Exact law for compressible pressure-anisotropic magnetohydrodynamic turbulence: Toward linking energy cascade and instabilities

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We derive an exact law for compressible pressure-anisotropic magnetohydrodynamic turbulence. For a gyrotropic pressure tensor, we study the double-adiabatic case and show the presence of new flux and source terms in the exact law, reminiscent of the plasma instability conditions due to pressure anisotropy. The Hall term is shown to bring ion-scale corrections to the exact law without affecting explicitly the pressure terms. In the pressure isotropy limit we recover all known results obtained for isothermal and polytropic closures. The incompressible limit of the gyrotropic system leads to a generalization of the Politano and Pouquet's law where a new *incompressible* source term is revealed and reflects exchanges of the magnetic and kinetic energies with the no-longer-conserved internal energy. We highlight the possibilities offered by the new laws to investigate potential links between turbulence cascade and instabilities widely observed in laboratory and astrophysical plasmas.

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I. INTRODUCTION

In recent years there has been a growing interest in deriving von Kármán–Howarth–Monin (vKHM) [1–3] equations that describe turbulent energy cascade in magnetized plasmas. Those equations present the double advantage of being fully nonlinear and of linking the turbulent energy cascade (or dissipation) rate to measurable fields [4–10]. The cascade rate is used to estimate energy dissipation from spacecraft data taken in the solar wind (SW) and the planetary plasma environments [11–16]. Efforts were thus put in generalizing the laws to more realistic conditions met in those plasmas at the cost of increasing complexity. Two main lines of research are pursued: one aiming at extending the range of the described scales, from magnetohydrodynamics (MHD), to Hall-MHD and two-fluids [17–24]; the second by incorporating density fluctuations described within isothermal or polytropic closures [25–34] or gravitational effects to study star formation in the interstellar medium [35,36].

Despite these important improvements, a key missing ingredient that none of the existing models can describe is the presence of pressure anisotropy (with respect to the background magnetic field $\mathbf{B}_0$). Indeed, while the existing laws do consider the presence of a background magnetic field, which allows one to study energy transfers along the parallel and perpendicular directions to $\mathbf{B}_0$ [37–41], they however all assume a scalar pressure, which is unrealistic to describe most of the magnetized collisionless astrophysical (or laboratory) plasmas where ion and electron pressure anisotropies are frequently reported from particle measurements [42–47].

In order to include pressure anisotropy in fluid modeling of magnetized plasmas, Chew *et al.* [48] introduced the double-adiabatic closure (known also as CGL). One of the main changes to the dynamics of the plasma brought up by pressure anisotropy in CGL-MHD equations is the presence of instabilities, which in the linear limit coincide with the firehose when $\frac{\beta_\parallel}{2}[1-a_p]>1$ and the mirror when $\beta_\parallel a_p+1<\frac{\beta_\parallel a_p^2}{6}$ ($\beta_\parallel$ is the ratio of the parallel thermal to the magnetic pressure, $a_p=T_\perp/T_\parallel$ is the ratio between the proton perpendicular and parallel temperatures [49–51]). These instabilities (or their kinetic counterparts) were shown to constrain part of the dynamics of the SW [45,46] and are thought to operate in laboratory devices [52], clusters of galaxies [53], and black holes' accretion disks [54]. However, the interplay between turbulence and instabilities remains an unsettled question although some hints were already reported. These include driving of sub-ion-scale turbulence [47,55,56], influencing the scaling of the high-frequency magnetic energy spectra in the SW [45], or linking unstable plasmas to high-energy cascade rates as measured in the near-Earth space [41,57], which remains to date not fully understood.

It is the goal of this paper to fill the existing gap by providing a self-consistent (fluid) theoretical framework to investigate the potential coupling between plasma turbulence and instabilities.

II. THEORETICAL MODEL

We use the classical MHD equations but assume a (symmetric) pressure tensor rather than a scalar one,

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}), \tag{1}$$

$$\partial_t(\rho \mathbf{v}) = \nabla \cdot (\rho \mathbf{v_A} \mathbf{v_A} - \rho \mathbf{v}\mathbf{v} - \overline{\overline{P_*}}) + \mathbf{d_k} + \mathbf{f}, \tag{2}$$

$$\partial_t(\rho \mathbf{v_A}) = \nabla \cdot (\rho \mathbf{v_A}\mathbf{v} - \rho \mathbf{v}\mathbf{v_A}) + \rho \mathbf{v}\nabla \cdot \mathbf{v_A} - \tfrac{1}{2}\rho \mathbf{v_A}\nabla \cdot \mathbf{v} + \mathbf{d_m}, \tag{3}$$

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where ρ is the mass density, $\mathbf{v}$ the velocity field, $\mathbf{v_A} = \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}}$ the Alfvén velocity with $\mathbf{B}$ the magnetic field, and $\overline{\overline{P_*}} = \overline{\overline{P}} + \overline{\overline{P}}_M$ is the total pressure tensor, i.e., the sum of the pressure tensor $\overline{\overline{P}}$ and the magnetic pressure tensor $\overline{\overline{P}}_M = P_M \overline{\overline{I}} = (\rho {v_A}^2/2)\overline{\overline{I}}$ ($\overline{\overline{I}}$ is the identity 3×3 matrix), $\mathbf{d_k}$ the kinetic viscous dissipation, $\mathbf{d_m}$ the magnetic diffusivity, and $\mathbf{f}$ a stationary homogeneous external force assumed to act on the largest scales.

Since we want to derive the exact law for the total energy of the system, Eqs. (1)–(3) are complemented by that of the (specific) internal energy u, which reads

$$\partial_t u = -\nabla \cdot (u\mathbf{v}) + u\nabla \cdot \mathbf{v} - \frac{\overline{\overline{P}}}{\rho} : \nabla\mathbf{v}, \tag{4}$$

since $\overline{\overline{P}}$ and $\overline{\overline{P_*}}$ are symmetrical tensors, i.e., $P_{ij} = P_{ji}$, the dual product between two such tensors $\overline{\overline{P}}$ and $\overline{\overline{A}}$ obeys $\overline{\overline{P}} : \overline{\overline{A}} = P_{ij}A_{ij} = P_{ji}A_{ij}$. Equation (4), valid for any symmetric pressure tensor when the heat flux is neglected, can be derived from thermodynamical considerations [58] or from the moments of the Vlasov-Maxwell equations [50]. For a scalar pressure, i.e., $\overline{\overline{P}} = P\overline{\overline{I}}$, we recover the equation of the internal energy used in Ref. [34].

Equations (1)–(4) will be used in the following section to derive the exact law of interest.

III. GENERAL EXACT LAW FOR COMPRESSIBLE PRESSURE-ANISOTROPIC MHD TURBULENCE

Following the standard approach used in statistical theories of fully developed turbulence [4–6,9], we define the spatial increment (or scale) $\boldsymbol{\ell}$ connecting two points $\mathbf{x}$ and $\mathbf{x}'$ as $\mathbf{x}' = \mathbf{x} + \boldsymbol{\ell}$ and introduce the notations, $\xi(\mathbf{x}) \equiv \xi$, its conjugate (i.e., taken at the position $\mathbf{x}'$) $\xi(\mathbf{x}') \equiv \xi'$, and the incremental quantity $\delta\xi \equiv \xi' - \xi$. These definitions impose that $\partial_x \xi' = \partial_{x'} \xi = 0$, while the hypothesis of space homogeneity implies the relations $\langle \nabla' \cdot \rangle = \nabla_{\boldsymbol{\ell}} \cdot \langle\,\rangle$ and $\langle \nabla \cdot \rangle = -\nabla_{\boldsymbol{\ell}} \cdot \langle\,\rangle$, where $\nabla_{\boldsymbol{\ell}}$ denotes the derivative operator along the increment vector $\boldsymbol{\ell}$ and $\langle\,\rangle$ an ensemble average.

We consider the mean correlation function of the total energy $R_{\text{tot}} = (R + R')/2$ with $R = \langle \rho\mathbf{v} \cdot \mathbf{v}'/2 + \rho\mathbf{v_A} \cdot \mathbf{v_A}'/2 + \rho u' \rangle = R_k + R_B + R_u$ a correlation function taken at the point $\mathbf{x}$ and R' its conjugate. We remark that if $\mathbf{x} = \mathbf{x}'$, $R_{\text{tot}} = E = \langle \rho v^2/2 + \rho {v_A}^2/2 + \rho u \rangle$, i.e., the mean total energy of the system.

Using the property $\partial_t \langle\rangle = \langle \partial_t \rangle$, Eqs. (1)–(4) written at the independent positions $\mathbf{x}$ then $\mathbf{x}'$ and multiplied by the appropriate variables [e.g., Eq. (2) multiplied by $\mathbf{v}'$] and the space homogeneity assumption (see Ref. [34] for more details), we obtain the temporal evolution of the kinetic, R_k, the magnetic, R_B, and the internal energy, R_u, correlators:

$$\begin{aligned}
2\partial_t R_k = &-\nabla_{\boldsymbol{\ell}} \cdot \langle \rho\mathbf{v} \cdot \mathbf{v}'\delta\mathbf{v} + \rho\mathbf{v_A} \cdot \mathbf{v}'\mathbf{v_A} - \rho\mathbf{v} \cdot \mathbf{v}'_\mathbf{A}\mathbf{v}'_\mathbf{A} \rangle \\
&+ \nabla_{\boldsymbol{\ell}} \cdot \left\langle \overline{\overline{P_*}} \cdot \mathbf{v}' - \frac{\rho}{\rho'}\overline{\overline{P'_*}} \cdot \mathbf{v} \right\rangle + \langle \rho\mathbf{v} \cdot \mathbf{v}'\nabla' \cdot \mathbf{v}' \rangle \\
&- \left\langle \frac{\rho}{\rho'}\mathbf{v} \cdot \overline{\overline{P'_*}} \cdot \frac{\nabla'\rho'}{\rho'} + 2\rho\mathbf{v} \cdot \mathbf{v}'_\mathbf{A}\nabla' \cdot \mathbf{v}'_\mathbf{A} \right\rangle + \mathcal{F} + \mathcal{D}_k,
\end{aligned} \tag{5}$$

$$\begin{aligned}
2\partial_t R_B = &-\nabla_{\boldsymbol{\ell}} \cdot \langle \rho\mathbf{v_A} \cdot \mathbf{v}'_\mathbf{A}\delta\mathbf{v} + \rho\mathbf{v} \cdot \mathbf{v}'_\mathbf{A}\mathbf{v_A} \rangle \\
&- \nabla_{\boldsymbol{\ell}} \cdot \langle \rho\mathbf{v_A} \cdot \mathbf{v}'\mathbf{v}'_\mathbf{A} \rangle + \left\langle \tfrac{1}{2}\rho\mathbf{v_A} \cdot \mathbf{v}'_\mathbf{A}(\nabla' \cdot \mathbf{v}' - \nabla \cdot \mathbf{v}) \right\rangle \\
&+ \langle \rho\mathbf{v} \cdot \mathbf{v}'_\mathbf{A}\nabla \cdot \mathbf{v_A} - \rho\mathbf{v_A} \cdot \mathbf{v}'\nabla' \cdot \mathbf{v}'_\mathbf{A} \rangle + \mathcal{D}_m,
\end{aligned} \tag{6}$$

$$\partial_t R_u = -\nabla_{\boldsymbol{\ell}} \cdot \langle \rho u'\delta\mathbf{v} \rangle + \left\langle \rho u'\nabla' \cdot \mathbf{v}' - \frac{\rho}{\rho'}\overline{\overline{P'}} : \nabla'\mathbf{v}' \right\rangle, \tag{7}$$

where the terms depending on the forcing, the kinetic, and magnetic dissipation are regrouped respectively in $\mathcal{F}$, $\mathcal{D}_k$, and $\mathcal{D}_m$. Then the temporal evolution of R_{tot} is the sum of the relations (5)–(7) and of their conjugates (written at position $\mathbf{x}'$). By recognizing the developed form of the structure functions $\langle \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v}\delta\mathbf{v} \rangle$, $\langle \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v_A}\delta\mathbf{v} \rangle$, $\langle \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v}\delta\mathbf{v_A} \rangle$, $\langle \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v_A}\delta\mathbf{v_A} \rangle$, $\langle \delta\rho\delta u\delta\mathbf{v} \rangle$, $\langle \delta\rho\delta(\overline{\overline{P}}/\rho) \cdot \delta\mathbf{v} \rangle$, and $\langle \delta\rho\delta(\overline{\overline{P}}_M/\rho) \cdot \delta\mathbf{v} \rangle$, the final expression for the temporal evolution of the total energy correlator reads

$$\begin{aligned}
4\partial_t R_{\text{tot}} = &\nabla_{\boldsymbol{\ell}} \cdot \left\langle [\delta(\rho\mathbf{v}) \cdot \delta\mathbf{v} + \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v_A} + 2\delta\rho\delta u]\delta\mathbf{v} - [\delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v} + \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v_A}]\delta\mathbf{v_A} - \delta\rho\delta\left(\frac{\overline{\overline{P_*}}}{\rho}\right) \cdot \delta\mathbf{v} \right\rangle \\
&+ \left\langle \left(\rho\mathbf{v} \cdot \delta\mathbf{v} + \frac{1}{2}\rho\mathbf{v_A} \cdot \delta\mathbf{v_A} - \frac{1}{2}\mathbf{v_A} \cdot \delta(\rho\mathbf{v_A}) + 2\rho\delta u\right)\nabla' \cdot \mathbf{v}' - 2\rho\delta\left(\frac{\overline{\overline{P}}}{\rho}\right) : \nabla'\mathbf{v}' \right\rangle \\
&+ \left\langle \left(-\rho'\mathbf{v}' \cdot \delta\mathbf{v} - \frac{1}{2}\rho'\mathbf{v}'_\mathbf{A} \cdot \delta\mathbf{v_A} + \frac{1}{2}\delta(\rho\mathbf{v_A}) \cdot \mathbf{v}'_\mathbf{A} - 2\rho'\delta u\right)\nabla \cdot \mathbf{v} + 2\rho'\delta\left(\frac{\overline{\overline{P}}}{\rho}\right) : \nabla\mathbf{v} \right\rangle \\
&+ \langle [-2\rho\mathbf{v} \cdot \delta\mathbf{v_A} - \rho\mathbf{v_A} \cdot \delta\mathbf{v} + \delta(\rho\mathbf{v}) \cdot \mathbf{v_A}]\nabla' \cdot \mathbf{v}'_\mathbf{A} + [2\rho'\mathbf{v}' \cdot \delta\mathbf{v_A} + \rho'\mathbf{v}'_\mathbf{A} \cdot \delta\mathbf{v} - \delta(\rho\mathbf{v}) \cdot \mathbf{v}'_\mathbf{A}]\nabla \cdot \mathbf{v_A} \rangle \\
&+ \left\langle \left[\delta\rho\frac{\overline{\overline{P_*}}}{\rho} \cdot \mathbf{v} - \rho\delta\left(\frac{\overline{\overline{P_*}}}{\rho}\right) \cdot \mathbf{v}\right] \cdot \frac{\nabla'\rho'}{\rho'} + \left[\rho'\delta\left(\frac{\overline{\overline{P_*}}}{\rho}\right) \cdot \mathbf{v}' - \delta\rho\frac{\overline{\overline{P'_*}}}{\rho'} \cdot \mathbf{v}'\right] \cdot \frac{\nabla\rho}{\rho} \right\rangle + \mathcal{F} + \mathcal{F}' + \mathcal{D}_k + \mathcal{D}'_k + \mathcal{D}_m + \mathcal{D}'_m.
\end{aligned} \tag{8}$$

From this relation and following the usual assumptions used in fully developed homogeneous turbulence, namely infinite kinetic and magnetic Reynolds numbers, stationary state, balance between forcing (at the largest scales), and dissipation (at the smallest ones) [5,6,25], we obtain the following exact law valid in the inertial range:

$$-4\varepsilon^{\mathrm{MHD}} = \nabla_{\boldsymbol{\ell}} \cdot \boldsymbol{\mathcal{F}}^{\mathrm{MHD}} + \mathcal{S}^{\mathrm{MHD}} + \mathcal{S}'^{\mathrm{MHD}},$$

with

$$\begin{aligned}
\boldsymbol{\mathcal{F}}^{\mathrm{MHD}} &= \left\langle [\delta(\rho\mathbf{v})\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v_A})\cdot\delta\mathbf{v_A} + 2\delta\rho\delta u]\delta\mathbf{v} - [\delta(\rho\mathbf{v_A})\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v})\cdot\delta\mathbf{v_A}]\delta\mathbf{v_A} - \delta\rho\delta\left(\frac{\overline{\overline{P_*}}}{\rho}\right)\cdot\delta\mathbf{v}\right\rangle,\\
\mathcal{S}^{\mathrm{MHD}} &= \left\langle \left(\rho\mathbf{v}\cdot\delta\mathbf{v} + \frac{1}{2}\rho\mathbf{v_A}\cdot\delta\mathbf{v_A} - \frac{1}{2}\mathbf{v_A}\cdot\delta(\rho\mathbf{v_A}) + 2\rho\delta u\right)\nabla'\cdot\mathbf{v}' - 2\rho\delta\left(\frac{\overline{\overline{P}}}{\rho}\right) : \nabla'\mathbf{v}'\right\rangle\\
&\quad + \langle[-2\rho\mathbf{v}\cdot\delta\mathbf{v_A} - \rho\mathbf{v_A}\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v})\cdot\mathbf{v_A}]\nabla'\cdot\mathbf{v}'_{\mathbf{A}}\rangle + \left\langle\left[\delta\rho\frac{\overline{\overline{P_*}}}{\rho}\cdot\mathbf{v} - \rho\delta\left(\frac{\overline{\overline{P_*}}}{\rho}\right)\cdot\mathbf{v}\right]\cdot\frac{\nabla'\rho'}{\rho'}\right\rangle,\\
\mathcal{S}'^{\mathrm{MHD}} &= \ \mathrm{conjugate}(\mathcal{S}^{\mathrm{MHD}}),
\end{aligned} \tag{9}$$

where $\varepsilon^{\mathrm{MHD}}$ is the classical mean energy dissipation rate by unit mass assumed to be equal to the injection rate due to the forcing, i.e., $\mathcal{F} + \mathcal{F}' \simeq 4\varepsilon^{\mathrm{MHD}}$, and to the cascade rate in the inertial range due to nonlinearities. The exact law (9) is the first main result of this paper. It is valid for any MHD flow with a (symmetric) pressure tensor when the heat flux is neglected.

As in other compressible exact laws, we can recognize the terms introduced by Ref. [29]: $\mathcal{F}^{\mathrm{MHD}}$ is the flux terms (increment derivative $\nabla_{\boldsymbol{\ell}}\cdot\langle\,\rangle$), and $\mathcal{S}^{\mathrm{MHD}}$ and its conjugate $\mathcal{S}'^{\mathrm{MHD}}$ are generally known as source terms (see below about the physical meaning of this terminology) where terms in $\langle\nabla\cdot\mathbf{v}\rangle$ reflect the role of velocity dilatation, terms in $\langle\nabla\cdot\mathbf{v_A}\rangle$ involve the (compressible) Alfvén speed dilatation, and terms in $\langle\nabla\rho\rangle$ contain density dilatation. Note that some hybrid and the β-dependent terms introduced in Ref. [29] are hidden in the new structure function $\langle\delta\rho\delta(\frac{\overline{\overline{P_*}}}{\rho})\cdot\delta\mathbf{v}\rangle$ and the terms in $\langle\nabla\rho\rangle$.

A. Extension to pressure anisotropic Hall-MHD

The extension of the previous MHD model to Hall-MHD flows can be readily obtained by noticing that the only change to the original model is to introduce the Hall term in Eq. (3), while the internal energy equation remains unchanged. Therefore, the changes to the exact law (9) will occur through the sole terms that depend on the current density, which were already derived in Ref. [30] for compressible isothermal MHD, without impacting pressure terms. The final exact law for Hall-MHD thus writes

$$\begin{aligned}
-4\varepsilon^{\mathrm{HMHD}} &= -4\varepsilon^{\mathrm{MHD}} + 2d_i\nabla_{\boldsymbol{\ell}}\cdot\langle\overline{\rho\mathbf{J_c}\times\mathbf{v_A}}\times\delta\mathbf{v_A} - \delta(\mathbf{J_c}\times\mathbf{v_A})\times\overline{\rho\mathbf{v_A}}\rangle\\
&\quad - \frac{d_i}{2}\langle(\delta\rho\mathbf{v}'_{\mathbf{A}}\cdot\mathbf{v_A})\nabla\cdot\mathbf{J_c} - (\delta\rho\mathbf{v}'_{\mathbf{A}}\cdot\mathbf{v_A})\nabla'\cdot\mathbf{J}'_{\mathbf{c}}\rangle + d_i\langle(\delta\rho\mathbf{J_c}\cdot\mathbf{v}'_{\mathbf{A}})\nabla\cdot\mathbf{v_A} - (\delta\rho\mathbf{J}'_{\mathbf{c}}\cdot\mathbf{v_A})\nabla'\cdot\mathbf{v}'_{\mathbf{A}}\rangle.
\end{aligned} \tag{10}$$

where $\varepsilon^{\mathrm{MHD}}$ is given by Eq. (9), d_i is the ion inertial length, and $\mathbf{J} = \rho\mathbf{J_c}$ is the current density in Alfvénic units.

B. In the isotropic pressure case

When considering a (total) scalar pressure $\overline{\overline{P}} = P\overline{\overline{I}}$ the MHD exact law (9) takes the form

$$-4\varepsilon^{\mathrm{MHD}} = \nabla_{\boldsymbol{\ell}} \cdot \boldsymbol{\mathcal{F}}^{\mathrm{MHD}} + \mathcal{S}^{\mathrm{MHD}} + \mathcal{S}'^{\mathrm{MHD}},$$

with

$$\begin{aligned}
\boldsymbol{\mathcal{F}}^{\mathrm{MHD}} &= \left\langle [\delta(\rho\mathbf{v})\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v_A})\cdot\delta\mathbf{v_A} + 2\delta\rho\delta u]\delta\mathbf{v} - [\delta(\rho\mathbf{v_A})\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v})\cdot\delta\mathbf{v_A}]\delta\mathbf{v_A} - \delta\rho\delta\left(\frac{P_*}{\rho}\right)\delta\mathbf{v}\right\rangle,\\
\mathcal{S}^{\mathrm{MHD}} &= \left\langle \left[\rho\mathbf{v}\cdot\delta\mathbf{v} + \frac{1}{2}\rho\mathbf{v_A}\cdot\delta\mathbf{v_A} - \frac{1}{2}\mathbf{v_A}\cdot\delta(\rho\mathbf{v_A}) + 2\rho\delta u - 2\rho\delta\left(\frac{P}{\rho}\right)\right]\nabla'\cdot\mathbf{v}'\right\rangle\\
&\quad + \langle[-2\rho\mathbf{v}\cdot\delta\mathbf{v_A} - \rho\mathbf{v_A}\cdot\delta\mathbf{v} + \delta(\rho\mathbf{v})\cdot\mathbf{v_A}]\nabla'\cdot\mathbf{v}'_{\mathbf{A}}\rangle + \left\langle\left[\delta\rho\frac{P_*}{\rho}\cdot\mathbf{v} - \rho\delta\left(\frac{P_*}{\rho}\right)\cdot\mathbf{v}\right]\cdot\frac{\nabla'\rho'}{\rho'}\right\rangle,\\
\mathcal{S}'^{\mathrm{MHD}} &= \ \mathrm{conjugate}(\mathcal{S}^{\mathrm{MHD}}).
\end{aligned} \tag{11}$$

One can notice in relation (11) the presence of a new flux term that was not recognized as such in the previous models derived for scalar pressure [26,29,34]: $\nabla_{\boldsymbol{\ell}}\cdot\langle-\delta\rho\delta(\frac{\overline{\overline{P_*}}}{\rho})\cdot\delta\mathbf{v}\rangle = \nabla_{\boldsymbol{\ell}}\cdot\langle-\delta\rho\delta(\frac{P_*}{\rho})\delta\mathbf{v}\rangle$. Using the first law of thermodynamics $\rho^2\nabla u = P\nabla\rho$, one can write the term in $\nabla'\rho'$ of $\mathcal{S}^{\mathrm{MHD}}$ of Eq. (11) as (the same holds for its conjugate)

$$\left\langle\left[\delta(\rho)\frac{P_*}{\rho}\mathbf{v} - \rho\delta\left(\frac{P_*}{\rho}\right)\mathbf{v}\right]\cdot\frac{\nabla'\rho'}{\rho'}\right\rangle = \left\langle\left[\delta\left(\frac{\rho^2}{P}\right)\frac{P_*}{\rho}\mathbf{v} - \delta\left(\frac{P_*}{P}\right)\rho\mathbf{v}\right]\cdot\nabla'u'\right\rangle = \left\langle\frac{P_*}{\rho}\mathbf{v}\cdot\nabla'\rho' - \frac{P'_*}{P'}\nabla'\cdot(\rho u'\mathbf{v})\right\rangle. \tag{12}$$

It is worth noting that the β-dependent term introduced by Ref. [29] is hidden in this line since $P_*/P = 1 + P_M/P = 1 + \beta^{-1}$. After some other manipulations, we recover the general exact law for isentropic flows derived in Ref. [34]:

$$\begin{aligned}
-4\varepsilon^{\mathrm{MHD}} =\ & \nabla_{\ell} \cdot \langle \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v}\delta\mathbf{v} + \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v_A}\delta\mathbf{v} + 2\delta\rho\delta u\delta\mathbf{v} - \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v}\delta\mathbf{v_A} - \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v_A}\delta\mathbf{v_A}\rangle \\
& + \nabla_{\ell} \cdot \left\langle \left(1 + \frac{\rho'}{\rho}\right)(P + P_M)\mathbf{v}' - \left(1 + \frac{\rho}{\rho'}\right)(P' + P_M')\mathbf{v} + \rho' u\mathbf{v}' - \rho u'\mathbf{v} \right\rangle \\
& + \left\langle (\nabla' \cdot \mathbf{v}')\left[\rho\mathbf{v} \cdot \delta\mathbf{v} + \rho\mathbf{v_A} \cdot \delta\mathbf{v_A} - \frac{1}{2}\rho'\mathbf{v_A}' \cdot \mathbf{v_A} - \frac{1}{2}\rho\mathbf{v_A} \cdot \mathbf{v_A}' + 2\rho\left(\delta u - \frac{P'}{\rho'}\right)\right]\right\rangle \\
& + \left\langle (\nabla \cdot \mathbf{v})\left[-\rho'\mathbf{v}' \cdot \delta\mathbf{v} - \rho'\mathbf{v_A}' \cdot \delta\mathbf{v_A} - \frac{1}{2}\rho\mathbf{v_A} \cdot \mathbf{v_A}' - \frac{1}{2}\rho'\mathbf{v_A}' \cdot \mathbf{v_A} - 2\rho'\left(\delta u + \frac{P}{\rho}\right)\right]\right\rangle \\
& - \langle (\nabla' \cdot \mathbf{v_A}')(2\rho\mathbf{v} \cdot \delta\mathbf{v_A} - \rho'\mathbf{v}' \cdot \mathbf{v_A} + \rho\mathbf{v_A} \cdot \mathbf{v}') - (\nabla \cdot \mathbf{v_A})(2\rho'\mathbf{v}' \cdot \delta\mathbf{v_A} + \rho\mathbf{v} \cdot \mathbf{v_A}' - \rho'\mathbf{v_A}' \cdot \mathbf{v})\rangle \\
& - \left\langle \frac{P_M'}{P'}\nabla' \cdot (\rho u'\mathbf{v}) + \frac{P_M}{P}\nabla \cdot (\rho' u\mathbf{v}')\right\rangle. \qquad (13)
\end{aligned}$$

It is worth recalling that this exact law is an extension of all scalar pressure models such as the isothermal and polytropic, which can be obtained by introducing the adequate state equation in relation (13) (i.e., specifying the relation between the pressure P and the density ρ) that are compatible with the isentropic hypothesis [34].

IV. COMPRESSIBLE MHD EXACT LAW WITH A GYROTROPIC PRESSURE

The gyrotropic exact law can be readily obtained from relation (9) by imposing the pressure tensor decomposition $\overline{\overline{P}} = P_\perp\overline{\overline{I}} + (P_\parallel - P_\perp)\mathbf{bb}$, with $\mathbf{b} = \mathbf{v_A}/|\mathbf{v_A}|$ the magnetic field direction [50]. These definitions yield the following form of the total pressure, $\overline{\overline{P_*}} = (P_\perp + P_M)\overline{\overline{I}} + (P_\parallel - P_\perp)\mathbf{bb}$. Using the tensor pressure equation, one can define the internal energy density as $\rho u = \frac{1}{2}\overline{\overline{P}} : \overline{\overline{I}} = \frac{1}{2}P_\parallel + P_\perp$ [50,58]. To highlight the terms in the exact law (9) that can be linked to known (linear) plasma instabilities [50], we further introduce the parameters $\beta_\parallel = \frac{P_\parallel}{P_M}$ and $a_p = \frac{P_\perp}{P_\parallel} = T_\perp/T_\parallel$. Injecting these relations in Eq. (9) yields the new gyrotropic-MHD exact law, which is the second main result of this paper:

$$-4\varepsilon^{\mathrm{GYR}} = \nabla_{\ell} \cdot \mathcal{F}^{\mathrm{GYR}} + \mathcal{S}^{\mathrm{GYR}} + \mathcal{S}'^{\mathrm{GYR}},$$

with

$$\begin{aligned}
\mathcal{F}^{\mathrm{GYR}} =\ & \langle \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v}\delta\mathbf{v} + \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v_A}\delta\mathbf{v} - \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v}\delta\mathbf{v_A} - \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v_A}\delta\mathbf{v_A}\rangle \\
& + \left\langle \delta\rho\delta\left(\frac{\mathbf{v_A}^2}{2}(\beta_\parallel[1 + a_p] - 1)\right)\delta\mathbf{v} - \delta\rho\delta\left(\frac{\beta_\parallel}{2}[1 - a_p]\mathbf{v_A}\mathbf{v_A}\right) \cdot \delta\mathbf{v}\right\rangle, \\
\mathcal{S}^{\mathrm{GYR}} =\ & \left\langle \left[\rho\mathbf{v} \cdot \delta\mathbf{v} + \frac{1}{2}\rho\mathbf{v_A} \cdot \delta\mathbf{v_A} - \frac{1}{2}\mathbf{v_A} \cdot \delta(\rho\mathbf{v_A}) + \rho\delta\left(\frac{\mathbf{v_A}^2\beta_\parallel}{2}\right)\right]\nabla' \cdot \mathbf{v}' - \rho\delta(\beta_\parallel[1 - a_p]\mathbf{v_A}\mathbf{v_A}) : \nabla'\mathbf{v}'\right\rangle \\
& + \langle [-2\rho\mathbf{v} \cdot \delta\mathbf{v_A} - \rho\mathbf{v_A} \cdot \delta\mathbf{v} + \delta(\rho\mathbf{v}) \cdot \mathbf{v_A}]\nabla' \cdot \mathbf{v}'_{\mathbf{A}}\rangle \\
& + \left\langle \left[(\delta\rho)\frac{\mathbf{v_A}^2}{2}[a_p\beta_\parallel + 1]\mathbf{v} - \rho\delta\left(\frac{\mathbf{v_A}^2}{2}[a_p\beta_\parallel + 1]\right)\mathbf{v}\right] \cdot \frac{\nabla'\rho'}{\rho'}\right\rangle \\
& + \left\langle \left[(\delta\rho)\frac{\beta_\parallel}{2}[1 - a_p]\mathbf{v_A}\mathbf{v_A} \cdot \mathbf{v} - \rho\delta\left(\frac{\beta_\parallel}{2}[1 - a_p]\mathbf{v_A}\mathbf{v_A}\right) \cdot \mathbf{v}\right] \cdot \frac{\nabla'\rho'}{\rho'}\right\rangle, \\
\mathcal{S}'^{\mathrm{GYR}} =\ & \text{conjugate}(\mathcal{S}^{\mathrm{GYR}}). \qquad (14)
\end{aligned}$$

Equation (14) shows the presence of new terms brought in by pressure anisotropy, which reveals how the turbulent cascade can be connected to the plasma instability conditions. For instance, the terms proportional to $1 - a_p$ will have either a positive or negative contribution to the cascade rate depending on the stability condition $a_p > 1$ or $a_p < 1$. In the case of a positive (respectively negative) contribution to the cascade rate, pressure anisotropy can be seen as a source of "free energy" (respectively a sink) that can reinforce (respectively diminish) the turbulence cascade. Furthermore, if the pressure anisotropy terms dominate the cascade, then the instability would impact both the value of the energy cascade rate and its "sense" (direct versus inverse). Equation (14), which can be used on simulation and spacecraft data, may thus provide a solid theoretical explanation of the results reported in Refs. [39,41] and to the overall prominent role of the instabilities (not necessarily linear) in controlling part of the dynamics in astrophysical plasmas [45,46,56,59].

In relation (14) the parameters $\beta_\parallel$ and a_p that depend on the pressure components $P_\parallel$ and $P_\perp$ are not yet determined since this relation derives from the internal energy equation (4), which constrains the sum of the two pressure components but not the individual ones. The latter can be determined by further introducing any closure equation compatible with the definition of the internal energy $\rho u = \frac{1}{2}\overline{\overline{P}} : \overline{\overline{I}}$ for each pressure component as done in the CGL-MHD theory.

We note finally that the Hall correction derived in Sec. III A remains valid with this gyrotropic version of the exact law.

A. Exact law for the CGL-MHD system

The CGL-MHD closure equations written in their conservative form [50] read

$$\frac{d}{dt}\left(\frac{P_\parallel B^2}{\rho^3}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{P_\perp}{\rho B}\right) = 0, \tag{15}$$

where d/dt is the total time derivative. Equations (15) lead to the integrated form of the pressures and, consequently, to the forms of the parameters $\beta_\parallel = 2C_\parallel \frac{\rho}{\mathbf{v_A}^4}$ and $a_p = C_p \frac{|\mathbf{v_A}|^3}{\rho^{1/2}}$, where the constants $C_\parallel$ and C_p guarantee the homogeneity. Injecting these integrated relations in Eq. (14) yields the new CGL-MHD exact law, which is the third result of this paper:

$$-4\varepsilon^{\mathrm{CGL}} = \nabla_{\boldsymbol{\ell}} \cdot \boldsymbol{\mathcal{F}}^{\mathrm{CGL}} + \mathcal{S}^{\mathrm{CGL}} + \mathcal{S}'^{\mathrm{CGL}},$$

with

$$\begin{aligned}
\boldsymbol{\mathcal{F}}^{\mathrm{CGL}} =& \ \langle \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v}\delta\mathbf{v} + \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v_A}\delta\mathbf{v} - \delta(\rho\mathbf{v_A}) \cdot \delta\mathbf{v}\delta\mathbf{v_A} - \delta(\rho\mathbf{v}) \cdot \delta\mathbf{v_A}\delta\mathbf{v_A}\rangle \\
&+ \left\langle \delta\rho\delta\left[\frac{\mathbf{v_A}^2}{2}\left(2C_\parallel \frac{\rho}{\mathbf{v_A}^4}\left[1 + C_p\frac{|\mathbf{v_A}|^3}{\rho^{1/2}}\right] - 1\right)\right]\delta\mathbf{v} - \delta\rho\delta\left(C_\parallel \frac{\rho}{\mathbf{v_A}^4}\left[1 - C_p\frac{|\mathbf{v_A}|^3}{\rho^{1/2}}\right]\mathbf{v_A}\mathbf{v_A}\right) \cdot \delta\mathbf{v}\right\rangle, \\
\mathcal{S}^{\mathrm{CGL}} =& \left\langle \left[\rho\mathbf{v} \cdot \delta\mathbf{v} + \frac{1}{2}\rho\mathbf{v_A} \cdot \delta\mathbf{v_A} - \frac{1}{2}\mathbf{v_A} \cdot \delta(\rho\mathbf{v_A}) + \rho\delta\left(\frac{C_\parallel\rho}{\mathbf{v_A}^2}\right)\right]\nabla' \cdot \mathbf{v}'\right\rangle \\
&- \left\langle 2\rho\delta\left(C_\parallel \frac{\rho}{\mathbf{v_A}^4}\left[1 - C_p\frac{|\mathbf{v_A}|^3}{\rho^{1/2}}\right]\mathbf{v_A}\mathbf{v_A}\right) : \nabla'\mathbf{v}' + [2\rho\mathbf{v} \cdot \delta\mathbf{v_A} + \rho\mathbf{v_A} \cdot \delta\mathbf{v} - \delta(\rho\mathbf{v}) \cdot \mathbf{v_A}]\nabla' \cdot \mathbf{v}'_{\mathbf{A}}\right\rangle \\
&+ \left\langle \left[(\delta\rho)\frac{\mathbf{v_A}^2}{2}\left[2C_pC_\parallel\frac{\rho^{1/2}}{|\mathbf{v_A}|} + 1\right]\mathbf{v} - \rho\delta\left(\frac{\mathbf{v_A}^2}{2}\left[2C_pC_\parallel\frac{\rho^{1/2}}{|\mathbf{v_A}|} + 1\right]\right)\mathbf{v}\right] \cdot \frac{\nabla'\rho'}{\rho'}\right\rangle \\
&+ \left\langle \left[(\delta\rho)C_\parallel\frac{\rho}{\mathbf{v_A}^4}\left[1 - C_p\frac{|\mathbf{v_A}|^3}{\rho^{1/2}}\right]\mathbf{v_A}\mathbf{v_A} \cdot \mathbf{v} - \rho\delta\left(C_\parallel\frac{\rho}{\mathbf{v_A}^4}\left[1 - C_p\frac{|\mathbf{v_A}|^3}{\rho^{1/2}}\right]\mathbf{v_A}\mathbf{v_A}\right) \cdot \mathbf{v}\right] \cdot \frac{\nabla'\rho'}{\rho'}\right\rangle, \\
\mathcal{S}'^{\mathrm{CGL}} =& \ \text{conjugate}(\mathcal{S}^{\mathrm{CGL}}).
\end{aligned} \tag{16}$$

In the isotropic limit $P_\parallel = P_\perp$ one finds the adiabatic (monoatomic) case with a polytropic index $\gamma = 5/3$ and $\rho u = 3P/2$. Note that in the CGL-Hall-MHD the pressure equations do not write in a conservative form as those of the CGL-MHD [see Eq. (15)] [50]. This prevents us from obtaining a reduced form of the exact law for the CGL-Hall-MHD as that of the CGL-MHD. Nevertheless, the exact law (10) is applicable to any CGL-Hall-MHD simulation data since the closure equations of the latter are compatible with the internal energy [Eq. (4)] used to derive the law (10) above.

B. The incompressible MHD with a gyrotropic pressure: A generalization of the Politano and Pouquet's law

In the incompressible limit, i.e., $\rho = \rho_0$ and $\nabla \cdot \mathbf{v} = 0$, Eq. (14) becomes

$$4\varepsilon^{\mathrm{IGYR}} = 4\varepsilon^{\mathrm{PP98}} + \rho_0\langle\delta(\beta_\parallel[1 - a_p]\mathbf{v_A}\mathbf{v_A}) : \delta(\nabla\mathbf{v})\rangle, \tag{17}$$

where $\varepsilon^{\mathrm{IGYR}}$ stands for the cascade rate of incompressible gyrotropic model and $-4\varepsilon^{\mathrm{PP98}} = \rho_0\nabla_{\boldsymbol{\ell}} \cdot \langle(\delta\mathbf{v} \cdot \delta\mathbf{v} + \delta\mathbf{v_A} \cdot \delta\mathbf{v_A})\delta\mathbf{v} - 2\delta\mathbf{v_A} \cdot \delta\mathbf{v}\delta\mathbf{v_A}\rangle$ is the so-called Politano and Pouquet's law [18], hereafter PP98. Interestingly, we evidence in Eq. (17) the presence of a new *source* term brought in by the anisotropy of the pressure tensor, which is written as a contraction of two increment tensors. Equation (17) is the fourth result of this paper. It generalizes PP98 to incompressible plasmas with a gyrotropic pressure and the notion of source terms. Indeed, so far the terminology of "source" terms introduced in Ref. [25] reflects compression (respectively dilatation) of the plasma that can sustain (respectively oppose) the cascade in the inertial range [33]. Here, we evidence a new source term in the incompressible gyrotropic limit that is not tied to plasma contraction/dilatation, but to pressure anisotropy. It reflects the exchange between the no-longer-conserved internal energy (unlike in incompressible pressure-isotropic flows [34]) with the sum of the magnetic and kinetic energies as can be seen in Eq. (4) where we have $-\frac{\overline{\overline{P}}}{\rho_0} : \nabla\mathbf{v} \neq \mathbf{0}$. This leads us to propose the following generalization of the notion of source: For compressible isentropic flows with a gyrotropic pressure tensor, the cascade of the kinetic and magnetic energies can be opposed/sustained by compression/dilatation of the fluid *and* by pressure anisotropy, the latter being relevant even in incompressible flows. For weakly compressible plasmas (e.g., SW), this result implies that the first-order correction to the PP98 law would not come from density fluctuations, but rather from (incompressible) pressure anisotropy.

Similarly to the compressible gyrotropic case discussed above, the parameters $\beta_\parallel$ and a_p remain undetermined. To determine the pressure components $P_\parallel$ and $P_\perp$ the internal energy equation (4) (now with $\rho = \rho_0$) is complemented by a new equation coming from imposing the incompressibility condition $\nabla \cdot \mathbf{v} = \mathbf{0}$ on the momentum equation (2) (with $\rho = \rho_0$, and $\mathbf{d_k} = \mathbf{f} = \mathbf{0}$ for simplicity), as done for incompressible isotropic hydrodynamics [6] or Hall-MHD [60]. This yields the generalized pressure balance equation for incompressible gyrotropic pressure tensor, namely,

$$\nabla \cdot \nabla \cdot (\rho \mathbf{v_A} \mathbf{v_A} - \rho \mathbf{v}\mathbf{v} - \overline{\overline{P_*}}) = \mathbf{0}. \tag{18}$$

Solving Eqs. (4) (with $\rho = \rho_0$) and (18) allows one to close the new incompressible gyrotropic MHD system proposed here and to self-consistently determine $P_\parallel$ and $P_\perp$. However, for nearly incompressible plasmas such as the SW, the exact law (17) can be directly applied to spacecraft data when $P_\parallel$ and $P_\perp$ are accessible to measurements assuming Eq. (18) to hold, as it has been done in all previous observational studies that used the PP98 model (assuming a scalar pressure).

Note finally that the new model of incompressible gyrotropic [whose exact law is given by Eq. (17)] admits the oblique firehose instability as a linear solution, which is the unstable version of the known shear Alfvén mode [50].

V. CONCLUSION

We derived general exact laws for homogeneous MHD and Hall-MHD turbulent flows that go beyond the pressure isotropy assumption, which make them more realistic to study strong turbulence in magnetized plasmas. By considering the specific case of a CGL closure, we showed that the new law involves new flux and source terms that potentially can reflect the impact of plasma instabilities on the turbulent cascade. In the limit of incompressible MHD with a gyrotropic pressure we provided a generalization of the Politano and Pouquet's law [17] to pressure anisotropic plasmas, where a new incompressible source term is revealed and highlights a fundamental difference between pressure isotropic and anisotropic plasmas: Internal energy is not conserved in the latter and pressure anisotropy can act as a source of free energy to supply the turbulent cascade with an additional energy. This work thus paves the road to different and more rigorous (albeit fluid) studies of the interplay between turbulent (fluid) cascade and plasma instabilities, both in numerical simulations and spacecraft observations when the full pressure tensor is accessible to measurements.

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[1] T. de Karman and L. Howarth, On the statistical theory of isotropic turbulence, Proc. R. Soc. London, Ser. A **164**, 192 (1938).

[2] A. S. Monin and A. M. Jaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 1 (MIT Press, Cambridge, MA, 1971).

[3] A. S. Monin and A. M. Jaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, 2nd ed., Vol. 2 (MIT Press, Cambridge, MA, 1975).

[4] A. N. Kolmogorov, Dissipation of energy in the locally isotropic turbulence, Proc. R. Soc. London, Ser. A **434**, 15 (1991).

[5] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Proc. R. Soc. London, Ser. A **434**, 9 (1991).

[6] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, U.K., 1995).

[7] R. A. Antonia, M. Ould-Rouis, F. Anselmet, and Y. Zhu, Analogy between predictions of Kolmogorov and Yaglom, J. Fluid Mech. **332**, 395 (1997).

[8] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I: Wave Turbulence*, Springer Series in Nonlinear Dynamics (Springer, Berlin, 1992).

[9] S. Galtier, *Introduction to Modern Magnetohydrodynamics* (Cambridge University Press, Cambridge, U.K., 2016).

[10] S. Galtier, *Physique de la Turbulence: des tourbillons aux ondes* (CNRS Editions/EDP Sciences (Savoirs Actuels), Les Ulis/Paris, 2021).

[11] C. W. Smith, K. Hamilton, B. J. Vasquez, and R. J. Leamon, Dependence of the dissipation range spectrum of interplanetary magnetic fluctuations on the rate of energy cascade, Astrophys. J. **645**, L85 (2006).

[12] L. Sorriso-Valvo, R. Marino, V. Carbone, A. Noullez, F. Lepreti, P. Veltri, R. Bruno, B. Bavassano, and E. Pietropaolo, Observation of Inertial Energy Cascade in Interplanetary Space Plasma, Phys. Rev. Lett. **99**, 115001 (2007).

[13] N. Andrés, F. Sahraoui, S. Galtier, L. Z. Hadid, R. Ferrand, and S. Y. Huang, Energy Cascade Rate Measured in a Collisionless Space Plasma with MMS Data and Compressible Hall Magnetohydrodynamic Turbulence Theory, Phys. Rev. Lett. **123**, 245101 (2019).

[14] L. Sorriso-Valvo, F. Catapano, A. Retino, O. Le Contel, D. Perrone, O. W. Roberts, J. T. Coburn, V. Panebianco, F. Valentini, S. Perri, A. Greco, F. Malara, V. Carbone, P. Veltri, O. Pezzi, F. Fraternale, F. Di Mare, R. Marino, B. Giles, T. E. Moore *et al.*, Turbulence-Driven Ion Beams in the Magnetospheric Kelvin-Helmholtz Instability, Phys. Rev. Lett. **122**, 035102 (2019).

[15] R. Bandyopadhyay, L. Sorriso-Valvo, A. Chasapis, P. Hellinger, W. H. Matthaeus, A. Verdini, S. Landi, L. Franci, L. Matteini, B. L. Giles, D. J. Gershman, T. E. Moore, C. J. Pollock, C. T. Russell, R. J. Strangeway, R. B. Torbert, and J. L. Burch, *In Situ* Observation of Hall Magnetohydrodynamic Cascade in Space Plasma, Phys. Rev. Lett. **124**, 225101 (2020).

[16] P. Quijia, F. Fraternale, J. E. Stawarz, C. L. Vsconez, S. Perri, R. Marino, E. Yordanova, and L. Sorriso-Valvo, Comparing turbulence in a Kelvin-Helmholtz instability region across the terrestrial magnetopause, Mon. Not. R. Astron. Soc. **503**, 4815 (2021).

[17] H. Politano and A. Pouquet, Dynamical length scales for turbulent magnetized flows, Geophys. Res. Lett. **25**, 273 (1998).

[18] H. Politano and A. Pouquet, von Kármán–Howarth equation for magnetohydrodynamics and its consequences on third-order longitudinal structure and correlation functions, Phys. Rev. E **57**, R21 (1998).

[19] S. Galtier, von Kármán–Howarth equations for Hall magnetohydrodynamic flows, Phys. Rev. E **77**, 015302(R) (2008).

[20] N. Andrés, S. Galtier, and F. Sahraoui, Exact scaling laws for helical three-dimensional two-fluid turbulent plasmas, Phys. Rev. E **94**, 063206 (2016).

[21] N. Andrés, P. D. Mininni, P. Dmitruk, and D. O. Gomez, von Kármán–Howarth equation for three-dimensional two-fluid plasmas, Phys. Rev. E **93**, 063202 (2016).

[22] S. Banerjee and S. Galtier, An alternative formulation for exact scaling relations in hydrodynamic and magnetohydrodynamic turbulence, J. Phys. A: Math. Theor. **50**, 015501 (2017).

[23] P. Hellinger, A. Verdini, S. Landi, L. Franci, and L. Matteini, von Kármán–Howarth equation for Hall magnetohydrodynamics: Hybrid simulations, Astrophys. J. **857**, L19 (2018).

[24] R. Ferrand, S. Galtier, F. Sahraoui, R. Meyrand, N. Andrés, and S. Banerjee, On exact laws in incompressible Hall magnetohydrodynamic turbulence, Astrophys. J. **881**, 50 (2019).

[25] S. Galtier and S. Banerjee, Exact Relation for Correlation Functions in Compressible Isothermal Turbulence, Phys. Rev. Lett. **107**, 134501 (2011).

[26] S. Banerjee and S. Galtier, Exact relation with two-point correlation functions and phenomenological approach for compressible magnetohydrodynamic turbulence, Phys. Rev. E **87**, 013019 (2013).

[27] S. Banerjee and S. Galtier, A Kolmogorov-like exact relation for compressible polytropic turbulence, J. Fluid Mech. **742**, 230 (2014).

[28] S. Banerjee, L. Z. Hadid, F. Sahraoui, and S. Galtier, Scaling of compressible magnetohydrodynamic turbulence in the fast solar wind, Astrophys. J., Lett. **829**, L27 (2016).

[29] N. Andrés and F. Sahraoui, Alternative derivation of exact law for compressible and isothermal magnetohydrodynamics turbulence, Phys. Rev. E **96**, 053205 (2017).

[30] N. Andrés, S. Galtier, and F. Sahraoui, Exact law for homogeneous compressible Hall magnetohydrodynamics turbulence, Phys. Rev. E **97**, 013204 (2018).

[31] N. Andrés, F. Sahraoui, S. Galtier, L. Z. Hadid, P. Dmitruk, and P. D. Mininni, Energy cascade rate in isothermal compressible magnetohydrodynamic turbulence, J. Plasma Phys. **84**, 21 (2018).

[32] S. Banerjee and N. Andrés, Scale-to-scale energy transfer rate in compressible two-fluid plasma turbulence, Phys. Rev. E **101**, 043212 (2020).

[33] R. Ferrand, S. Galtier, F. Sahraoui, and C. Federrath, Compressible turbulence in the interstellar medium: New insights from a high-resolution supersonic turbulence simulation, Astrophys. J. **904**, 160 (2020).

[34] P. Simon and F. Sahraoui, General exact law of compressible isentropic magnetohydrodynamic flows: Theory and spacecraft observations in the solar wind, Astrophys. J. **916**, 49 (2021).

[35] S. Banerjee and A. G. Kritsuk, Exact relations for energy transfer in self-gravitating isothermal turbulence, Phys. Rev. E **96**, 053116 (2017).

[36] S. Banerjee and A. G. Kritsuk, Energy transfer in compressible magnetohydrodynamic turbulence for isothermal self-gravitating fluids, Phys. Rev. E **97**, 023107 (2018).

[37] B. T. MacBride, C. W. Smith, and M. A. Forman, The turbulent cascade at 1 AU: Energy transfer and the third-order scaling for MHD, Astrophys. J. **679**, 1644 (2008).

[38] J. E. Stawarz, C. W. Smith, B. J. Vasquez, M. A. Forman, and B. T. MacBride, The turbulent cascade and proton heating in the solar wind at 1 AU, Astrophys. J. **697**, 1119 (2009).

[39] K. T. Osman, M. Wan, W. H. Matthaeus, J. M. Weygand, and S. Dasso, Anisotropic Third-Moment Estimates of the Energy Cascade in Solar Wind Turbulence Using Multispacecraft Data, Phys. Rev. Lett. **107**, 165001 (2011).

[40] L. Z. Hadid, F. Sahraoui, and S. Galtier, Energy cascade rate in compressible fast and slow solar wind turbulence, Astrophys. J. **838**, 9 (2017).

[41] L. Z. Hadid, F. Sahraoui, S. Galtier, and S. Y. Huang, Compressible Magnetohydrodynamic Turbulence in the Earth's Magnetosheath: Estimation of the Energy Cascade Rate Using *In Situ* Spacecraft Data, Phys. Rev. Lett. **120**, 055102 (2018).

[42] D. Griffel and L. Davis, The anisotropy of the solar wind, Planet. Space Sci. **17**, 1009 (1969).

[43] S. P. Gary, R. M. Skoug, J. T. Steinberg, and C. W. Smith, Proton temperature anisotropy constraint in the solar wind: ACE observations, Geophys. Res. Lett. **28**, 2759 (2001).

[44] J. C. Kasper, A. J. Lazarus, and S. P. Gary, Wind/SWE observations of firehose constraint on solar wind proton temperature anisotropy, Geophys. Res. Lett. **29**, 20 (2002).

[45] S. D. Bale, J. C. Kasper, G. G. Howes, E. Quataert, C. Salem, and D. Sundkvist, Magnetic Fluctuation Power Near Proton Temperature Anisotropy Instability Thresholds in the Solar Wind, Phys. Rev. Lett. **103**, 211101 (2009).

[46] P. Hellinger, P. Trvnek, J. C. Kasper, and A. J. Lazarus, Solar wind proton temperature anisotropy: Linear theory and WIND/SWE observations, Geophys. Res. Lett. **33**, L09101 (2006).

[47] F. Sahraoui, G. Belmont, L. Rezeau, N. Cornilleau-Wehrlin, J. L. Pincon, and A. Balogh, Anisotropic Turbulent Spectra in the Terrestrial Magnetosheath as Seen by the Cluster Spacecraft, Phys. Rev. Lett. **96**, 075002 (2006).

[48] G. F. Chew, M. Goldberger, and F. E. Low, The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions, Proc. R. Soc. London, Ser. A **236**, 112 (1956).

[49] T. Passot and P. L. Sulem, Collisionless magnetohydrodynamics with gyrokinetic effects, Phys. Plasmas **14**, 082502 (2007).

[50] P. Hunana, A. Tenerani, G. P. Zank, E. Khomenko, M. L. Goldstein, G. M. Webb, P. S. Cally, M. Collados, M. Velli, and L. Adhikari, An introductory guide to fluid models with anisotropic temperatures. Part 1. CGL description and collisionless fluid hierarchy, J. Plasma Phys. **85**, 205850602 (2019).

[51] L. N. Hau and B. J. Wang, On MHD waves, fire-hose and mirror instabilities in anisotropic plasmas, Nonlinear Processes Geophys. **14**, 557 (2007).

[52] E. E. Scime, P. A. Keiter, M. M. Balkey, R. F. Boivin, J. L. Kline, M. Blackburn, and S. P. Gary, Ion temperature anisotropy limitation in high beta plasmas, Phys. Plasmas **7**, 2157 (2000).

[53] A. Schekochihin, S. Cowley, R. Kulsrud, G. Hammett, and P. Sharma, Magnetised plasma turbulence in clusters of galaxies, in *The Magnetized Plasma in Galaxy Evolution*, edited by K. T. Chyzy, K. Otmianowska-Mazur, M. Soida, and R.-J. Dettmar (Jagiellonian University, Kraków, 2005), pp. 86–92.

[54] P. Sharma, G. W. Hammett, and E. Quataert, Shearing box simulations of the MRI in a collisionless plasma, Astrophys. J. **637**, 952 (2006).

[55] F. Sahraoui, G. Belmont, J. L. Pincon, L. Rezeau, A. Balogh, P. Robert, and N. Cornilleau-Wehrlin, Magnetic turbulent spectra in the magnetosheath: new insights, Ann. Geophys. **22**, 2283 (2004).

[56] M. W. Kunz, A. A. Schekochihin, and J. M. Stone, Firehose and Mirror Instabilities in a Collisionless Shearing Plasma, Phys. Rev. Lett. **112**, 205003 (2014).

[57] K. T. Osman, W. H. Matthaeus, K. H. Kiyani, B. Hnat, and S. C. Chapman, Proton Kinetic Effects and Turbulent Energy Cascade Rate in the Solar Wind, Phys. Rev. Lett. **111**, 201101 (2013).

[58] R. D. Hazeltine, S. M. Mahajan, and P. J. Morrison, Local thermodynamics of a magnetized, anisotropic plasma, Phys. Plasmas **20**, 022506 (2013).

[59] A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, E. Quataert, and T. Tatsuno, Astrophysical gyrokinetics: Kinetic and fluid turbulent cascades in magnetized weakly collisional plasma, Astrophys. J., Suppl. Series **182**, 310 (2009).

[60] F. Sahraoui, S. Galtier, and G. Belmont, On waves in incompressible Hall magnetohydrodynamics, J. Plasma Phys. **73**, 723 (2007).