

Generalized Lyapunov exponent for the one-dimensional Schrödinger equation with Cauchy disorder: Some exact results

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We consider the one-dimensional Schrödinger equation with a random potential and study the cumulant generating function of the logarithm of the wave function $\psi(x)$, known in the literature as the “generalized Lyapunov exponent”; this is tantamount to studying the statistics of the so-called “finite-size Lyapunov exponent.” The problem reduces to that of finding the leading eigenvalue of a certain nonrandom non-self-adjoint linear operator defined on a somewhat unusual space of functions. We focus on the case of Cauchy disorder, for which we derive a secular equation for the generalized Lyapunov exponent. Analytical expressions for the first four cumulants of $\ln |\psi(x)|$ for arbitrary energy and disorder are deduced. In the universal (weak-disorder and high-energy) regime, we obtain simple asymptotic expressions for the generalized Lyapunov exponent and for all the cumulants. The large deviation function controlling the distribution of $\ln |\psi(x)|$ is also obtained in several limits. As an application, we show that, for a disordered region of size L , the distribution \mathcal{W}_L of the conductance g exhibits the power-law behavior $\mathcal{W}_L(g) \sim g^{-1/2}$ as $g \rightarrow 0$.

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I. INTRODUCTION

Solvable models of disorder in one dimension have played an important role in the theory of Anderson localization, providing exact results which have improved our understanding of the underlying physical mechanisms. Some of the milestones in the development of the theory of one-dimensional localization are the conjecture by Mott and Twose [1] that, in the presence of disorder, every eigenstate becomes localized; the first numerical simulations [2,3]; the proof of the localization of the high-energy states [3]. Following the development of Furstenberg’s theory of products of random matrices [4], a complete proof of the Mott-Twose conjecture was given a few years later in Refs. [5,6]. The pure point nature of the spectrum was demonstrated in Refs. [7,8].

Despite the apparent simplicity of one-dimensional disordered systems, exact results are scarce. Explicit formulas for the density of states and the localization length have been obtained only for a few models. In order to explain the scope of the present paper, it is useful to review briefly these solvable models.

A. Discrete models

The first exact result can be found in Dyson’s seminal paper [9], where the spectral density for the one-dimensional tight-binding model with random hopping, distributed according to a gamma law, was obtained [10]. Another famous solvable case is the Lloyd model (tight-binding model with a Cauchy-distributed on-site potential), for which the Green’s function at coinciding points can be obtained in any dimension [11] (see also Refs. [6,12] for a detailed discussion of the

one-dimensional case). A third type of distribution leading to exact solution for the tight-binding model is the symmetric exponential [13].

B. Continuous models with gaussian white noises

Amongst the continuous models, those where the disorder takes the form of a Gaussian white noise are the most straightforward, as the disorder is then characterized by only two parameters (the mean value and the weight of the two-point correlator). A simple example is provided by the Schrödinger equation, with Hamiltonian $H = -\partial_x^2 + V(x)$ (the mean value plays no role in this case). Its density of states was obtained by Halperin [14] and its localization length in Refs. [15–17]. Another model relevant in several physical contexts is the Dirac equation with random mass, $\mathcal{H}_D = i\sigma_2 \partial_x + \sigma_1 m(x)$, where σ_i are the Pauli matrices (see Ref. [18] for a brief review); here the mean value of the mass cannot be removed. This model is related to the supersymmetric Schrödinger operators $H_{\pm} = -\partial_x^2 + m(x)^2 \pm m'(x)$ (the two partners appear in \mathcal{H}_D^2), which plays a central role in the problem of classical diffusion in a random force field (the so-called Sinai problem). The density of states for this model was found by Erikmann and Ovchinnikov [19] and rediscovered independently by Bouchaud *et al.* [20], who also worked out the localization length. Those results were reproduced later by a different method in Ref. [21].

C. Continuous models with non-gaussian white noises

Exact results can also be obtained for models involving non-Gaussian white noises, when the disorder is modeled

by using delta interactions with random uncorrelated positions and random weights. A well-known example is the Frisch-Lloyd model [22]—also known as the “liquid alloy” model [6,16]—which is the Schrödinger equation with a potential of the form $V(x) = \sum_n v_n \delta(x - x_n)$. Some solvable cases, involving exponentially distributed weights, were found by Nieuwenhuizen [16,23,24]. The Frisch-Lloyd model was generalized in Ref. [24]; in particular, the supersymmetric case where the disorder is modeled by a general Lévy process was discussed in Refs. [25] and [26], and some solvable cases were found.

D. Disorder with spatial correlations

The case of disorder with spatial correlation is more difficult to analyze. A solution was obtained for the Dirac or supersymmetric Hamiltonian with a mass that takes the form of a random telegraph noise (with exponentially decaying correlations) [27]. This result has found an application to spin chain models [28,29]. The lattice model for spatially correlated Cauchy disorder was studied, showing that the averaged local Green’s function is independent of the correlation length [30].

E. Mixed (continuous) models

The fact that the Schrödinger and supersymmetric Hamiltonians exhibit very different properties has motivated the study of the mixed Hamiltonian $H = -\partial_x^2 + m(x)^2 + m'(x) + V(x)$. The case where V and m are two Gaussian white noises (uncorrelated or correlated) was solved in Ref. [31] (see also Ref. [32]). The mixed case where $m(x)$ is a Gaussian white noise and $V(x)$ is a non-Gaussian white noise consisting of delta interactions with positive weights v_n can be used to study absorption in the Sinai problem [33]; the problem was also analyzed in the $v_n \rightarrow \infty$ limit with the Real Space Renormalization Group method in Ref. [34]. It was shown in Ref. [35] that the problem becomes solvable when the mass disorder strength g and the mean value of the weights satisfy $g = 2\overline{v_n}$. Let us finally mention that certain continuum limits of random 2×2 matrices lead to models that combine up to three Gaussian white noises; the scaling forms that the Lyapunov exponent and the density of states can take were classified in Ref. [32].

So far, our review has been concerned exclusively with analytical results for the density of states and the Lyapunov exponent. When we look beyond these basic quantities, exact results become very rare indeed. One can mention the individual energy-level distributions—the extreme value spectral statistics—for various models [18,36,37], or various properties of the Dirac or supersymmetric model at the band center (for instance, wave function correlations and moments [38,39], the distribution of the transmission probability [40], or the distribution of the Wigner time delay [40–42]). Results that are exact in some asymptotic sense are also known for various models (see the reviews [12,15,43] and for the Sinai model [44]).

A conjecture which has generated much discussion among researchers working on Anderson localization is that of “single parameter scaling” (SPS). This conjecture, introduced in

Ref. [45], says that distribution functions of physical observables, like the conductance, are effectively controlled by a single parameter [46,47]. A proper examination of the validity (or otherwise) of this conjecture requires the study of *fluctuations*. It was first investigated from a phenomenological point of view, based on a composition law for the transmission amplitude and an *ad hoc* random phase assumption [46,48].

The fluctuations may be studied by computing the *cumulants*

$$\gamma_n = \lim_{x \rightarrow \infty} \frac{1}{x} \langle [\ln |\psi(x)|]^n \rangle_c. \quad (1)$$

Here $\psi(x)$ is the wave function that solves the initial value problem associated with the model, and $\langle \dots \rangle$ denotes averaging over the disorder ($\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$, etc.). The existence of the limit (1) is a nontrivial matter but, for models that can be formulated in terms of products of random matrices, one can use the theory expounded in Refs. [49,50]. The first cumulant γ_1 is the Lyapunov exponent: according to Borland’s conjecture [3], it provides a measure of the localization of the eigenfunctions that solve the (Sturm-Liouville) spectral problem associated with the model. For the Lloyd model, Deych, Lisyanski, and Altshuler were able to derive an analytic formula for the variance γ_2 [51,52]. For the same model, Titov and Schomerus obtained a complicated analytical form for the third cumulant, and weak-disorder estimates for γ_3 , γ_4 , and γ_5 [53]. They also developed a recursive approach for a different model, equivalent to the Schrödinger equation $-\psi''(x) + V(x)\psi(x) = E\psi(x)$ with a Gaussian white noise potential (Halperin’s model), leading to formulas for the cumulants in terms of multiple integrals [54]. A simpler representation of the variance γ_2 was obtained in Ref. [55] for Halperin’s model, and also for the Dirac equation with a random mass. More recently, a representation of the variance in terms of a *single* integral, valid for the Schrödinger equation with any random potential exhibiting local correlations, was derived in [56], thus simplifying and extending previous results; this is the formula contained in Eq. (43) below.

We stress an important difference between the approaches used in Refs. [53,54] and that used in Refs. [56,57] (initiated in Ref. [58]). In both cases, the starting point is the formulation of a certain spectral problem in which the leading eigenvalue is precisely the cumulant generating function

$$\Lambda(q) = \lim_{x \rightarrow \infty} \frac{1}{x} \ln \langle |\psi(x)|^q \rangle = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!} q^n \quad (2)$$

known also as the *generalized Lyapunov exponent (GLE)* [59]. In Refs. [53,54] the GLE is computed perturbatively in the parameter q , starting from a standard equation of the Fokker-Planck type with stationary solution [67]. On the other hand, [56–58] aims at calculating the GLE *nonperturbatively* for a finite value of the parameter q . This is obviously a much more difficult undertaking; in particular, it requires a careful consideration of the function space in which the eigenfunction (of the nonrandom transfer operator) corresponding to the GLE should be sought. The correct choice of “boundary conditions,” proposed in Ref. [58] for a specific model, was identified in great generality in Refs. [56,57], by relating the problem to a certain representation of the group $SL(2, \mathbb{R})$. The

relevance of group-theoretical considerations in this context comes from the fact that, in the disordered models considered, the solution of the initial-value problem can be expressed in terms of a product of random matrices in $SL(2, \mathbb{R})$ [24–26].

Despite this progress, the computation of the GLE remains, in general, an extremely difficult problem and, to the best of our knowledge—apart from a somewhat trivial case corresponding to a product of triangular 2×2 matrices (cf. Subsection 7.4 of Ref. [56])—no tractable model has yet been found. The aim of this article is to present and analyze a model which is “almost solvable,” in the sense that one can write down explicitly a secular equation for the generalized Lyapunov exponent [68].

As in our previous works [56,57], the present paper addresses the problem of computing the GLE, identified as the leading eigenvalue of a certain nonrandom, non-self-adjoint linear operator. This spectral problem was precisely stated in Refs. [56,57] in the more general context of products of random 2×2 matrices. The general formalism introduced in these two papers was applied to several models for which the spectral problem remained unsolvable, hence we proceeded through a perturbative approach in the conjugated parameter q : the two first terms of the expansion in powers of q were obtained, providing access to the growth rate and to the variance of the logarithm of the matrix products. More recently, models with power-law disorder were studied by one of us [69], where the GLE was studied by a different perturbative approach, in the disorder strength, leading to a weak-disorder expression of the GLE for those models. The essential novelty of the present paper is that, by focusing exclusively on the case of Cauchy disorder, and by exploiting its special features, we are able to obtain results that are *nonperturbative* in both the parameter q and the disorder strength. Hence we get much more information on the large deviations of the wave function. At present, we know of no other model for which such a thorough analysis is feasible.

F. Some physical motivations for Cauchy disorder

Among the models with Cauchy disorder, the Lloyd model, i.e., the one-dimensional tight-binding model $-\psi_{n+1} + V_n \psi_n - \psi_{n-1} = \varepsilon \psi_n$, with potentials V_n that are independent and identically distributed (i.i.d.) with probability density $P(V) = (b/\pi)/[V^2 + b^2]$, has received the most attention. As mentioned earlier, from the theoretical point of view, this model is remarkable for the fact that the Lyapunov exponent and the density of states are relatively easy to compute [6,12,51]. On the hand, from the physical point of view, the model is somewhat pathological since the second moment $\langle V_n^2 \rangle$ is infinite. Nevertheless it is relevant in various contexts.

The first is the quantum kicked rotor, a model exhibiting the phenomenon of dynamical localization (localization of the wave function in momentum space): the Floquet eigenstates in momentum space can be shown to obey the same equation as the wave functions of the tight-binding model with Cauchy disorder [70,71].

The second application appeared recently in connection with various models of disordered ladders. Even though the disorder in the ladder has finite moments $\langle V_n^2 \rangle < \infty$, it was shown in Ref. [72] that, when the ladder is such that its

spectrum exhibits a flat band, the problem can be mapped onto the one-dimensional tight-binding model with an effective potential V_n^{eff} distributed according to the Cauchy law, i.e., such that $\langle (V_n^{\text{eff}})^2 \rangle = \infty$.

Finally, the solvability of the Lloyd model has been used in Ref. [73] in order to determine the various topological phases of a disordered Kitaev chain.

G. A continuous model

Instead of considering lattice models, we find it convenient to work here with a continuous model. We start with the Schrödinger equation

$$-\psi''(x) + V(x) \psi(x) = E \psi(x) \quad (3)$$

with a potential

$$V(x) = \sum_n v_n \delta(x - x_n) \quad (4)$$

consisting of impurities distributed along the real line at random uncorrelated positions x_n with a mean uniform density ρ . In this expression, the weights v_n are drawn independently from some distribution whose probability density function is denoted $p(v)$.

Under these assumptions, the integral of the potential $W(x) = \int_0^x dt V(t)$ is a so-called compound Poisson process, and so we can introduce the Lévy exponent $\mathcal{L}(s)$ [25,35,76], defined implicitly by

$$\langle e^{-is \int_0^x dt V(t)} \rangle = e^{-x \mathcal{L}(s)}. \quad (5)$$

Equivalently, the generating functional of the disordered potential takes the form [35]

$$G[h] \stackrel{\text{def}}{=} \langle e^{-i \int dx h(x) V(x)} \rangle = e^{-\int dx \mathcal{L}(h(x))}. \quad (6)$$

The Lévy exponent is given explicitly by

$$\mathcal{L}(s) = \rho [1 - \hat{p}(s)] \quad \text{where } \hat{p}(s) = \int dv p(v) e^{-ivs}. \quad (7)$$

In the case where the weights are Cauchy-distributed, we have $p(v) = (b/\pi)/[v^2 + b^2]$, and so

$$\mathcal{L}(s) = \rho(1 - e^{-b|s|}). \quad (8)$$

The resulting model is still too difficult to solve. However, a simplification occurs by considering the high-density limit with vanishing weights

$$\rho \rightarrow \infty \quad \text{and} \quad b \rightarrow 0 \quad \text{with} \quad \rho b = c \quad \text{fixed}. \quad (9)$$

In this limit, the Lévy exponent becomes

$$\mathcal{L}(s) = c |s|. \quad (10)$$

It is the Lévy exponent associated with the so-called α -stable Lévy process $W(x) = \int_0^x dt V(t)$ (with $\alpha = 1$). In other words, in this limit, $W(x)$ is distributed according to the Cauchy law

$$p_x(W) = \frac{c x / \pi}{W^2 + (c x)^2}. \quad (11)$$

The parameter c is the strength of the disorder. This is the model studied here. This continuous model differs from the discrete (tight-binding and Kronig-Penney) models with

Cauchy disorder considered in Refs. [6,12,51–53]. Nevertheless, they all exhibit similar properties in the universal high-energy and weak-disorder regime (i.e., at the band edge of the lattice model with vanishing disorder).

H. Main results

One key result of the paper is the derivation of the following secular equation for the generalized Lyapunov exponent $\Lambda(q)$:

$$k_c^{q+1} \frac{\Gamma(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1)}{\Gamma(\frac{i\Lambda}{2k_c} - \frac{q}{2})} = (k_c^*)^{q+1} \frac{\Gamma(\frac{-i\Lambda}{2k_c^*} + \frac{q}{2} + 1)}{\Gamma(\frac{-i\Lambda}{2k_c^*} - \frac{q}{2})},$$

where $k_c^2 = E + ic$ combines the energy E and the disorder strength c . Several exact results are deduced from its analysis. First, we derive analytical expressions for the first four cumulants, valid for arbitrary energy and disorder strength; see Eqs. (50), (53), (69), and (70). Turning then to the high-energy limit, we obtain the following expression for the GLE:

$$\lim_{E/c \rightarrow +\infty} \frac{\Lambda(q)}{\gamma_1} = \frac{2}{\pi} (1 + q) \tan\left(\frac{\pi q}{2}\right)$$

for $q \in]-3, 1[$. We also find expressions for all the cumulants: for n even and $E \gg c$, one has

$$\gamma_n \simeq 4\pi^{n-2} (2^n - 1) |B_n| \gamma_1 \quad \text{and} \quad \gamma_{n-1} \simeq \gamma_n/n,$$

where the B_n are the Bernoulli numbers and $\gamma_1 \simeq c/(2\sqrt{E})$ is the Lyapunov exponent. The divergence of the GLE, i.e., of the moments $\langle |\psi(x)|^q \rangle$, for $q \rightarrow 1^-$ and $q \rightarrow -3^+$ is characteristic of power-law disorder. Using a Legendre transform of $\Lambda(q)$, we then deduce the large deviation function controlling the distribution of $\ln |\psi(x)|$. We show that the distribution of the conductance for a disordered region of size L exhibits the power-law singular behavior

$$\mathcal{W}_L(g) \underset{g \rightarrow 0}{\sim} g^{-1/2}.$$

This is in agreement with recent numerical simulations [77]. The high-energy and weak-disorder results obtained in the paper are expected to be *universally valid* for every model with a disordered potential characterized by the power-law tail $p(V) \sim V^{-2}$. Finally, we also derive various results away from the universal regime.

I. Outline

Section II gives a brief and simplified presentation of the general formalism of Ref. [56], adapted to the specific case considered here. We characterize the GLE $\Lambda(q)$ as the leading eigenvalue of a nonrandom linear operator. As a warm-up exercise, in Sec. III we use the perturbative approach (in powers of q) described in Refs. [56,57]. In Sec. IV we go beyond the perturbative analysis and obtain the secular equation satisfied by the GLE for fixed values of q . The exact analytical expressions for the first four cumulants are deduced in Sec. V. The universal (weak-disorder) regime is discussed in Sec. VI. The zero-energy limit and the limit of large negative energy are studied in Sec. VII. The GLE is a particular eigenvalue of a spectral problem: the full spectrum of eigenvalues is discussed in Sec. VIII. The study of the wave function fluctuations and

of the distribution of the conductance is carried out in Sec. IX. Finally, in Sec. X we indicate briefly how our somewhat unusual spectral problem relates to some recent works on the spectral problem for non-self-adjoint generalizations of the Schrödinger equation with a Coulomb potential.

II. The GLE AS THE LEADING EIGENVALUE OF A NONRANDOM LINEAR OPERATOR

In this section, we recall the main formalism introduced in Refs. [56,57] to study the generalized Lyapunov exponent (GLE) of products of random matrices in the group $SL(2, \mathbb{R})$. We focus here on a specific case, which leads to a simple derivation of the main equation for the spectral problem.

A. Formulation in terms of a product of random matrices

As is well-known from elementary quantum mechanics, for the Kronig-Penney potential Eq. (4), the solution of the initial-value problem for the Schrödinger Eq. (3) on the half-line $x > 0$ can be expressed as a product of random transfer matrices. More precisely, if we set $E = k^2$ and recast the equation as a first-order system of two equations for the unknown vector $(\psi'(x), k\psi(x))^T$, then, by considering the equation in each of the intervals $[x_n, x_{n+1}[$, we see that the effect of the delta potential and of the free evolution corresponds to multiplication by the matrices

$$N(u_n) = \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K(\theta_n) = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \tag{12}$$

respectively. Both these 2×2 matrices have unit determinant. Hence they, and any repeated product of them, belong to the group $SL(2, \mathbb{R})$. [In the case $E = -k^2$, the matrix K must be replaced by another matrix in $SL(2, \mathbb{R})$ with entries involving the hyperbolic functions; see [24,55].] The angle of rotation is proportional to the interval length: $\theta_n = k\ell_n$, with $\ell_n = x_{n+1} - x_n$, and the upper off-diagonal coefficient in the matrix N is proportional to the weight of the impurity at x_n : $u_n = v_n/k$. In our particular model, the number of impurities in the interval $[0, x]$ is a Poisson process, say, $\mathcal{N}(x)$, of intensity ρ , i.e., $\text{Proba}\{\mathcal{N}(x) = n\} = e^{-\rho x} (\rho x)^n / n!$. Therefore, the ℓ_n are independent and exponentially distributed with mean $1/\rho$, so that $\text{Proba}\{\ell_n > \ell\} = e^{-\rho \ell}$. Introducing the random matrices $M_n = K(\theta_n)N(u_n)$, the q th moment of the wave function ψ that solves the initial-value problem can then be written as

$$\langle |\psi(x)|^q \rangle \sim \langle \|\Pi_{\mathcal{N}(x)} \vec{x}_0\|^q \rangle, \tag{13}$$

where

$$\Pi_n = M_n \cdots M_2 M_1. \tag{14}$$

$\|\vec{x}\|$ is the usual Euclidean vector norm. \vec{x}_0 is a vector of unit length on which acts the product Π_n , expressing the initial conditions. For example, $\vec{x}_0 = (1, 0)^T$ corresponds to imposing the initial conditions $\psi'(0) = 1$ and $\psi(0) = 0$.

B. A spectral problem

A general formalism for the study of the moments $\langle \|\Pi_N \vec{x}_0\|^q \rangle$, when N is nonrandom, was developed in Refs. [56,57], and the necessary adjustments that are needed

to cater for the case (13), where $N = \mathcal{N}(x)$ is random, were indicated in Ref. [56]. In what follows, we provide a simplified description of this formalism, adapted to our particular class of models.

The matrices in $SL(2, \mathbb{R})$ act on \mathbb{R}^2 by multiplication, and we have expressed the wave function in terms of a matrix acting on some initial vector of unit length. Every vector of unit length can be identified with a *direction*, which can be parametrized either in terms of the angle, say, $\theta \in [0, \pi)$, it makes with the horizontal axis, or else by the ratio, say, $z \in \mathbb{R}$, of the Cartesian coordinates of the vector; in the first case, we speak of the *projective semicircle*, and in the second of the *projective line*. In the latter parametrization, the action of a matrix M on the projective line corresponds to a Möbius map [49]:

$$z \mapsto \mathcal{M}(z) = \frac{az + b}{cz + d} \quad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (15)$$

Instead of following the vector $(\psi'(x), k\psi(x))^T$, it will be simpler to track the Riccati variable $z(x) = \psi'(x)/\psi(x)$. Accordingly, we define

$$\begin{aligned} \mathcal{P}_x(z|z_0; q) &\stackrel{\text{def}}{=} \langle \delta(z - z(x)) |\psi(x)|^q \rangle & (16) \\ &= \langle \delta(z - z(x)) e^{q \int_0^x dt z(t)} \rangle, & (17) \end{aligned}$$

which determines the moments

$$\langle |\psi(x)|^q \rangle = \int dz \mathcal{P}_x(z|z_0; q). \quad (18)$$

How it evolves during an infinitesimal interval of “time” $[x, x + dx]$ can be deduced from the following considerations:

(1) If $[x, x + dx]$ contains exactly one impurity, then the Riccati variable makes a jump $z(x_n^+) = z(x_n^-) + v_n$. This occurs with probability ρdx .

(2) If the interval $[x, x + dx]$ contains no impurity, then the evolution of ψ —and hence also of z —is free. In that case, $z(x + dx) \simeq z(x) - [E + z(x)^2] dx$ so that, if we introduce $\tilde{z} = z + (E + z^2) dx$, we can write $d\mathcal{P}_x(z|z_0; q) = d\tilde{z} \mathcal{P}_x(\tilde{z}|z_0; q)$. This scenario occurs with probability $1 - \rho dx$.

(3) During the interval $[x, x + dx]$, the exponential in Eq. (17) grows by a factor $1 + qz dx$.

(4) The probability that $[x, x + dx]$ contains two or more impurities is $o(dx)$ and may be neglected.

As a result

$$\begin{aligned} \mathcal{P}_{x+dx}(z|z_0; q) &\simeq \rho dx \langle \mathcal{P}_x(z - v|z_0; q) \rangle_v + (1 - \rho dx) \\ &\quad \times (1 + 2z dx) \mathcal{P}_x(z + [E + z^2] dx|z_0; q) \\ &\quad \times (1 + qz dx), \end{aligned} \quad (19)$$

where $\langle \dots \rangle_v$ denotes averaging over the impurity weights with distribution $p(v)$. This leads to

$$\partial_x \mathcal{P}_x(z|z_0; q) = \mathcal{L}_q \mathcal{P}_x(z|z_0; q), \quad (20)$$

where \mathcal{L}_q is a nonrandom linear operator, defined by

$$\begin{aligned} \mathcal{L}_q \phi(z) &\stackrel{\text{def}}{=} [\partial_z(E + z^2) + qz + \rho \langle (e^{-v \partial_z} - 1) \rangle] \phi(z) \\ &= [\partial_z(E + z^2) + qz] \phi(z) + \rho \langle [\phi(z - v)]_v - \phi(z) \rangle. \end{aligned} \quad (21)$$

This operator acts in a certain q -dependent space of functions whose properties are discussed below. Equations (20) and (21) agree with Eq. (6.11) of Ref. [56], specialized to our particular case. Equation (20) makes clear that the large- x behavior of $\mathcal{P}_x(z|z_0; q)$ can be obtained from a spectral analysis of the operator \mathcal{L}_q : if we assume a discrete spectrum, denoted by $\{\Lambda_n(q)\}_{n \in \mathbb{Z}}$, where $\Lambda_0(q)$ is the leading eigenvalue, we expect the behavior $\mathcal{P}_x(z|z_0; q) \sim e^{x \Lambda_0(q)}$. From Eq. (18), we then deduce that

$$\Lambda(q) = \Lambda_0(q), \quad (22)$$

so that the GLE can be obtained from the spectral analysis of \mathcal{L}_q . This approach relies on (i) a precise definition of the underlying spectral problem, and in particular of the nature of the space of functions on which the operator \mathcal{L}_q acts, and (ii) the existence of a spectral gap—that is, the discrete spectrum must be such that the leading eigenvalue $\Lambda_0(q)$ is isolated from the rest of the spectrum. We proceed to discuss these two important points.

The case $q = 0$ has been studied extensively in the literature pertaining to products of random matrices and is well understood: $\mathcal{P}_x(z|z_0; 0)$ is the distribution of the process $z(x)$ and, under broad conditions, it has a limit law [74]: $\mathcal{P}_x(z|z_0; 0) \rightarrow f(z)$ as $x \rightarrow \infty$, where the stationary probability density satisfies $\mathcal{L}_0 f(z) = 0$ or, more explicitly,

$$\partial_z [(E + z^2) f(z)] + \rho [(f(z - v))_v - f(z)] = 0. \quad (23)$$

In this case, f is a right-eigenfunction corresponding to the eigenvalue 0 and, under the same broad conditions, it may be shown that it is the leading eigenvalue, so that $\Lambda(0) = 0$. The integro-differential equation Eq. (23) is the form taken by the so-called Dyson-Schmidt equation for our particular model. This equation makes clear that the stationary density f exhibits the asymptotic behavior

$$\lim_{z \rightarrow -\infty} (-z)^2 f(z) = \lim_{z \rightarrow +\infty} z^2 f(z). \quad (24)$$

It expresses that the probability current associated with the stationary distribution takes the same value at $+\infty$ and $-\infty$. The current, i.e., the value of the limit Eq. (24), coincides with the integrated density of states (IDoS) $\mathcal{N}(E)$ of the random Schrödinger operator [12,15,24]: this is the well-known Rice formula [78]. We proceed to argue that a relation of the type Eq. (24), suitably generalized for $q \neq 0$, provides the correct auxiliary condition that must be imposed on every right-eigenfunction in order to complete the definition of the spectral problem.

For arbitrary q , but small density ρ (or small weights v_n), the operator \mathcal{L}_q can be viewed as a perturbation of the differential operator $\mathcal{D}_K(q) = \partial_z(k^2 + z^2) + qz$ and so, it is at least plausible that the functional setting should be the same for both problems. Now, the spectral problem for $\mathcal{D}_K(q)$ takes a particularly simple form if, instead of working on the projective line, we go over to the projective semicircle, which uses $\theta = \text{arccotg}(z/k)$ as the independent variable. In this alternative parametrization, $\mathcal{D}_K(q)$ is transformed into $\tilde{\mathcal{D}}_K(q) = -\partial_\theta + q \cotg \theta$ and the natural domain for this operator is the space of π -periodic functions. It follows easily (see [56], Appendix B) that it has a discrete spectrum consisting of the eigenvalues $\Lambda_n^{(0)}(q) = -2ink$ (with $n \in \mathbb{Z}$), with

corresponding right-eigenfunctions $e^{2in\theta} (\sin \theta)^q$. The spectrum does not depend on how we parametrize the space of directions. Expressed as functions on the projective line, the corresponding right-eigenfunctions of $\mathcal{D}_K(q)$ are

$$\varphi_n^R(z; q) = \frac{1}{\sqrt{\pi}} \left(\frac{z + ik}{z - ik} \right)^n (z^2 + k^2)^{-1-q/2} \quad (25)$$

and, together, form a countable basis for the “correct” space of functions on the projective line. These basis functions all exhibit the asymptotic behavior $\varphi_n^R(z; q) \simeq A |z|^{-2-q}$. The key observation is that coefficient of the power law is the same at $z \rightarrow +\infty$ and $z \rightarrow -\infty$.

These remarks allow us to complete the definition of the spectral problem for

$$\mathcal{L}_q \phi(z; \Lambda) = \Lambda \phi(z; \Lambda) \quad (26)$$

by imposing that the two limits

$$A_{\pm}(\Lambda) \stackrel{\text{def}}{=} \lim_{z \rightarrow \pm\infty} (\pm z)^{2+q} \phi(z; \Lambda) \quad (27)$$

must exist and coincide:

$$A_+(\Lambda) = A_-(\Lambda). \quad (28)$$

This equation is precisely the secular equation satisfied by the eigenvalues of the non-self-adjoint operator \mathcal{L}_q .

We close the section with several remarks:

(1) The condition (28) is dictated by representation-theoretical considerations whose relevance to the correct formulation of the spectral problem was an important outcome of [56,57]. For products of random matrices in $SL(2, \mathbb{R})$, the spectral problem involves a nonrandom operator which depends—in a usually very complicated way—on $\mathcal{D}_K(q)$ and on two other operators which, together, span a representation of the Lie algebra of $SL(2, \mathbb{R})$. The operators in this Lie algebra act on functions defined on the projective line and, roughly speaking, condition of the type (27,28) characterises the function spaces associated with a particularly important family, indexed by q , of irreducible representations of the group; see, for instance, Chapter 2 of [79]. In this connection, we remark that the case $q = -1$ corresponds to a unitary representation of the group $SL(2, \mathbb{R})$; the eigenfunctions Eq. (25) for this case appear in Ref. [80].

(2) Since the operator \mathcal{L}_q is not self-adjoint, each eigenvalue is associated with a pair of right- and left-eigenfunctions. Above, we have worked exclusively with the auxiliary condition satisfied by the right-eigenfunctions. If we work instead with left-eigenfunctions, we find that they satisfy an auxiliary condition like Eq. (27), but with q replaced by $-q^* - 2$ [57]. We then obtain a secular equation that is equivalent.

(3) We will denote by $\Lambda_n(q)$ the solutions of the equation (28) and by $\Phi_n^R(z; q)$ the related right-eigenvectors of \mathcal{L}_q , which thus satisfy

$$\begin{aligned} [\partial_z(E + z^2) + qz]\Phi_n^R(z; q) + \rho[(\Phi_n^R(z - v; q))_v - \Phi_n^R(z; q)] \\ = \Lambda_n(q) \Phi_n^R(z; q). \end{aligned} \quad (29)$$

(4) For products of random matrices, the existence of a spectral gap below the leading eigenvalue, when q is small,

has been investigated; see Ref. [49], Chapter V. For Cauchy disorder, the spectral gap is discussed in Sec. VIII below.

(5) For the specific case of a Gaussian white noise potential, Eq. (26) reduces to a differential equation: it suffices to take the limit $\rho \rightarrow \infty$ and $v_n \rightarrow 0$ with $\langle v_n \rangle = 0$ and $\sigma = \rho \langle v_n^2 \rangle$ fixed, which yields $\rho[\langle \Phi_n^R(z - v; q) \rangle_v - \Phi_n^R(z; q)] \rightarrow (\sigma/2)\partial_z^2 \Phi_n^R(z; q)$ in Eq. (29), leading to Eq. (105) of Ref. [58]. In that reference, Eq. (28) was used to determine the GLE numerically to a great accuracy.

C. The spectral problem in Fourier space: From an integro-differential to a differential operator

Unless the integro-differential equation Eq. (26) can be solved explicitly for $\phi(z; \Lambda)$, Eq. (28) appears to be of limited use at this stage. The problem becomes more tractable if we look instead for the Fourier transform [81]

$$\widehat{\phi}(s; \Lambda) = \int_{-\infty}^{+\infty} dz e^{-isz} \phi(z; \Lambda). \quad (30)$$

In Fourier space, Eq. (26) takes the form

$$is \left[-\frac{d^2}{ds^2} + \frac{q}{s} \frac{d}{ds} + E - \frac{\mathcal{L}(s)}{is} \right] \widehat{\phi}(s; \Lambda) = \Lambda \widehat{\phi}(s; \Lambda), \quad (31)$$

where the Lévy exponent $\mathcal{L}(s)$ was defined by Eqs. (5) and (7).

To proceed, we must explain how, knowing the Fourier transform $\widehat{\phi}(s; \Lambda)$, we can make concrete use of the secular equation Eq. (28) to determine the spectrum. The explanation consists of two parts. First, we recall that, for an arbitrary Λ , the function $\phi(z; \Lambda)$ denotes any nonzero solution of Eq. (26) such that the limits Eq. (27) exist. From the existence of these limits, we deduce the following:

(1) For $q > -3/2$, $\phi(z; \Lambda)$ is square-integrable, and so $\widehat{\phi}(s; \Lambda)$ must decay at $\pm\infty$.

(2) Unless the limits $A_{\pm}(\Lambda)$ vanish, $\phi(z; \Lambda)$ exhibits an algebraic behavior at infinity. This implies that $\widehat{\phi}(s; \Lambda)$ is not smooth at $s = 0$.

In order to analyze the local behavior of $\widehat{\phi}(s; \Lambda)$ at $s = 0$, it will be convenient to consider in the first instance the case $q > -1$, so that $\phi(z; \Lambda)$ is integrable and its Fourier transform continuous. There is then no loss of generality in assuming that $\widehat{\phi}(0; \Lambda) = 1$, and we can write

$$\widehat{\phi}(s; \Lambda) = \begin{cases} \phi_-(s; \Lambda) & \text{if } s < 0 \\ \phi_+(s; \Lambda) & \text{if } s > 0, \end{cases} \quad (32)$$

where $\phi_{\pm}(s; \Lambda)$ is the solution of

$$\left[-\frac{d^2}{ds^2} + \frac{q}{s} \frac{d}{ds} + E - \frac{\mathcal{L}(s) + \Lambda}{is} \right] \phi_{\pm}(s; \Lambda) = 0, \quad \pm s > 0, \quad (33)$$

which satisfies $\phi_{\pm}(0; \Lambda) = 1$ and vanishes at $\pm\infty$. Consider first the “+” equation: in the limit $s \rightarrow 0^+$, $\mathcal{L}(s)/s$ has a limit [75] and so three terms dominate in the differential equation, namely, $-\phi_+'' + (q/s)\phi_+' + (i\Lambda/s)\phi_+ \simeq 0$. Retaining only the last two leads to the behavior $\phi_+(s; \Lambda) \simeq 1 - (i\Lambda/q)s$, while retaining the first two produces a nonanalytic contribution s^{q+1} . Thus, for $|\text{Re } q| < 1$, the first terms of the

small s expansion are

$$\phi_+(s; \Lambda) = 1 - \frac{i\Lambda}{q} s + \Omega_+(\Lambda) s^{q+1} + \dots \quad \text{for } s \rightarrow 0^+. \quad (34)$$

Similar considerations hold for the solution ϕ_- ; thus

$$\phi_-(s; \Lambda) = 1 - \frac{i\Lambda}{q} s + \Omega_-(\Lambda) (-s)^{q+1} + \dots \quad \text{for } s \rightarrow 0^-. \quad (35)$$

We arrive at the second part of our explanation. We can write

$$\widehat{\phi}(s; \Lambda) = \int dz \phi(z; \Lambda) - \int dz \phi(z; \Lambda) (1 - e^{-isz}). \quad (36)$$

Let us now suppose that Λ is an eigenvalue, so that Eq. (28) holds and the limits are *equal*. In the limit $s \rightarrow 0$, the second integral is then dominated by the tail of the function $\phi(z; \Lambda) \simeq A |z|^{-2-q}$ for $z \rightarrow \infty$, with $A \equiv A_+(\Lambda) = A_-(\Lambda)$. More precisely, using the result in Appendix, we can relate the power-law decay for $z \rightarrow \pm\infty$ to the $s \rightarrow 0$ behavior

$$\widehat{\phi}(s; \Lambda) \underset{s \rightarrow 0}{\simeq} 1 + \alpha_1 s - 2A \Gamma(-1-q) \sin\left(\frac{\pi q}{2}\right) |s|^{q+1} \quad (37)$$

for $-1 < \text{Re}(q) < 1$, where α_1 is some coefficient. Comparing this with Eqs. (34) and (35), we deduce that the secular equation Eq. (28) for the eigenvalues can be expressed in the equivalent form

$$\Omega_+(\Lambda) = \Omega_-(\Lambda). \quad (38)$$

We expect this to remain true for a larger interval of the parameter q than that assumed here, so that, in particular, the Fourier transform of the right-eigenfunction presents the small- s behavior

$$\widehat{\Phi}_n^R(s; q) \underset{s \rightarrow 0}{=} \underbrace{1 - \frac{i\Lambda_n(q)}{q} s + O(s^2)}_{\text{analytic}} + \underbrace{\omega_q |s|^{q+1} + O(s^{q+2})}_{\text{nonanalytic}}, \quad (39)$$

where $\omega_q = \Omega_{\pm}(\Lambda_n(q))$.

Since $\Lambda(q) = \Lambda_0(q)$, the case $n = 0$ will be our prime concern in what follows and calls for a few remarks:

(1) For q real, from the definition Eq. (2), the leading eigenvalue $\Lambda_0(q) = \Lambda(q)$ must be real. As a result, the right-eigenfunction $\Phi_0^R(z; q)$ is real, and hence also the coefficient ω_q ; see Appendix C of Ref. [56] where the behavior Eq. (39) was derived.

(2) Equation (31) reduces in the case $\Lambda = q = 0$ to the equation for the Fourier transform $\widehat{f}(s)$ of the invariant probability density $f(z)$, the normalized solution of the Dyson-Schmidt equation Eq. (23). After dividing both sides by s , the equation takes the form

$$\left[-\frac{d^2}{ds^2} + E - \frac{\mathcal{L}(s)}{is} \right] \widehat{f}(s) = 2\pi \mathcal{N}(E) \delta(s), \quad (40)$$

where $\mathcal{N}(E)$ is the IDoS mentioned earlier in connection with the equality Eq. (24) [24,35,56,78]. It is clear from the equation that $\widehat{f}(-s) = \widehat{f}(s)^*$ and $\pi \mathcal{N}(E) = -\text{Im}[\widehat{f}'(0^+)] = -\omega_0$. Furthermore, the first cumulant $\gamma_1 = \Lambda'(0)$ may be expressed as

$$\gamma_1 = \int dz z f(z) = -\text{Im}[\widehat{f}'(0^+)]. \quad (41)$$

By putting these results together, we deduce

$$i \widehat{f}'(0^+) = \gamma_1 - i\pi \mathcal{N}. \quad (42)$$

The quantity on the right-hand side is sometimes referred to as the “complex Lyapunov exponent” or the “characteristic function” associated with the model [12,16,24,25,32,35].

(3) In Ref. [56], the spectral problem Eq. (31) was studied by a perturbative approach (in powers of q) for a general Schrödinger equation with disorder, leading to a compact formula for the variance

$$\gamma_2 = \int_0^\infty \frac{ds}{s} \text{Re} \left[\left(2\gamma_1 - i \frac{d}{ds} \right) \widehat{f}(s)^2 \right] \quad (43)$$

in terms of the Fourier transform of the invariant probability density, the solution of Eq. (40).

The remainder of the paper will be concerned with the application of the formalism developed in this section to the particular model with Cauchy disorder introduced earlier in Sec. I G.

III. WARM-UP: PERTURBATIVE APPROACH (IN q) AND THE CUMULANTS γ_1 AND γ_2

It is useful to begin by working out the first two cumulants perturbatively—that is, following the method of [56], by making use of Formulas Eqs. (40)–(43). This will provide a useful check for the results that will be derived subsequently from an analysis of the secular equation. The strategy is then to solve this differential equation Eq. (40) for $s > 0$, identify the solution vanishing at $+\infty$ and impose $\widehat{f}(0) = 1$. For the case of Cauchy disorder, this is extremely simple. Equation (40) yields

$$\left[-\frac{d^2}{ds^2} + E + ic \right] \widehat{f}(s) = 0 \quad \text{for } s > 0. \quad (44)$$

The solution is

$$\widehat{f}(s) = e^{-k_c s} \quad \text{for } s > 0, \quad (45)$$

where

$$(k_c)^2 = E + ic = |k_c|^2 e^{i\theta_c} \quad \text{with } \theta_c \in]0, \pi[. \quad (46)$$

For $s < 0$, we use $\widehat{f}(s) = \widehat{f}(-s)^*$. From Eq. (42), we deduce

$$k_c = \pi \mathcal{N} + i\gamma_1, \quad (47)$$

so that k_c coincides with the complex Lyapunov exponent associated with the model Eqs. (3) and (10). By considering the real and imaginary parts, we arrive at

$$\pi \mathcal{N} = |k_c| \cos(\theta_c/2) = \sqrt{\frac{E^2 + c^2 + E}{2}} \quad (48)$$

$$\simeq \begin{cases} \sqrt{E} + \frac{c^2}{8E^{3/2}} & \text{for } E \gg c \\ \sqrt{c/2} & \text{for } |E| \ll c \\ \frac{c}{2\sqrt{-E}} & \text{for } -E \gg c \end{cases} \quad (49)$$

and

$$\gamma_1 = |k_c| \sin(\theta_c/2). \quad (50)$$

Thus, for Cauchy disorder we have the property

$$\gamma_1(E) = \pi \mathcal{N}(-E). \quad (51)$$

In particular, we see that the power-law decay $\mathcal{N} \simeq c/(2\pi\sqrt{-E})$ of the IDoS in the limit $E \rightarrow -\infty$ is related to the power-law decay $\gamma_1 \simeq c/(2\sqrt{E})$ of the Lyapunov exponent as $E \rightarrow +\infty$. The slow power-law decay of the IDoS for $E \rightarrow -\infty$ shows that Cauchy disorder shifts states to very large negative energies. This contrasts with the case where the disorder has finite moments, where the IDoS exhibits Lifshitz tails that decay exponentially.

We obtain the variance straightforwardly from Eq. (43):

$$\begin{aligned} \gamma_2 &= -2\pi\mathcal{N} \int_0^\infty \frac{ds}{s} \text{Im}[\hat{f}(s)] \\ &= 2\pi\mathcal{N} \int_0^\infty ds \frac{\sin(2\gamma_1 s)}{s} e^{-2\pi\mathcal{N}s} = 2\pi\mathcal{N} \operatorname{arccot}\left(\frac{\pi\mathcal{N}}{\gamma_1}\right) \end{aligned} \quad (52)$$

or, equivalently,

$$\begin{aligned} \gamma_2 &= |k_c| \theta_c \cos(\theta_c/2) \\ &= \sqrt{2(\sqrt{E^2 + c^2} + E)} \operatorname{arccot}\left(\sqrt{\frac{\sqrt{E^2 + c^2} + E}{\sqrt{E^2 + c^2} - E}}\right). \end{aligned} \quad (53)$$

The formulas Eqs. (50) and (53) make it easy to compare the limiting behaviors of the first two cumulants as $E \rightarrow +\infty$ (then $\theta_c \rightarrow 0$). We get

$$\gamma_2 \simeq 2\gamma_1 \quad \text{for } E \gg c. \quad (54)$$

We close the section with some remarks:

(1) The asymptotic behavior of the IDoS for $E \rightarrow -\infty$ and the decay of the Lyapunov exponent for $E \rightarrow +\infty$ are quite different from the ones obtained in the more standard case where the second moment of the disordered potential is finite, usually leading to Lifshitz tails and a faster power-law decay for the Lyapunov exponent $\gamma_1 \sim 1/E$ [12,15,82]. The origin of the unusual power-law decay was already identified in Ref. [83] for a power-law disorder characterized by $p(v) \sim |v|^{-1-\alpha}$ for $v \rightarrow \pm\infty$ with $\alpha \in]0, 2[$: as was made clear in that paper, the increase of $\ln|\psi(x)|$ due to an impurity is $\sim \ln|v_n/\sqrt{E}|$, so that $\ln|\psi(x)|$ obeys a generalized central limit theorem when the second moment of $\ln|v_n|$ is finite. However, the signature of the power-law disorder can be seen in the energy decay of the Lyapunov exponent, namely, $\gamma_1 \sim E^{-\alpha/2}$ for $\alpha \in]0, 2[$.

(2) The relation Eq. (54) was obtained for the discrete tight-binding model with Cauchy disorder in Refs. [51–53]. It is a manifestation of the “single parameter scaling” (SPS) property, i.e., the fact that the distribution of $\ln|\psi(x)|$ (or the distribution of the conductance of a disordered slice) is controlled by a unique scale. Cauchy disorder is responsible for an additional factor of 2 compared to the standard case of disorder with finite second moment. A broader perspective was given in Ref. [69] where the factor 2 was related to the exponent of the tail of the disorder distribution $p(v) \sim |v|^{-1-\alpha}$ (or equivalently, the exponent controlling the Lévy exponent $\mathcal{L}(s) \sim |s|^\alpha$ for $s \rightarrow 0$): Eq. (54) then generalizes to $\gamma_2 \simeq (2/\alpha)\gamma_1$. Furthermore, the relation was shown to be a particular case of a more general relation between even and

odd cumulants in the case of power-law disorder

$$\gamma_{2m} \simeq \frac{2m}{\alpha} \gamma_{2m-1} \quad \text{for } E \gg \text{disorder strength} \quad (55)$$

with $m \in \mathbb{N}^*$. The model that we study in this paper corresponds to the case $\alpha = 1$, and, by computing the cumulants explicitly, we shall verify in due course that this property does indeed hold for our model.

IV. BEYOND PERTURBATIONS: EXPLICIT FORM OF THE SECULAR EQUATION

As explained in Sec. II, our approach is to reduce the spectral problem to the solution of the secular Eq. (38), where the coefficients $\Omega_+(\Lambda)$ and $\Omega_-(\Lambda)$ are to be found by computing $\phi_+(s; \Lambda)$ and $\phi_-(s; \Lambda)$, respectively. For the particular Lévy exponent Eqs. (10) and (33) reads

$$\left[-\frac{d^2}{ds^2} + \frac{q}{s} \frac{d}{ds} + E \pm ic + \frac{i\Lambda}{s}\right] \phi_\pm(s; \Lambda) = 0 \quad \text{for } \pm s > 0. \quad (56)$$

We begin by considering the “+” case: a simple substitution reduces it to the confluent hypergeometric equation or alternatively, to the Whittaker equation [84]. The solution that decays at $+\infty$ is

$$\begin{aligned} \phi_+(s; \Lambda) &= a (2k_c s)^{q/2} W_{-i\Lambda/(2k_c), (q+1)/2}(2k_c s) \\ &= a (2k_c s)^{q+1} e^{-k_c s} \Psi\left(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1, q + 2; 2k_c s\right), \end{aligned} \quad (57)$$

where Ψ is the confluent hypergeometric (Kummer) function [84]. Assuming $q > -1$, we select a so that $\phi_+(0; \Lambda) = 1$. For $q = 0$, the identity $\Psi(1, 2; z) = 1/z$ leads to $\phi_+(s; 0) = \hat{f}(s) = e^{-k_c s}$, so this is consistent with the result obtained in the previous section.

To compute $\Omega_+(\Lambda)$, it is more convenient to rewrite ϕ_+ in terms of the regular Kummer function $\Phi(a, c; z) = \sum_{n=0}^\infty z^n (a)_n / [(c)_n n!]$. This is readily achieved by using Formula (9.210) of [84]; the result is

$$\begin{aligned} \phi_+(s; \Lambda) &= e^{-k_c s} \left[\Phi\left(\frac{i\Lambda}{2k_c} - \frac{q}{2}, -q; 2k_c s\right) \right. \\ &\quad + (2k_c s)^{q+1} \frac{\Gamma(-q-1) \Gamma(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1)}{\Gamma(q+1) \Gamma(\frac{i\Lambda}{2k_c} - \frac{q}{2})} \\ &\quad \left. \times \Phi\left(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1, q + 2; 2k_c s\right) \right]. \end{aligned} \quad (58)$$

This expression provides a concrete example of the general expansion Eq. (34). Upon inserting the MacLaurin expansion for Φ , we immediately deduce

$$\Omega_+(\Lambda) = (2k_c)^{q+1} \frac{\Gamma(-q-1) \Gamma(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1)}{\Gamma(q+1) \Gamma(\frac{i\Lambda}{2k_c} - \frac{q}{2})}. \quad (59)$$

As it is clear from Eq. (56), the equation for ϕ_- can be deduced from the one for ϕ_+ by changing the signs of $s, c,$

and Λ . Hence

$$\Omega_+(\Lambda) \xrightarrow{c \rightarrow -c} \Omega_-(\Lambda), \quad (60)$$

and this leads to the following explicit form of the secular Eq. (38):

$$k_c^{q+1} \frac{\Gamma\left(\frac{i\Lambda}{2k_c} + \frac{q}{2} + 1\right)}{\Gamma\left(\frac{i\Lambda}{2k_c} - \frac{q}{2}\right)} = (k_c^*)^{q+1} \frac{\Gamma\left(\frac{-i\Lambda}{2k_c^*} + \frac{q}{2} + 1\right)}{\Gamma\left(\frac{-i\Lambda}{2k_c^*} - \frac{q}{2}\right)}. \quad (61)$$

The fundamental result of this paper is that the roots of this transcendental equation yield the eigenvalues $\Lambda_n(q)$ of the operator \mathcal{L}_q .

A. A symmetry property of the GLE

The secular Eq. (61) exhibits an obvious symmetry: it is invariant under the transformation

$$q + 1 \rightarrow -q^* - 1 \quad \text{and} \quad \Lambda \rightarrow \Lambda^*. \quad (62)$$

This property implies a symmetry of the full spectrum of eigenvalues $\{\Lambda_n(q)\}_{n \in \mathbb{Z}} = \{\Lambda_n(-q^* - 2)\}_{n \in \mathbb{Z}}$. In fact, for a very natural ordering of the eigenvalues, the analysis to follow in Sec. VIII suggests the more precise relationship

$$\Lambda_n(q) = (\Lambda_{-n}(-q^* - 2))^*, \quad (63)$$

and we expect this relationship to hold for more general models [56,57]. For $n = 0$ and q real, it reduces to

$$\Lambda(q) = \Lambda(-q - 2). \quad (64)$$

In Ref. [85] it was argued that the symmetry property $\Lambda(q) = \Lambda(-q - 2m)$ should hold quite generally for products of random $2m \times 2m$ symplectic matrices. Its occurrence here comes from the fact that, as explained in the introduction, our model is formulated as a continuum limit of a product of matrices in the group $SL(2, \mathbb{R})$, which coincides with the 2×2 symplectic group $Sp(2, \mathbb{R})$. Some counterexamples, however, exist [56,86] and the precise conditions under which the relation Eq. (64) is true remain to be clarified.

V. EXACT EXPRESSIONS FOR THE FIRST FOUR CUMULANTS

It is clear from Eqs. (2) and (29) that the cumulants γ_n can in principle be found by treating q as a perturbation parameter. In the traditional approach, illustrated in Sec. III, one needs to keep track of the corresponding eigenfunction, and this usually results in formulas involving (multiple) integrals [54,66]. In this section, we apply the perturbative approach to the secular equation itself; the problem of computing the eigenvalues and that of computing the eigenfunctions are decoupled, and this produces formulas for the cumulants that are free of integrals.

We can simplify the analysis by using the fact that $\Lambda(q)$ is real for q real, so that the secular Eq. (61) can be expressed as $\text{Im}[\Omega_+(\Lambda)] = 0$, a form used in Refs. [56,69]. Here it yields

$$\text{Im} \left[\left(\frac{i\Lambda(q)}{q} - k_c \right) k_c^q \frac{\Gamma\left(1 + \frac{q}{2} + \frac{i\Lambda(q)}{2k_c}\right)}{\Gamma\left(1 - \frac{q}{2} + \frac{i\Lambda(q)}{2k_c}\right)} \right] = 0. \quad (65)$$

The idea is then to expand the left-hand side in powers of q . Equating the coefficient of the q^n term to zero then provides an equation for γ_{n+1} in terms of lower cumulants.

Now, for ξ independent of q , we can write

$$\begin{aligned} \frac{\Gamma\left(1 + \frac{q}{2} + \xi\right)}{\Gamma\left(1 - \frac{q}{2} + \xi\right)} &= 1 + \psi(1)q + \left(\psi'(1)q\xi + \frac{\psi(1)^2}{2}q^2 \right) \\ &+ \left\{ \frac{\psi''(1)}{2}q\xi^2 + \psi(1)\psi'(1)q^2\xi + \left[\frac{\psi''(1)}{24} + \frac{\psi(1)^3}{6} \right]q^3 \right\} \\ &+ O(q^4), \end{aligned} \quad (66)$$

where $\psi(z)$ here denotes the digamma function. We use $\psi(1) = -\mathbf{C}$, where $\mathbf{C} \simeq 0.577 \dots$ is the Euler-Mascheroni constant, $\psi'(1) = \zeta(2) = \pi^2/6$ and $\psi''(1) = \zeta(3)$, where $\zeta(x)$ is the Riemann zeta function. By setting $\xi = i\Lambda(q)/(2k_c)$ and expressing $\Lambda(q)$ in terms in the cumulants, we obtain the desired expansion for the left-hand side of Eq. (65) in powers of q . As is typical of such calculations, the complexity increases rapidly with n . The results for the first four cumulants are summarized below.

The Lyapunov exponent and the variance. The q^0 term of the secular Eq. (65) is obviously

$$\text{Im} [i\gamma_1 - k_c] = 0. \quad (67)$$

Using the fact that $i\gamma_1 - k_c = -\pi \mathcal{N}$ is real, we see that the q^1 term of Eq. (65) yields

$$\gamma_2 = \pi \mathcal{N} \text{Im} [2 \ln k_c]. \quad (68)$$

Since $2 \ln k_c = 2 \ln |k_c| + i\theta_c$, these formulas for γ_1 and γ_2 agree with our previous calculations, which used the perturbative approach of [56,57]; see Eqs. (50) and (53).

Third cumulant: Calculation of the q^2 term leads to

$$\gamma_3 = \frac{\pi^2 |k_c|}{2} \cos^2 \left(\frac{\theta_c}{2} \right) \sin \left(\frac{\theta_c}{2} \right) = \frac{\pi^2 c}{4} \sqrt{\frac{\sqrt{E^2 + c^2} + E}{2(E^2 + c^2)}}. \quad (69)$$

In particular, $\gamma_3 \simeq (\pi^2/2)\gamma_1$ as $E \rightarrow +\infty$.

Fourth cumulant: For the q^3 term, some complicated algebra eventually leads to

$$\begin{aligned} \gamma_4 &= |k_c| \cos \left(\frac{\theta_c}{2} \right) \left[\pi^2 \theta_c \cos^2 \left(\frac{\theta_c}{2} \right) \right. \\ &\left. + \theta_c^3 - 6\zeta(3) \sin(\theta_c) \sin^2 \left(\frac{\theta_c}{2} \right) \right]. \end{aligned} \quad (70)$$

In particular, $\gamma_4 \simeq 2\pi^2 \gamma_1$ for $E \rightarrow +\infty$.

The first four cumulants are plotted against the energy E in Fig. 1.

We end by discussing briefly two limits:

(1) $E = +k^2 \rightarrow +\infty$: For large positive energy, our calculations yield

$$\gamma_1 \sim \gamma_2 \sim \gamma_3 \sim \gamma_4 \sim O(c/k) \quad (71)$$

and so the fluctuations are *non-Gaussian*. This is very different from the case of disorder with finite moments $\langle v_n^2 \rangle < \infty$ [i.e., Lévy exponent $\mathcal{L}(s) \sim s^2$ for $s \rightarrow 0$], for which we have $\gamma_n \ll \gamma_1 \simeq \gamma_2$ for $n > 2$ in the weak-disorder limit [54,69,82,87].

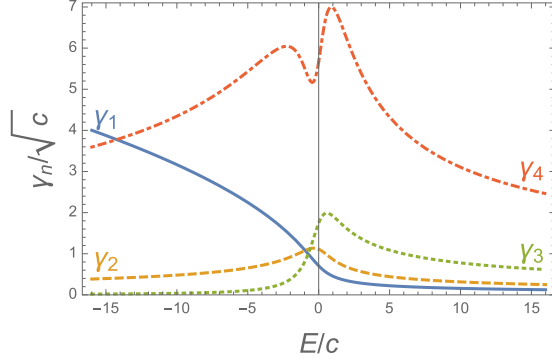


FIG. 1. The four first cumulants as a function of the energy.

(2) $E = -k^2 \rightarrow -\infty$: For large negative energy, we have instead $\gamma_1 \simeq \sqrt{|E|}$, $\gamma_2 \simeq (\pi/2)c|E|^{-1/2}$, $\gamma_3 \simeq (\pi^2/8)c^2|E|^{-3/2}$, and $\gamma_4 \simeq (\pi^3/2)c|E|^{-1/2}$. Thus

$$\gamma_1 - \sqrt{|E|} \sim \gamma_3 \sim O(c^2/k^3) \ll \gamma_2 \sim \gamma_4 \sim O(c/k). \quad (72)$$

In this limit, the fluctuations are symmetric about the mean, but again non-Gaussian.

VI. THE UNIVERSAL ($E/c \rightarrow +\infty$) REGIME

A. Explicit limiting form of the GLE

In the high-energy regime, the previous analysis suggests that the cumulants all scale in the same way with the energy $\gamma_n \sim \gamma_1 \simeq c/(2\sqrt{E})$. This motivates the introduction of the dimensionless rescaled GLE

$$\lambda^+(q) \stackrel{\text{def}}{=} \lim_{E/c \rightarrow +\infty} \frac{\Lambda(q)}{\gamma_1} \quad (73)$$

so that $\Lambda(q) \simeq \lambda^+(q)c/(2\sqrt{E})$, and of the dimensionless parameter $\epsilon = c/E$. In terms of these new variables, the secular Eq. (65) takes the form

$$\text{Im} \left[\left(\frac{i\epsilon \lambda^+(q)}{2} - \sqrt{1+i\epsilon} \right) (1+i\epsilon)^{q/2} \frac{\Gamma(1 + \frac{q}{2} + \frac{i\epsilon \lambda^+(q)}{4\sqrt{1+i\epsilon}})}{\Gamma(1 - \frac{q}{2} + \frac{i\epsilon \lambda^+(q)}{4\sqrt{1+i\epsilon}})} \right] = 0. \quad (74)$$

In the previous section, we found expressions for the cumulants γ_n by expanding this equation in powers of q , and we ascertained their high-energy behavior by taking the limit $\epsilon = c/E \rightarrow 0$ in the resulting expressions. Here, in the spirit of [69], we reverse the order of the two limits: we first let $\epsilon \rightarrow 0$ and obtain an expression for the GLE in the universal high-energy regime; in the next subsection, we use the expression obtained to deduce the asymptotics of the cumulants in this regime.

To proceed, we remark that Eq. (74) can be expanded in powers of ϵ by using $\Gamma(1 \pm q/2 + \xi) = \Gamma(1 \pm q/2) [1 + \xi \psi(1 \pm q/2) + O(\xi^2)]$. The ϵ^0 term of the secular equation yields

$$\lambda^+(q) = \frac{2(1+q)}{\psi(1 - \frac{q}{2}) - \psi(\frac{q}{2})}. \quad (75)$$

This can be expressed in terms of elementary functions by using the identity

$$\psi(1-z) - \psi(z) = \pi \cotg \pi z, \quad (76)$$

which follows from the complement formula $\Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z)$. The upshot is

$$\lambda^+(q) = \frac{2}{\pi} (1+q) \tan\left(\frac{\pi q}{2}\right). \quad (77)$$

In [69], this formula was obtained as a limiting case of a more general formula for power-law disorder, itself derived by a perturbative method in the weak-disorder limit. Here we followed a different route and recovered this expression from our exact secular Eq. (61) corresponding to Cauchy disorder. We observe that the right-hand side blows up when q is an odd integer. So the largest interval containing 0 for which this formula makes sense is

$$q \in]-3, 1[. \quad (78)$$

We see in particular that the GLE is positive for $q \in]-3, -2[\cup]0, 1[$ —implying an exponential *growth* of the moment $\langle |\psi(x)|^q \rangle$ with x and negative for $q \in]-2, 0[$, corresponding to an exponential *decay* of the moment.

The most striking feature of our formula is obviously the fact that the GLE—equivalently the moment $\langle |\psi(x)|^q \rangle$ —blows up at the endpoints of the finite interval $]-3, 1[$; see Fig. 3 below. This behavior, characteristic of power-law disorder, is in sharp contrast with that observed in the case of the Halperin model, where the potential is a Gaussian white noise corresponding to the Lévy exponent $\mathcal{L}(s) = (\sigma/2)s^2$. In the Halperin model, the GLE exists for every q , and it was shown in Refs. [56,58] that it behaves asymptotically like $\Lambda^{(\text{Halp})}(q) \simeq (3/4)(\sigma/2)^{1/3} |q|^{4/3}$ as $q \rightarrow \pm\infty$. This behavior was identified earlier by Bouchaud *et al.* [88], who made use of the replica trick.

The simple analytic formula Eq. (77) is another important result of this paper. It will enable us to derive a general formula for the cumulants, in the high-energy limit, and to deduce the large deviation function controlling the distribution of the wave function. Furthermore, the formula provides a limiting form of the GLE that is expected to be *universally* valid in the weak-disorder and high-energy limit for models in which the disorder is characterized by the power-law tail $p(V) \sim V^{-2}$ for $V \rightarrow \pm\infty$ [89]. For instance, we shall see in the next section that it correctly reproduces the few results concerning the cumulants that are known for the Lloyd lattice model.

B. Cumulants in the universal regime

The rescaled GLE

$$\lambda^+(q) = \sum_{n=1}^{\infty} \frac{\kappa_n^+}{n!} q^n \quad (79)$$

is the generating function of the rescaled cumulants κ_n^+ ; they are related to the γ_n via

$$\gamma_n \simeq \kappa_n^+ \gamma_1 \simeq \frac{c}{2\sqrt{E}} \kappa_n^+ \quad (80)$$

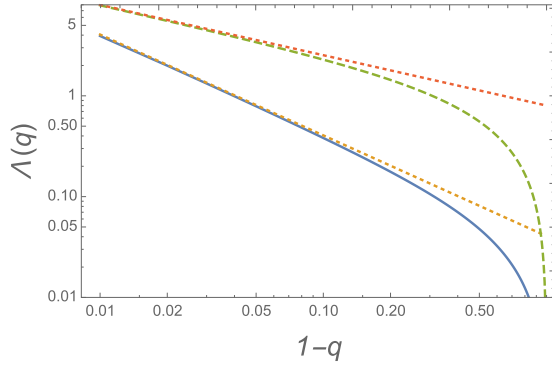


FIG. 2. Divergence of the GLE as $q \rightarrow 1^-$, for $k = 10$ (blue continuous line) and $k = 0$ (green dashed line). The dotted lines are fits with $\Lambda(q) \simeq 4c/[\pi^2 k(1-q)]$ (orange) and $\Lambda(q) \simeq 0.8\sqrt{c/(1-q)}$ (red).

for $E/c \rightarrow +\infty$. The MacLaurin expansion of the tangent function involves the Bernoulli numbers [84]:

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1}. \quad (81)$$

Thus

$$\kappa_n^+ = \begin{cases} 4\pi^{n-2}(2^n-1)|B_n| & \text{for } n \text{ even} \\ \frac{\kappa_{n+1}^+}{n+1} & \text{for } n \text{ odd} \end{cases} \quad (82)$$

($\kappa_1^+ = 1$ by definition). Using $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, etc., we recover the relation $\kappa_2^+ = 2 \simeq \gamma_2/\gamma_1$, i.e., Eq. (54), first obtained in Refs. [51,52]. For the third, fourth, and fifth cumulants, Titov and Schomerus [53] found the numerical estimates $\kappa_3^+ \simeq 5$, $\kappa_4^+ \simeq 20$, and $\kappa_5^+ \simeq 100$. Equation (82) gives $\kappa_3^+ = \pi^2/2 \simeq 4.93$, $\kappa_4^+ = 2\pi^2 \simeq 19.7$, and $\kappa_5^+ = \pi^4 \simeq 97.4$ in agreement with these estimates.

The large- n behavior of the cumulants can also be obtained by using Formula (24.11.1) of [91] $B_{2n} \sim (-1)^{n+1} 2(2n)! (2\pi)^{-2n}$, leading to the asymptotics

$$\kappa_n^+ \underset{n \rightarrow \infty}{\simeq} \frac{8n!}{\pi^2}. \quad (83)$$

The series Eq. (79) has a unit radius of convergence. This is related to the fact that the distribution of $\Upsilon(x) = \ln |\psi(x)|$ has the exponential tail $e^{-\Upsilon}$ as $\Upsilon \rightarrow +\infty$ for Cauchy disorder.

VII. BEYOND THE UNIVERSAL REGIME

A. The limit $E \rightarrow 0$

The study of this limit is more involved; here, we content ourselves with a brief discussion of some results obtained by solving the secular equation Eq. (65) numerically. These results suggest that the nature of the singularity at $q \rightarrow 1^-$ [and thus, due to the symmetry (64), also as $q \rightarrow 3^+$] depends on whether E is large or small. For $E = k^2 \gg c$, one verifies that the behavior $\Lambda(q) \simeq 4c/[\pi^2 k(1-q)]$ as $q \rightarrow 1^-$ —which follows easily from Eq. (77)—is confirmed by the numerical analysis (see Fig. 2). The nature of the singularity changes when $k = 0$: the numerics (Fig. 2) suggest the

behavior

$$\Lambda(q) \underset{q \rightarrow 1^-}{\simeq} \sqrt{\frac{c}{1-q}} \quad \text{for } k = 0 \quad (84)$$

and an analogous behavior in the limit $q \rightarrow -3^+$. We do not yet have an analytical proof for this limiting behavior.

B. The $E/c \rightarrow -\infty$ regime

The limit of large negative energy is also of interest. The fact that, in this limit, $\gamma_4 \simeq \pi^2 \gamma_2 \gg \gamma_1 - \sqrt{|E|} \sim \gamma_3$, suggests the rescaling

$$\lambda^-(q) \stackrel{\text{def}}{=} \lim_{E/c \rightarrow -\infty} \frac{\Lambda(q) - q\gamma_1}{\gamma_2} = \lim_{E/c \rightarrow -\infty} \frac{\Lambda(q) - q\sqrt{|E|}}{\pi c / (2\sqrt{|E|})}. \quad (85)$$

Using $k_c = i\sqrt{|E|}\sqrt{1-i\epsilon}$, where $\epsilon = c/|E| \ll 1$, some simple algebra shows that the secular Eq. (65) assumes the form

$$\begin{aligned} & \text{Re} \left[i^q \left(1 - \sqrt{1-i\epsilon} + \epsilon \frac{\pi \lambda^-(q)}{2q} \right) (1-i\epsilon)^{\frac{q}{2}} \right. \\ & \times \left. \frac{\Gamma\left(1 + \frac{q}{2} + \frac{q+(\epsilon\pi/2)\lambda^-(q)}{2\sqrt{1-i\epsilon}}\right)}{\Gamma\left(1 - \frac{q}{2} + \frac{q+(\epsilon\pi/2)\lambda^-(q)}{2\sqrt{1-i\epsilon}}\right)} \right] = 0. \end{aligned} \quad (86)$$

In the limit of small ϵ , this yields

$$\lambda^-(q) = \frac{q}{\pi} \tan\left(\frac{\pi q}{2}\right) \quad \text{for } q \in]-1, 1[. \quad (87)$$

The interval of validity of this formula prompts us to make the following remark: expressed in terms of the rescaled GLE, the symmetry relation Eq. (64) reads

$$\lambda^-(-q-2) \simeq \lambda^-(q) + \frac{4|E|}{\pi c} (q+1). \quad (88)$$

Now, Eq. (87) says that $\lambda_-(q)$ is finite for $q \in]-1, +1[$. It follows from the symmetry relation that, for $q \in]-3, -1[$, $\lambda_-(q)$ cannot have a limit as $|E|/c \rightarrow \infty$.

To end our discussion of this regime, we note that the rescaled cumulants κ_n^- in the expansion

$$\lambda^-(q) = \sum_{n=1}^{\infty} \frac{\kappa_n^-}{n!} q^n \quad (89)$$

are easily computed. Indeed, the rescaled cumulants are related to the original ones by $\gamma_n \simeq \gamma_2 \kappa_n^-$ for $E/c \rightarrow -\infty$. Using, as before, the MacLaurin expansion of the tangent function, we deduce

$$\kappa_n^- = \begin{cases} 2\pi^{n-2}(2^n-1)|B_n| & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}. \quad (90)$$

In particular, we see that $\kappa_2^- = 1$ and $\kappa_4^- = \pi^2$, as they should, given the results obtained at the end of Sec. V. We also note the identity

$$\kappa_n^+ = 2\kappa_n^- \quad (n \text{ even}) \quad (91)$$

relating the cumulants' behavior as $E \rightarrow +\infty$ to their behavior as $E \rightarrow -\infty$.

VIII. THE FULL SPECTRUM OF EIGENVALUES AND THE SPECTRAL GAP

The study of the full spectrum of the operator \mathcal{L}_q is interesting as it controls the spectral representation of the “propagator” Eq. (17)

$$\mathcal{P}_x(z|z_0; q) = \sum_n \Phi_n^R(z; q) \Phi_n^L(z_0; q) e^{x \Lambda_n(q)}, \quad (92)$$

where $\Phi_n^L(z; q)$ is the left-eigenvector associated with the problem adjoint to Eq. (29). The most important feature in this respect is the existence of a spectral gap

$$\Delta(q) = \text{Re}[\Lambda_0(q) - \Lambda_1(q)] > 0. \quad (93)$$

This is a crucial requirement in our approach: the “propagator” then behaves like

$$\mathcal{P}_x(z|z_0; q) \simeq e^{x \Lambda_0(q)} [\Phi_0^R(z; q) \Phi_0^L(z_0; q) + O(e^{-x \Delta(q)})] \quad (94)$$

in the limit $x \rightarrow \infty$, and Eq. (22), which equates the GLE with the leading eigenvalue, is then justified.

A. The spectrum for $q = 0$

For this simple case, we find that the solution set of the secular Eq. (61) consists of the complex numbers

$$\Lambda_n(0) = \begin{cases} -2ik_c^* n & \text{for } n \geq 0 \\ -2ik_c n & \text{for } n < 0 \end{cases}. \quad (95)$$

The fact that $\Lambda_{-n}(0) = \Lambda_n(0)^*$ shows that the eigenvectors $\Phi_n^R(z; 0)$ and $\Phi_{-n}^R(z; 0)$ form a complex conjugate pair. The

only real eigenvalue is $\Lambda_0(0) = 0$. The gap is

$$\Delta(0) = 2|k_c| \sin\left(\frac{\theta_c}{2}\right). \quad (96)$$

By definition, θ_c is the argument of $E + ic$ and so the gap is strictly positive since $c > 0$.

B. The universal regime

In this regime, we can look for solutions of the secular Eq. (61) in powers of $\epsilon = c/(2E)$:

$$\Lambda_n = \Lambda_n^{(0)} + \frac{c}{2k} \ell_n + O(c^2/k^3), \quad (97)$$

where $E = k^2$ and n labels the different solutions, i.e., the different eigenvalues of \mathcal{L}_q . At lowest order ϵ^0 (i.e., setting $c = 0$), Eq. (61) reduces to

$$\frac{\Gamma(1 + \frac{i\Lambda}{2k} + \frac{q}{2})}{\Gamma(\frac{i\Lambda}{2k} - \frac{q}{2})} = \frac{\Gamma(1 - \frac{i\Lambda}{2k} + \frac{q}{2})}{\Gamma(-\frac{i\Lambda}{2k} - \frac{q}{2})}. \quad (98)$$

Hence

$$\sin \pi \left(\frac{q}{2} - \frac{i\Lambda}{2k} \right) = \sin \pi \left(\frac{q}{2} + \frac{i\Lambda}{2k} \right) \quad (99)$$

and the solutions of this equation are

$$\Lambda_n^{(0)} = -2in k \quad \text{with } n \in \mathbb{Z}. \quad (100)$$

As discussed in Sec. II B, these are the eigenvalues of the operator $\mathcal{D}_K(q) = \partial_z(k^2 + z^2) + qz$ appearing in Eq. (26) (see Refs. [56], Appendix B, and [57]).

We can refine our calculation by inserting the Eq. (97) in the secular Eq. (61):

$$\begin{aligned} & (1 + i\epsilon)^{\frac{q+1}{2}} \frac{\Gamma(1 + \frac{q}{2} + n) (1 + i\epsilon[-\frac{n}{2} + \frac{\ell_n}{4}]) \psi(1 + \frac{q}{2} + n) + \dots}{\Gamma(-\frac{q}{2} + n) (1 + i\epsilon[-\frac{n}{2} + \frac{\ell_n}{4}]) \psi(-\frac{q}{2} + n) + \dots} \\ & = (1 - i\epsilon)^{\frac{q+1}{2}} \frac{\Gamma(1 + \frac{q}{2} - n) (1 - i\epsilon[\frac{n}{2} + \frac{\ell_n}{4}]) \psi(1 + \frac{q}{2} - n) + \dots}{\Gamma(-\frac{q}{2} - n) (1 - i\epsilon[\frac{n}{2} + \frac{\ell_n}{4}]) \psi(-\frac{q}{2} - n) + \dots}. \end{aligned} \quad (101)$$

If we retain only the terms of order ϵ^1 , we deduce the following equation for ℓ_n :

$$\begin{aligned} & q + 1 + \frac{n}{2} \left[-\psi\left(1 + \frac{q}{2} + n\right) + \psi\left(-\frac{q}{2} + n\right) + \psi\left(1 + \frac{q}{2} - n\right) - \psi\left(-\frac{q}{2} - n\right) \right] \\ & = \ell_n \left[-\psi\left(1 + \frac{q}{2} + n\right) + \psi\left(-\frac{q}{2} + n\right) - \psi\left(1 + \frac{q}{2} - n\right) + \psi\left(-\frac{q}{2} - n\right) \right]. \end{aligned} \quad (102)$$

To simplify the resulting formula, consider first the case $n > 0$. By using Eq. (76) and also

$$\psi(z + n) - \psi(z - n) = (2z - 1) \sum_{r=1}^n \frac{1}{(z + r - 1)(z - r)}, \quad (103)$$

we eventually find

$$\ell_n = \frac{2}{\pi} (q + 1) \tan(\pi q/2) \left[1 - n \sum_{r=1}^n \frac{1}{(\frac{q}{2} + r)(\frac{q}{2} + 1 - r)} \right]. \quad (104)$$

For $n < 0$, a similar simplification can be achieved; we omit the details. The upshot is the following formula for the

eigenvalues, with an error of order $O(\epsilon^2)$:

$$\begin{aligned} \Lambda_n(q) & \simeq -2in k + \frac{c}{\pi k} (q + 1) \tan(\pi q/2) \\ & \times \left[1 - |n| \sum_{r=1}^{|n|} \frac{1}{(\frac{q}{2} + r)(\frac{q}{2} + 1 - r)} \right] \end{aligned} \quad (105)$$

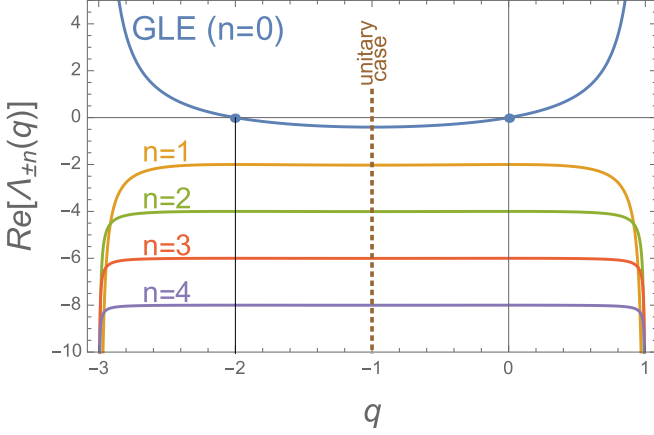


FIG. 3. The first five eigenvalues for $k = 10$ (weak-disorder and high-energy universal regime).

with $n \in \mathbb{Z}$. As a check, we consider the limit $q \rightarrow 0$: the expression Eq. (105) become $\Lambda_n(0) \simeq -2in k - (c/k)|n|$, in agreement with Eq. (95). Equation (105) obviously satisfies the symmetry Eq. (63). At the symmetry point $q = -1$, we get

$$\Lambda_n(-1) \simeq -2in k - \frac{2c}{\pi^2 k} \left[1 + |n| \sum_{r=1}^{|n|} \frac{1}{(r-1/2)^2} \right]. \quad (106)$$

In Fig. 3 we have plotted the right-hand side of Eq. (105) against q for $0 \leq n \leq 4$. All the eigenvalues, except for the leading one, have a negative real part for every q . The gap is

$$\Delta(q) \simeq \frac{c}{k\pi} \frac{4(q+1) \tan(\pi q/2)}{q(q+2)} > 0, \quad (107)$$

and this is obviously positive in the interval $q \in]-3, 1[$.

C. Unitary case ($q = -1$)

As mentioned at the end of Sec. II B, the case $q = -1$ is of special significance because it corresponds to a case where the representation of $\text{SL}(2, \mathbb{R})$ underlying the spectral problem can be made unitary. It is also the symmetry point of the relation discussed in Sec. IV A.

When we put $q = -1$ in the secular Eq. (61), it becomes a trivial identity. In order to extract some information, we set $q = -1 + \epsilon$ and expand the equation in powers of ϵ . At order ϵ^1 , we get

$$\psi\left(\frac{1}{2} + \frac{i\Lambda}{2k_c}\right) - \psi\left(\frac{1}{2} - \frac{i\Lambda}{2k_c^*}\right) = -i \arctan(c/k^2), \quad (108)$$

where we have used $\ln k_c - \ln k_c^* = i\theta_c$. The secular equation thus takes a simpler form in this case. For $c \rightarrow 0$, the ratio $\Lambda_0/k_c \simeq \Lambda_0/k$ is almost real, so that we can use $\psi(1/2 + ix) - \psi(1/2 - ix) = i\pi \tanh(\pi x)$. We get $\Lambda_0(-1) \simeq -2c/(\pi^2 k)$ in agreement with the perturbative result, Eq. (106). For $n = 0$, the fact that $\Lambda \equiv \Lambda_0 \in \mathbb{R}$ makes Eq. (108) easier to analyze. In particular one readily finds the low-energy behavior $\Lambda(-1) \simeq a + bE$ as $E \rightarrow 0$ (the two coefficients can be determined numerically; the result is $a \simeq -0.312\sqrt{c}$ and $b \simeq 0.196/\sqrt{c}$). The GLE is plotted in Fig. 4.

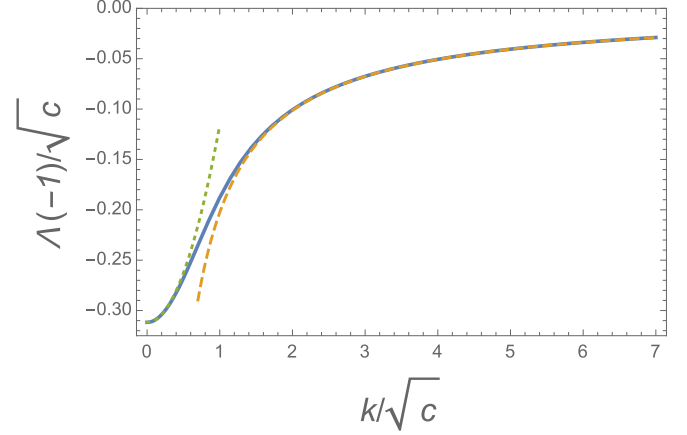


FIG. 4. $\Lambda(-1)$ as a function of $k = \sqrt{E}$. The orange dashed line is the perturbative result $\Lambda(-1) \simeq -2c/(\pi^2 k)$ and the green dotted line $\Lambda(-1) \simeq a + bk^2$.

IX. WAVE FUNCTION DISTRIBUTION AND LARGE DEVIATION FUNCTION

The GLE is the cumulant generating function for the logarithm of the wave function [90]

$$\Upsilon(x) = \ln |\psi(x)|. \quad (109)$$

We can therefore find its distribution by inverting a Laplace transform:

$$P_x(\Upsilon) \stackrel{\text{def}}{=} \langle \delta(\Upsilon - \ln |\psi(x)|) \rangle = \int_{-i\infty}^{+i\infty} \frac{dq}{2i\pi} e^{-q\Upsilon} \langle |\psi(x)|^q \rangle \\ \underset{x \rightarrow \infty}{\sim} \int_{-i\infty}^{+i\infty} \frac{dq}{2i\pi} e^{-q\Upsilon + x\Lambda(q)}. \quad (110)$$

This suggests the large deviation form

$$P_x(\Upsilon) \underset{x \rightarrow \infty}{\sim} \exp[-x F(\Upsilon/x)] \quad (111)$$

with

$$F(\xi) = \min_q [q\xi - \Lambda(q)]. \quad (112)$$

The symmetry relation Eq. (64) implies [56]

$$F(-\xi) = F(\xi) + 2\xi. \quad (113)$$

Note that the singularity of the GLE as $q \rightarrow 1^-$ corresponds to an exponential tail $P_x(\Upsilon) \sim e^{-\Upsilon}$. Indeed, assume a singularity of the form

$$\Lambda(q) \simeq \frac{A}{\eta-1} (q_0 - q)^{1-\eta} \quad \text{for } q \rightarrow q_0^-, \quad (114)$$

with $\eta > 1$. Then the minimum in Eq. (112) is attained at $q_* \simeq q_0 - (A/\xi)^{1/\eta}$, and so

$$F(\xi) \simeq q_0 \xi - \frac{\eta A^{1/\eta}}{\eta-1} \xi^{1-1/\eta} \quad \text{for } \xi \rightarrow +\infty. \quad (115)$$

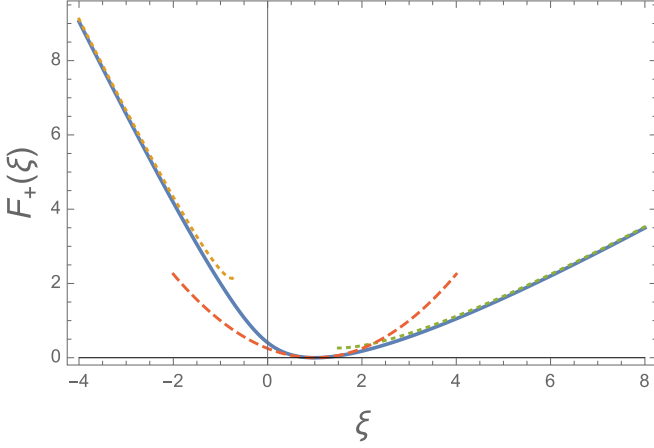


FIG. 5. The large deviation function controlling the distribution of $\ln |\psi(x)|$ in the regime $E \rightarrow +\infty$. The dotted lines correspond to the limiting behaviors discussed in the paper.

A. Universal weak-disorder regime ($E/c \rightarrow +\infty$)

Using the result $\Lambda(q) \simeq \gamma_1 \lambda^+(q)$, we expect the large deviation form

$$P_x(\Upsilon) \underset{x \rightarrow \infty}{\simeq} \exp \left[-\gamma_1 x F_+ \left(\frac{\Upsilon}{\gamma_1 x} \right) \right], \quad (116)$$

where

$$F_+(\xi) = \min_q [q \xi - \lambda^+(q)] \quad (117)$$

is the dimensionless large deviation function. The minimum is attained at q_* , the solution of

$$\xi = \frac{2}{\pi} \tan(\pi q_*/2) + \frac{q_* + 1}{\cos^2(\pi q_*/2)}. \quad (118)$$

Hence

$$F_+(\xi) = \left(\frac{q_* + 1}{\cos(\pi q_*/2)} \right)^2 - \xi \quad (119)$$

$$= \frac{q_*(q_* + 1)}{\cos^2(\pi q_*/2)} - \frac{2}{\pi} \tan(\pi q_*/2). \quad (120)$$

The function $F_+(\xi)$ is plotted in Fig. 5.

1. Typical values

In the limit $q_* \rightarrow 0$, we can write

$$\xi = 1 + 2q_* + \frac{\pi^2}{4} q_*^2 + O(q_*^3) \quad (121)$$

and

$$\begin{aligned} F_+(\xi) &= 1 + 2q_* + q_*^2 + \frac{\pi^2}{4} q_*^2 + \dots - \xi \\ &= q_*^2 + \dots \simeq \frac{1}{4} (\xi - 1)^2. \end{aligned} \quad (122)$$

We recover the expected result for the variance, namely, $\kappa_2^+ = 2$.

2. Large deviations

Setting $q = 1 - \epsilon$, some algebra gives the small- ϵ asymptotics

$$\xi = \frac{8}{\pi^2 \epsilon^2} + \frac{2}{3} + O(\epsilon), \quad (123)$$

from which one deduces $\epsilon \simeq \frac{4}{\pi} / \sqrt{2(\xi - 2/3)}$. Thus

$$F_+(\xi) \underset{\xi \rightarrow +\infty}{\simeq} \frac{8}{\pi^2 \epsilon^2} - \frac{16}{\pi^2 \epsilon} + \frac{2}{3} + \frac{4}{\pi^2} + O(\epsilon) \quad (124)$$

$$\simeq \xi - \frac{4}{\pi} \sqrt{2(\xi - 2/3)} + \frac{4}{\pi^2}. \quad (125)$$

The tail associated with the limit $\xi \rightarrow -\infty$ can be analyzed in a similar way by setting $q = -3 + \epsilon$.

3. Summary

The three limiting behaviors are

$$F_+(\xi) \simeq \begin{cases} -3\xi - \frac{4}{\pi} \sqrt{2(-\xi - 2/3)} + \frac{4}{\pi^2} & \text{for } \xi \rightarrow -\infty \\ \frac{1}{4} (\xi - 1)^2 & \text{for } \xi \sim 1 \\ \xi - \frac{4}{\pi} \sqrt{2(\xi - 2/3)} + \frac{4}{\pi^2} & \text{for } \xi \rightarrow +\infty \end{cases}. \quad (126)$$

These limiting behaviors are consistent with the symmetry Eq. (113).

The two behaviors $P_x(\Upsilon) \sim e^{+3\Upsilon}$ as $\Upsilon \rightarrow -\infty$ and $P_x(\Upsilon) \sim e^{-\Upsilon}$ as $\Upsilon \rightarrow +\infty$ are quite different from the ones obtained in Ref. [58] for the Halperin model with a Gaussian white noise potential. In this case the distribution is $P_x(\Upsilon) \sim \exp[-x F^{(\text{Halp})}(\Upsilon/x)]$ where $F^{(\text{Halp})}(\xi)$ is the Legendre transform of the GLE of the model; with the limiting behavior recalled at the end of Sec. VIA for $\Lambda^{(\text{Halp})}(q)$, one gets $F^{(\text{Halp})}(\xi) \simeq \xi^4/(8\sigma)$ as $\xi \rightarrow \pm\infty$; that is, $P_x^{(\text{Halp})}(\Upsilon) \sim \exp[-\Upsilon^4/(8\sigma x^3)]$ as $\Upsilon \rightarrow \pm\infty$.

B. Application: distribution of the conductance in the universal regime

We deduce from Eq. (126) the following limiting behaviors for the distribution of modulus of the wave function $\mathcal{P}_x(\psi) = \langle \delta(\psi - |\psi(x)|) \rangle = \frac{1}{\psi} P_x(\ln \psi)$:

$$\mathcal{P}_x(\psi) \sim \begin{cases} \psi^2 & \text{as } \psi \rightarrow 0 \\ \psi^{-1} e^{-(\ln \psi - \gamma_1 x)^2 / (4\gamma_1 x)} & \text{as } \ln \psi \sim \gamma_1 x. \\ \psi^{-2} & \text{as } \psi \rightarrow +\infty \end{cases} \quad (127)$$

Let us elaborate the implication of this last formula for the distribution of the conductance. According to the Borland conjecture [3], the probability g of transmission through a disordered sample of length L —the dimensionless conductance of the sample—should be related to the solution $\psi(x)$ of the initial value problem via $g \sim |\psi(L)|^{-2}$. This holds only for configurations with small transmission probability $g \ll 1$; it does not describe the atypical configurations where the transmission probability is large, $g \lesssim 1$. Equation (127) implies the following for the distribution $\mathcal{W}_L(g)$ of the conductance:

$$\mathcal{W}_L(g) \sim \begin{cases} g^{-1} e^{-(\ln g + 2\gamma_1 L)^2 / (16\gamma_1 L)} & \text{as } \ln g \sim -2\gamma_1 L. \\ g^{-1/2} & \text{as } g \rightarrow 0 \end{cases}. \quad (128)$$

The large deviation tail as $g \rightarrow 1^-$ requires a different analysis. The power-law behavior as $g \rightarrow 0$ was demonstrated numerically in Ref. [53], but the precise value of the exponent was not determined. The behavior Eq. (128) is in agreement with the numerical results of Mendez-Bermudez *et al.* [77] who conjectured the behavior $\mathcal{W}_L(g) \sim g^{-1+\alpha/2}$ for the power-law disorder $p(v) \sim |v|^{-1-\alpha}$. This behavior was later demonstrated for arbitrary α by an analytic calculation in Ref. [69]. The case of Cauchy disorder corresponds to taking $\alpha = 1$.

Although, in all cases, the typical values of the conductance are exponentially small, namely, $g \sim e^{-2\gamma_1 L}$, the power-law singularity of $\mathcal{W}_L(g)$ at $g = 0$ is completely different from that expected for the more standard case of disorder with finite moments. To give a concrete example, for the Halperin model, we can deduce from the large deviation function $F^{(\text{Halp})}(\xi)$ stated earlier that the conductance distribution exhibits *suppression* as $g \rightarrow 0$: $\mathcal{W}_L^{(\text{Halp})}(g) \sim (1/g) \exp[-(\ln g)^4 / (128\sigma L^3)]$. Although the precise behavior will depend on the details of the model, we expect such a suppression to be generic for disorder with finite second moment.

C. Large deviation function for $E/c \rightarrow -\infty$

In the limit $E \rightarrow -\infty$, the GLE assumes the form $\Lambda(q) \sim q\sqrt{|E|} + \gamma_2 \lambda^-(q)$. Hence we expect the large deviation form

$$P_x(\Upsilon) \underset{x \rightarrow \infty}{\sim} \exp\left[-\gamma_2 x F_-\left(\frac{\Upsilon - \sqrt{|E|x}}{\gamma_2 x}\right)\right], \quad (129)$$

where

$$F_-(\xi) = \min_q [q\xi - \lambda^-(q)]. \quad (130)$$

Applying the Legendre transform yields

$$\xi = \frac{1}{\pi} \tan\left(\frac{\pi q_*}{2}\right) + \frac{q_*}{2 \cos^2(\pi q_*/2)} \quad (131)$$

and

$$F_-(\xi) = \frac{1}{2} \left(\frac{q_*}{\cos(\pi q_*/2)}\right)^2. \quad (132)$$

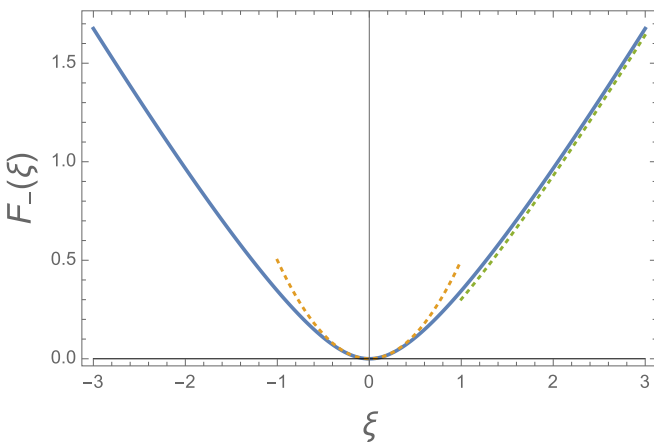


FIG. 6. The large deviation function controlling the distribution of $\ln |\psi(x)|$ in the regime $E \rightarrow -\infty$. The dotted lines correspond to the asymptotics discussed in the paper.

Some algebra then gives the limiting behaviors

$$F_-(\xi) \simeq \begin{cases} \frac{1}{2}\xi^2 & \text{for } |\xi| \ll 1 \\ \left|\xi\right| - \frac{2}{\pi}\sqrt{2|\xi|} + \frac{2}{\pi^2} + O(\xi^{-1/2}) & \text{for } |\xi| \gg 1 \end{cases} \quad (133)$$

This translates into $P_x(\Upsilon) \sim \exp[-|\Upsilon - \gamma_1 x|]$ for large Υ . The function $F_-(\xi)$ is plotted in Fig. 6.

X. THE SPECTRAL PROBLEM IN TERMS OF GENERALIZED COULOMB PROBLEMS

The spectral problem Eqs. (31)–(38) is unusual in many respects. The purpose of this section is to explain its relationship with some recent work on certain generalizations of the spectral problem for the Schrödinger equation with a Coulomb potential [92,93].

The calculation of the coefficients Ω_+ and Ω_- that appear in the secular equation relied on the explicit solution of Eq. (56). For definiteness, consider the “+” case and set $\phi_+(s; \Lambda) = s^{q/2} \chi_+(s)$, so that the equation for the new unknown χ_+ is

$$L_{m,\beta} \chi_+(s) = -k^2 \chi_+(s) \quad \text{for } s > 0, \quad (134)$$

where

$$L_{m,\beta} = -\frac{d^2}{ds^2} + \left(m^2 - \frac{1}{4}\right) \frac{1}{s^2} - \frac{\beta}{s}, \quad (135)$$

and, in order to conform to the notation used in Ref. [93], we have introduced

$$m = \frac{q+1}{2}, \quad \beta = -i\Lambda, \quad \text{and } k^2 = E + ic.$$

This has the same form as the differential equation for the radial part of the wave function describing a quantum particle in a Coulomb potential [94]. The classical example is the hydrogen atom, where the spectral problem of interest consists of finding the values of the energy k^2 such that χ_+ is square-integrable. However, the problem discussed in the present paper differs from the classical Coulomb problem in three respects:

(1) In the Coulomb problem, the point spectrum is the set $\{-k_n^2\}$ such that $\chi^+(s)$ is square-integrable. By contrast, in our case $k^2 = E + ic$ is just a given parameter, and the spectrum consists of the numbers Λ_n for which the operator $L_{m,\beta}$, subject to some boundary conditions, has $-k^2$ as one of its eigenvalues. Therefore, our problem can be described as a *spectral problem in the coupling constant*.

(2) Some of the coefficients—in particular the coupling constant $\beta = -i\Lambda$ —are complex, so that the problem cannot be treated within the standard framework of self-adjoint operator theory.

(3) A third difference is the fact that our problem is defined on the whole real line.

Nevertheless, it was shown recently in Refs. [92,93] that the familiar self-adjoint theory works almost as well in the complex case, provided the boundary condition at $s = 0$ is chosen from a certain family parametrized by the number κ . In particular, for the so-called “holomorphic” family characterized by the triplet $\{\beta, m, \kappa\}$ with $\beta, \kappa \in \mathbb{C}$ and $-1 < \text{Re}(m) < 1$, the operator $L_{m,\beta}$, supplemented with the

boundary condition

$$\chi^+(s) \sim s^{1/2+m} \left(1 - \beta \frac{s}{1+2m}\right) + \kappa s^{1/2-m} \left(1 - \beta \frac{s}{1-2m}\right) \text{ for } s \rightarrow 0^+ \quad (136)$$

is a closed operator in the space of square-integrable functions on the positive half-line. Furthermore, it is shown in Ref. [93] that the spectral problem for this operator is well posed, with a complex point spectrum that coincides with the solution set of the transcendental equation

$$\kappa = (2k)^{-2m} \frac{\Gamma(2m) \Gamma[1/2 - m - \beta/(2k)]}{\Gamma(-2m) \Gamma[1/2 + m - \beta/(2k)]}. \quad (137)$$

In other words, for a fixed triplet $\{\beta, m, \kappa\}$, the eigenvalues k^2 of the operator correspond to the values of k that solve this equation.

The relevance of this theory to the calculation of the coefficient Ω_+ becomes clear when, upon comparing Eq. (136) and (34), we realize that $\kappa = 1/\Omega_+$. By requiring that ϕ_+ decay at infinity, we are effectively imposing that $E + ic$ must be an eigenvalue of $L_{m,\beta}$, for the boundary condition Eq. (136). So Formula Eq. (59) could have been deduced from Formula Eq. (137). Similar considerations apply to the coefficient Ω_- .

We end by pointing out another interesting interpretation of the coefficient Ω_{\pm} in terms of the *Weyl-Titchmarsh coefficient* associated with a certain singular Sturm-Liouville operator on a half-line [95]. Recall that, for the operator $H = -\partial_x^2 + V(x)$, with a potential that is well behaved at $x = 0$, we may express any solution of $H\psi = \lambda\psi$ as a linear combination of two particular solutions, say, $f(x, \lambda)$ and $g(x, \lambda)$, satisfying

$$f(0, \lambda) = g'(0, \lambda) = 0 \quad \text{and} \quad g(0, \lambda) = -f'(0, \lambda) = 1. \quad (138)$$

By definition, the Weyl-Titchmarsh coefficient is the unique number $w(\lambda)$ that makes the linear combination

$$\varphi(x, \lambda) = g(x, \lambda) - w(\lambda)f(x, \lambda) \quad (139)$$

square-integrable. When, as in the Coulomb case, the potential $V(x)$ is singular, a Weyl-Titchmarsh coefficient may still be defined, provided we adjust the conditions that $f(x, \lambda)$ and $g(x, \lambda)$ must satisfy at $x = 0$ appropriately. The Weyl-Titchmarsh coefficient $w(\lambda)$ computed in Ref. [95] is essentially the coefficient κ defined in Eq. (137). It is also clear that our analysis of the local behavior of the solution of Eq. (33), valid for a generic Lévy process, is also consistent with the interpretation of Ω_{\pm} in such terms, so that the secular Eq. (38) can be understood as the equality between two Weyl-Titchmarsh coefficients: one associated with the “+” part of Eq. (56), and the other associated with the “−” part. This interpretation could shed some light on the spectral properties of the operator \mathcal{L}_q .

XI. CONCLUSION

In this paper, we have addressed the question of wave function fluctuations for the one-dimensional Schrödinger equation with a random potential. We studied in great detail a case where the random potential is a Lévy noise, and the underlying Lévy process is the Cauchy process—resulting

in a model with power-law disorder. By building on recent progress in our understanding of the role of representation theory in the study of products of random matrices in $SL(2, \mathbb{R})$, we have been able to derive an explicit transcendental equation for the cumulant generating function—also known as the generalized Lyapunov exponent (GLE)—that describes the fluctuations. This led to analytical expressions for the first few cumulants. We have also obtained the limiting form of the GLE and of all the cumulants in the high-energy and weak-disorder regime. Considering the scarcity of exact results in this area, we view these findings as a significant progress.

Although much of the paper has been devoted to a particular model, our results have wider implications for the “single parameter scaling” (SPS) conjecture. In the more standard case where the moments of the disorder are finite, SPS manifests itself in the fact that $\gamma_1 \simeq \gamma_2 \gg \gamma_n$ for $n > 2$, and so the large deviations (the atypical fluctuations) involve a scale that is different from that which controls the typical fluctuations. For example, for the Schrödinger equation with a Gaussian white noise potential (the Halperin model), with ψ the corresponding wave function, the mean value $\langle \ln |\psi(x)| \rangle \simeq \gamma_1 x$ and the typical fluctuations $(\ln |\psi(x)|)_{\text{typ.}} \sim \sqrt{\gamma_1 x}$ are both controlled by the Lyapunov exponent, while the large deviation that characterizes atypically large fluctuations yields $(\ln |\psi(x)|)_{\text{atyp.}} \sim (\sigma^{1/3} x)^{3/4}$. This result is in sharp contrast with the case of Cauchy disorder studied in the present paper: we have shown that all the cumulants scale in the same way with energy and disorder strength in the high-energy and weak-disorder regime: $\gamma_n \sim c/\sqrt{E}$. As a consequence both typical and atypical fluctuations are controlled by the same scale. This is a very strong manifestation of SPS, characteristic of models with power-law disorder [69]. Indeed, by studying our continuous model, we have recovered the few known results from previous studies of the well-known tight-binding Lloyd model. This is a strong confirmation of the universal character of our results.

Our study has focused on the case of Cauchy disorder. It would be interesting to find other cases that could be solved via a secular equation: for models where the noise arises from a Lévy process, this requires finding a Lévy exponent $\mathcal{L}(s)$ such that the Weyl-Titchmarsh coefficients associated with each of the differential equations in (33) can be computed explicitly.

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APPENDIX: RELATION BETWEEN (28) AND (37)

We start from Eq. (36), where $\phi(z; \Lambda)$ is bounded and decays as $\phi(z; \Lambda) \simeq A |z|^{-2-q}$ asymptotically. The first integral in Eq. (36) is a constant. We now introduce a large positive number, say, z_c , and write the second integral as a sum of two integrals: one over the interval $-z_c < z < z_c$ and the other over its complement. The first of these integrals clearly exhibits an analytic behavior $\int_{|z| < z_c} dz \phi(z; \Lambda) (1 - e^{-isz}) =$

$\alpha_1 s + \alpha_2 s^2 + \dots$ as $s \rightarrow 0$. As for the second of the integrals, we assume that z_c is so large that the power tail dominates in the integrand:

$$\begin{aligned} A \int_{|z|>z_c} dz (1 - e^{-isz}) |z|^{-2-q} \\ = A |s|^{q+1} \int_{|y|>z_c|s|} dy (1 - e^{-iy}) |y|^{-2-q}. \end{aligned} \quad (\text{A1})$$

This last integral is convergent for $q > -1$. Obviously, when $q < 1$ the integral over y has a limit as $s \rightarrow 0$. For finite s , the expansion of the integral in the right-hand side of Eq. (A1) produces the nonanalytic series $\beta_1 |s|^{q+1} + \beta_2 |s|^{q+2} + \dots$. This establishes the correspondence between the power-law tail of $\phi(z; \Lambda)$ as $z \rightarrow \pm\infty$ and the nonanalytic $|s|^{q+1}$ term in its Fourier transform. Such a relation, which characterizes Fourier reciprocity, is known as a ‘‘Tauberian theorem’’ in the mathematical literature.

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