



**Enhanced cooperation in multiplayer snowdrift games with random and dynamic groupings**Chen Xu <sup>\*</sup>*School of Physical Science and Technology, Soochow University, Suzhou, 215006, China*Pak Ming Hui <sup>†</sup>*Department of Physics and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong, China*

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An analytically tractable generalization of the  $N$ -person snowdrift (NSG) game that illustrates how cooperation can be enhanced is proposed and studied. The number of players competing within a NSG varies from one time step to another. Exact equations governing the frequency of cooperation  $f_c(r)$  as a function of the cost-to-benefit ratio  $r$  within an imitation strategy updating scheme are presented. For group sizes  $g$  uniformly distributed within the range  $g \in [1, g_m]$ , an analytic formula for the critical value  $r_c(g_m)$ , below which the system evolves into a totally cooperative (AllC) state, is derived. In contrast, a fixed group size NSG does not support an AllC state. The result  $r_c(g_m)$  requires the presence of sole-player groups and involves the inverse of the harmonic numbers and, more generally, the inverse first moment of the group size distribution. For  $r > r_c(g_m)$ , the equation that determines the dynamical mixed states  $f_c(r)$  is given, with exact solutions existing for  $g_m \leq 5$ . The exact treatment allows the study of the phase boundary between the AllC state and the mixed states. The analytic results are checked against simulation results and exact agreements are demonstrated. The analytic form of the critical  $r_c(g_m)$  illustrates the necessity of having groups of a sole player in the evolutionary process. This result is supported by simulations with group sizes excluding the sole groups for which no AllC state emerges. A physically transparent picture of the importance of the sole players in inducing an AllC state is further presented based on the last surviving pattern before the AllC state is attained. The exact expression  $r_c(g_m)$  turns out to remain valid for nonuniform group-size distributions. Our analytical tractable generalization, therefore, sheds light on how a competing environment with variable group sizes could enhance cooperation and induce an AllC state.

DOI: [10.1103/PhysRevE.105.054309](https://doi.org/10.1103/PhysRevE.105.054309)**I. INTRODUCTION**

It is important to study how cooperation emerges in a community from the interactions among selfish individuals [1–5], as evidenced by the wealth of literature on the subject. A typical setting is that of a cooperative individual needs to do some work and thus pay a cost for getting a benefit, and a noncooperative individual gets a free ride of sharing the benefit. An obvious question is “why cooperate?” The evolutionary game theory has been established as a powerful tool for understanding cooperative behavior [6,7], with many game theoretical settings designed to study the emergence and extent of cooperation in competitive populations [8–14]. Important rules for the emergence of cooperation were discovered [15]. A detailed review on human cooperation within the scope of statistical physics was given by Perc *et al.* [16]. As human psychology and behavior are a complicated subject, the emergence of cooperation may also be caused by personal and moral preferences beyond the social preferences as reviewed recently by Capraro and Perc [17]. Games typically involve two-person competitions, but they have been

generalized to involve  $N$ -person competitions. The prisoner’s dilemma (PD), as a paradigm, has been widely used to study the cooperative behavior among selfish individuals via two-person interactions [1–3,6,8,18]. The  $N$ -person generalization of the PD, called the public goods game (PGG), exhibits features due to group interactions when  $N$ -person interactions are incorporated [19–22]. These  $N$ -person interactions can be incorporated into a population in a well-mixed manner [20,23], with spatial relationship [24,25], or under mechanisms that involve punishments and/or rewards [23–28]. Other  $N$ -person extensions of two-person games have also been studied [29–35].

In general, the  $N$ -person scenario involves a group of  $N$  individuals, who come together momentarily within a bigger population and decide on being cooperative or not. Many previous works on  $N$ -person games have the number  $N$  being fixed throughout. When the group size varies, cooperation could be sustained or enhanced in the PGG [20–22]. In investigating problems in physics and applied mathematics, much insight will be gained by having an analytically tractable problem. Here, we present an evolutionary  $N$ -person snowdrift game, in which the group size  $g$  is varying from time step to time step within a range  $g \in [1, g_m]$ , that can be treated analytically. The analytic approach leads to an expression for the critical cost-to-benefit parameter, below which a totally

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cooperative population results. It also gives a physically transparent picture on how the varying group sizes serve to enhance cooperation.

We motivate the problem by the setup in the two-person snowdrift game (SG) [11]. The scenario is one with a snowdrift on a road obstructing two drivers traveling in opposite directions. It takes some labor to remove the snowdrift and thus a cost  $c$  ( $c > 0$ ) to complete the task. There are two strategies: to cooperate ( $C$ ) or not-to-cooperate ( $D$ ). When both drivers cooperate, they can share the cost. A noncooperative driver, who refuses to shovel the snowdrift, is called a defector ( $D$ ). A defector can enjoy the work of a cooperater without paying a cost. As long as there is one cooperater, both drivers can continue their journey and each enjoys a benefit  $b$ . When both drivers are cooperaters, each gets a payoff  $b - c/2$ . When a cooperater meets a defector, the cooperater gets a payoff  $b - c$  and the defector gets  $b$ . When the drivers are both noncooperative, they get stuck and each gets a zero payoff. For the SG to make sense,  $b > c$  so that the cooperater still gets a payoff that is higher than the case when two defectors meet. For  $c/2 < b < c$ , the scenario becomes that of the prisoner's dilemma (PD) [36]. For the SG, the two Nash equilibria are to take  $C$  when the opponent takes  $D$ , and to take  $D$  when the opponent takes  $C$  [7]. The structure or the organization of the population also matters. Here are two examples. In a well-mixed population, a stable mixed state with coexisting  $C$  and  $D$  strategies emerges when strategy updating is carried out by imitating the opponent's strategy of better performance [11]. A networked population supports a totally cooperative (AllC) state when a synchronous strategy updating rule is applied [11,37].

There are tasks in real-life situations that involve bigger groups than two persons. The famous public goods game [38] for studying group interactions in experimental economics is a representative  $N$ -person design. We proposed the  $N$ -person snowdrift game (NSG) as an extension of the two-person SG and studied its cooperation and dynamics [39–41]. In NSG, the cooperaters in a group can share the labor and thus the cost, and so the payoffs to the cooperaters become dependent on the number of cooperaters in the  $N$ -group. These  $N$ -groups are formed momentarily as the strategies of the players evolve. In a well-mixed population, strategy updates based on the replicator dynamics shows that cooperation is suppressed as the group size increases [39]. The players tend to take the  $D$ -strategy and hope that someone in the  $N$ -group could complete the task. As a result, the population does not support an AllC state. The NSG was subsequently modified by Souza *et al.* [42] to accommodate situations in which the task can only be completed by at least having a certain number of cooperaters in the  $N$ -group. Another modification was to include the extra benefit that a task can be completed sooner when more cooperaters are present in the  $N$ -group [43]. Comparing with the original NSG, cooperation is enhanced and an infinite population can evolve into an AllC state under proper conditions.

In the NSG and its variations, however, the number  $N$  of participating players in a group often takes on a fixed number during the strategy evolutionary process. In such a situation, the players are often competing in a well-mixed population, in a static spatial relationship defined by an underlying structure

[44,45], or in preassigned groups of a given size [39,42]. In the PGG, enhanced cooperation was found when the competing group size was allowed to vary [21,46]. We are, therefore, motivated to study the effects of having a varying group size *during* the evolutionary process. First, it is always important to have a nontrivial but exactly solvable model so that further complications can be studied subsequently in a controllable manner. Second, it has been realized that the dynamical grouping of players, often called agents in different contexts, is crucial in understanding many real-life phenomena across different disciplines. Such a grouping mechanism of agents often involves dynamical formation, growth, and dissociation of groups of various sizes. An agent will then belong to different groups with different members at different times. Diversified phenomena including patterns observed in market trading [47–49], human groups formation in online guilds and offline gangs [50], variations in foreign-exchange rates and the spread of a disease among school children [51], insurgent and terrorists dynamics [52,53], and more recently on online hate ecology [54], all have dynamical grouping being a crucial element. Closer to daily-life, a person may interact with other passengers on a train, with colleagues in a company, with other customers in a restaurant, and with friends at a pub in their daily routines. Here, we propose and study a model in which the group size can vary from one time step to another. Assuming, for simplicity, that the group size  $g$  can vary within a uniform distribution with  $g \in [1, g_m]$ , we study the model analytically for the existence of an AllC state. We find an analytic expression of the critical value  $r_c$  of the cost-to-benefit ratio below which an AllC state is supported. This is in sharp contrast with the original NSG in which no AllC state is supported. For the mixed state for  $r > r_c$  with coexisting  $C$  and  $D$  strategies, we present the governing equations for the cooperation level  $f_c(r)$  for which analytic solutions are possible for  $g_m \leq 5$ . The phase diagram in the  $g_m - r$  space is presented. The analytic results are further checked against simulation results. We also give a physical picture for the importance of the  $g = 1$  groups in the evolutionary process in promoting the AllC state.

## II. MODEL

We consider a large population with  $M$  players. The context under consideration is a multiplayer SG with a *variable number* of players in the group momentarily engaging in a SG. We refer to our model as the *variable*  $N$ -person SG (VNSG). The special case of a fixed and predetermined group size reduces to the NSG [39–41] previously studied. In the present model, the number of players participating in the SG is not fixed but instead is randomly chosen within a range in each time step. The group size affects the number of attending cooperaters in the group. The payoff to a player, therefore, depends not only on the player's strategy but also on the instantaneous group size and the number of cooperaters in the group. The model can be applied to scenarios involving the varying interests among a population, e.g., a market in which the number of investors varies from time to time. Specifically, let  $g$  be the number of players involved in a VNSG at a time. The value of  $g$  lies within the range  $[1, g_m]$ , with  $g_m$  being the maximum size of the attending group. The case of  $g = 1$  is

included because a single player could also complete a task and receive a positive payoff. We specify the payoff assignments to the  $g$  players first, before describing the evolutionary process from one time step to another. After choosing  $g$ , the group of  $g$  players are involved in a round of NSG after which the payoff to a cooperator is

$$v_c = b - \frac{c}{n}, \quad (1)$$

with  $n$  being the number of cooperators in the group, and the payoff to a defector is

$$v_d = \begin{cases} b, & n > 0, \\ 0, & n = 0. \end{cases} \quad (2)$$

The important parameter is the cost-to-benefit ratio  $r = c/b$  with  $0 < r < 1$ . This can be seen by dividing Eqs. (1) and (2) by  $b$ , letting  $V_c \equiv v_c/b$  and  $V_d \equiv v_d/b$  and rewriting the payoffs as

$$V_c = 1 - \frac{r}{n} \quad (3)$$

and

$$V_d = \begin{cases} 1, & n > 0, \\ 0, & n = 0. \end{cases} \quad (4)$$

The effects of the variable group sizes enter through the evolutionary process as follows. At each time step, a target player  $i$  is randomly selected from the whole population and a value of group size  $g_i$  is chosen within  $[1, g_m]$ . The player will participate in a NSG with the other  $(g_i - 1)$  players randomly selected from the whole population. After the round of NSG, the player  $i$  gets a payoff  $V_i$  according to his current strategy based on Eqs. (3) or (4). For carrying out a possible evolutionary process, a player  $j$  is randomly selected from the population as a reference player and a group size  $g_j \in [1, g_m]$  is randomly chosen. The player  $j$  then engages in a NSG with other  $(g_j - 1)$  randomly selected players in the population and attains a payoff  $V_j$ . Note that  $g_j$  is, in general, different from  $g_i$ . The target player  $i$  will consider a switch in its strategy by comparing  $V_i$  with  $V_j$ . If  $V_i \geq V_j$ , then the player  $i$  will keep his strategy. If  $V_i < V_j$ , then the player  $i$  has a probability  $w_{ij} = (V_j - V_i)/\Omega$  to imitate player  $j$ 's strategy, where  $\Omega$  ensures that  $w_{ij} \leq 1$ . Using the reduced payoffs allows us to simply take  $\Omega = 1$ . This strategy updating rule is a learning mechanism that helps the strategy with a higher payoff to prevail in the population [11,39]. As the competing players are randomly selected in each time step from the whole population, the players interact effectively in a well-mixed situation. Note that the reference player is selected globally from the population and is not restricted to be within the local competing group of the target player. Such global referencing was also employed in multiplayer ultimatum games and  $N$ -person PGG in evolving the fitness of the players [21,32]. The strategy updating scheme is based on the idea that there is a tendency among players to mimic what the more successful players are doing, but without knowing the details of the reference's players competing group. Depending on the samplings of the target player's group members and the reference player's group members, the difference between the competing environments of the two players varies from one time step to another. In a social context, learning

beyond one's close circle of neighbors and even learning from someone one does not know has become much easier in the information age, when public information is readily available through various social media. The questions of interest in our model are the levels of cooperation in the population as a function of the payoff parameter  $r$  and the range of group sizes characterized by  $g_m$ , and the transitions between a totally cooperative state that exists even without a networked structure [44] among the players and a mixed-strategy state with the two strategies coexisting. The present model allows analytic treatment to these interesting questions, as we now show.

### III. THEORY

Let  $f_c(t)$  be the frequency of cooperation, i.e., the fraction of  $C$ -players, in the population at the time step  $t$ . According to the model, the probability  $p_c(g, n)$  of randomly selecting a  $C$ -player to participate in a  $g$ -sized NSG given that there are  $n$  cooperators in it is given by

$$p_c(g, n) = \frac{f_c}{g_m} \binom{g-1}{n-1} f_c^{n-1} (1-f_c)^{g-n}. \quad (5)$$

Here,  $f_c$  is the probability of randomly selecting a  $C$ -player and the factor  $1/g_m$  is the probability that the  $C$ -player participating in a  $g$ -size group because  $g_m$  gives the possible values of the group size. The remaining factors in the right-hand side of Eq. (5) represent the binomial distribution that there are  $n-1$  other  $C$ -players in the remaining  $g-1$  players in the group. Similarly, the probability  $p_d(g, n)$  of randomly selecting a  $D$ -player to participate in a  $g$ -sized NSG given that there are  $n$  cooperators in it is given by

$$p_d(g, n) = \frac{1-f_c}{g_m} \binom{g-1}{n} f_c^n (1-f_c)^{g-1-n}. \quad (6)$$

We then proceed to construct a dynamical equation of  $f_c(t)$  in the system. Based on the strategy updating rule, there are two situations. When a target  $C$ -player having a payoff  $1 - r/n_i$  is selected for an updating process with the probability  $p_c(g_i, n_i)$ , he has a probability  $p_d(g_j, n_j)$  ( $n_j \geq 1$ ) to make reference to a  $D$ -player having a payoff 1. The target  $C$ -player, therefore, has a probability  $r/n_i$  to switch his strategy. The number of  $C$ -players will then drop by 1 and the frequency of cooperation  $f_c$  will drop by an amount of  $1/M$ . When a target  $D$ -player having a payoff 0 is selected for an updating process with the probability  $p_d(g_i, 0)$ , he has a probability  $p_c(g_j, n_j)$  to make reference to a  $C$ -player having a payoff  $(1 - r/n_j)$ . The target  $D$ -player, therefore, has a probability  $(1 - r/n_j)$  to switch his strategy. The number of  $C$ -player will then increase by 1 and the frequency of cooperation  $f_c$  will increase by  $1/M$ . It is worth noting that a target  $D$ -player selected with probabilities  $p_d(g_j, n_j)$  ( $n_j \geq 1$ ) for an updating process will not make a switch since such a  $D$ -player has the highest payoff. Accounting for all possible events that would lead to a change in the frequency of cooperation, the

dynamical equation of  $f_c(t)$  is given by

$$\begin{aligned} \frac{df_c}{dt} = & -\frac{1}{M} \sum_{g_j=2}^{g_m} \sum_{n_j=1}^{g_j-1} p_d(g_j, n_j) \sum_{g_i=1}^{g_m} \sum_{n_i=1}^{g_i} p_c(g_i, n_i) \frac{r}{n_i} \\ & + \frac{1}{M} \sum_{g_j=1}^{g_m} \sum_{n_j=1}^{g_j} p_c(g_j, n_j) \sum_{g_i=1}^{g_m} p_d(g_i, 0) \left(1 - \frac{r}{n_j}\right). \end{aligned} \quad (7)$$

Here, the first (second) term on the right-hand side includes all the possibilities that a target  $C$ -player ( $D$ -player) will switch into using the strategy  $D$  ( $C$ ) by comparing payoffs with a referencing  $D$ -player ( $C$ -player), as detailed above. In general, by setting  $df_c/dt = 0$ , we can immediately get the two solutions with  $f_c = 1$  and  $f_c = 0$  for the homogeneous states. There is another solution for a mixed state ( $0 < f_c < 1$ ) that can be solved for given  $r$  and  $g_m$ . Defining

$$\Delta_1(f_c, g_m) = \frac{1}{M} \sum_{g_j=2}^{g_m} \sum_{n_j=1}^{g_j-1} p_d(g_j, n_j) \sum_{g_i=1}^{g_m} \sum_{n_i=1}^{g_i} \frac{p_c(g_i, n_i)}{n_i}, \quad (8)$$

$$\Delta_2(f_c, g_m) = \frac{1}{M} \sum_{g_j=1}^{g_m} \sum_{n_j=1}^{g_j} p_c(g_j, n_j) \sum_{g_i=1}^{g_m} \frac{p_d(g_i, 0)}{n_j}, \quad (9)$$

$$\Delta_3(f_c, g_m) = \frac{1}{M} \sum_{g_j=1}^{g_m} \sum_{n_j=1}^{g_j} p_c(g_j, n_j) \sum_{g_i=1}^{g_m} p_d(g_i, 0), \quad (10)$$

it follows from Eq. (7) that the value of  $f_c$  in the steady mixed state is given implicitly by

$$r = \frac{\Delta_3(f_c, g_m)}{\Delta_1(f_c, g_m) + \Delta_2(f_c, g_m)}, \quad (0 < f_c < 1). \quad (11)$$

Equation (11) is a  $(g_m - 1)$ -th order equation for  $f_c(r, g_m)$ . Analytic solutions can be found for  $g_m \leq 5$ . For example, when  $g_m = 3$ , Eq. (11) gives a quadratic equation for  $f_c$  in the form of

$$r = \frac{6(f_c^2 - 3f_c + 3)}{2f_c^2 - 9f_c + 18}, \quad (12)$$

and thus the solution

$$\begin{aligned} f_c(r, g_m = 3) = & \frac{3[6 - 3r - \sqrt{-7r^2 + 28r - 12}]}{4(3 - r)}, \\ & (6/11 < r < 1). \end{aligned} \quad (13)$$

Note that the result gives a mixed state with  $0 < f_c < 1$  only in the range of the cost-to-benefit ratio  $r \in (6/11, 1)$ . For  $r \leq 6/11$  and  $g_m = 3$ , the system will evolve to an absorbing AllC state with  $f_c = 1$ . Therefore, there exists a critical value of  $r_c$  for a given  $g_m$  below which the system attains an AllC state. The solution of  $f_c = 0$ , i.e., an AllD state, exists only when  $r \geq 1$  and it is not the focus of discussion here. Even for values of  $g_m$  that Eq. (11) can only be solved numerically

for  $f_c(r, g_m)$ , the equation can be used to locate the critical value  $r_c$  for all values of  $g_m$  analytically. Noting that the phase transition from a mixed state to an AllC state is continuous in  $r$ , the value of  $r_c$  is the first value of  $r$  that  $f_c = 1$  for a given value of  $g_m$  and thus can be obtained from Eq. (11) by setting  $f_c = 1$ . As an example, Eq. (11) for  $g_m = 6$  is

$$r = \frac{60(f_c^5 - 6f_c^4 + 15f_c^3 - 20f_c^2 + 15f_c - 6)}{10f_c^5 - 72f_c^4 + 255f_c^3 - 400f_c^2 + 450f_c - 360}. \quad (14)$$

The critical value  $r_c(g_m = 6)$  then follows by setting  $f_c = 1$  in Eq. (14), giving the analytic result  $r_c = 60/147$ . In contrast, the mixed state  $f_c(r, g_m = 6)$  for  $r_c < r < 1$  can only be found by solving Eq. (14) numerically. Interestingly, the same approach using Eq. (11) with Eqs. (8)–(10) gives  $r_c$  analytically for all values of  $g_m$  after some algebra. The result is that  $r_c$  is given by the inverse of the  $g_m$ -th harmonic number for a given  $g_m$ , or equivalently

$$\frac{1}{r_c(g_m)} = \sum_{g=1}^{g_m} \frac{1}{g}. \quad (15)$$

The simple form of the result asks for a physical argument, which we will discuss in the following section.

#### IV. RESULTS AND DISCUSSIONS

The validity of the analytic results can be checked against results obtained by numerical simulations on the model. In what follows, the simulation results are obtained in a population of  $M = 10^4$  players for different values of the upper bound  $g_m$  in the group size, with an initial frequency of cooperation of 0.5. After the system equilibrates, the frequency of cooperation is obtained by sampling the population every  $10^4$  steps for  $10^4$  times. The chosen population size is sufficiently large that finite-size effects are not noticeable in the results.

Figure 1 shows the frequency of cooperation  $f_c$  as a function of  $r$  at four different values of  $g_m = 2, 3, 4, 6$ . The simulation results (symbols) and the theoretical results (curves) are in excellent agreement for all the cases. For  $g_m = 2$ , the results are given analytically by  $r_c(g_m = 2) = 2/3$  and  $f_c(r) = 4(1 - r)/(2 - r)$  for  $2/3 < r < 1$ . For  $g_m = 4$ , Eq. (11) gives the following cubic equation to solve for  $f_c(r)$ :

$$r = \frac{12(f_c^3 - 4f_c^2 + 6f_c - 4)}{3f_c^3 - 16f_c^2 + 36f_c - 48}. \quad (16)$$

Analytic solution exists, but the expression is too lengthy to be shown here. The critical value  $r_c(g_m = 4) = 12/25$  follows from Eq. (15). The theoretical results of  $g_m = 3$  and 6 can be obtained by solving Eqs. (12) and (14), respectively. The general behavior is that  $r_c$  gradually drops as  $g_m$  increases. In the mixed phase,  $f_c(r)$  drops with  $r$  and approaches zero as  $r$  approaches unity in a way governed by Eq. (11).

A phase diagram on the  $g_m - r$  plane is shown in Fig. 2. The phase boundary (curve) separating the mixed  $C + D$  phase and the AllC phase is calculated using Eq. (15) and confirmed by the simulation results (symbols). The behavior of the harmonic numbers gives  $r_c$  a gradually lower sensitivity



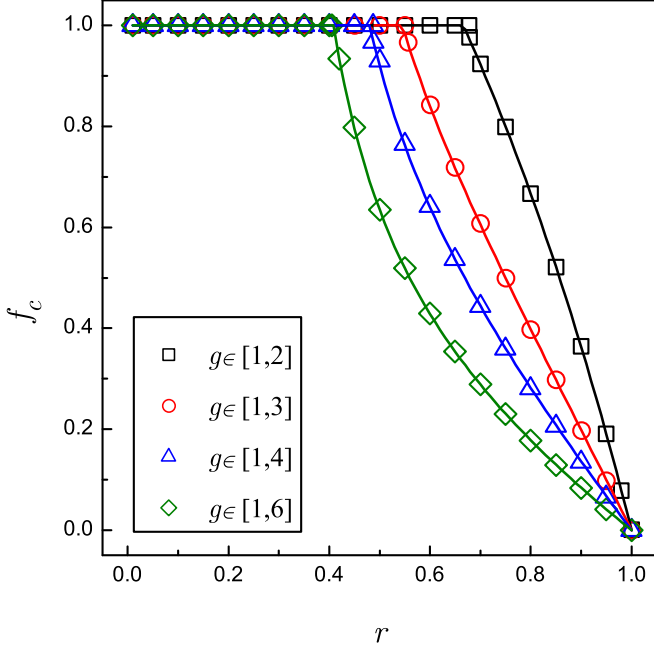


FIG. 1. The dependence of  $f_c$  on  $r$  for four different values of  $g_m$ . The competing group sizes are randomly selected within the range of  $g = 1$  to  $g = g_m$  during evolutionary process. Results obtained by the theory (curves) and results obtained by simulations (symbols) are shown for comparison. There is an AllC phase with  $f_c = 1$  for  $r < r_c$ , in contrast to the NSG.

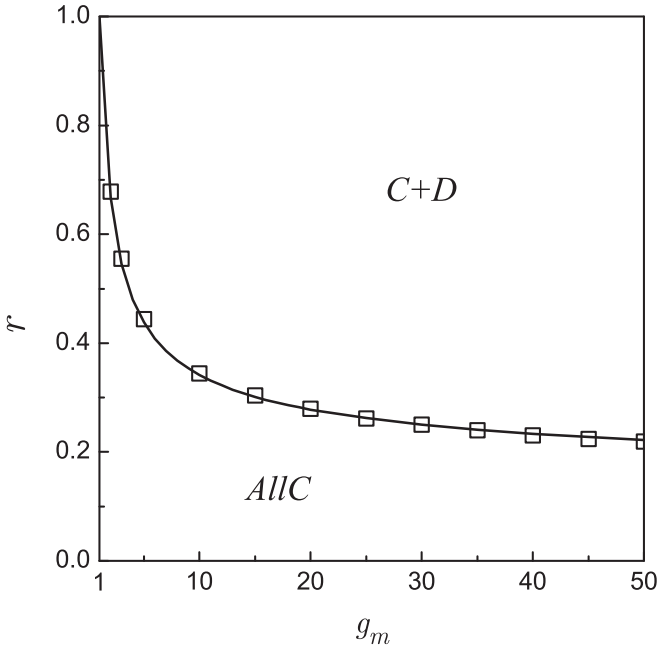


FIG. 2. Phase diagram on the  $g_m - r$  plane. The phase boundary as obtained by theory (curve) and by simulations (symbols) are shown. The phase boundary separates a homogeneous phase with a totally cooperative population (AllC phase) from a mixed dynamical phase characterized by a coexistent population with cooperators and defectors ( $C + D$  phase).

on  $g_m$  as  $g_m$  increases, leading to the existence of an AllC state even for very large  $g_m$ . The interesting physics here is the importance of the presence of the  $g = 1$  groups in inducing the AllC state in the evolutionary process. In sharp contrast, the NSG with any fixed value of group size  $N \geq 2$  does *not* support an AllC phase for  $r > 0$  [39]. Our results indicate, however, that even for  $g \in [1, g_m = 1000]$ , we have an AllC state for  $r < 0.133$  even the probability of having the  $g = 1$  groups in the evolutionary process is only 0.001. The mere presence of the  $g = 1$  groups in the dynamics, therefore, must be responsible for the emergence of AllC state. To understanding the point, it is instructive to consider the last surviving pattern [37], from which a physically transparent picture behind Eq. (15) and the role played by the  $g = 1$  groups emerges. As a system evolves to an AllC state, the last surviving pattern is that of a single  $D$ -player in an otherwise cooperative population. When the probability  $\mathcal{P}_{c \rightarrow d}$  for the number of  $D$ -players to grow is smaller than the probability  $\mathcal{P}_{d \rightarrow c}$  for the number to drop, the last surviving  $D$ -player will be replaced and an AllC state will result. This sets a criterion for the value  $r_c$ . For  $r < r_c$ ,  $\mathcal{P}_{c \rightarrow d} < \mathcal{P}_{d \rightarrow c}$ , and for  $r > r_c$ ,  $\mathcal{P}_{c \rightarrow d} > \mathcal{P}_{d \rightarrow c}$ . The value of  $r_c$  can thus be estimated by setting  $\mathcal{P}_{c \rightarrow d} = \mathcal{P}_{d \rightarrow c}$ . For a large population, the probability  $\mathcal{P}_{c \rightarrow d}$  is given by

$$\mathcal{P}_{c \rightarrow d} = p_{c,d} \left( 1 - \frac{1}{g_m} \right) \sum_{g=1}^{g_m} \frac{1}{g_m} \frac{r}{g}, \quad (17)$$

where  $p_{c,d}$  is the probability of selecting a target  $C$ -player and a reference  $D$ -player in the evolutionary process. The factor  $1/g_m$  in the summation is approximately the probability of the target  $C$ -player being in a totally cooperative group of size  $g$ ; and the factor  $(1 - 1/g_m)$  is the probability of the referencing  $D$ -player gaining a payoff of 1 in a group of size  $g > 1$ . Similarly, the probability  $\mathcal{P}_{d \rightarrow c}$  is given by

$$\mathcal{P}_{d \rightarrow c} = p_{d,c} \frac{1}{g_m} \sum_{g=1}^{g_m} \frac{1}{g_m} \left( 1 - \frac{r}{g} \right), \quad (18)$$

where  $p_{d,c}$  is the probability of selecting a target  $D$ -player and a reference  $C$ -player in the evolutionary process. The factor  $1/g_m$  in the summation is the probability of the target  $D$ -player being in a group of size  $g = 1$  and having a payoff 0, and the factor  $1/g_m$  is again the probability of the reference  $C$ -player being in a totally cooperative group of size  $g$ . The critical value  $r_c$  can be estimated by equating  $\mathcal{P}_{c \rightarrow d}$  and  $\mathcal{P}_{d \rightarrow c}$ , with the resulting expression  $r_c = 1 / (\sum_{g=1}^{g_m} \frac{1}{g})$  happens to be identical to Eq. (15). It should, however, be noted that Eq. (15) in Sec. III was obtained by rigorous mathematical manipulations of the governing equations. The physical picture emerged from the discussion can be summarized in simple terms. A single  $D$ -player competing in any group of sizes  $g \geq 2$  will attain the highest payoff and therefore will not make a switch in strategy and the single  $D$ -player cannot be made extinct. The presence of  $g = 1$  groups facilitates the extinction of  $D$ -players, because the occurrence of a single  $D$ -player in a  $g = 1$  group will definitely switch to the  $C$ -strategy due to their zero payoff.

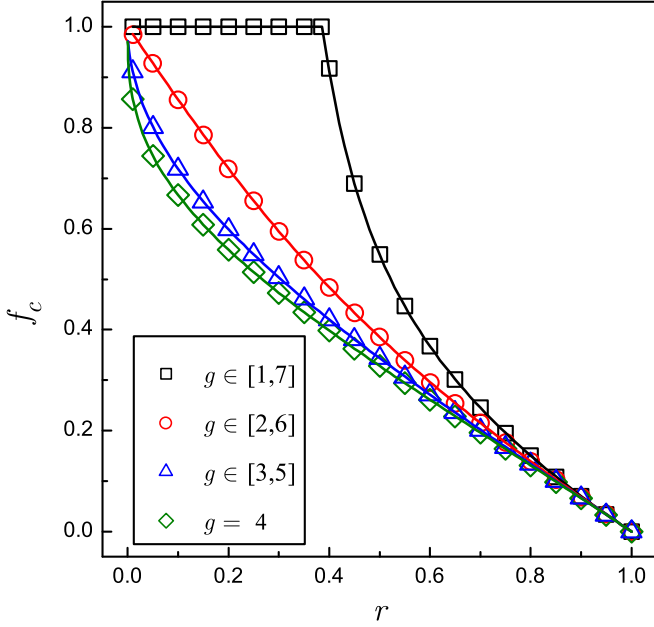


FIG. 3. The dependence of  $f_c$  on  $r$  for several distributions with identical mean group size of 4. Results obtained by the theory (curves) and results obtained by simulations (symbols) are shown for comparison. The case of  $g = 4$  refers to the case of NSG with a fixed group size of 4. The presence of  $g = 1$  groups induces the extinction of  $D$ -strategy for  $r < r_c$  and leads to an AllC state.

Equation (15) is also mathematically illuminating. For the uniform distribution of group sizes considered so far, every

group size occurs with the probability  $w(g) = 1/g_m$  for all values of  $g \in [1, g_m]$ . We can rewrite Eq. (15) for  $r_c(g_m)$  as

$$\frac{1}{r_c(g_m)} = g_m \sum_{g=1}^{g_m} \frac{1}{g_m g} = \frac{1}{w(1)} \sum_{g=1}^{g_m} w(g) \frac{1}{g} = \frac{1}{w(1)} \left\langle \frac{1}{g} \right\rangle, \tag{19}$$

where  $w(1)$  is the probability of having  $g = 1$  groups and  $\langle \dots \rangle$  is an average over the distribution  $w(g)$ . The last form implies that there exists a finite  $r_c < 1$  as long as  $w(1) \neq 0$ . This echoes the argument of the extinction of  $D$ -players induced by the presence of the  $g = 1$  groups. Most interestingly, it can be shown that the last expression in Eq. (19) remains valid for *nonuniform* distributions with values of  $g \in [1, g_m]$  by following the approach given in Sec. III. We skip the proof here, but the result indicates that our physical argument on the importance of the  $g = 1$  groups is valid for both uniform and nonuniform group-size distributions. As many dynamical grouping models exhibit nonuniform group-size distributions, it will be interesting to extend our study in conjunction with group formation and dissociation dynamics in the future.

Equation (19) indicates that it is the nonvanishing value of  $w(1)$  and the inverse first moment of the group-size distribution, instead of the first moment, that determine  $r_c$ . To illustrate this point further, we study three distributions with the same mean group size of  $\langle g \rangle = 4$ , namely, uniform distributions with  $g \in [1, 7]$ ,  $g \in [2, 6]$ , and  $g \in [3, 5]$ . The results are shown in Fig. 3. We also included the results of NSG with a fixed group size of 4 for comparison. For  $g \in [1, 7]$ ,  $r_c = 140/363$  and  $f_c(r)$  in the mixed state is obtained by solving

$$r = \frac{420(f_c^6 - 7f_c^5 + 21f_c^4 - 35f_c^3 + 35f_c^2 - 21f_c + 7)}{60f_c^6 - 490f_c^5 + 1764f_c^4 - 3675f_c^3 + 4900f_c^2 - 4410f_c + 2940}. \tag{20}$$

For the cases with the minimum group size larger than one, Eq. (11) can still be used to solve for  $f_c(r)$  after the summations over  $g_i$  and  $g_j$  in the  $\Delta_{1,2,3}$  [see Eqs. (8)–(10)] are properly modified to start from the corresponding lower limit. For  $g \in [2, 6]$ ,  $f_c(r)$  satisfies the equation

$$r = \frac{60(f_c^5 - 6f_c^4 + 15f_c^3 - 20f_c^2 + 15f_c - 5)}{10f_c^5 - 72f_c^4 + 255f_c^3 - 400f_c^2 + 450f_c - 300}. \tag{21}$$

The numerical solution of  $f_c(r)$  (curve) is shown in Fig. 3. There is no AllC state, as argued after Eq. (19). For  $g \in [3, 5]$ ,  $f_c(r)$  follows the equation

$$r = \frac{60(f_c - 1)^2(f_c^2 - 3f_c + 3)}{12f_c^4 - 75f_c^3 + 200f_c^2 - 270f_c + 180}, \tag{22}$$

and for  $g = 4$ ,

$$r = \frac{4(f_c - 1)^3}{f_c^3 - 4f_c^2 + 6f_c - 4}. \tag{23}$$

The results obtained by Eqs. (22) and (23) are also shown in Fig. 3, together with the simulation results (symbols). Again, the AllC state disappeared. In comparing the results among

$g \in [2, 6]$ ,  $g \in [3, 5]$ , and  $g = 4$ , we see that the NSG with fixed group size attains a lowering level of cooperation for the same value of the cost-to-benefit ratio  $r$ .

### V. SUMMARY

Motivated by the  $N$ -person snowdrift game (NSG) and the absence of a totally cooperative (AllC) phase in NSG together with the importance of having dynamical groups of various sizes in many phenomena, we proposed and studied the evolution of cooperation in a population in which the players compete within a NSG with the group size varying randomly from time step to time step within a range. The model can be treated analytically and exact equations governing the frequency of cooperation  $f_c(r)$  are derived, within an imitation strategy updating rule. The theoretical results are confirmed by numerical simulations. As exactly solvable models in this research area are rather unusual, our work adds to the literature on providing an exactly solvable model of a competing population with dynamical group-size  $N$ -player interactions. We, therefore, explained the analytic approach in details and provided a physical picture based on the analysis of the last

surviving patterns for the analytic results. A particularly important result is that as long as  $g = 1$  groups are allowed in the evolutionary process,  $r_c \neq 0$  and an AllC phase exists. In contrast to NSG, for which an AllC phase does not exist, the presence of the  $g = 1$  groups in the evolutionary process serves to induce the AllC phase. A physically transparent argument for the mathematical expression of  $r_c(g_m)$  based on the last surviving pattern was given. For other models with a phase transition between a homogeneous phase and an inhomogeneous phase, the technique of analyzing the last surviving patterns used here should also be useful. A more speculative comment of our results is that a society should provide the environment for nurturing cooperative individuals and they could become the seeds in building up a largely cooperative community. This could refer to different contexts, including the narrower scopes of online trading in which honesty (cooperative behavior) is needed for a trading platform to grow, and the broader scope of a harmonious society in general. We further checked that for systems in which the  $g = 1$  groups are forbidden, an AllC phase is not supported. Nonetheless, allowing a spread in group sizes during strategy updates enhances  $f_c(r)$  relative to a NSG with the same mean group size. This indicates that the existence of small groups in the population helps to sustain a relatively high cooperation level. The finding is consistent with that was observed in experiments involving PGG interaction [19].

Finally, the exactly solvable model and the analytic approach reported here shed light on how cooperation is promoted and an AllC phase is supported when the competing group size varies in  $g \in [1, g_m]$ . The model and the approach can be readily extended to include cases in which the time-varying group sizes following a nonuniform distribution. Indeed, the expression of  $r_c$  in the form of Eq. (19) can be proven to be valid for *nonuniform* group-size distributions. The latter case is important as, more often than not, dynamical grouping models show nonuniform group-size distributions. The analytic approach can be applied to extract the critical behavior of  $f_c$  near the critical point  $r_c$  for different values of  $g_m$ . It is also of interest to extend the study to the effects of varying group sizes in other  $N$ -person games. For instance, the analytic approach here can be applied to study dynamical groups with PGG-type interactions on questions such as how the group-size distribution would affect the cooperative behavior. An additional value of having an analytically solvable model in hand is that one can use it to gauge the effects of new features in the grouping, game, and updating mechanisms when they are added to the model. This is analogous to adding a perturbation to an exactly solvable problem in many areas in physics. In summary, we illustrated that allowing for the possibility of  $g = 1$  groups in the evolutionary process helps promote cooperation and induces an AllC phase within a model that can be studied analytically.

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