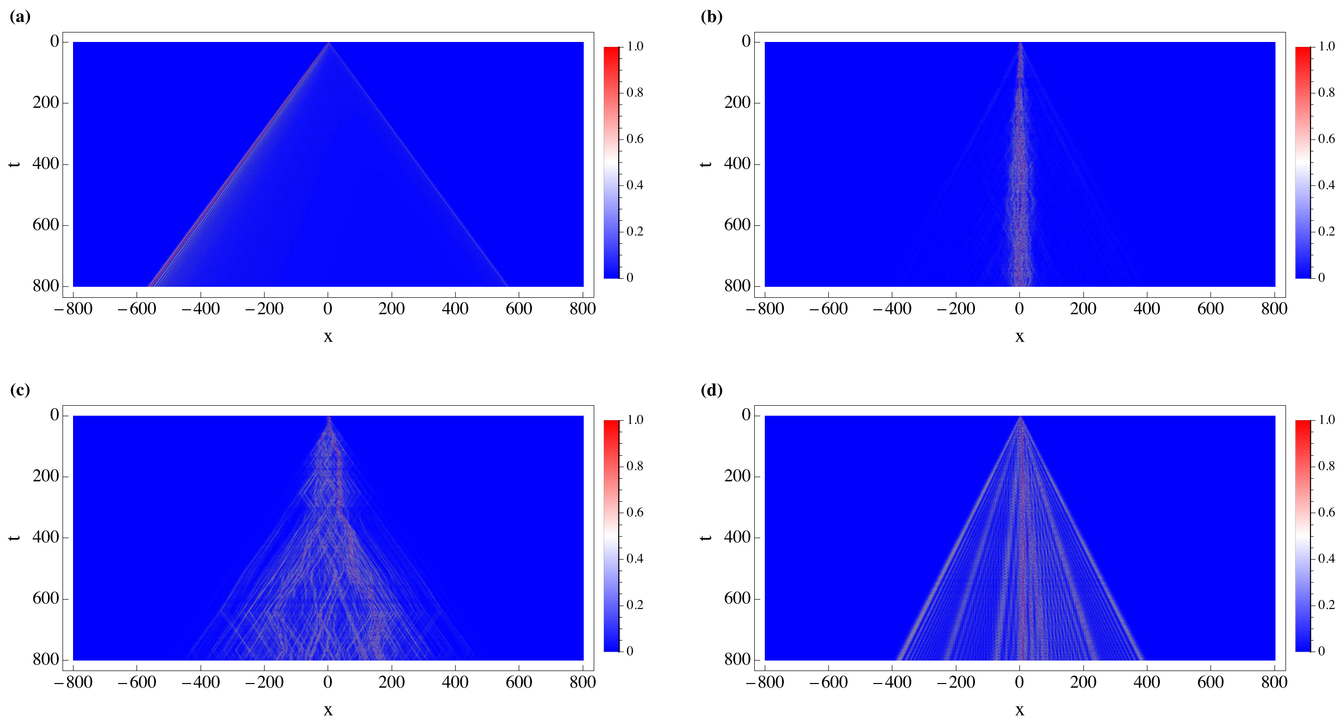

FIG. 1. The space-time evolution of $p(x,t)/\max_x p(x,t)$. The plot (a) is for DTQWs with the coin operator C_A, C_B, and C_C, respectively, understood as DTQWs with the period-one sequence of the coin operators C_A, C_B, and C_C, respectively. The plots (b) and (c) are for DTQWs with deterministic aperiodic sequences of the coin operators C_A, C_B, and C_C given by Eqs. (19) and (20), and the plot (d) is for a DTQW with a deterministic periodic sequence of the coin operators C_A, C_B, and C_C given by Eq. (21).

walker, the average position of the walker, and the walker-coin quantum entanglement are exactly the same for each of the above DTQWs [see Figs. 1(a), 2(a) and 3(a)]. The time evolution of the walker-coin quantum entanglement for each of the above DTQWs is in agreement with the result showing that, in the case of such DTQWs, $\lim_{t\to\infty} S_E(t) = \frac{1}{2} + (\frac{1}{\sqrt{2}} - 1)\log_2(\sqrt{2}-1) \approx 0.872\,429$ [73] [see Fig. 3(a)]. The probability distribution of the position of the walker for each of the above DTQWs has a bias toward negative values of x [see Fig. 1(a)] such that the average position of the walker is nonpositive [see Fig. 2(a)]. In the next section, we show that the direction of the bias can be reversed, leading to Parrondo's paradox.

III. PARADOXICAL DTQWs

Let us introduce now three DTQWs with the following sequences of the coin operators C_A, C_B, and C_C:

$$C_A C_B C_C C_B C_C C_A C_B C_C C_A C_B C_C C_A C_A C_A C_B \dots, \tag{19}$$

$$C_A C_B C_C C_B C_B C_C C_C C_A C_A C_C C_C C_C C_C C_A C_A \dots, \tag{20}$$

$$C_C C_C C_A C_B C_C C_C C_C C_A C_B C_C C_C C_C C_A C_B C_C \dots, \tag{21}$$

where C_A, C_B, and C_C are defined by Eqs. (15)–(17), and in each of the above DTQWs the walker-coin initial state is given by Eq. (18). Let us note that the first two sequences are aperiodic while the last sequence is periodic with period 5, and moreover each of the above sequences is deterministic. The first two sequences are deterministic because the coin operators $C_B C_C$ and C_A are at positions of prime and nonprime numbers, respectively, for the sequence (19) while the sequence (20) is obtained by starting with $C_A C_B C_C$ and successively C_A is replaced by $C_B C_B$ each time but C_B is replaced by $C_C C_C$ and C_C is replaced by $C_A C_A$. The first few steps of this procedure are as follows:

$$\text{step 1:}\;\; C_A C_B C_C, \tag{22}$$

$$\text{step 2:}\;\; C_B C_B C_C C_C C_A C_A, \tag{23}$$

$$\text{step 3:}\;\; C_C C_C C_C C_C C_A C_A C_A C_A C_B C_B C_B C_B. \tag{24}$$

The probability distribution of the position of the walker for the above DTQWs has a bias toward positive values of x [see Figs. 1(b)–1(d)] such that the average position of the walker is non-negative [see Figs. 2(b)–2(d)]. Thus, each of the above DTQWs exhibits the genuine Parrondo's effect. The time evolution of the walker-coin quantum entanglement for these DTQWs [see Figs. 3(b)–3(d)] shows that, in these cases, the walker-coin quantum entanglement is greater, on average, than in the cases of DTQWs with the period-one sequence of the coin operators C_A, C_B, and C_C, respectively [see Fig. 3(a)]. Interestingly, the walker-coin quantum entanglement for the DTQWs presented in Refs. [60,62] exhibits a similar behavior.

Finally, it should be emphasized that the sequences of the coin operators C_A, C_B, and C_C introduced above [see Eqs. (19)–(21)] differ substantially from each other. Therefore, there is no straightforward explanation why DTQWs with these particular sequences of the coin operators, and that particular walker-coin initial state, exhibit the genuine Parrondo's effect.

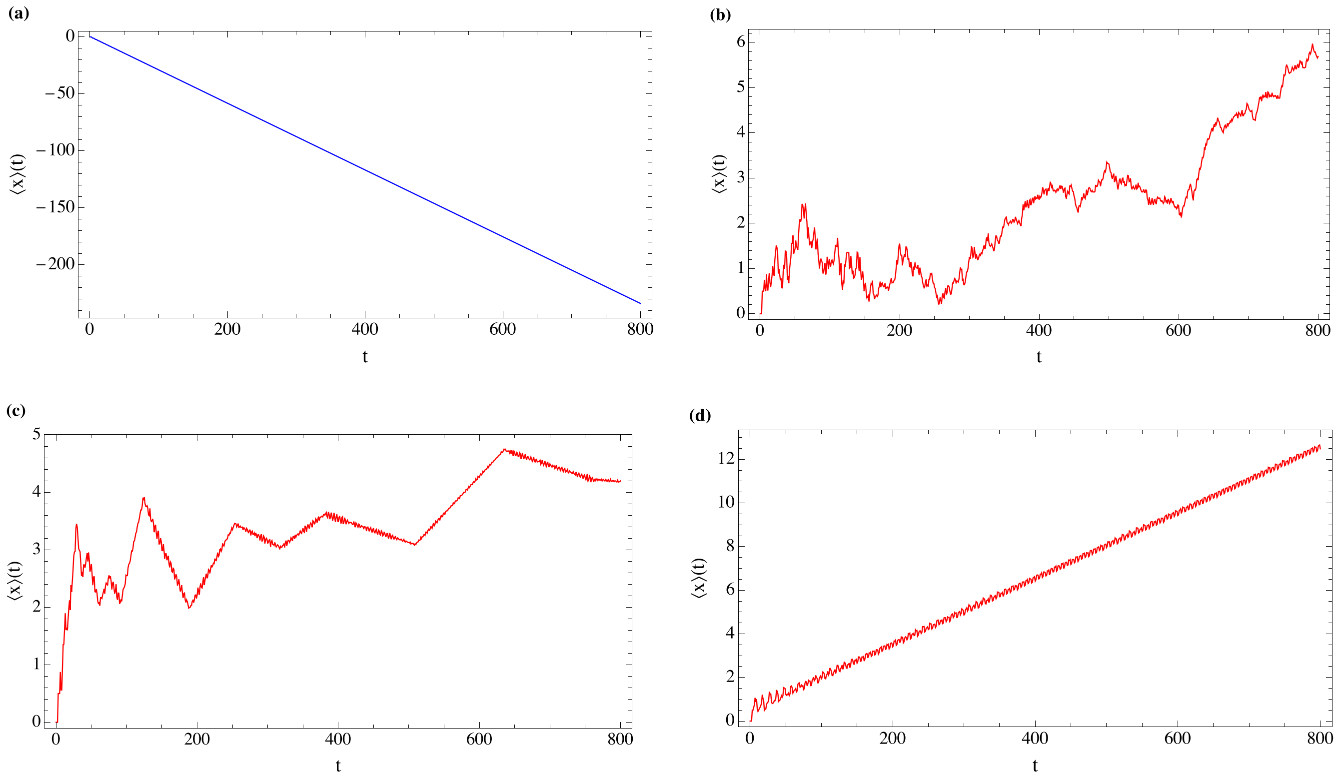

FIG. 2. The time evolution of the average position of the walker. The plot (a) is for DTQWs with the coin operator C_A, C_B, and C_C, respectively, understood as DTQWs with the period-one sequence of the coin operators C_A, C_B, and C_C, respectively. The plots (b) and (c) are for DTQWs with deterministic aperiodic sequences of the coin operators C_A, C_B, and C_C given by Eqs. (19) and (20), and the plot (d) is for a DTQW with a deterministic periodic sequence of the coin operators C_A, C_B, and C_C given by Eq. (21).

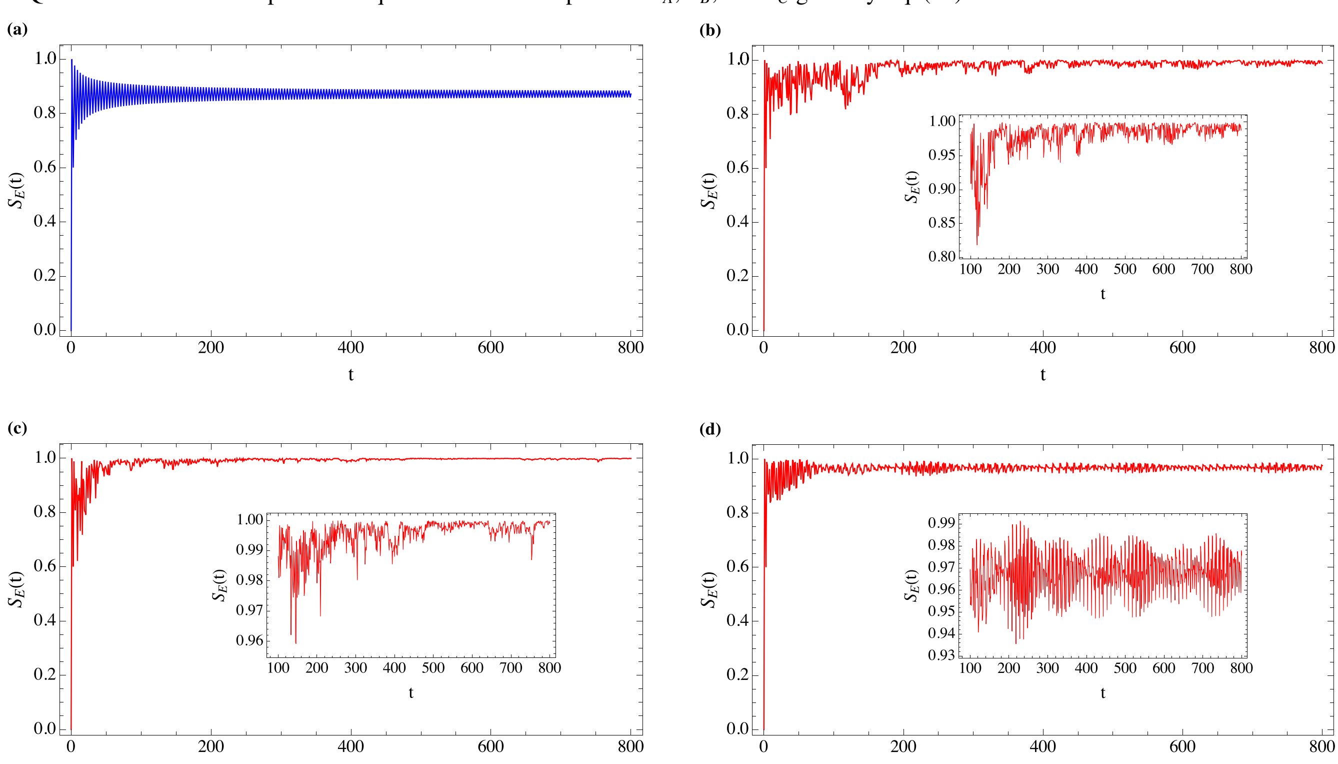

FIG. 3. The time evolution of the walker-coin quantum entanglement. The plot (a) is for DTQWs with the coin operator C_A, C_B, and C_C, respectively, understood as DTQWs with the period-one sequence of the coin operators C_A, C_B, and C_C, respectively. The plots (b) and (c) are for DTQWs with deterministic aperiodic sequences of the coin operators C_A, C_B, and C_C given by Eqs. (19) and (20), and the plot (d) is for a DTQW with a deterministic periodic sequence of the coin operators C_A, C_B, and C_C given by Eq. (21).

IV. CONCLUSION

We have shown that the genuine Parrondo's effect can occur in the case of DTQWs with a deterministic aperiodic as well as a periodic sequence of three, and not only two, two-state quantum coins by providing the first examples of such DTQWs. Moreover, we have shown that the occurrence of the genuine Parrondo's effect in the case of DTQWs with a deterministic sequence of three two-state quantum coins affects the time evolution of the walker-coin quantum entanglement in a similar way as the occurrence of this effect in the case of DTQWs with a deterministic sequence of two two-state quantum coins. More precisely, the genuine Parrondo's effect leads to the enhancement of the walker-coin quantum entanglement. Interestingly, this enhancement can be greater in the case of DTQWs with a deterministic sequence of three two-state quantum coins than in the case of DTQWs with a deterministic sequence of two two-state quantum coins (cf. Refs. [60,62]). Moreover, the enhancement of the walker-coin quantum entanglement can be greater in the case of DTQWs with a deterministic aperiodic sequence of three two-state quantum coins than in the case of DTQWs with a deterministic periodic sequence of three two-state quantum coins [cf. the insets in Figs. 3(b)–3(d)]. Interestingly, this is in line with the results presented in Refs. [74,75], where it has been shown that DTQWs with a deterministic aperiodic sequence of two two-state quantum coins generate maximally entangled quantum states in the asymptotic limit, outperforming the entangling power of DTQWs with a deterministic periodic sequence of two two-state quantum coins. We believe that our results can be experimentally verified in the quantum optics framework and will be helpful in designing new quantum algorithms, including new protocols for generating highly entangled quantum states.

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