Random walks with asymmetric time delays

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It is usually expected that time delays cause oscillations in dynamical systems—there might appear cycles around stationary points. Here we study random walks with asymmetric time delays. In our models, the probability of a walker to move to the right or to the left depends on the difference between two state-dependent fitness functions evaluated at two different times. We observe a different behavior—a dependence of the mean position of the walker on time delays. Moreover, the effect of time delays is reversed when one shifts (in a symmetric way) fitness functions. This is a joint effect of both stochasticity and time delays present in the system. The position of a random walker may be interpreted as a frequency of a given strategy in discrete replicator dynamics of evolutionary games. Then our results show that the effect of asymmetric time delays on the equilibrium structure of a population depends not only on time delays, but also on details of the fitness functions.

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I. INTRODUCTION

Many social and biological processes can be described by deterministic population dynamics [1,2]. It is usually assumed that interactions between individuals take place instantaneously and their effects are immediate. In reality, all processes take a certain amount of time. Results of biological interactions between individuals may appear in the future, and in social models, individuals or players may act, that is, choose appropriate strategies, on the basis of the information concerning events in the past. It is natural therefore to introduce time delays to describe such processes. It is well known that time delays may cause oscillations in dynamical systems [3–6]. One usually expects that interior equilibria of evolving populations—describing a coexistence of strategies or behaviors—are asymptotically stable for small time delays and above a critical time delay, where the Hopf bifurcation appears, they become unstable.

The effects of time delays in replicator dynamics describing the evolution of populations of individuals interacting through playing games [7] were discussed in [8–21] for games with an interior stable equilibrium (an evolutionarily stable strategy [22]). Recent models have studied strategy-dependent time delays. In particular, Moreira *et al.* [15] discussed a multiplayer Stag Hunt game with time delays, Khalifa *et al.* [19] investigated asymmetric games in interacting communities, and Wesson and Rand [16] studied Hopf bifurcations in two-strategy delayed replicator dynamics. A systematic analysis of two-player games with two strategies and strategy-dependent time delays is presented in [21]. A different behavior—the continuous dependence of equilibria on time delays—was observed.

In finite populations, one has to take into account stochastic fluctuations [2]. One of the simplest models involving both stochasticity and time delays is a delayed random walk [23,24]. In such a walk, transition probabilities depend on the position of the walker at some earlier time. In [23], a delayed random walk was considered, where, in the absence of delays, the transition toward the origin (a stable state) is more probable than the outward transition. The authors show that the mean square displacement of the walker, that is, the variance, approaches a stationary value in an oscillatory manner for large time delays and in a monotonic way for small ones. Moreover, the stationary value of the variance is a linear function of the delay and the coefficient of the proportionality is a linear function of the transition probability.

Here we consider a modification of the above model. Namely, transition probabilities depend on the difference between two functions (which can be interpreted as fitness functions in discrete replicator dynamics) evaluated at different times in the past. To simplify our model, we set one of the time delays to zero. We will use hyperbolic-tangent and step functions. In the absence of time delays and stochasticity, our models are discrete (in time and space) replicator-type dynamics of evolutionary games with an interior asymptotically stable stationary state.

It should be noted here that such dynamics are not Markovian. However, we may restore Markovianity if we consider transition probabilities not between states, but between histories of states. Such Markov chains are called higher-order Markov chains [25]. Although a stationary probability distribution is defined on histories, we will still be interested in

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the stationary probability of visiting particular states of the physical space. Our main object of study is the expected value of such a stationary probability distribution, its dependence on time delays, and other parameters of our models. We have performed stochastic simulations for hyperbolic-tangent and step-function models and we have derived analytical formulas in the step-function case for some small time delays.

We report a different behavior—the dependence of stationary probability distribution on time delays.

II. DETERMINISTIC DYNAMICS WITH TIME DELAYS

The state space S of the walker is the set of all integers. We introduce two fitness functions on S, i.e., f_A and f_B, such that f_A is a nonincreasing decreasing function of the position of the walker on S and f_B is a nondecreasing one, $f_A(x) = f_B(-x)$, and hence $f_A(0) = f_B(0)$, $f_A \geqslant f_B$ on $(-\infty, 0]$, and $f_A \leqslant f_B$ on $[0, \infty)$. Deterministic dynamics is given by the following rules: If $f_A[x(t-\tau)] - f_B[x(t)] > 0$, then the walker moves to the right at $t+1$; in the case of the reverse inequality, it moves to the left; and in the case of the equality, it stays at its current position. We have to specify the initial conditions. For systems with time delays, it is a history $\{x(-\tau), x(-\tau+1), \dots, x(-1), x(0)\}$.

Obviously, the origin is a stationary state of such dynamics (if the walker stayed there for τ previous steps). However, it is unstable. One can see that if the walker moves to the right or to the left of the origin, then there immediately appears a cycle around the origin, with the period $2\tau + 2$ and the amplitude 1. It is also easy to see that if we start with initial conditions entirely on the right or on the left of the origin, then in a finite number of steps the system develops a cycle with the period $2\tau + 2$ and the amplitude equal to the smallest integer bigger or equal to $\tau/2$. Because the set of reachable states is finite for any initial condition, after a finite number of steps, the system moves along a cycle; there are no other stationary states besides the origin.

Now we will show that any such cycle has the period $2\tau + 2$; it is symmetric around the origin and therefore the average walker position along its trajectory is equal to 0.

We introduce a state space of our system: $\Omega = \{(x(t-\tau), x(t)\} = \{(m, n) \in Z^2, |m-n| \leqslant \tau\}$. The dynamics can be represented by walks over edges or diagonals of elementary squares of Ω; see Fig. 1. Let $T = \{(m, n), |m+n| \leqslant 1\}$. It follows from the phase portrait in Fig. 1 that $T \cap \Omega$ is the attractor and the invariant set of the dynamics.

We will show that if the system moves on $T \cap \Omega$ from (i, j) to (k, l) in one step, then $l = -i$. It follows from the dynamics on $T \cap \Omega$ that either $i = -j-1$, $i = -j$, or $i = -j+1$. Moreover, monotonicity and their relative symmetry gives us that if $i = -j-1$, then $l = j+1 = -i$; if $i = -j$, then $l = j = -i$; and if $i = -j+1$, then $l = j-1 = -i$. From the definition of the phase space, we have that after τ steps, the system moves from (m, n) to (n, p) for some m, n, and p.

Assume now that the system moves from (i, j) to $(m, -i)$ in a single step. Then it moves to (j, n) in $\tau - 1$ steps. It follows that it has to be at the state $(-i, -j)$ in the next step. It can be written schematically as

$$(i, j) \xrightarrow{1} (m, -i) \xrightarrow{\tau-1} (j, n) \xrightarrow{1} (-i, -j). \quad (1)$$

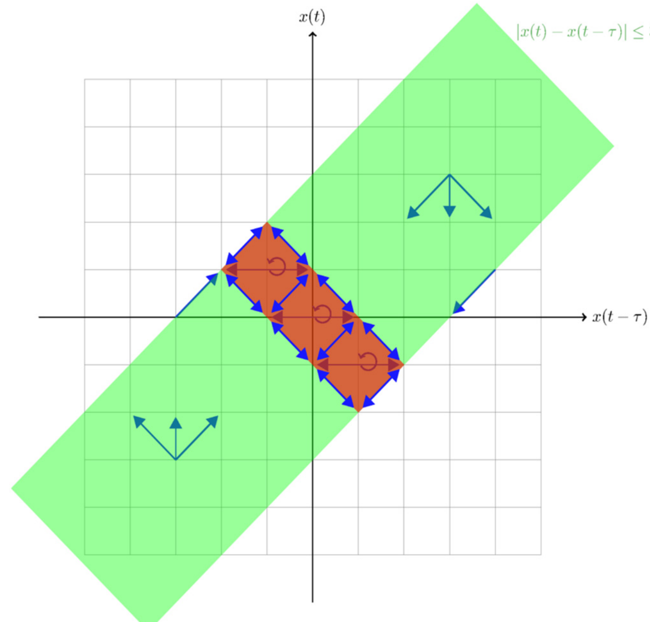

FIG. 1. Phase portrait in the state space $\Omega = \{(x(t-\tau), x(t)\} = \{(m, n) \in Z^2, |m-n| \leqslant \tau\}$ for the deterministic dynamics with $\tau = 3$. Arrows show the directions in which the system can move. The attractor is shown in a darker shade.

Hence, every cycle is symmetric around the origin and the average position of the walker is 0. After $2(\tau + 1) = 2\tau + 2$ steps, the system returns to the initial state. Hence any cycle must have a period which is a divisor of $2\tau + 2$.

III. STOCHASTIC DYNAMICS WITH TIME DELAYS

In the stochastic dynamics without time delays, we assume that a probability of the walker to move to the center, being at a position x, is proportional to the absolute value of the difference $f_A(x) - f_B(x)$ and such that a probability of moving to the center is higher than the one of moving outward. In this way, we constructed an ergodic Markov chain with a unique stationary probability distribution symmetric around the origin and with the expected value of the walker position equal to zero.

It was shown in [13] that when one introduces a discrete time delay τ, that is, the transition probabilities depend on the difference $f_A[x(t-\tau)] - f_B[x(t-\tau)]$, then there appears a cycle around the origin with the amplitude τ and the time period $4\tau + 2$ which is stochastically stable [26–28]. It means that the stationary probability is concentrated on the cycle with the probability converging to one in the limit of zero stochasticity.

Here we will consider asymmetric time delays—we assume that transition probabilities depend on the difference $f_A[x(t-\tau)] - f_B[x(t)]$. Formally, our Markov chain is described by transition probabilities on the set of consistent histories, $\{x(t-\tau), x(t-\tau+1), \dots, x(t-1), x(t)\}$. We define the transition probabilities in the following way. When the walker is at state x, then the probability of moving to the

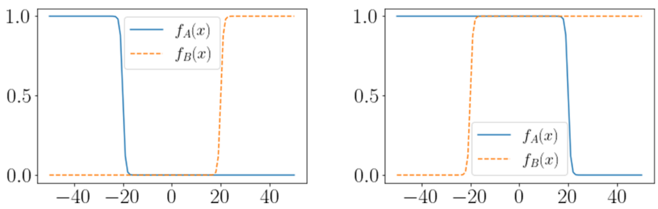

FIG. 2. Hyperbolic-tangent fitness functions; $d = -20$ on the left and $d = 20$ on the right.

right is given by

$$p_+[x(t)] = \frac{1}{2} + \omega\frac{f_A[x(t-\tau)] - f_B[x(t)]}{2}, \qquad (2)$$

where $\omega \leqslant 1$ measures the strength of selection—the importance of fitness functions: In the case of $\omega = 0$, our dynamics becomes an unbiased random walk. From now on, we will set $\omega = 1$. The probability of moving to the left, $p_-[x(t)]$, is given by $1 - p_+[x(t)]$.

For fitness functions, we use hyperbolic tangents:

$$f_A(x) = \tfrac{1}{2}[\tanh(x-d) + 1], \qquad (3a)$$

$$f_B(x) = \tfrac{1}{2}[\tanh(x+d) + 1], \qquad (3b)$$

where d is the length of the half of the interval of an almost nonbiased random walk—where transition probabilities are close to $1/2$. Figure 2 shows plots of the fitness functions for $d = -20, 20$.

In this way, we defined a Markov chain on histories with a unique stationary probability distribution which we denote by π^h. Although states of our Markov chain are time histories, we are interested in frequencies in which particular physical states in S are visited in the long run—in the stationary probability distribution. We denote by π a stationary probability of visiting physical states in S.

We performed stochastic simulations of our dynamics; the frequencies of the positions of the walker are presented in Fig. 3. One can see that they are skewed for nonzero d's, for negative d's, the walker visits positive integers more frequently, and the situation is reversed for negative d's.

In Fig. 4, we show the dependence of the expected value of the walker's position on the time delay τ. It is an increasing function of τ for $d = -20$ and a decreasing one for $d > 0$. This can be understood in the following way. Assume that $d > 0$. If a random walker is on the right of $-d$, it essentially performs the unbiased random walk. However, on the left of d, with the history to the right of it, the walker moves to the left with a probability close to one. As a consequence of this,

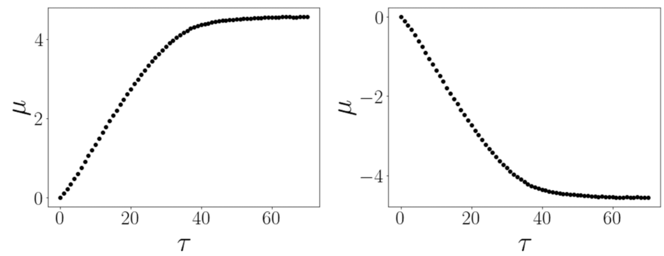

FIG. 4. The dependence of the expected value of the walker's position μ on the time delay τ with hyperbolic-tangent fitness functions $\tau = 0, 1, 2, \ldots, 68$. $d = -20$ on the left and $d = 20$ on the right.

the walker spends more time around $-d$ than around d, and hence its mean position is negative. The situation is reversed for negative d's.

In Fig. 5, we show the dependence of the expected value of the walker's position on d, the size of the space region with an almost unbiased random walk. We see that the expected value is a decreasing function of d.

We see in Figs. 4 and 5 that the expected value of the walker position is symmetric with respect to $d : \mu(d) = -\mu(-d)$ and hence $\mu(0) = 0$. This result follows immediately from symmetries of fitness functions, namely, $f_A(x, d) = 1 - f_A(-x, -d)$ and $f_B(x, d) = 1 - f_B(-x, -d)$.

Our results show a different behavior not observed in any previous models. The shift of the expected value of the stationary probability distribution is a joint effect of both time delays and stochasticity; compare the frequencies in Fig. 3 and the results for the corresponding deterministic dynamics with time delays.

IV. ANALYTICAL RESULTS

Here we present some analytical results for fitness step functions which approximate hyperbolic tangents:

$$f_A(x) = \begin{cases} 1 & \text{for } x < d \\ \frac{1}{2} & \text{for } x = d \\ 0 & \text{for } x > d, \end{cases} \qquad (4a)$$

$$f_B(x) = \begin{cases} 0 & \text{for } x < -d \\ \frac{1}{2} & \text{for } x = -d \\ 1 & \text{for } x > -d. \end{cases} \qquad (4b)$$

Figure 6 shows the plots of the functions for $d = -20, 20$.

It is easy to see that it is enough to consider the physical space state $S' = \{-(d+1), \ldots, d+\tau+1\}$ for $d > 0$ and $S' = \{d-\tau-1, \ldots, -d+1\}$ for $d < 0$, of the position of the walker which is the attractive and invariant set of dynam-

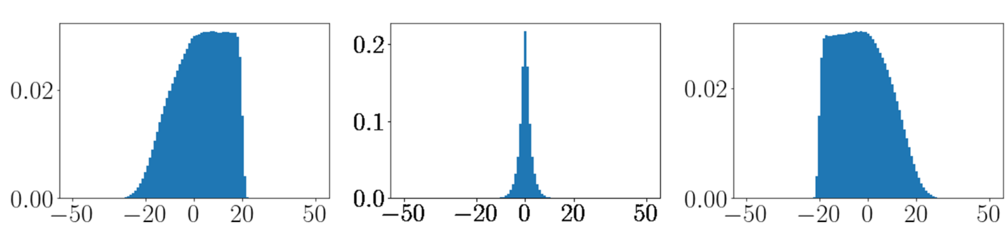

FIG. 3. Frequencies of positions of the walker for the hyperbolic-tangent models for $\tau = 20$ and 10^8 simulation steps in each case: $d = -20$ on the left, $d = 0$ in the middle, and $d = 20$ on the right.

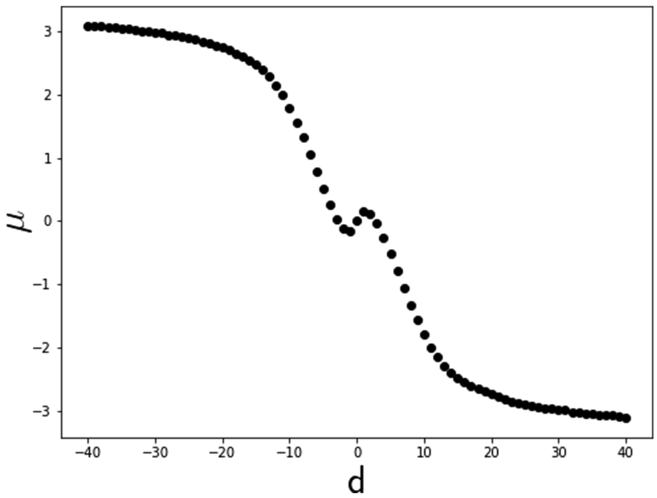

FIG. 5. The dependence of the expected value of the walker's position μ on d with hyperbolic-tangent fitness functions with $\tau = 20$, $d = -40, \dots, -1, 0, 1, \dots, 40$.

ics defined in Eq. (2). Stochastic simulations with the above fitness step functions provide the same qualitative dependence of the expected value of the walker's position on τ and d as in the hyperbolic-tangent case.

Let us first discuss the higher-order Markov chain on histories. Let us consider the case of $\tau = 1$ and $d > 0$. The state space Ω' consists of pairs: A position of a walker in S' and its position one unit time in the past. Namely, $\Omega' = \{(x, -), (x, +); -d \leqslant x \leqslant d+1, (-d-1, +), (d+2, -)\}$. The transition probabilities of such Markov chain are given in Eq. (2) with $\omega = 1$ and therefore assume the following values: 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. We denote by π^h the stationary probability distribution of our higher-order Markov chain, which can be calculated as usual by solving the system of linear algebraic equations—total probability formulas present in the definition of the stationary probability distribution. Having π^h, one can easily calculate the stationary probability distribution on the physical state S'—the stationary frequency of visiting states by the walker. Technical details are provided in the Appendix. Then one can calculate the expected value of the walker's position as a function of d for $\tau = 1$,

$$\mu(d) = \frac{7\text{sgn}(d) - 6d}{48|d| + 10} \text{ for } d \in \mathbb{Z}. \quad (5)$$

In Fig. 7, we graph the above function and compare it to the simulation results.

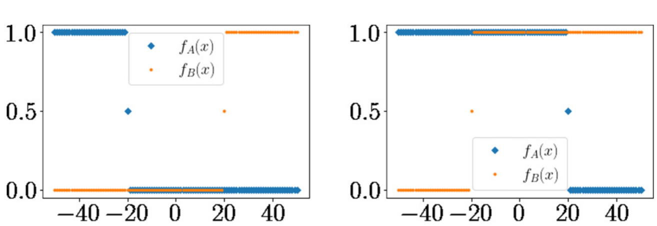

FIG. 6. Step functions with an interval of a nonbiased random walk in the middle; $d = -20$ on the left and $d = 20$ on the right.

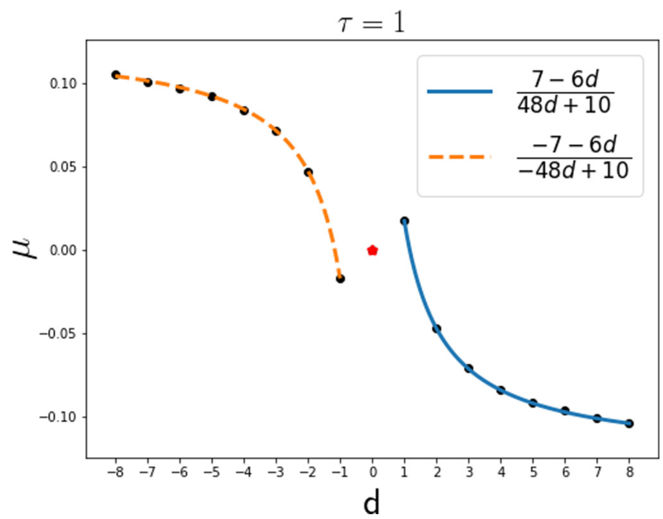

FIG. 7. The expected value of the position of the walker, $\mu(d)$, for the fitness step function given in Eq. (4), $\tau = 1$: Graph of Eq. (5) and results of simulations. Points • show the results of the simulations. The dashed orange line represents the formula in (5) for $d < 0$. The star ⋆ represents the value in (5) for $d = 0$; it coincides with the simulation data point. The blue unbroken line represents the formula in (5) for $d > 0$.

V. DISCUSSION

We studied random walks with asymmetric time delays. In our models, the probability of a walker to move to the right or to the left depends on the difference between two state-dependent fitness functions f_A and f_B evaluated at states of the walker at two different times. We observed a different behavior—a dependence of the mean position of the walker on time delays. Moreover, the effect of time delays is reversed when one shifts (in a symmetric way) fitness functions.

If both fitness functions are almost (or exactly in the step-function case) equal to 1 on the interval of the length $2d$ around the origin, then the mean position of the walker is negative and a decreasing function of the time delay. It is also a decreasing function of d. However, if both fitness functions are almost equal to 0 on this interval, then the mean position of the walker is positive, an increasing function of the time delay, and an increasing function of d.

This is a joint effect of both stochasticity and time delays present in the system. In the deterministic version of our model (where the walker moves deterministically), there appear symmetric cycles around the stationary point so the mean position of the walker is equal to the stationary point of the corresponding dynamics without time delays. In the stochastic version without time delays, the expected value of the position of the walker is given by the stationary point of the deterministic dynamics.

One can interpret our models as discrete replicator dynamics with two strategies A and B, fitness functions given by f_A and f_B, and strategy-dependent time delays. This can be done exactly in models with a finite state space as in our models with fitness step functions. The position of a random walker may then be interpreted as a frequency of a given strategy. Our results show that the effects of strategy-dependent time delays on the equilibrium structure of populations may depend not only on time delays, but also on the details of fitness functions.

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APPENDIX: DERIVATION OF STATIONARY FREQUENCIES FOR $\tau = 1$

For the random walk with fitness functions given by Eq. (4a), and transition probabilities by Eq. (2) with $\omega = 1$ and $\tau = 1$, $S' = \{-d-1, \dots, d+2\}$ is the state space of the walker. We denote, by $\pi^h(x,-)$ and $\pi^h(x,+)$, the stationary probabilities of visiting the state x at time t and, respectively, the state $x-1$ and $x+1$ at time $t-1$. π^h satisfy the system of linear algebraic equations—total probability formulas present in the definition of the stationary probability distribution—and it is given by

$$\begin{bmatrix}\pi^h(-d-1,+)\\ \pi^h(-d,-)\\ \pi^h(-d,+)\\ \pi^h(-d+1,-)\\ \pi^h(-d+1,+)\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & \frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix} \begin{bmatrix}\pi^h(-d-1,+)\\ \pi^h(-d,+)\\ \pi^h(-d,-)\\ \pi^h(-d+1,-)\\ \pi^h(-d+1,+)\\ \pi^h(-d+2,-)\\ \pi^h(-d+2,+)\end{bmatrix}, \tag{A1a}$$

$$\begin{bmatrix}\pi^h(d-1,-)\\ \pi^h(d-1,+)\\ \pi^h(d,-)\\ \pi^h(d,+)\\ \pi^h(d+1,-)\\ \pi^h(d+1,+)\\ \pi^h(d+2,-)\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 1 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0\end{bmatrix} \begin{bmatrix}\pi^h(d-2,-)\\ \pi^h(d-2,+)\\ \pi^h(d-1,-)\\ \pi^h(d-1,+)\\ \pi^h(-d+2,-)\\ \pi^h(-d+2,+)\\ \pi^h(d,-)\\ \pi^h(d,+)\\ \pi^h(d+1,-)\\ \pi^h(d+1,+)\end{bmatrix}, \tag{A1b}$$

and, for $-d+2 \leqslant x \leqslant d-1$,

$$\pi^h(x,+) = 1/2\pi^h(x+1,+) + 1/2\pi^h(x+1,-), \tag{A2a}$$

$$\pi^h(x,-) = 1/2\pi^h(x-1,+) + 1/2\pi^h(x-1,-). \tag{A2b}$$

The stationary probability distribution π on S' and the stationary probability distribution π^h on histories are related by standard expressions,

$$\pi(x) = \pi^h(x-1,x) + \pi^h(x+1,x), \quad -d \leqslant x \leqslant d+1, \tag{A3a}$$

$$\pi(-d-1) = \pi^h(-d,-d-1), \tag{A3b}$$

$$\pi(d+2) = \pi^h(d+1,d+2). \tag{A3c}$$

One can solve Eqs. (A1), (A2) and, using Eqs. (A3a), (A3c), one gets

$$\pi(-d-1) = \frac{2}{24d+5}, \tag{A4a}$$

$$\pi(-d) = \frac{8}{24d+5}, \tag{A4b}$$

$$\pi(-d < x < d-1) = \frac{12}{24d+5}, \tag{A4c}$$

$$\pi(d-1) = \frac{10}{24d+5}, \tag{A4d}$$

$$\pi(d) = \frac{6}{24d+5}, \tag{A4e}$$

$$\pi(d+1) = \frac{5}{48d+10}, \tag{A4f}$$

$$\pi(d+2) = \frac{1}{48d+10}. \tag{A4g}$$

[1] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, 1998).
[2] M. A. Nowak, *Evolutionary Dynamics: Exploring the Equations of Life* (Harvard University Press, Cambridge, 2006).
[3] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications* (Clarendon, Oxford, 1991).
[4] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population* (Springer, New York, 1992).
[5] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics* (Academic Press, San Diego, 1993).
[6] T. Erneux, *Applied Delay Differential Equations* (Springer, New York, 2009).
[7] P. D. Taylor and L. B. Jonker, Evolutionarily stable strategy and game dynamics, Math. Biosci. **40**, 145 (1978).
[8] Tao Yi and Z. Wang, Effect of time delay and evolutionarily stable strategy, J. Theor. Biol. **187**, 111 (1997).
[9] J. Alboszta and J. Miękisz, Stability of evolutionarily stable strategies in discrete replicator dynamics with time delay, J. Theor. Biol. **231**, 175 (2004).
[10] H. Oaku, Evolution with delay, Japan Econ. Rev. **53**, 114 (2002).
[11] R. Iijima, Heterogeneous information lags and evolutionary stability, Math. Soc. Sci. **61**, 83 (2011).
[12] R. Iijima, On delayed discrete evolutionary dynamics, J. Theor. Biol. **300**, 1 (2012).
[13] J. Miękisz and S. Wesołowski, Stochasticity and time delays in evolutionary games, Dyn. Games Appl. **1**, 440 (2011).
[14] J. Miękisz, M. Matuszak, and J. Poleszczuk, Stochastic stability in three-player games with time delays, Dyn. Games Appl. **4**, 489 (2014).
[15] J. A. Moreira, F. L. Pinheiro, N. Nunes, and J. M. Pacheco, Evolutionary dynamics of collective action when individual fitness derives from group decisions taken in the past, J. Theor. Biol. **298**, 8 (2012).
[16] E. Wesson, R. Rand, and E. Hopf, bifurcations in delayed rock-paper-scissors replicator dynamics, Dyn. Games Appl. **6**, 139 (2016).
[17] E. Wesson, R. Rand, and D. Rand, Hopf bifurcations in two-strategy delayed replicator dynamics, Intl. J. Bifurc. Chaos **26**, 1650006 (2016).
[18] N. Ben-Khalifa, R. El-Azouzi, and Y. Hayel, Discrete and continuous distributed delays in replicator dynamics, Dyn. Games Appl. **8**, 713 (2018).
[19] N. Ben Khalifa, R. El-Azouzi, Y. Hayel, and I. Mabrouki, Evolutionary games in interacting communities, Dyn. Games Appl. **7**, 131 (2017).
[20] M. Bodnar, J. Miękisz, and R. Vardanyan, Three-player games with strategy-dependent time delays, Dyn. Games Appl. **10**, 664 (2020).
[21] J. Miękisz and M. Bodnar, Evolution of populations with strategy-dependent time delays, Phys. Rev. E **103**, 012414 (2021).
[22] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, 1982).
[23] T. Ohira and J. G. Milton, Delayed random walks, Phys. Rev. E **52**, 3277 (1995).
[24] T. Ohira, Oscillatory correlation of delayed random walks, Phys. Rev. E **55**, R1255(R) (1997).
[25] A. E. Raftery, A model for high-order Markov chains, J. R. Statist. Soc. B **47**, 528 (1985).
[26] M. Freidlin and A. Wentzell, *Random Perturbations of Dynamical Systems* (Springer Verlag, New York, 1984).
[27] H. P. Young, The evolution of conventions, Econometrica **61**, 57 (1993).
[28] J. Miękisz, Evolutionary Game Theory and Population Dynamics, in *Multiscale Problems in the Life Sciences*, edited by V. Capasso and M. Lachowicz, Lecture Notes in Mathematics, Vol. 1940 (Springer, Berlin, Heidelberg, 2008), pp. 269–316.